Entanglement generation in quantum thermal machines

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We show that in a linear quantum machine, a driven quantum system that evolves while coupled with thermal reservoirs, entanglement between the reservoir modes is unavoidably generated. This phenomenon, which occurs at sufficiently low temperatures and is at the heart of the third law of thermodynamics, is a consequence of a simple process: the transformation of the energy of the driving field into pairs of excitations in the reservoirs. For a driving with frequency ω_d we show entanglement exists between environmental modes whose frequencies satisfy the condition $\omega_i + \omega_j = \omega_d$. We show that this entanglement can persist for temperatures that can be significantly higher than the lowest achievable ones with sideband resolved cooling methods.

I. INTRODUCTION

Quantum thermodynamics [1–3] is an emerging field whose goal is to study the exchange of heat and work in the quantum domain. In recent years novel thermal machines operating at the atomic scale have been built using various technologies. These include, most notably, ion traps [4–6] and superconducting qubits [7]. Although a first-principle description of such devices needs to be based on quantum laws, a study of their performance revealed that they satisfy the same constraints imposed by classical thermodynamics (e.g., their efficiency is bounded by the Carnot limit) [8–14]. Although it is reassuring that classical results are reobtained from a quantum treatment, this naturally raises a troubling question: What is quantum in quantum thermodynamics [15]? More precisely, are there thermodynamical tasks that are possible (or impossible) because of quantum effects—or are there quantum effects that are unavoidably associated with thermodynamical cycles? Even though, in this context, the role of quantum coherences [16], correlations [17], and entanglement [18–20] as thermodynamical resources have been recently studied, in our opinion, the answers to the above questions are still inconclusive. Here we show that quantum entanglement plays a central role in thermodynamics: we prove that when work is performed on a system $\mathcal{S}$, which is in contact with reservoirs $\mathcal{E}_R$ and $\mathcal{E}_L$, time-extensive entanglement between environmental degrees of freedom is unavoidable at sufficiently low temperatures.

Hints about the existence of this entanglement, induced by the driving, were found in [21,22] while studying the nature of the heat flow between the environments. It was shown that in the stationary regime the energy stored in each environment varies due to two processes. First, the resonant absorption (or emission) from (or into) the driving field can transport an excitation in a mode with frequency ω_i to a mode with frequency $\omega_i + k\omega_d$, where ω_d is the driving frequency and k is an integer number. Naturally, this process is not present at zero temperature, where there are no excitations to be transported around. Thus, at very low temperature, the environmental energy varies because of a different process: the energy of the driving is dumped into two modes whose frequencies satisfy the condition $\omega_i + \omega_j = k\omega_d$. In [21,22] this was interpreted as the nonresonant creation of a pair of excitations, one in each mode, a process similar to the one underlying the dynamical Casimir effect [23,24]. However, the arguments in [21,22] are a conjecture and not a proof of the existence of entangled pairs. Here we present a rigorous proof of this fact by showing that the environmental modes become entangled below a certain temperature. Moreover, we show that entanglement can persist for temperatures which are higher than the lowest achievable ones by physically relevant cooling methods (analyzing both the resolved sideband and Doppler limits). Although our proof is restricted to linear quantum open systems [25] (see below), as the process responsible for the generation of entanglement is the same enforcing the validity of the third law of thermodynamics [21,22], it is natural to conjecture that our result is not a mere consequence of linearity but a general one.

The paper is organized as follows: In Sec. II we present our model, which is a generalization of the standard quantum Brownian motion (QBM) model including a time-dependent driving field, and present its formal solution. In Sec. III we show how to explicitly solve this model for the case of a periodic driving using Floquet theory. In Sec. IV we use the previous results to compute correlation functions between different environmental modes and we discuss their properties in the long-time regime. In Sec. V we study the entanglement between environmental modes at zero and nonzero environmental temperature. We provide a simple analytical expression to compute an entanglement measure (the logarithmic negativity) that shows entanglement is created at a constant speed (time extensivity) in a physically relevant range of parameters. We also provide a simple formula to

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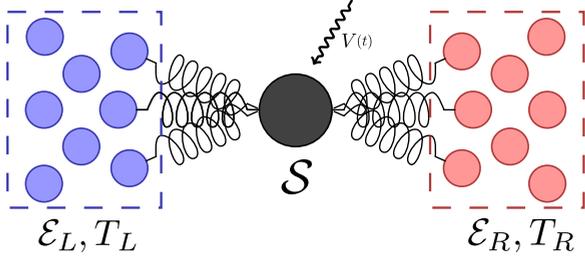
FIG. 1. A parametric oscillator $\mathcal{S}$ driven by $V(t)$ is coupled with two environments $\mathcal{E}_L$ and $\mathcal{E}_R$ at temperatures T_L and T_R, respectively.

predict the temperature at which entanglement vanishes due to thermal excitations. We summarize our results in Sec. VI.

II. THE MODEL

We consider the following generalization of the quantum Brownian motion model [26,27]: a system $\mathcal{S}$ (a parametric oscillator with coordinate x) is coupled with an environment $\mathcal{E}$ formed by independent oscillators (whose coordinates are denoted as q_i, with $i = 1, \ldots, N$). The environment is divided into two pieces $\mathcal{E}_\alpha$, where $\alpha = R, L$, each of which consists of the oscillators q_i with $i \in \mathcal{E}_\alpha$. Each $\mathcal{E}_\alpha$ is initially prepared in a thermal state with temperature T_α. This model represents the physical situation shown in Fig. 1.

