POLYAK'S THEOREM ON HILBERT SPACES

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Abstract. We extend to infinite dimensional Hilbert spaces a celebrated result, due to B. Polyak, about the convexity of the joint image of quadratic functions. We show sufficient conditions which assure that the joint image is also closed. However, we prove that the closedness part of Polyak's theorem does not hold in general in the infinite dimensional setting. Finally, we give some applications to S-lemma type results.

1. Introduction

In [17], Polyak extended a well-known theorem of Dines [8], by providing a convexity property related to non-homogeneous quadratic functions. Consider the functions

$$\phi_i(x) = \langle A_i x, x\rangle + \langle x, a_i\rangle + b_i,$$

where A_i is a $n \times n$ symmetric matrix, $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$ for $i = 1, 2$. Polyak's result [17] states that if $n \geq 2$ and there exists $(\mu_1, \mu_2) \in \mathbb{R}^2$ such $\mu_1 A_1 + \mu_2 A_2 > 0$ then the set

$$\{(\phi_1(x), \phi_2(x)) : x \in \mathbb{R}^n\}$$

is closed and convex. Here, the notation $\mu_1 A_1 + \mu_2 A_2 > 0$ means that the matrix $\mu_1 A_1 + \mu_2 A_2$ is positive definite. Polyak also proved that the joint image of three homogeneous quadratic forms in $\mathbb{R}^n$ is a closed and convex cone of $\mathbb{R}^3$ if and only if there is a positive definite linear combination of the operators determining the three quadratic forms.

In [2] an extension of Polyak's theorems to quadratic forms defined by compact operators on infinite dimensional separable Hilbert spaces was investigated. However, in [2, Theorems 2.1 and 2.3], some compact operators are assumed to be bounded below, so unfortunately,their main results are only applicable to finite dimensional spaces (see the comments after Corollary 2.2). Moreover, Example 2.3 shows that the joint image can be non-closed, even for quadratic functions determined by compact positive definite operators. This shows that additional hypothesis must be considered in order to prove the closedness part of Polyak's theorem.

In this work we extend Polyak's convexity result to an arbitrary infinite dimensional Hilbert space $\mathcal{H}$. Moreover, we show that if A_1 is a compact operator on $\mathcal{H}$ with 0 in its numerical range and A_2 is a positive invertible definite operator, then the joint image of two non necessarily homogeneous quadratic forms determined by A_1 and A_2, is also closed. We finish this work with some applications to S-lemma type results.

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2. Extension of Polyak's results: the homogeneous case

In this section we prove the convexity of the joint image of three homogeneous quadratic forms on a Hilbert space. Let us first introduce some notations.

Throughout $\mathcal{H}$ and $\mathcal{K}$ denote real inner product spaces. The range and nullspace of any given mapping A are denoted by $R(A)$ and $N(A)$, respectively. Also, $L(\mathcal{H},\mathcal{K})$ stands for the space of the bounded linear operators defined on $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{H}=\mathcal{K}$ we write, for short, $L(\mathcal{H})$. Given a linear operator T on $\mathcal{H}$ (possibly densely defined) we say that T is *positive definite* or $T>0$ if T is *symmetric* (i.e., $\langle Tx,y\rangle = \langle x,Ty\rangle$ for every x,y in the domain of T) and $\langle Tx,x\rangle > 0$ for every $x\neq 0$ in the domain of T. The group of invertible operators in $L(\mathcal{H})$ is denoted by $GL(\mathcal{H})$ and $GL(\mathcal{H})^+$ denotes the set of positive definite and invertible operators in $L(\mathcal{H})$. For a closed subspace $\mathcal{M}$, $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. Finally, $S_{\mathcal{H}}$ and $B_{\mathcal{H}}$ denote the unit sphere and the open unit ball of $\mathcal{H}$, respectively.

A key tool used in the proof of Polyak's theorems is a result on the joint real numerical range of real symmetric matrices due to Brickman [3]. This result can be seen as the real analogue of the classical Toeplitz-Hausdorff Theorem (and implies it, see e.g. [14]). Brickman's result was extended to infinite dimensional inner product spaces, [15, 13] (see also [12, Theorem 2]):

Theorem 2.1 (Brickman's convexity). *Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a real inner product space, $3\leq dim(\mathcal{H})\leq\infty$. Let A_1,A_2 be (not necessarily bounded) linear endomorphisms on $\mathcal{H}$. Then the set*

$$W_{\mathbb{R}}(A_1,A_2) := \{(\langle A_1x,x\rangle,\langle A_2x,x\rangle)\in\mathbb{R}^2 \,:\, \|x\|=1\}$$

is a convex subset of $\mathbb{R}^2$.

As a consequence of Brickman's convexity theorem, it is easy to show that a similar result holds considering two different inner products in $\mathcal{H}$. The following corollary will be useful to prove our convexity result (see Theorem 2.10). In order to include examples of densely defined unbounded operators (e.g. the differentiation operator on $L^2(\mathbb{R})$) we state the next corollary for linear mappings from an inner product space to its completion.

Corollary 2.2. *Let $\mathcal{H}$ be a real vector space and let $\langle\cdot,\cdot\rangle,\langle\cdot,\cdot\rangle_*$ be two inner products on $\mathcal{H}$ and $3\leq dim(\mathcal{H})\leq\infty$. Consider A_1,A_2 (not necessarily bounded) linear transformations from $\mathcal{H}$ to $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ denote the completion of $\mathcal{H}$ with respect to the inner product $\langle\cdot,\cdot\rangle$. Then the set*

$$\{(\langle A_1x,x\rangle,\langle A_2x,x\rangle)\in\mathbb{R}^2 \,:\, \|x\|_*=1\}$$

is a convex subset of $\mathbb{R}^2$, where $\|\cdot\|_*$ is the norm associated to the inner product $\langle\cdot,\cdot\rangle_*$.

Proof. As in the proof of [13, Theorem 2.2], we first consider $\mathcal{H}=\mathbb{R}^3$. In this case $\mathcal{H}=\tilde{\mathcal{H}}=\mathbb{R}^3$ and, since $\langle\cdot,\cdot\rangle$ is continuous on $(\mathbb{R}^3,\langle\cdot,\cdot\rangle_*)$, there exists $B\in L(\mathbb{R}^3)$ such that $\langle x,y\rangle=\langle Bx,y\rangle_*$ for every $x,y\in\mathbb{R}^3$. Thus, by Theorem 2.1 for $\mathcal{H}=\mathbb{R}^3$, the set

$$\{(\langle A_1x,x\rangle,\langle A_2x,x\rangle)\in\mathbb{R}^2 \,:\, \|x\|_*=1\} = \{(\langle BA_1x,x\rangle_*,\langle BA_2x,x\rangle_*)\in\mathbb{R}^2 \,:\, \|x\|_*=1\}$$

is convex.

