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ON COMPLETENESS OF INTEGRAL MANIFOLDS OF NULLITY DISTRIBUTIONS

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ABSTRACT. We give a conceptual proof of the fact that if M is a complete submanifold of a space form, then the maximal integral manifolds of the nullity distribution of its second fundamental form through points of minimal index of nullity are complete.

1. Introduction

Let M be a submanifold of a space form and let $\mathcal{N}$ be the nullity distribution of its second fundamental form. The index of nullity of M at p is the dimension of $\mathcal{N}_p$. It is well known, from the Codazzi equation, that $\mathcal{N}$ is an autoparallel distribution restricted to the open and dense subset of M where the index of nullity is locally constant.

If M is complete and one restricts to the open subset U of points of M where the index of nullity is minimal, then the integral manifolds of $\mathcal{N}$ through points of U are also complete, from a result of Ferus [1]. We will give a conceptual proof of this result, as a corollary of a general theorem, whose proof involves very simple geometric ideas.

2. Main results

Lemma 2.1. *Let M be a Riemannian manifold and let $f : M \to N$ be a differentiable function of constant rank such that $f(M)$ is an embedded submanifold of N (this can always be assumed locally). Assume that the distribution $\ker(df)$ is autoparallel and let Σ be an integral manifold of $\ker(df)$. Let $\gamma(t)$ be a geodesic in Σ, $v \in T_{f(\gamma(0))}f(M)$ and let $J(t)$ be the horizontal lift of v along γ, i.e., $J(t) \in \ker(df_{\gamma(t)})^{\perp}$ and $df\,J(t) = v$. Then $J(t)$ is a Jacobi vector field along γ.*

Proof. Let $c(s)$ be a (short) curve in $f(M)$ such that $c'(0) = v$. The horizontal lift of $c(s)$ through points of Σ gives rise to a perpendicular variation $\widetilde{c}_q(s)$ by totally geodesic submanifolds, which must be by isometries. Therefore $\widetilde{c}_{\gamma(t)}(s)$ is a variation by geodesics whose associated variation field is $J(t)$. □

Theorem 2.2. *Let M be a complete Riemannian manifold, $f : M \to N$ be a differentiable function and let U be the open subset of M where the rank of f is maximal. Assume that $\ker(df)_{|U}$ is autoparallel. Then its integral manifolds are complete.*

Proof. Let Σ be a totally geodesic integral manifold of $\ker(df)$ through a point $p \in U$ and let $\gamma : [0, b) \to \Sigma$ be a maximal geodesic in Σ.

Observe that f has maximal rank in a neighborhood of each point of Σ. From the local form of maps of constant rank it is not difficult to see that given $t_1,\ t_2 \in [0, b)$ there are open neighborhoods V_1 and V_2 of $\gamma(t_1)$, $\gamma(t_2)$ such that $f(V_1)$ and $f(V_2)$ are embedded submanifolds of N and $f(V_1) \cap f(V_2)$ contains an open neighborhood of $f(\gamma(t_1)) = f(\gamma(t_2))$ in both $f(V_1)$ and $f(V_2)$. In particular, $T_{f(\gamma(t_1))}f(V_1) = T_{f(\gamma(t_2))}f(V_2) =: \mathbb{V}$.

Let $v \in \mathbb{V}$ and apply the previous lemma to define a Jacobi field J along γ that projects down to v. Since M is complete $\gamma(b)$ and $J(b)$ are well defined and $J(b)$, by the continuity of df, also projects down to v. Then $df_{\gamma(b)}(T_{\gamma(b)}M)$ contains $\mathbb{V}$. So, $\mathrm{rank}(df_{\gamma(b)}) = \mathrm{rank}(df_{\gamma(0)})$ and therefore $\gamma(b) \in \Sigma$. □

If M^n is a submanifold of the Euclidean space $\mathbb{R}^{n+k}$, the Gauss map of M is the map $G : M \to G_k(\mathbb{R}^{n+k})$ defined by $p \mapsto \nu_p M$, where $\nu_p M$ denotes the normal space of M at p. If M^n is a submanifold of the sphere $\mathbb{S}^{n+k} \subset \mathbb{R}^{n+k+1}$, then the Gauss map of M is defined to be the map $G : M \to G_k(\mathbb{R}^{n+k+1})$ that sends each point to its normal space in the sphere, regarded as a subspace of $\mathbb{R}^{n+k+1}$ (see the remark below). A similar construction can be made for a submanifold M^n of the hyperbolic space H^{n+k}, regarded as a submanifold of the Lorentz space $\mathbb{R}^{n+k,1}$.

It is well known that in the three cases, the nullity distribution of M coincides with the kernel of its Gauss map. Therefore we get:

Corollary 2.3. *Let M be a complete submanifold of a space form. Then any maximal integral manifold of the nullity distribution of M through a point of minimal index of nullity is complete.*

Remark 2.4. *Let M be a submanifold of a space form and let ν_1 be a parallel sub-bundle of the normal bundle. One can regard to the nullity of the second fundamental form projected to this sub-bundle. This is equivalent to regard the common kernel of all shape operators of vectors in this sub-bundle. This generalized nullity space coincides with the kernel of the generalized Gauss map $p \mapsto \nu_1(p)$. So, as in the corollary, if M is complete one has completeness of the integral manifolds where the kernel has minimal dimension.*

References

[1] D. Ferus. On the completeness of nullity foliations. *Michigan Math. J.*, 18:61–64, 1971. 89

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