Thus, our model describes the closed universe formed by the system $\mathcal{S}$ and the environment $\mathcal{E}$, whose dynamics is governed by the total Hamiltonian $H_T = H_{\mathcal{S}} + H_{\mathcal{E}} + H_{\mathcal{S},\mathcal{E}}$. Here, the system's Hamiltonian is $H_{\mathcal{S}} = p^2/2m + mV(t)x^2/2$ while the environmental and interaction terms are, respectively, $H_{\mathcal{E}} = \sum_i (p_i^2/2m_i + m_i\omega_i^2 q_i^2/2)$ and $H_{\mathcal{S},\mathcal{E}} = x \sum_i \lambda_i q_i$. An important feature of the environment is its spectral density $I(\omega) = \sum_i \lambda_i^2 \delta(\omega - \omega_i)/m_i\omega_i$ [the spectral density of each $\mathcal{E}_\alpha$ is denoted as $I_\alpha(\omega)$ and is defined in the same way, restricting the summation to $i \in \alpha$]. As we are interested in studying correlations between the environmental modes, we will solve this model in a way which is different from the one used in standard treatments of QBM [26]. Thus, we solve the full Heisenberg equations of motion of the coupled system, which read

$$\ddot{q}_i + \omega_i^2 q_i = -\lambda_i x/m_i, \quad i = 1, \ldots, N$$

$$\ddot{x} + V(t)x = -\sum_{i=1}^{N} \lambda_i q_i/m. \quad (1)$$

Interestingly, the above equations can be formally solved and the solutions expressed in terms of two decoupled sets of operators. One of them, which we denote $q_i^h(t)$, acts on the environmental state space, and the other one, which we denote $x_h(t)$, acts on the system state space. The operators $q_i^h(t)$ are the free Heisenberg operators of the environmental modes

$$q_i^h(t) = q_{i,0}\cos(\omega_i t) + p_{i,0}\sin(\omega_i t)/m_i\omega_i, \quad (2)$$

where $q_{i,0}$ and $p_{i,0}$ are Schrödinger operators. In turn, $x_h(t)$ is a dressed operator for $\mathcal{S}$ that satisfies the linear equation

$$\ddot{x}_h + V_R(t)x_h + \gamma * \dot{x}_h = 0, \quad (3)$$

where the notation $F * f = \int_0^t dt' F(t,t')f(t')$ is used. Above, γ is the dissipation kernel defined as $\gamma(t,t') =$

$\int d\omega\, I(\omega)\cos[\omega(t-t')]/m\omega$, and $V_R(t) = V(t) - \gamma(0)$ is the renormalized potential. The solution of the full Heisenberg equations (1) is

$$q_i = q_i^h + K_{ij}^{(1)} * q_j^h + K_i^{(2)} * x_h,$$

$$x = x_h + K_j^{(3)} * q_j^h, \quad (4)$$

where a summation over repeated indices is implicit. The kernel $K^{(1)}$, which plays a central role in our calculations, is

$$K_{ij}^{(1)}(t,t') = \lambda_i\lambda_j \int_{t'}^{t} d\tau\, \sin[\omega_i(t-\tau)]G(\tau,t')/mm_i\omega_i, \quad (5)$$

where G is the Green's function of Eq. (3) above. The other two kernels are

$$K_i^{(2)}(t,t') = -\lambda_i\sin[\omega_i(t-t')]/m_i\,\omega_i,$$

$$K_j^{(3)}(t,t') = -\lambda_j G(t,t')/m. \quad (6)$$

So far the expressions we obtained for $q(t)$ and $x(t)$ are exact. To compute them explicitly we need to solve the equation for G, a task that is feasible for certain spectral densities and for simple forms of $V(t)$. [In fact, once we do this we can express $x_h(t)$ as a linear combination of the Schrödinger operators x_0 and p_0]. In the case of constant $V(t)$, the expressions presented in Eq. (4) can be used to obtain all the known results for the standard QBM model, and also to compute environmental correlations. Below we will discuss the solution for a time-periodic driving field.

III. SOLUTION FOR PERIODIC DRIVING USING FLOQUET THEORY

In order to deal with the explicit time dependence induced by $V(t)$, we use Floquet theory. For a periodic driving with $V(t) = \sum_k V_k e^{ik\omega_d t}$, $G(t,t')$ can always be written as

$$G(t,t') = \sum_k A_k(t-t')e^{ik\omega_d t}, \quad (7)$$

where A_k vanishes for negative arguments. For G to be a Green's function of Eq. (3), A_k must satisfy a linear set of coupled differential equations. Instead of explicitly writing this system, we will simply present an equivalent one involving their Laplace transform $\tilde{A}_k(s) = \int_0^\infty dt\, A_k(t)e^{-st}$:

$$\tilde{g}^{-1}(s + ik\omega_d)\tilde{A}_k(s) + \sum_{n\neq 0} V_n\tilde{A}_{k-n}(s) = \delta_{k0}. \quad (8)$$

Here, $\tilde{g}$ is the Laplace transform of the static Green's function [obtained when $V(t) = V_0$], which is $\tilde{g}^{-1}(s) = s^2 + \omega_r^2 + s\tilde{\gamma}(s)$ [where $\omega_r^2 = V_0 - \gamma(0)$ is the renormalized frequency and $\tilde{\gamma}$ is the Laplace transform of γ]. The above system of equations can be simply solved by using a perturbative series expansion which is valid when the Fourier coefficients of the potential $|V_k|$ are small. The validity of this approximation also requires the frequency of the driving ω_d to be detuned from the parametric resonance. Thus, the solution of (8) satisfies the following recurrence relation:

$$A_k^{(m)}(s) = \tilde{g}(s + ik\omega_d)\left[\delta_{k0} - \sum_{n\neq 0} V_n\tilde{A}_{k-n}^{(m-1)}(s)\right], \quad (9)$$

for $m \geqslant 1$, with $A_k^{(0)}(s) = \tilde{g}(s + ik\omega_d)\delta_{k0}$. It is interesting to note that the coefficients $\tilde{A}_k$ satisfy certain properties which can be interpreted as a generalization of the static fluctuation-dissipation relation to the driven case. After some algebraic manipulations one can show that

$$\mathrm{Im}[\tilde{A}_0(i\omega)] = -\pi \sum_k I(\omega - k\omega_d)|\tilde{A}_k[i(\omega - k\omega_d)]|^2/2m. \tag{10}$$

In the static case, when $V_k = 0 \; \forall |k| \geqslant 1$, it reduces to

$$\mathrm{Im}[\tilde{g}(i\omega)] = -\pi I(\omega)|\tilde{g}(i\omega)|^2/2m, \tag{11}$$

which is the well-known expression for the fluctuation-dissipation relation in the absence of a driving field [26]. Below we will use the above equations to compute correlations between environmental modes. It is worth noticing that the generalized fluctuation-dissipation relation considerably simplifies the expressions for the correlators and determines some of their most interesting properties.