Now suppose that $3\leq\dim(\mathcal{H})\leq\infty$.

Let $y_1 := (\langle A_1x_1,x_1\rangle,\langle A_2x_1,x_1\rangle)$ and $y_2 := (\langle A_1x_2,x_2\rangle,\langle A_2x_2,x_2\rangle)$, with $\|x_1\|_*=\|x_2\|_*=1$, be any two different points in $\{(\langle A_1x,x\rangle,\langle A_2x,x\rangle)\in\mathbb{R}^2 \,:\, \|x\|_*=1\}$. Take any orthonormal basis $\{w_1,w_2\}$

of the space $(\operatorname{span}\{x_1, x_2\}, \langle \cdot, \cdot \rangle)$ and take another vector w_3 such that $\langle w_3, w_1 \rangle = \langle w_3, w_2 \rangle = 0$ and $\|w_3\|_* = 1$. Set $W := \operatorname{span}\{w_1, w_2, w_3\}$ and consider the operators $\tilde{A}_l := P_W(A_l)|_W : W \to W$, for $l = 1, 2$. Then by the first part of the proof, $\{(\langle \tilde{A}_1 x, x\rangle, \langle \tilde{A}_2 x, x\rangle) : x \in W, \|x\|_* = 1\}$ is convex.

Moreover, since $\langle \tilde{A}_l x, x\rangle = \langle A_l x, x\rangle$ for any $x \in W$, we have that

$$\{(\langle \tilde{A}_1 x, x\rangle, \langle \tilde{A}_2 x, x\rangle) : x \in W, \|x\|_* = 1\} \subset \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle) : x \in \mathcal{H}, \|x\|_* = 1\}.$$

Finally, since $x_1, x_2 \in W$, we conclude that for every $\lambda \in [0, 1]$,

$$(1-\lambda)y_1 + \lambda y_2 \in \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle) \in \mathbb{R}^2 : \|x\|_* = 1\}.$$

□

In [2] the authors tried to extend Polyak's theorems to quadratic forms defined by compact operators on infinite dimensional separable real Hilbert spaces. For example, Theorem 2.1 in [2] was intended to show the closedness part of Polyak's theorem. There it is assumed that $A_1, A_2 \in L(\mathcal{H})$ are compact operators and that there exist scalars $\mu_1, \mu_2 \in \mathbb{R}$ such that $C := \mu_1 A_1 + \mu_2 A_2$ satisfies that for some $\alpha > 0$,

$$\langle Cx, x \rangle \geq \alpha \|x\|^2 \text{ for every } x \in \mathcal{H}. \tag{1}$$

The reader should be aware that in [2] an operator C satisfying (1) is denoted by $C > 0$. It is well known that there are no compact operators on infinite dimensional Hilbert spaces that satisfy (1). Indeed, consider $(x_n)_{n\geq 1} \subseteq \overline{B_{\mathcal{H}}}$ (the closed unit ball). Since $(x_n)_{n\geq 1}$ is bounded and $\overline{B_{\mathcal{H}}}$ is a closed subset of $\mathcal{H}$, then there exists a subsequence $(x_{n_k})_{k\geq 1} \subseteq \overline{B_{\mathcal{H}}}$ and $x_0 \in \overline{B_{\mathcal{H}}}$ such that $(x_{n_k})_{k\geq 1}$ converges weakly to x_0. Since C is compact, it follows that $\lim_{k\to\infty} \|Cx_{n_k} - Cx_0\| = 0$. Therefore

$$\|x_{n_k} - x_0\|^2 \leq \frac{1}{\alpha} \langle C(x_{n_k} - x_0), x_{n_k} - x_0 \rangle \leq \frac{1}{\alpha} \|Cx_{n_k} - Cx_0\| \|x_{n_k} - x_0\| \underset{k\to\infty}{\longrightarrow} 0.$$

Then $\overline{B_{\mathcal{H}}}$ is norm compact. Therefore $\mathcal{H}$ is finite dimensional.

The following examples show that the closedness part of Polyak's theorem does not hold neither for pairs of compact positive definite operators (Example 2.3) nor for pairs of bounded below operators (Example 2.4) on infinite dimensional spaces. Also, it is not difficult to extend both examples to k-tuples of operators.

Example 2.3. *Take any sequence $(\alpha_n)_n$ of positive real numbers converging to 0. Consider on ℓ_2 (the usual Hilbert space of square summable sequences with orthonormal basis $(e_n)_{n\in\mathbb{N}}$) a pair of diagonal operators defined by*

$$A_0(x) = (\alpha_n x_n)_n, \quad A_1(x) = \big(\alpha_n(1 + \frac{1}{n})x_n\big)_n.$$

Note that A_0, A_1 are both positive definite. Since $\alpha_n \to 0$, then A_0, A_1 can be uniformly approximated by finite range operators, so they are both compact operators. Moreover, note that for $j = 0, 1$,

$$\langle A_j \alpha_n^{-1/2} e_n, \alpha_n^{-1/2} e_n\rangle = \alpha_n^{-1}\langle A_j e_n, e_n\rangle = 1 + \frac{j}{n} \to 1, \quad \text{as } n \to \infty.$$

This means that $(1, 1)$ is in the closure of $\{(\langle A_0 x, x\rangle, \langle A_1 x, x\rangle) : x \in \ell_2\}$ in $\mathbb{R}^2$.

But on the other hand, $(1, 1) \notin \{(\langle A_0 x, x\rangle, \langle A_1 x, x\rangle) : x \in \ell_2\}$ because for any $x \neq 0$,

$$\langle A_0 x, x\rangle = \sum_n \alpha_n x_n^2 < \sum_n \alpha_n (1 + \frac{1}{n}) x_n^2 = \langle A_1 x, x\rangle.$$

Therefore the image of the quadratic form determined by A_0, A_1 is not closed.

Example 2.4. *Take any sequence $(\alpha_n)_n$ of positive real numbers converging to $\alpha > 0$. As in Example 2.3, let A_0, A_1 be operators on $\mathcal{H} = \ell_2$ defined by,*

$$A_0(x) = (\alpha_n x_n)_n, \quad A_1(x) = \big(\alpha_n(1 + \frac{1}{n})x_n\big)_n.$$

Then $A_0, A_1 \in GL(\mathcal{H})^+$ because $\alpha > 0$ (in particular, both operators satisfy (1)). Moreover, proceeding as in the previous example, it follows that $(1,1)$ is in the closure of the image $\{(\langle A_0x, x\rangle, \langle A_1x, x\rangle) : x \in \ell_2\}$ in $\mathbb{R}^2$, but not in the image of the quadratic form determined by A_0, A_1.

Remark 2.5. It is known that the numerical range of a compact operator is not necessarily closed on infinite dimensional Hilbert spaces: take for example on ℓ_2 the operator $(x_n)_n \mapsto (\frac{x_n}{n})_n$, then the numerical range is $(0,1]$ (see [11, Problem 212]). Thus, the image of the unit sphere by pairs of quadratic forms (i.e. the joint numerical range) is not closed in general for infinite dimensional spaces, even for compact operators. On the other hand, since a quadratic form determined by a compact operator is weakly continuous on bounded sets, and the closed unit ball is weakly compact, we immediately conclude the following: given $\{A_1, \cdots, A_n\}$, any collection of compact operators, the set

$$\{(\langle A_1x, x\rangle, \cdots, \langle A_nx, x\rangle) \in \mathbb{R}^n : \|x\| \leq 1\}$$

is closed.

Next, we give some conditions under which the joint image of three quadratic forms is closed and convex. First we need the following lemma, which is an extension of a result in [6] and shows, using the same ideas, that under certain conditions the joint numerical range of compact operators on real Hilbert spaces is closed.

Lemma 2.6. *Consider A_1, A_2 compact selfadjoint operators on a real Hilbert space $\mathcal{H}$. Suppose that $(0,0) \in W_{\mathbb{R}}(A_1, A_2)$ then $W_{\mathbb{R}}(A_1, A_2)$ is closed.*

Proof. Let $\lambda \in \overline{W_{\mathbb{R}}(A_1, A_2)}$. Since the closed ball is weakly compact then

$$\lambda = \lim_{\alpha}(\langle A_1x_\alpha, x_\alpha\rangle, \langle A_2x_\alpha, x_\alpha\rangle)$$

for some net $(x_\alpha)_\alpha$ with $\|x_\alpha\| = 1$ weakly convergent to some x with $\|x\| \leq 1$. Moreover, since the operators A_1, A_2 are compact, it is easy to see that $\lambda = (\langle A_1x, x\rangle, \langle A_2x, x\rangle)$. If $\lambda = (0,0)$ there is nothing to prove. Otherwise, $x \neq 0$, and thus $\frac{\lambda}{\|x\|^2}$ belongs to $W_{\mathbb{R}}(A_1, A_2)$. Finally, since $\|x\| \leq 1$ and $(0,0) \in W_{\mathbb{R}}(A_1, A_2)$, we conclude that $\lambda \in W_{\mathbb{R}}(A_1, A_2)$ by Theorem 2.1. □ □

Theorem 2.7. *Let $F(x) = (\langle A_1x, x\rangle, \langle A_2x, x\rangle, \langle A_3x, x\rangle)$ be a quadratic mapping determined by bounded operators A_1, A_2, A_3 on a real Hilbert space $\mathcal{H}$. Suppose that there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that $\mu_1A_1 + \mu_2A_2 + \mu_3A_3 \in GL(\mathcal{H})^+$, A_1, A_2 are compact and $(0,0) \in W_{\mathbb{R}}(A_1, A_2)$. Then $F(\mathcal{H})$ is closed.*

Proof. We may assume that A_1, A_2, A_3 are selfadjoint and we assume that $\mathcal{H}$ is infinite dimensional because the finite dimensional case was proved by Polyak [17, Theorem 2.1].

We first assert that it is sufficient to prove the case when A_1, A_2 are compact and $A_3 = I$. In fact, consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(r, s, t) = (r, s, \mu_1 r + \mu_2 s + \mu_3 t)$. Since $\mu_3 \neq 0$, T is invertible and preserve closedness. Then it suffices to prove that

$$T(F(\mathcal{H})) = \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \langle \tilde{A}_3 x, x\rangle) \in \mathbb{R}^3 \,:\, x \in \mathcal{H}\}$$

is closed, where $\tilde{A}_3 := \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$.

Since $\tilde{A}_3 \in GL(\mathcal{H})^+$, the inner product $\langle x, y\rangle_* := \langle \tilde{A}_3 x, y\rangle$ makes $(\mathcal{H}, \langle\cdot,\cdot\rangle_*)$ a Hilbert space. Denote by $\|\cdot\|_*$ the induced norm, which is equivalent to $\|\cdot\|$.

Then $\langle x, y \rangle = \left\langle \tilde{A}_3 \tilde{A}_3^{-1} x, y \right\rangle = \langle \tilde{A}_3^{-1} x, y\rangle_*$, for every $x, y \in \mathcal{H}$, and

$$T(F(\mathcal{H})) = \{(\langle \tilde{A}_3^{-1} A_1 x, x\rangle_*, \langle \tilde{A}_3^{-1} A_2 x, x\rangle_*, \|x\|_*^2) \in \mathbb{R}^3 \,:\, x \in \mathcal{H}\}.$$

Finally note that, in $(\mathcal{H}, \langle\cdot,\cdot\rangle_*)$, we have that $\tilde{A}_3^{-1} A_1, \tilde{A}_3^{-1} A_2$ are compact operators and $(0, 0) \in W_{\mathbb{R}}(\tilde{A}_3^{-1} A_1, \tilde{A}_3^{-1} A_2)$.

Suppose then that $A_3 = I$ and take $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \overline{F(\mathcal{H})}$. Then

$$\lambda = \lim_n F(x_n) = \lim_n (\langle A_1 x_n, x_n\rangle, \langle A_2 x_n, x_n\rangle, \|x_n\|^2)$$

for some sequence $(x_n)_n \subseteq \mathcal{H}$. If $\lambda_3 = 0$ then $0 = \lambda_3 = \lim_n \|x_n\|^2$. So that $\lambda = 0 \in F(\mathcal{H})$.