IV. CORRELATION FUNCTIONS BETWEEN ENVIRONMENTAL BANDS

We use Eqs. (4) to compute all correlation functions between two environmental bands: one of them consists of oscillators whose frequencies are distributed around ω_i (with a bandwidth $\Delta\omega$) and the other (disjoint) one is centered around ω_j. Expressions below will depend on the product $I(\omega)\Delta\omega$ which, for sufficiently small values of $\Delta\omega$, plays the role of an effective coupling strength between the reservoir band and $\mathcal{S}$ [since $I(\omega)\Delta\omega \approx \lambda^2/m\omega$]. We can show that the environmental correlators can be expressed as the sum of a term that depends on the initial state of $\mathcal{S}$ and another one that depends on the initial correlations within $\mathcal{E}$. In fact, when a stable stationary regime exists, the dependence on the initial conditions for x becomes irrelevant and the dynamics is dominated by the term involving the initial correlations within $\mathcal{E}$. The existence of such regime requires the energy pumped into $\mathcal{S}$ to be dissipated by $\mathcal{E}$, which can be achieved for small driving amplitudes, provided that ω_d is detuned from the parametric resonance. In Appendix A we include the general form of the equal time correlators $\langle\{q_i(t), q_j(t)\}\rangle$, both at zero and nonzero environmental temperatures which, together with the expression for the momentum and the cross correlators, will be used below to compute a measure of entanglement between environmental bands. Here, for the sake of simplicity, we show just an expression that enables us to discuss a feature that is common to all correlators which in a physically relevant regime become time extensive. Thus, we compute the energy E_i stored in the band ω_i, with $i \in \alpha$, which is the sum of the two diagonal correlators. It has the following form in the long-time limit:

$$E_i(t) \to [1/2 + n_\alpha(\omega_i)]\omega_i + \dot{\mathcal{Q}}_i \times t, \tag{12}$$

with

$$\frac{\dot{\mathcal{Q}}_i}{\Delta\omega} = \sum_k \sum_{\alpha'} \omega_i \big\{ \Theta(\omega_{i,k}) p_{\alpha,\alpha'}^{(k)}(\omega_i)[n_{\alpha'}(\omega_{i,k}) - n_\alpha(\omega_i)]$$
$$+ \Theta(-\omega_{i,k}) p_{\alpha,\alpha'}^{(k)}(\omega_i)[n_{\alpha'}(|\omega_{i,k}|) + n_\alpha(\omega_i) + 1] \big\}, \tag{13}$$
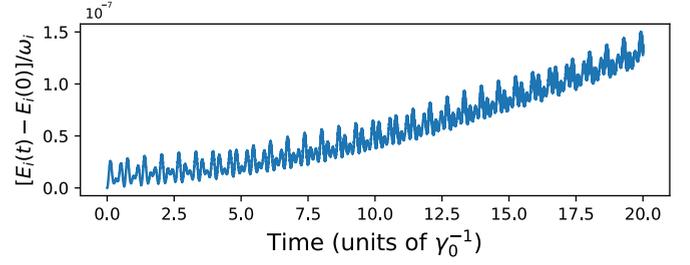
FIG. 2. Energy of the mode ω_i as a function of time at zero temperature, for $\omega_i = \omega_d - \delta$. Here, $\omega_d = \omega_r - \delta$ with $\delta = 10\gamma_0$, $\omega_r = 800\gamma_0$, $V = \omega_r^2/32$, $m = 10m_i$, $t = 20\gamma_0$, and $\gamma_0 = 0.005$.

where $\omega_{i,k} = \omega_i - k\omega_d$, Θ is the step function, $n_\alpha(\omega)$ is the Planck distribution with temperature T_α, and $p_{\alpha,\alpha'}^{(k)}(\omega_i) = \pi I_\alpha(\omega_i) I_{\alpha'}(|\omega_{i,k}|)|\tilde{A}_k(i\omega_{i,k})|^2/2m^2$ is the probability that mode ω_i in $\mathcal{E}_\alpha$ interacts with mode $\omega_{i,k}$ in $\mathcal{E}_{\alpha'}$ through $\mathcal{S}$. Remarkably, E_i is a linear function of time, a feature that is common to all correlators. This is really a consequence of the continuum hypothesis for $\mathcal{E}$ since any discrete environment has recurrence times $t_{\mathrm{rec}} \approx 1/\delta\omega$, where $\delta\omega$ is the smallest frequency splitting in $\mathcal{E}$. Therefore, although the above expression is valid for long times, it is not valid for arbitrarily long times. (The continuum hypothesis for arbitrarily long times, which is equivalent to assuming an infinite heat capacity for each band, yields nonphysical covariance matrices.) Technically, time extensivity is obtained in two steps: (i) we write all second order correlators as frequency integrals [using Eq. (4) and transforming the summations over discrete modes into integrals over ω, weighted by $I(\omega)$]; and (ii) we collapse the frequency integrals using the fact that terms involving the kernels $K_{ij}^{(1)}$ become highly peaked functions of ω in the long-time limit (that can be treated as Dirac δ functions until the time for which the discreteness of the environmental spectrum becomes relevant).