If $\lambda_3 \neq 0$, then $\lambda_3 = \lim_n \|x_n\|^2$. Therefore,

$$\lim_n \langle A_j \frac{x_n}{\|x_n\|}, \frac{x_n}{\|x_n\|}\rangle = \frac{\lambda_j}{\lambda_3} \text{ for } j = 1, 2.$$

Then $(\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}, 1) \in \overline{F(S_{\mathcal{H}})}$ and $(\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}) \in \overline{W_{\mathbb{R}}(A_1, A_2)} = W_{\mathbb{R}}(A_1, A_2)$, where we used Lemma 2.6. Hence, there exists $z \in S_{\mathcal{H}}$ such that

$$(\frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_3}) = (\langle A_1 z, z\rangle, \langle A_2 z, z\rangle).$$

Then $F(\lambda_3^{1/2} z) = (\lambda_3 \langle A_1 z, z\rangle, \lambda_3 \langle A_2 z, z\rangle, \lambda_3) = \lambda$, so that $\lambda \in F(\mathcal{H})$. □ □

Remark 2.8. Modifying Example 2.3, it can be seen that the assumption $(0, 0) \in W_{\mathbb{R}}(A_1, A_2)$ cannot be dropped in the above theorem. Indeed, take $A_j(x) = (\frac{j}{n} x_n)_n$, for $j = 1, 2$, $A_3 = I$. Then $(0, 0, 1) = \lim_n F(e_n)$ is in $\overline{F(\mathcal{H})}$ but not in $F(\mathcal{H})$.

With a similar proof we may show the following more general result.

Corollary 2.9. *Let $F(x) = (\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \langle A_3 x, x\rangle)$ be a quadratic mapping determined by operators A_1, A_2, A_3 on a real Hilbert space $\mathcal{H}$. Suppose that there are linear combinations*

$$\tilde{A}_i := \mu_{i1} A_1 + \mu_{i2} A_2 + \mu_{i3} A_3 \textit{ for } i = 1, 2, 3$$

such that the 3×3 matrix of real numbers $\mu = (\mu_{ij})_{i,j=1}^3$ is not singular, $\tilde{A}_3 \in GL(\mathcal{H})^+$, $\tilde{A}_1, \tilde{A}_2$ are compact and $(0, 0) \in W_{\mathbb{R}}(\tilde{A}_1, \tilde{A}_2)$. Then $F(\mathcal{H})$ is closed.

We prove now the extension of Polyak convexity theorem [17, Theorem 2.1] to not necessarily bounded linear operators on inner product spaces.

Theorem 2.10. *Let $\mathcal{H}$ be a real inner product space, $3 \leq dim(\mathcal{H}) \leq \infty$. Let A_1, A_2, A_3 be linear transformations from $\mathcal{H}$ to its completion $\tilde{\mathcal{H}}$ such that there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ with $\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 > 0$. Then the set*

$$F(\mathcal{H}) = \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \langle A_3 x, x\rangle) \in \mathbb{R}^3 \,:\, x \in \mathcal{H}\}$$

is a convex cone in $\mathbb{R}^3$.

Proof. We may suppose that $\mu_3 \neq 0$ (otherwise we interchange the order of the operators).

As in the proof of Theorem 2.7, consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(r,s,t) = (r, s, \mu_1 r + \mu_2 s + \mu_3 t)$. Then T is invertible and preserves convexity. Therefore it suffices to prove that

$$T(F(\mathcal{H})) = \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \langle \tilde{A}_3 x, x\rangle) \in \mathbb{R}^3 \,:\, x \in \mathcal{H}\}$$

is convex, where $\tilde{A}_3 := \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3$. Since $\tilde{A}_3 > 0$, the bilinear form $\langle \cdot, \cdot \rangle_* := \langle \tilde{A}_3 \cdot, \cdot\rangle$ makes $\mathcal{H}_* := (\mathcal{H}, \langle \cdot, \cdot\rangle_*)$ an inner product space with norm denoted by $\|\cdot\|_*$. Then

$$T(F(\mathcal{H})) = \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \|x\|_*^2) \in \mathbb{R}^3 \,:\, x \in \mathcal{H}\}.$$

By Corollary 2.2, the set $\{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \|x\|_*^2) \in \mathbb{R}^3 \,:\, \|x\|_* = 1\} = T(F(S_{\mathcal{H}_*}))$ is convex. Hence by homogeneity, $T(F(\mathcal{H}))$ is a convex cone because

$$T(F(\mathcal{H})) = \bigcup_{t \geq 0} t \cdot \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \|x\|_*^2) \in \mathbb{R}^3 \,:\, \|x\|_* = 1\}.$$

□ □

3. The non-homogeneous case

Using the closedness of the joint image of a pair of non necessarily homogeneous quadratic forms, it was proved in [2, Theorem 2.2] that this image is also convex. We will now prove Polyak's theorem for non-homogeneous quadratic forms without assuming that it is closed.

Proposition 3.1. *Let* $\mathcal{H}$ *be a real inner product space,* $3 \leq dim(\mathcal{H}) \leq \infty$. *Let* $A_1, A_2 \in L(\mathcal{H})$ *be such that* $\mu_1 A_1 + \mu_2 A_2 > 0$ *for some* $\mu_1, \mu_2 \in \mathbb{R}$, $a_1, a_2 \in \mathcal{H}$ *and* $b_1, b_2 \in \mathbb{R}$. *Let* $\Phi = (\phi_1, \phi_2)$ *be the non-homogeneous quadratic form defined by* $\phi_j(x) = \langle A_j x, x\rangle + \langle x, a_j\rangle + b_j$, $j = 1, 2$. *Then*

$$\Phi(\mathcal{H}) = \{(\phi_1(x), \phi_2(x)) \in \mathbb{R}^2 \,:\, x \in \mathcal{H}\}$$

is convex.

Proof. Let $t, s \in \Phi(\mathcal{H})$, with $t \neq s$, then there exist $x, y \in \mathcal{H}$ such that

$$t = \Phi(x) \text{ and } s = \Phi(y).$$

Consider $\tilde{\mathcal{H}} := \operatorname{span}\{w, x, y\}$, where $w \in \mathcal{H}$ is linearly independent to x and y. Note that $2 \leq dim(\tilde{\mathcal{H}}) \leq 3$. Let $\langle \cdot, \cdot\rangle_{\tilde{\mathcal{H}}}$ be the restriction of $\langle \cdot, \cdot\rangle$ to $\tilde{\mathcal{H}}$. Let $P_{\tilde{\mathcal{H}}}$ denote the orthogonal projection onto the finite dimensional Hilbert space $\tilde{\mathcal{H}}$. Set $\tilde{\Phi} := \Phi|_{\tilde{\mathcal{H}}} = (\tilde{\phi_1}, \tilde{\phi_2})$ where $\tilde{\phi_j} := \phi_j|_{\tilde{\mathcal{H}}}$ for $j = 1, 2$. Then, $\tilde{\phi_j} : \tilde{\mathcal{H}} \to \mathbb{R}$, $t = \tilde{\Phi}(x)$, $s = \tilde{\Phi}(y)$ and, for $z \in \tilde{\mathcal{H}}$,

$$\begin{aligned}
\tilde{\phi_j}(z) &= \langle A_j z, z \rangle + \langle a_j, z \rangle + b_j \\
&= \langle A_j|_{\tilde{\mathcal{H}}} z, P_{\tilde{\mathcal{H}}} z \rangle + \langle a_j, P_{\tilde{\mathcal{H}}} z \rangle + b_j \\
&= \langle P_{\tilde{\mathcal{H}}} A_j|_{\tilde{\mathcal{H}}} z, z \rangle_{\tilde{\mathcal{H}}} + \langle P_{\tilde{\mathcal{H}}} a_j, z \rangle_{\tilde{\mathcal{H}}} + b_j.
\end{aligned}$$

Let $\tilde{A}_j := P_{\tilde{\mathcal{H}}} A_j|_{\tilde{\mathcal{H}}}$ for $j = 1, 2$. Then $\mu_1 \tilde{A}_1 + \mu_2 \tilde{A}_2 > 0$. In fact, for $z \in \tilde{\mathcal{H}}$ we have

$$\left\langle (\mu_1 \tilde{A}_1 + \mu_2 \tilde{A}_2) z, z \right\rangle_{\tilde{\mathcal{H}}} = \langle (\mu_1 P_{\tilde{\mathcal{H}}} A_1|_{\tilde{\mathcal{H}}} + \mu_2 P_{\tilde{\mathcal{H}}} A_2|_{\tilde{\mathcal{H}}}) z, z \rangle = \langle (\mu_1 A_1 + \mu_2 A_2) z, z \rangle > 0.$$

Then, by Polyak's Theorem, $\tilde{\Phi}(\tilde{\mathcal{H}})$ is a convex set. Therefore, for every $\alpha \in [0,1]$,

$$\alpha t + (1-\alpha)s \in \tilde{\Phi}(\tilde{\mathcal{H}}) \subseteq \Phi(\mathcal{H}).$$

Hence $\Phi(\mathcal{H})$ is a convex set. □ □

Remark 3.2. We may actually prove the convexity of the image of Φ under the hypothesis of A_1, A_2 being non-degenerate (that is, if $\langle A_1 u, u \rangle = 0 = \langle A_2 u, u \rangle$ then $u = 0$). For infinite dimensional Hilbert spaces, this is a strictly weaker assumption, see e.g. [4].

Proof. Using the notation as in the proof of Proposition 3.1, it is clear that $\tilde{A}_1, \tilde{A}_2$ is a non-degenerate pair. If the 2-homogeneous part of $\tilde{\Phi}$ is not surjective, then by [8, Corollary 1], there are $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_1 \tilde{A}_1 + \mu_2 \tilde{A}_2 > 0$. Then, by Polyak's Theorem, $\tilde{\Phi}(\tilde{\mathcal{H}})$ is a convex set.

On the contrary, if the 2-homogeneous part of $\tilde{\Phi}$ is surjective, then by [10, Lemma 4.10], $\tilde{\Phi}(\tilde{\mathcal{H}}) = \mathbb{R}^2$. Therefore, $\tilde{\Phi}(\tilde{\mathcal{H}})$ is a convex set. Then, for every $\alpha \in [0,1]$, Therefore $\hat{A}_1$ and $\hat{A}_3$ are compact operators and $(0,0) \in W_{\mathbb{R}}(\hat{A}_1, \hat{A}_3)$. Hence $\Phi(\mathcal{H})$ is a convex set. □ □

Proposition 3.3. *Let $\mathcal{H}$ be a real Hilbert space, $3 \le dim(\mathcal{H}) \le \infty$. Let $A_1, A_2 \in L(\mathcal{H})$ be selfadjoint operators, $a_1, a_2 \in \mathcal{H}$ and $b_1, b_2 \in \mathbb{R}$. Let $\Phi = (\phi_1, \phi_2)$ be the non-homogeneous quadratic form defined by $\phi_j(x) = \langle A_j x, x \rangle + \langle x, a_j \rangle + b_j$, $j = 1, 2$. Suppose that there are linear combinations*

$$\tilde{A}_1 := \alpha_1 A_1 + \alpha_2 A_2, \quad \tilde{A}_2 := \beta_1 A_1 + \beta_2 A_2$$

such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, $\tilde{A}_1$ is compact, $\langle \tilde{A}_1 x, x \rangle = 0$ for some $x \neq 0$ and $\tilde{A}_2 \in GL(\mathcal{H})^+$. Then $\Phi(\mathcal{H})$ is convex and closed.

In particular if 0 is in the numerical range of A_1, A_1 is compact and $A_2 \in GL(\mathcal{H})^+$. Then $\Phi(\mathcal{H})$ is convex and closed.

Proof. Since $\tilde{A}_2 \in GL(\mathcal{H})^+$, by Proposition 3.1, $\Phi(\mathcal{H})$ is convex.

Now we are going to show that $\Phi(\mathcal{H})$ is closed. As in the proof of Theorem 2.7, consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(r,t) = (\alpha_1 r + \alpha_2 t, \beta_1 r + \beta_2 t)$. Since $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, T is invertible and preserves closedness. Therefore it suffices to prove that

$$T(\Phi(\mathcal{H})) = \{(\langle \tilde{A}_1 x, x\rangle + \langle x, \tilde{a}_1\rangle + \tilde{b}_1, \langle \tilde{A}_2 x, x\rangle + \langle x, \tilde{a}_2\rangle + \tilde{b}_2) \in \mathbb{R}^2 \,:\, x \in \mathcal{H}\}$$

is closed, where $\tilde{A}_1 := \alpha_1 A_1 + \alpha_2 A_2$ is compact, $0 = \langle \tilde{A}_1 x, x\rangle$ for some $x \in S_{\mathcal{H}}$, $\tilde{A}_2 := \beta_1 A_1 + \beta_2 A_2 \in GL(\mathcal{H})^+$, $\tilde{a}_1 := \alpha_1 a_1 + \alpha_2 a_2$, $\tilde{a}_2 := \beta_1 a_1 + \beta_2 a_2$, $\tilde{b}_1 := \alpha_1 b_1 + \alpha_2 b_2$ and $\tilde{b}_2 := \beta_1 b_1 + \beta_2 b_2$.