In Eq. (13) we can see the hints pointing towards the existence of entanglement that motivated our current study. The first term in its right-hand side involves the interaction of a mode ω_i and another one with frequency $\omega_i - k\omega_d$, and describes resonant transport (which can induce either heating or cooling of the modes, depending on the relation between their populations). To the contrary, the second term, which being positive is always associated with heating, involves the interaction between modes whose frequencies add up to a multiple of ω_d. This splitting of the driving energy between two modes motivated us to ask if entanglement is generated between them. We will show below that this is indeed the case. Before that, let us mention that our exact formula (which takes into account the initial transient and does not involve any time averaging) can be used to obtain Fig. 2, where we show the behavior of $E_i(t)$ for an Ohmic spectral density $I(\omega) = 2m\gamma_0\omega\Lambda^2/\pi(\omega^2 + \Lambda^2)$ and a harmonic driving $V_R(t) = \omega_r^2 + V\cos(\omega_d t)$. The plot (that corresponds to the case where $T_R = T_L = 0$) shows a linear behavior for long times, with a modulation with frequency ω_d.

In the next section we will discuss entanglement between environmental modes focusing on the long- (but not infinitely long) time limit, where time extensivity dominates.

V. ENTANGLEMENT BETWEEN ENVIRONMENTAL BANDS

Here we will study entanglement between two environmental bands which are respectively centered around frequencies ω_i and ω_j, with $i \in \mathcal{E}_R$ and $j \in \mathcal{E}_L$. For this we will compute a standard measure of entanglement, which is obtained from the reduced density matrix of the two modes i and j. We will use the logarithmic negativity $E_{\mathcal{N}}(t)$ which is a good measure of entanglement since the two-mode state in the stationary regime is Gaussian [28]. $E_{\mathcal{N}}(t)$ is obtained, as it is well known [28,29], from the smallest symplectic eigenvalue of the covariance matrix corresponding to the partially transposed reduced density matrix of i and j. We will obtain our results using two complementary methods. On the one hand, we will derive analytic expressions for $E_{\mathcal{N}}(t)$, which are valid in the limit of small driving and weak coupling. On the other hand, we will numerically evaluate our exact analytic expressions and compare them with the above approximation. In the following sections we will first discuss the case where the environmental temperatures vanish and show that entanglement exists only for modes satisfying the condition $\omega_i + \omega_j = \omega_d$. Then we will present the generalization of this result for the nonzero temperature case. In both cases we will analyze in detail the nature of the results for physically relevant ranges of parameters (ω_d, ω_r, and γ_0) describing the most interesting cases that correspond to the resolved sideband and Doppler limits (see below). In the last section we present a formula that allows us to compute the maximum temperature at which entanglement between the environmental bands is still present.

A. Entanglement at zero temperature

We compute the long-time limit of the average value of $E_{\mathcal{N}}(t)$ between two bands whose frequencies add up to a multiple of the driving frequency ω_d (where the average is taken over a driving cycle). Using the above expressions we show that

$$E_{\mathcal{N}}(t) = \Gamma_0 \times t, \tag{14}$$

where

$$\Gamma_0 = \frac{1}{m} \Delta\omega |V| \sqrt{I_R(\omega_i) I_L(\omega_j)} |\mathrm{Re}[\tilde{g}_i \tilde{g}_j^*]|, \tag{15}$$

with $\tilde{g}_l = \tilde{g}(i\omega_l)$. This formula, which is valid to leading order in V in the weak damping regime, shows that the driving creates entanglement at a constant speed. This is one of the central results of this paper. Instead, if $\omega_i + \omega_j = \omega_d + \epsilon$ with $\epsilon \ll \omega_d$, we find that $E_{\mathcal{N}}(t) \propto |\mathrm{sinc}(\epsilon t)| t$, which is a rapidly decaying function of ϵ whose amplitude does not grow with time. It is interesting to compare the time dependence of the entanglement with that of the energy. Although they are both linear in time they display significant differences. Thus, in the same regime described above, Eq. (12) can be rewritten as $E_i(t) = \omega_i/2 + \dot{\mathcal{Q}}_i \times t$ where $\dot{\mathcal{Q}}_i = \pi \omega_i \Delta\omega |V|^2 I_R(\omega_i) I(\omega_j) |\tilde{g}_i \tilde{g}_j^*|^2 / 2m^2$. Notably, the speed with which the energy grows is of second order in V. Instead, as the entanglement grows when cross correlations develop, Γ_0 turns out to be of first order in the driving amplitude. It is worth mentioning that $E_{\mathcal{N}}(t)$ is proportional to the
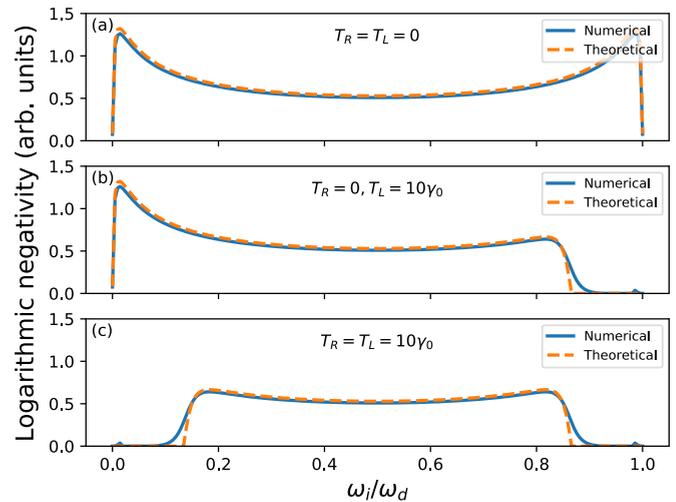
FIG. 3. Dependence of the logarithmic negativity $E_{\mathcal{N}}/E_0$ on the frequency ω_i in the resolved sideband limit when $\omega_i + \omega_j = \omega_d$. (a) When both environments are at zero temperature, two peaks and a broadband plateau are clearly visible. (b) When T_L is raised to $T_L = 10\gamma_0$, the peak at the right of the figure (that couples with low frequency oscillators in $\mathcal{E}_L$) disappears. (c) When $T_R = T_L = 10\gamma_0$, only the central plateau is left. In all cases the exact result (blue line) coincides with the analytical estimate given in (15) and (17). We used the parameters $\omega_d = \omega_r - \delta$ with $\delta = 10\gamma_0$, $\omega_r = 800\gamma_0$, $V = \omega_r^2/32$, $m = 10m_i$, $t = 20/\gamma$, and $\gamma_0 = 0.005$.

environmental bandwidth $\Delta\omega$. This is a natural result since entanglement is established between modes satisfying the matching condition $\omega_i + \omega_j = \omega_d$. Therefore, $E_{\mathcal{N}}(t)$ should be proportional to the number of entangled pairs, which is linear in $\Delta\omega$.