Let $\tilde{\mathcal{H}} := \mathcal{H} \times \mathbb{R}$ and define the following 2-homogeneous forms on $\tilde{\mathcal{H}}$:

$$\begin{aligned} f_j(x,t) =& \langle \tilde{A}_j x, x\rangle + t\langle x, \tilde{a}_j\rangle + t^2 \tilde{b}_j, \qquad j = 1,2 \\ f_3(x,t) =& t^2. \end{aligned}$$

Then, the homogeneous quadratic form f_j is determined by the selfadjoint operators

$$\hat{A}_j := \begin{pmatrix} \tilde{A}_j & \frac{\tilde{a}_j}{2} \\ \langle \cdot, \frac{\tilde{a}_j}{2} \rangle & \tilde{b}_j \end{pmatrix} \text{ for } j = 1,2 \text{ and } \hat{A}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $\left\langle \hat{A}_3 \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} \right\rangle = t^2 = f_3(x,t)$, and for $j = 1, 2$,

$$\left\langle \hat{A}_j \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \tilde{A}_j x + t\frac{\tilde{a}_j}{2} \\ \langle x, \frac{\tilde{a}_j}{2}\rangle + t\tilde{b}_j \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} \right\rangle = \langle \tilde{A}_j x, x\rangle + t\langle x, \tilde{a}_j\rangle + t^2\tilde{b}_j = f_j(x,t).$$

Also, $\hat{A}_1$ and $\hat{A}_3$ are compact operators and $(0,0) \in W_{\mathbb{R}}(\hat{A}_1, \hat{A}_3)$.

Let $\mu_3 \in \mathbb{R}$ be such that

$$\mu_3 > \|\tilde{A}_2^{-1/2}\frac{\tilde{a}_2}{2}\|^2 - \tilde{b}_2$$

then $\hat{A}_2 + \mu_3\hat{A}_3 \in GL(\tilde{\mathcal{H}})^+$. In fact,

$$Z := \hat{A}_2 + \mu_3\hat{A}_3 = \begin{pmatrix} \tilde{A}_2 & \frac{\tilde{a}_2}{2} \\ \langle \cdot, \frac{\tilde{a}_2}{2}\rangle & \tilde{b}_2 + \mu_3 \end{pmatrix} = \begin{pmatrix} \tilde{A}_2 & d \\ d^* & \tilde{b}_2 + \mu_3 \end{pmatrix},$$

where $d : \mathbb{R} \to \mathcal{H}$ is the operator defined by $d(t) := t\frac{\tilde{a}_2}{2}$. Then $d^* = \left\langle \cdot, \frac{\tilde{a}_2}{2} \right\rangle$, $d = \tilde{A}_2^{1/2}(\tilde{A}_2^{-1/2}d)$ and $g := \tilde{A}_2^{-1/2}d$ is the (reduced) solution of the equation $d = \tilde{A}_2^{1/2}z$, see [9]. Then

$$g^*g = d^*\tilde{A}_2^{-1}d = \|\tilde{A}_2^{-1/2}\frac{\tilde{a}_2}{2}\|^2.$$

Hence, $\tilde{b}_2+\mu_3 = g^*g+t$ with $t := \tilde{b}_2+\mu_3-\|\tilde{A}_2^{-1/2}\frac{\tilde{a}_2}{2}\|^2 > 0$. Then, by [1, Theorem 3], $Z = \hat{A}_2+\mu_3\hat{A}_3 \geq 0$. Also, $z := \tilde{b}_2 + \mu_3 - g^*g = \tilde{b}_2 + \mu_3 - d^*\tilde{A}_2^{-1}d = t > 0$. Then $z^{-1} = (\tilde{b}_2 + \mu_3 - d^*\tilde{A}_2^{-1}d)^{-1} \in \mathbb{R}$ and, it can be checked that

$$Z^{-1} = \begin{pmatrix} \tilde{A}_2^{-1} + \tilde{A}_2^{-1}dz^{-1}d^*\tilde{A}_2^{-1} & -\tilde{A}_2^{-1}dz^{-1} \\ -z^{-1}d^*\tilde{A}_2^{-1} & z^{-1} \end{pmatrix} \in L(\tilde{\mathcal{H}}).$$

Therefore $Z = \hat{A}_2 + \mu_3\hat{A}_3 \in GL(\tilde{\mathcal{H}})^+$.

Set $F := (f_1, f_2, f_3)$. Then, by Theorem 2.7, $F(\tilde{\mathcal{H}})$ is closed. Then

$$F(\tilde{\mathcal{H}}) \cap \{(a,b,c) \in \mathbb{R}^3 : c = 1\} = F(\mathcal{H} \times \{-1,1\}) = F(\mathcal{H} \times \{1\}),$$

where we used that $F(x,-1) = F(-x,1)$ for every $x \in \mathcal{H}$. Therefore, the set $F(\mathcal{H}\times\{1\})$ is closed because the set $\{(a,b,c) \in \mathbb{R}^3 : c = 1\}$ is closed. Finally, note that the projection of $F(\mathcal{H} \times \{1\})$ to $\mathbb{R}^2$ is exactly $\Phi(\mathcal{H})$. □ □

4. Applications

Let $\mathcal{H}$ be a real Hilbert space, $A \in L(\mathcal{H})$, $b \in \mathcal{H}$ and $\rho > 0$. Consider the function $G : \mathcal{H} \to \mathbb{R}$ given by

$$G(x) := \frac{\|Ax - b\|^2}{1 + \|x\|^2} + \rho\|x\|^2.$$

In [5, Proposition 4.13], we apply the following version of an S-lemma in order to give a method for finding the infimum of G. In that work, we give a characterization of such infimum and we present sufficient conditions for the existence of solution of a related total least squares problem.

Lemma 4.1. *Let $\mathcal{H}$ be a real Hilbert space. Let $\phi_j(x) = \langle A_jx, x\rangle + \langle x, a_j\rangle + b_j$, with $A_j \in L(\mathcal{H})$, $a_j \in \mathcal{H}$, $b_j \in \mathbb{R}$, $j = 1, 2$. Suppose that $\mu_1A_1 + \mu_2A_2 > 0$ for some $\mu_1, \mu_2 \in \mathbb{R}$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined as*

$$F(z) = \langle \Theta z, z\rangle + \langle z, v\rangle - t,$$

where Θ is a real symmetric nonnegative 2×2 matrix, $v = (v_1, v_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Then the following are equivalent:

(i) *$F(\phi_1(x),\phi_2(x)) \geq 0$ for every $x \in \mathcal{H}$.*

(ii) *There exist $\alpha, \beta \in \mathbb{R}$ such that for every $x \in \mathcal{H}$ and every $z = (z_1, z_2) \in \mathbb{R}^2$,*

$$F(z) + \alpha(\phi_1(x) - z_1) + \beta(\phi_2(x) - z_2) \geq 0.$$

Moreover,

(1) *if A_1 is not bounded below and $A_2 \in GL^+(\mathcal{H})$ then $\beta \geq 0$. Likewise, if A_2 is not bounded below and $A_1 \in GL^+(\mathcal{H})$ then $\alpha \geq 0$;*

(2) *if either $\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}$ and $v_1 > 0$, or $\Theta = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ and $v_2 < 0$ then $\alpha \geq 0$.*

In order to prove the above result we need the following $\mathbb{R}^2$ version of Farkas' Theorem (see for example [16], [7], [18, section 6.10]):

Let $F, h : \mathbb{R}^2 \to \mathbb{R}$ be convex functions and suppose that there exists $\bar{x} \in \mathbb{R}^2$ such that $h(\bar{x}) \leq 0$. Then $F(z) \geq 0$ for every $z \in \mathbb{R}^2$ such that $h(z) \leq 0$ if and only if there exists $\lambda \geq 0$ such that $F(z) + \lambda h(z) \geq 0$ for every $z \in \mathbb{R}^2$.

of Lemma 4.1. By Proposition 3.1, $D := \{(\phi_1(x), \phi_2(x)) : x \in \mathcal{H}\}$ is convex. Since $\Theta \geq 0$, the set $\{z \in \mathbb{R}^2 : F(z) < 0\}$ is also convex. Moreover, by (i), $D \cap \{z : F(z) < 0\} = \emptyset$. Thus, we can separate these sets by a hyperplane in $\mathbb{R}^2$, i.e., there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$z \in D \Rightarrow \alpha z_1 + \beta z_2 + \gamma \geq 0 \qquad \text{and,} \tag{2}$$
$$F(z) < 0 \Rightarrow \alpha z_1 + \beta z_2 + \gamma < 0.$$

Thus, $F(z) \geq 0$ for every $z = (z_1, z_2)$ such that $\alpha z_1 + \beta z_2 + \gamma \geq 0$. By the Farkas' Theorem, there exists $\lambda \geq 0$ such that for every $z \in \mathbb{R}^2$,

$$F(z) - \lambda(\alpha z_1 + \beta z_2 + \gamma) \geq 0.$$

From this inequality and (2) we conclude that,

$$F(z) + \lambda\alpha(\phi_1(x) - z_1) + \lambda\beta(\phi_2(x) - z_2) = F(z) - \lambda(\alpha z_1 + \beta z_2 + \gamma) + \lambda(\alpha\phi_1(x) + \beta\phi_2(x) + \gamma) \geq 0,$$

for every $z = (z_1, z_2) \in \mathbb{R}^2$ and every $x \in \mathcal{H}$. The converse is straightforward.

Moreover,

(1) Suppose that A_1 is not bounded below, $A_2 \in GL^+(\mathcal{H})$ and $\beta < 0$. By (2), it holds that $z_2 \leq -\frac{\alpha}{\beta} z_1 - \frac{\gamma}{\beta}$, for every $z \in D$. Then the set D must be below a line with finite slope. We will now prove that this is not possible. Let $0 < \varepsilon < \delta \frac{|\beta|}{|\alpha|}$, where $\delta > 0$ is such that $\langle A_2 x, x\rangle \geq \delta \|x\|^2$ for every $x \in \mathcal{H}$.

Since A_1 is not bounded below, given $r > 0$, there exists $x \in \mathcal{H}$ such that $\|x\| = r$ and $|\langle A_1 x, x\rangle| < \epsilon r^2$. Then

$$\begin{aligned} \delta r^2 - \|a_2\| r - |b_2| \leq \phi_2(x) &\leq -\frac{\alpha}{\beta}\phi_1(x) - \frac{\gamma}{\beta} \\ &< \frac{|\alpha|}{|\beta|}(\varepsilon r^2 + \|a_1\| r + b_1) + \frac{|\gamma|}{|\beta|}. \end{aligned}$$

Thus, for every r we should have that

$$(\delta - \frac{|\alpha|}{|\beta|}\varepsilon) r^2 - (\frac{|\alpha|}{|\beta|}\|a_1\| + \|a_2\|) r - |b_2| - \frac{|\alpha|}{|\beta|} b_1 - \frac{|\gamma|}{|\beta|} < 0.$$

This is a contradiction because $\delta - \frac{|\alpha|}{|\beta|}\varepsilon > 0$.

(2) Suppose now that $\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}$, $v_1 > 0$ and $\alpha < 0$. By (2), it holds that $z_1 > -\frac{\beta}{\alpha} z_2 - \frac{\gamma}{\alpha}$, for every $z \in \mathbb{R}^2$ such that $F(z) < 0$.

Since $\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}$ and $v_1 > 0$, then $F(z) = \rho z_2^2 + v_1 z_1 + v_2 z_2 - t$. Therefore, $\{z : F(z) < 0\}$ is the convex set determined by the parabola $z_1 = -\frac{\rho}{v_1} z_2^2 - \frac{v_2}{v_1} z_2 + \frac{t}{v_1}$; so that it can not be on the right side of a straight line (for example, if $z_1 < \min\{-\frac{\gamma}{\alpha}, \frac{t}{v_1}\}$, then $(z_1, 0) \in \{z : F(z) < 0\}$ but does not satisfies (2)).

The other case follows similarly.