The amount of entanglement produced at a certain time depends on the frequencies of both modes. By analyzing the dependence of the entanglement (as measured by the logarithmic negativity) as a function of (ω_i, ω_j) we can understand the relevance of our result in the various regimes defined by the relation between ω_d, ω_r, and γ_0. For example, if $\omega_d = \omega_r - \delta$ with $\delta \gg \gamma_0$, a maximum of entanglement is expected for ω_i close to $\omega_r - \omega_d = \delta$. This is because oscillators in such band are in the "resolved sideband" limit as the decay rate is $\gamma_0 \ll \omega_i$ ($\omega_r \gg \gamma_0$ is implicit). In this regime the dominant process is the creation of a pair of phonons, one in the mode ω_i and another one in the mode $\omega_j = \omega_d - \omega_i = \omega_r - 2\delta$. This is the pair production mechanism that, as discussed in [22], sets the limit for the lowest achievable temperature in sideband resolved cooling methods. This intuitive picture is confirmed by an exact numerical evaluation of the entanglement, shown in Fig. 3(a) where these spectral peaks can be clearly seen. Dimensional analysis of Eq. (15) provides us with a natural unit of entanglement $E_0 = \gamma_0 \Delta\omega V/\omega_r^3$, which we use to normalize our plots. The agreement between the exact numerical evaluation and the above analytic estimate is remarkable.

In turn, when the value of δ is reduced so that $\delta \ll \gamma_0$, $E_{\mathcal{N}}$ has peaks at a band centered around $\omega_i \approx \gamma_0$, which is not in the resolved sideband regime (but, as discussed in [22], can be associated with the Doppler limit of laser cooling). We will present further details of the behavior of entanglement

in other regimes elsewhere [30] and focus here on studying the robustness of the entanglement in the nonzero temperature regime.

B. Entanglement at nonzero temperature

When the initial environmental temperatures are arbitrary, we can use our expressions to show that the logarithmic negativity is the maximum between zero and

$$E_{\mathcal{N}}(t) = -S_{i,j} + \Gamma_{\mathcal{N}} \times t, \quad (16)$$

where

$$\Gamma_{\mathcal{N}} = \Gamma_0 e^{-2S_{i,j}} \frac{\nu_{R,i} + \nu_{L,j}}{2} \left| \frac{\nu_{R,i}\tilde{g}_i\tilde{g}_j^* + \nu_{L,j}\tilde{g}_i^*\tilde{g}_j}{\tilde{g}_i\tilde{g}_j^* + \tilde{g}_i^*\tilde{g}_j} \right| \quad (17)$$

and $2S_{i,j} = \ln[(\nu_{R,i}^2 + \nu_{L,j}^2)/2]$, with $\nu_{\alpha,i} = 1 + 2n_\alpha(\omega_i)$ and Γ_0 defined in (15). Therefore, at nonzero temperature the entanglement is also created at a constant speed, which is smaller than that corresponding to the zero temperature case. Moreover, for any finite temperature there is a latency time t_{ent} until entanglement is generated. This is fixed by $S_{i,j}$ and is simply defined as $t_{\text{ent}} = S_{i,j}/\Gamma_{\mathcal{N}}$. Also, $S_{i,j}$, which is independent of the driving field, establishes a lower bound to the amount of entanglement that is unavoidably lost due to thermal fluctuations. Interestingly, it is fixed by the entropy of the environmental bands: For example, in the high temperature limit a lower bound for $S_{i,j}$ is simply given by the average Von Neumann entropy as $S_{i,j} \gtrsim \{\ln(n_R[\omega_i]) + \ln(n_L[\omega_j])\}/2$. Figures 3(b) and 3(c) show the dependence of $E_{\mathcal{N}}$ on frequency at nonzero temperature in the resolved sideband regime. Again, the agreement between the analytical estimate and the exact numerical result is quite remarkable.

C. Entanglement-breaking temperature

Entanglement can persist for a relevant range of temperatures. For example, when $T_R = T_L$ [the case shown in Fig. 3(c)], the entanglement persists up to temperatures which are 15 times higher than the lowest temperature that can be achieved using sideband resolved cooling methods. In fact, such methods can cool the band ω_i up to an occupation number $n_{\min} \approx (\gamma_0/2\omega_i)^2$ which, in our case corresponds to a temperature $T_{\min} \approx 1.7\gamma_0$ but we can show that entanglement persists up to $T \simeq 30\gamma_0$. On the other hand, when only the temperature T_R is zero, the entanglement can persist even for higher values of T_L. Thus, in this case [studied in Fig. 3(b), with $T_L = 10\gamma_0$] we found that the entanglement can persist for temperatures up to $T \approx 70\gamma_0$, which are 40 times higher than the lowest achievable one (notice that, in this case, the entanglement is lost first for high frequencies in $\mathcal{E}_R$, which

correspond to low frequencies in $\mathcal{E}_L$, as expected). It is possible to derive an analytic estimate for the temperature above which entanglement vanishes (the proof is shown in Appendix B). Thus, it turns out that entanglement persists only if

$$n_\alpha(\omega_i) < \frac{2}{\pi} \frac{\gamma_0}{2\omega_i} \frac{\Delta\omega}{\omega_i} \frac{m}{m_i}. \quad (18)$$

This simple formula, notably, accurately predicts the bounds mentioned above (and clearly implies that the temperatures above which entanglement is lost can be significantly higher than the minimum cooling temperatures achievable by sideband resolved methods).

VI. CONCLUSIONS

In this paper we studied the creation of quantum correlations between environmental modes in a generalization of the usual QBM model that includes a time-dependent driving enforced on the system $\mathcal{S}$. The method we used enabled us to solve the model focusing on the behavior of the environment $\mathcal{E}$ rather than on that of the driven system $\mathcal{S}$. Using our analytical expressions we computed a measure of the entanglement between environmental bands, centering our attention in the long-time regime. We showed that with a periodically driven system $\mathcal{S}$ with a frequency ω_d, entanglement between environmental modes satisfying the matching condition $\omega_i + \omega_j = \omega_d$ is always created at low temperatures. This implies that entanglement is in fact unavoidable for driven thermal machines operating at sufficiently low temperatures. We also showed that the entanglement can persist for temperatures which are significantly higher than the lowest achievable ones for realistic cooling methods. Interestingly, these quantum correlations are created by the same process enforcing the dynamical third law of thermodynamics in linear quantum refrigerators, namely, the pair creation induced by the driving. Our results are consistent with recent findings [31,32] that showed that the entropy production in the stationary regime of nonequilibrium systems is dominated by the creation of intraenvironment correlations. Notably, we showed that in a quantum thermodynamical system, such correlations have an intrinsic quantum nature in a relevant regime.

APPENDIX A: POSITION CORRELATION FUNCTIONS

Here we will present the general expression for the position correlation function for two modes $i \in \mathcal{E}_R$ and $j \in \mathcal{E}_L$. In order to obtain the correlation function for just one mode (e.g., $\langle\{q_i(t), q_i(t)\}\rangle$ instead of $\langle\{q_i(t), q_j(t)\}\rangle$), we just need to make the replacement $\{i, R\} \to \{j, L\}$, and use the generalized fluctuation-dissipation relation when possible. The correlator at zero temperature is

$$\langle\{q_i(t), q_j(t)\}\rangle = \frac{1}{m_i\omega_i}\delta_{ij} + \frac{1}{2m}\frac{1}{\sqrt{m_i\omega_i}}\frac{1}{\sqrt{m_j\omega_j}}\Delta\omega\sqrt{I_R(\omega_i)I_L(\omega_j)}\,\text{Im}\big[\mathcal{J}(\omega_i, \omega_j, t)e^{-i(\omega_i-\omega_j)t} - \mathcal{J}(\omega_i, -\omega_j, t)e^{-i(\omega_i+\omega_j)t}\big]$$

$$+ \frac{1}{2m}\frac{1}{\sqrt{m_i\omega_i}}\frac{1}{\sqrt{m_j\omega_j}}\Delta\omega\sqrt{I_R(\omega_i)I_L(\omega_j)}\,\text{Im}\big[\mathcal{J}(\omega_j, \omega_i, t)e^{i(\omega_i-\omega_j)t} - \mathcal{J}(\omega_j, -\omega_i, t)e^{-i(\omega_i+\omega_j)t}\big]$$

$$+ \frac{1}{\sqrt{m_i\omega_i}}\frac{1}{\sqrt{m_j\omega_j}}\Delta\omega\sqrt{I_R(\omega_i)I_L(\omega_j)}\int_0^t dt_1 \int_0^t dt_2 \sin[\omega_i(t-t_1)]\sin[\omega_j(t-t_2)]\langle\{x^h(t_1), x^h(t_2)\}\rangle$$

$$+ \frac{1}{4m^2} \frac{1}{\sqrt{m_i \omega_i}} \frac{1}{\sqrt{m_j \omega_j}} \Delta\omega \sqrt{I_R(\omega_i) I_L(\omega_j)} \sum_\alpha \int_0^\infty d\omega \, I_\alpha(\omega) \mathrm{Re}\big[\mathcal{J}(\omega, \omega_i, t) \mathcal{J}^*(\omega, \omega_j, t) e^{i(\omega_i - \omega_j)t}$$

$$+ \mathcal{J}(\omega, -\omega_i, t) \mathcal{J}^*(\omega, -\omega_j, t) e^{-i(\omega_i - \omega_j)t} - \mathcal{J}(\omega, \omega_i, t) \mathcal{J}^*(\omega, -\omega_j, t) e^{i(\omega_i + \omega_j)t}$$

$$- \mathcal{J}(\omega, -\omega_i, t) \mathcal{J}^*(\omega, \omega_j, t) e^{-i(\omega_i + \omega_j)t} \big], \tag{A1}$$

and at nonzero environmental temperature,

$$\langle \{ \hat{q}_i(t), \hat{q}_j(t) \} \rangle = \frac{1}{m_i \omega_i} [2n_R(\omega_i) + 1] \delta_{ij} + \frac{1}{2m} \frac{1}{\sqrt{m_i \omega_i}} \frac{1}{\sqrt{m_j \omega_j}} \Delta\omega \sqrt{I_R(\omega_i) I_L(\omega_j)} \, [2n_R(\omega_i) + 1]$$

$$\times \mathrm{Im}\big[\mathcal{J}(\omega_i, \omega_j, t) e^{-i(\omega_i - \omega_j)t} - \mathcal{J}(\omega_i, -\omega_j, t) e^{-i(\omega_i + \omega_j)t} \big]$$

$$+ \frac{1}{2m} \frac{1}{\sqrt{m_i \omega_i}} \frac{1}{\sqrt{m_j \omega_j}} \Delta\omega \sqrt{I_R(\omega_i) I_L(\omega_j)} \, [2n_L(\omega_j) + 1]$$

$$\times \mathrm{Im}\big[\mathcal{J}(\omega_j, \omega_i, t) e^{i(\omega_i - \omega_j)t} - \mathcal{J}(\omega_j, -\omega_i, t) e^{-i(\omega_i + \omega_j)t} \big]$$

$$+ \frac{1}{\sqrt{m_i \omega_i}} \frac{1}{\sqrt{m_j \omega_j}} \Delta\omega \sqrt{I_R(\omega_i) I_L(\omega_j)} \int_0^t dt_1 \int_0^t dt_2 \, \sin[\omega_i(t - t_1)] \sin[\omega_j(t - t_2)] \langle \{ x^h(t_1), x^h(t_2) \} \rangle$$