□ □

4.1. **S-Procedure.** In [17], Polyak gave several applications of his convexity theorem. Most of them can be extended to infinite dimensional spaces using our result. In this final subsection we briefly present as an example one of these extensions. Let $\mathcal{H}$ be a real Hilbert space and $A_0, A_1, A_2 \in L(\mathcal{H})$. Given two quadratic forms

$$f_i(x) = \langle A_i x, x \rangle, \quad i = 1, 2$$

in $\mathcal{H}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$; the problem is to characterize all $f_0(x) = \langle A_0 x, x \rangle, \alpha_0 \in \mathbb{R}$ such that

$$f_0(x) \leq \alpha_0 \text{ for every } x \in \mathcal{H} \text{ such that } f_1(x) \leq \alpha_1,\ f_2(x) \leq \alpha_2. \tag{3}$$

Proposition 4.2. *Let $\mathcal{H}$ be a real Hilbert space, $3 \leq dim(\mathcal{H}) \leq \infty$. Suppose that there exist $\mu_1, \mu_2 \in \mathbb{R}, x^0 \in \mathcal{H}$ such that*

$$\mu_1 A_1 + \mu_2 A_2 > 0, \tag{4}$$

$$f_1(x^0) < \alpha_1,\ f_2(x^0) < \alpha_2. \tag{5}$$

Then, (3) holds if and only if there exist $\tau_1 \geq 0,\ \tau_2 \geq 0$ such that

$$A_0 \leq \tau_1 A_1 + \tau_2 A_2, \tag{6}$$

$$\alpha_0 \geq \tau_1 \alpha_1 + \tau_2 \alpha_2. \tag{7}$$

Proof. Consider

$$F := \{f(x) : x \in \mathcal{H}\}, \quad f(x) := (f_0(x), f_1(x), f_2(x)).$$

Then, all the assumptions of Theorem 2.10 hold; hence F is convex. Then, the results follows using the same arguments as those found in the proof of [17, Theorem 4.1]. □

Examples 4.1, 4.2 and 4.3 of [17] show that all the conditions of Theorem 4.2 are necessary.

A version of Proposition 4.2 where one of the inequalities $f_i(x) \leq \alpha_i$ is replaced by an equality can be proven with an extra condition. See also [17, Proposition 4.1].

Proposition 4.3. *Let $\mathcal{H}$ be a real Hilbert space, $3 \leq dim(\mathcal{H}) \leq \infty$ and $\alpha_2 \neq 0$. Suppose that there exist $\mu_1, \mu_2 \in \mathbb{R}$ satisfying (4), $x^0 \in \mathcal{H}$ such that*

$$f_1(x^0) < \alpha_1,\ f_2(x^0) = \alpha_2. \tag{8}$$

Then,

$$f_0(x) \leq \alpha_0 \text{ for every } x \in \mathcal{H} \text{ such that } f_1(x) \leq \alpha_1, \; f_2(x) = \alpha_2,$$

if and only if there exists $\tau_1 \geq 0$ *such that* (6) *and* (7) *hold.*

5. Conclusions

An important result due to Polyak [17] states that the joint image of two non-homogeneous quadratic forms defined on $\mathbb{R}^n$ is a convex closed set of $\mathbb{R}^2$. This class of result has many applications, for instance to S-lemma type results.

In this article we extend the convexity part of Polyak's result to an arbitrary infinite dimensional real Hilbert space $\mathcal{H}$, see Theorem 2.10 and Proposition 3.1.

We present examples involving diagonal operators showing that the closedness part of Polyak's theorem does not hold on infinite dimensional spaces for quadratic forms determined by (compact or invertible) positive definite operators. Moreover, we show that if A_1 is a compact operator on $\mathcal{H}$ with 0 in its numerical range and A_2 is a positive and invertible definite operator, then the joint image of two non necessarily homogeneous quadratic forms determined by A_1 and A_2 is closed, see Proposition 3.3.

For further research, it would be interesting to find necessary and sufficient conditions that allow to prove the closedness part of Polyak's theorem in the infinite dimensional setting.

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References

[1] W.N. Anderson and G.E. Trapp, *Shorted operators II*, SIAM J. Appl. Math., **28**(1975), 60–71.

[2] A. Baccari and B. Samet, *An extension of Polyak's theorem in a Hilbert space*, J. Optim. Theory Appl., **140** (2009), 409–418.

[3] L. Brickman, *On the field of values of a matrix*, Proc. Amer. Math. Soc., **12** (1961), 61–66.

[4] E. Calabi, *Linear systems of real quadratic forms*, Proc. Amer. Math. Soc., **15** (1964), 844–846.

[5] M. Contino, G. Fongi, A. Maestripieri, S. Muro, *Total least squares problems on infinite dimensional spaces*, Inverse Problems, **37**, 045008.

[6] G. de Barra, J.R. Giles, and B. Sims, *On the numerical range of compact operators on Hilbert spaces*, J. London Math. Soc., **2** (1972), 704–706.

[7] E. de Klerk, C. Roos, and T. Terlaky, *Nonlinear Optimization*, Lecture Notes, University of Technology, Delft, The Netherlands (2003), 1–65.

[8] L.L. Dines, *On the mapping of quadratic forms*, Bull. Am. Math. Soc., **47** (1941), 494–498.

[9] R.G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc., **17** (1966), 413–416.

[10] F. Flores-Bazán and F. Opazo, *Characterizing the convexity of joint-range for a pair of inhomogeneous quadratic functions and strong duality*, Minimax Theory Appl., **1**(2016), 257–290.

[11] P.R. Halmos, *Hilbert Space Problem Book*, 2nd Edition, Springer-Verlag, New York, 1982.

[12] M.R. Hestenes, *Pairs of quadratic forms*, Linear Algebra Appl., **1** (1968), 397–407.

[13] J.E. Martínez-Legaz, *On Brickman's theorem*, J. Convex Anal., **12** (2005), 139–143.

[14] A. McIntosh, *The Toeplitz-Hausdorff theorem and ellipticity conditions*, Amer. Math. Monthly, **85** (1978), 475–477.

[15] C.A. Micchelli, *Optimal estimation of linear operators from inaccurate data: a second look*, Numer. Algorithms, **5** (1993), 375–390.

[16] I. Pólik and T. Terlaky, *A survey of the S-lemma*, SIAM review, **49** (2007), 371–418.

[17] B.T. Polyak, *Convexity of quadratic transformations and its use in control and optimization*, J. Optim. Theory Appl., **99** (1998), 553–583.

[18] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, vol. 163, Springer Science & Business Media, Berlin, 2012.

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