$$+ \frac{1}{4m^2} \frac{1}{\sqrt{m_i \omega_i}} \frac{1}{\sqrt{m_j \omega_j}} \Delta\omega \sqrt{I_R(\omega_i) I_L(\omega_j)} \sum_\alpha \int_0^\infty d\omega \, I_\alpha(\omega) [2n_\alpha(\omega) + 1]$$

$$\times \mathrm{Re}\big[\mathcal{J}(\omega, \omega_i, t) \mathcal{J}^*(\omega, \omega_j, t) e^{i(\omega_i - \omega_j)t} + \mathcal{J}(\omega, -\omega_i, t) \mathcal{J}^*(\omega, -\omega_j, t) e^{-i(\omega_i - \omega_j)t}$$

$$- \mathcal{J}(\omega, \omega_i, t) \mathcal{J}^*(\omega, -\omega_j, t) e^{i(\omega_i + \omega_j)t} - \mathcal{J}(\omega, -\omega_i, t) \mathcal{J}^*(\omega, \omega_j, t) e^{-i(\omega_i + \omega_j)t} \big], \tag{A2}$$

where the $\mathcal{J}$ function is defined as

$$\mathcal{J}(\omega, \omega_i, t) = \sum_k \int_0^t dt' e^{i(\omega - \omega_i + k\omega_d)t'} \int_0^{t'} dt'' A_k(t'') e^{-i\omega t''}, \tag{A3}$$

which can be formally solved as

$$\mathcal{J}(\omega, \omega_i, t) = \sum_k \big\{ t \, \mathrm{sinc}[(\omega - \omega_i + k\omega_d)t/2] a_k(i\omega)$$

$$\times \, e^{i(\omega - \omega_i + k\omega_d)t/2} + F_k(\omega, \omega_i) \big\}, \tag{A4}$$

where we used

$$a_k(i\omega) = \int_0^t dt' A_k(t') e^{-i\omega t'},$$

$$F_k(\omega, \omega_i) = \frac{a_k(i\omega) - a_k[i(\omega_i - k\omega_d)]}{i(\omega - \omega_i + k\omega_d)}. \tag{A5}$$

Note that F_k is always finite, and $a_k(i\omega) \to \tilde{A}_k(i\omega)$ in the long-time limit.

APPENDIX B: PROOF OF THE ENTANGLEMENT-BREAKING TEMPERATURE FORMULA

From now on we will consider two environmental bands $i \in \mathcal{E}_R$ and $j \in \mathcal{E}_L$ centered around frequencies ω_i and ω_j, respectively, such that $\omega_i + \omega_j = \omega_d$. We begin our proof by defining the dimensionless quantity

$$\varepsilon_i = \frac{\Delta\omega}{m_i \omega_i^3} I_R(\omega_i) \tag{B1}$$

for mode i, and the analog ε_j for mode j. We will work in the weak-coupling regime where $\varepsilon_{i,j} \ll 1$. Using this, and

remembering the canonical form of the two-mode covariance matrix

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}, \tag{B2}$$

the matrices α and β can be written as

$$\alpha = \nu_{R,i} \mathbb{1}/2 + \varepsilon_i \tilde{\alpha}, \quad \beta = \nu_{L,j} \mathbb{1}/2 + \varepsilon_j \tilde{\beta}. \tag{B3}$$

Now we will make our first approximation, which is $\varepsilon_i \simeq \varepsilon_j$. Although this is, in fact, an approximation, it is not too far off from reality: $\varepsilon_{i,j}$ is a measure of the coupling of the modes (i, j) with the system $\mathcal{S}$, which is roughly the same for entangled i and j modes. From now on we will simply write ε without a subindex. Under this approximation, we have for all matrices involved,

$$\alpha \simeq \nu_{R,i} \mathbb{1}/2 + \varepsilon\tilde{\alpha}, \quad \beta \simeq \nu_{L,j} \mathbb{1}/2 + \varepsilon\tilde{\beta}, \quad \gamma \simeq \epsilon\tilde{\gamma} \tag{B4}$$

and

$$\sigma \simeq \frac{1}{2} \begin{pmatrix} \nu_{R,i} & 0 \\ 0 & \nu_{L,j} \end{pmatrix} + \varepsilon \begin{pmatrix} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\gamma}^T & \tilde{\beta} \end{pmatrix}. \tag{B5}$$

Using the Caley-Hamilton theorem one can write the determinant of a matrix in terms of traces of powers of such matrix. For example, for α and β we have

$$\det(\alpha) \simeq \nu_{R,i}^2/4 + \varepsilon\nu_{R,i}\mathrm{tr}(\tilde{\alpha})/2 + \varepsilon^2 \, [\mathrm{tr}(\tilde{\alpha})^2 - \mathrm{tr}(\tilde{\alpha}^2)]/2,$$

$$\det(\beta) \simeq \nu_{L,j}^2/4 + \varepsilon\nu_{L,j}\mathrm{tr}(\tilde{\beta})/2 + \varepsilon^2 \, [\mathrm{tr}(\tilde{\beta})^2 - \mathrm{Tr}(\tilde{\beta}^2)]/2. \tag{B6}$$

In order for this expansion to be exact for σ, it should be up to order $O(\varepsilon^4)$. Here is where we will make our second approximation: we will expand $\det(\sigma)$ up to $O(\varepsilon^2)$, and assume the next two orders do not significantly contribute to our

computations. Thus, we write

$$\det(\sigma) \simeq \tfrac{1}{16}\nu_{R,i}^2\nu_{L,j}^2 + \tfrac{1}{8}\varepsilon\nu_{R,i}\nu_{L,j}[\nu_{R,i}\mathrm{tr}(\tilde{\alpha}) + \nu_{L,j}\mathrm{tr}(\tilde{\beta})]$$
$$+ \tfrac{1}{8}\varepsilon^2\nu_{L,j}^2[\mathrm{tr}(\tilde{\alpha})^2 - \mathrm{tr}(\tilde{\alpha}^2)]$$
$$+ \tfrac{1}{8}\varepsilon^2\nu_{R,i}^2[\mathrm{tr}(\tilde{\beta})^2 - \mathrm{tr}(\tilde{\beta}^2)]$$
$$+ \tfrac{1}{4}\varepsilon^2\nu_{R,i}\nu_{L,j}[\mathrm{tr}(\tilde{\alpha})\mathrm{tr}(\tilde{\beta}) - \mathrm{tr}(\tilde{\gamma}\tilde{\gamma}^T)]. \quad (\mathrm{B7})$$

For γ we make no approximations: $\det(\gamma) \simeq \varepsilon^2\det(\tilde{\gamma})$. If we analyze the smallest symplectic eigenvalue of the covariance matrix associated with the partially transposed reduced density matrix, we conclude modes i and j are entangled if, and only if, the following inequality holds true:

$$0 > \tfrac{1}{2}\left(\nu_{R,i}^2 - 1\right)\left(\nu_{L,j}^2 - 1\right) + \varepsilon(\nu_{R,i}\nu_{L,j} - 1)$$
$$\times [\nu_{R,i}\mathrm{tr}(\tilde{\alpha}) + \nu_{L,j}\mathrm{tr}(\tilde{\beta})]$$
$$+ \varepsilon^2\left(\nu_{L,j}^2 - 1\right)[\mathrm{tr}(\tilde{\alpha})^2 - \mathrm{tr}(\tilde{\alpha}^2)]$$
$$+ \varepsilon^2\left(\nu_{R,i}^2 - 1\right)[\mathrm{tr}(\tilde{\beta})^2 - \mathrm{tr}(\tilde{\beta}^2)]$$
$$+ \varepsilon^2\nu_{R,i}\nu_{L,j}[\mathrm{tr}(\tilde{\alpha})\mathrm{tr}(\tilde{\beta}) + 2\det(\tilde{\gamma}) - \mathrm{tr}(\tilde{\gamma}\tilde{\gamma}^T)]$$
$$- 2\varepsilon^2(\nu_{R,i}\nu_{L,j} - 1)\det(\tilde{\gamma}). \quad (\mathrm{B8})$$

Now let us suppose that $\nu_{R,i}^2 - 1 = 2n_R(\omega_i)$ and $\nu_{L,j}^2 - 1 = 2n_L(\omega_j)$ are of order ε^r with $r \in \mathbb{R} > 1$ to be fixed later. If this is the case, then we have $n_{R,L}(\omega_{i,j}) \simeq \epsilon^r/2 < \epsilon/2$. Rewriting the last equation, we obtain

$$0 > \varepsilon^{2r}/2 + \varepsilon^{1+r}(1 + \varepsilon^r/2)[\mathrm{tr}(\tilde{\alpha}) + \mathrm{tr}(\tilde{\beta})]$$
$$+ \varepsilon^{2+r}[\mathrm{tr}(\tilde{\alpha})^2 - \mathrm{tr}(\tilde{\alpha}^2)] + \varepsilon^{2+r}[\mathrm{tr}(\tilde{\beta})^2 - \mathrm{tr}(\tilde{\beta}^2)]$$
$$+ \varepsilon^2(1 + \varepsilon^r)[\mathrm{tr}(\tilde{\alpha})\mathrm{tr}(\tilde{\beta}) + 2\det(\tilde{\gamma}) - \mathrm{tr}(\tilde{\gamma}\tilde{\gamma}^T)]$$
$$- 2\varepsilon^{2+r}\det(\tilde{\gamma}). \quad (\mathrm{B9})$$

Since $r > 1$, we can throw away terms of order ε^{2r+1} and ε^{2+r} that are smaller than ε^3. Writing $r = 1 + \delta$ we get

$$0 > \varepsilon^{2\delta} + 2\varepsilon^{\delta}[\mathrm{tr}(\tilde{\alpha}) + \mathrm{tr}(\tilde{\beta})]$$
$$+ 2[\mathrm{tr}(\tilde{\alpha})\mathrm{tr}(\tilde{\beta}) + 2\det(\tilde{\gamma}) - \mathrm{tr}(\tilde{\gamma}\tilde{\gamma}^T)]. \quad (\mathrm{B10})$$

We want to prove that there is at least one pair of bands (i, j) that fulfill this inequality for any $\delta > 0$. Since bands i and j are entangled at zero environmental temperature (because $\omega_i + \omega_j = \omega_d$), we know that $0 > \mathrm{tr}(\tilde{\alpha})\mathrm{tr}(\tilde{\beta}) + 2\det(\tilde{\gamma}) - \mathrm{tr}(\tilde{\gamma}\tilde{\gamma}^T)$ holds (that is the condition for bands to be entangled at zero temperature). To prove (B10) still holds with the two extra terms, we will consider the case $\omega_i = \omega_j = \omega_d/2$. We choose this particular case because, when the environmental temperature is higher than zero, entanglement vanishes from the highest frequencies to the lowest ones. Therefore, the entanglement between these modes ($\omega_i = \omega_j = \omega_d/2$) is the last one to disappear, and this will give us an upper bound for the temperature at which entanglement vanishes for all bands. Thus, in this case, we have $\tilde{\alpha} = \tilde{\beta} = \tilde{\gamma}$, and (B10) becomes

$$0 > \varepsilon^{2\delta} + 4\varepsilon^{\delta}\mathrm{tr}(\tilde{\alpha}) + 8\det(\tilde{\alpha}). \quad (\mathrm{B11})$$

For long enough times, $\mathrm{tr}(\tilde{\alpha}) \sim \varepsilon t/\Delta\omega$, and $\det(\tilde{\alpha}) \sim -t^2$, and therefore the inequality is valid. In short, we proved that if the temperature T_R is such that

$$\boxed{n_R(\omega_i) < \tfrac{1}{2}\varepsilon = \frac{1}{2}\frac{\Delta\omega}{m_i\omega_i^3}I_R(\omega_i)} \quad (\mathrm{B12})$$

and the analog for $n_L(\omega_j)$ and T_L, then those modes are entangled for long enough times.

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