Houston Journal of Mathematics
© 2013 University of Houston
Volume 39, No. 1, 2013

NATURALLY REDUCTIVE PSEUDO-RIEMANNIAN 2-STEP NILPOTENT LIE GROUPS

GABRIELA P. OVANDO

Communicated by Robert M. Hardt

Abstract. This paper deals with naturally reductive pseudo-Riemannian 2-step nilpotent Lie groups for which the metric is invariant under a left action. The case of nondegenerate center is characterized as follows. The simply connected Lie group can be constructed starting from a real representation of a certain Lie algebra which carries an ad-invariant metric. Also a naturally reductive homogeneous structure is given and applications are shown.

1. Introduction

In this work we focus on naturally reductive pseudo-Riemannian 2-step nilpotent Lie groups. The 2-step nilpotent Lie groups equipped with a left-invariant metric have been extensively investigated in the Riemannian situation for a long time; in the case of indefinite metrics, there are significant advances as showed in [2, 6, 7, 11, 15, 16, 17, 27] but there are still several open problems.

Our aim here is to find conditions on the underlying algebraic structure for the existence of a naturally reductive pseudo-Riemannian 2-step nilpotent Lie group. Important studies concerning the structure of a naturally reductive Riemannian Lie group G were given by D'Atri and Ziller [8] and Gordon [14].

In the nilpotent case Gordon proved that a naturally reductive Riemannian nilmanifold N must be at most 2-step. Lauret [20] exploited this result to afford a classification of naturally reductive Riemannian connected simply connected

2000 *Mathematics Subject Classification.* 53C50 22E25 53B30 53C30.

Key words and phrases. Naturally reductive pseudo-Riemannian, 2-step nilpotent, isometry groups, representations.

The author is partially supported by DAAD, ANPCyT, SCyT - UNRosario.

nilmanifolds. According to Wilson's results [32] such a manifold can be realized as a 2-step nilpotent Lie group equipped with a left-invariant metric.

Here we characterize the naturally reductive 2-step nilpotent Lie group with nondegenerate center $(N, \langle\,,\,\rangle)$. The starting point is a result of Cordero and Parker [6] stating that the isometry subgroup fixing the identity for metrics with nondegenerate center coincides with the group of isometric automorphisms. This enables to specify the action of the isometry group in our case. Let $\mathfrak{n}$ denote the corresponding 2-step nilpotent Lie algebra furnished with the metric $\langle\,,\,\rangle$ for which the center is nondegenerate. Then $\mathfrak{n}$ can be decomposed into an orthogonal direct sum

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \qquad \text{with } \mathfrak{v} = \mathfrak{z}^{\perp} \tag{1.1}$$

and the Lie bracket on $\mathfrak{n}$ induces for $x \in \mathfrak{z}$ the linear map $j(x) : \mathfrak{v} \to \mathfrak{v}$ given by

$$\langle [u, v], x\rangle = \langle j(x)u, v\rangle \qquad \text{for } x \in \mathfrak{z}, u, v \in \mathfrak{v},$$

which is skew-adjoint with respect to $\langle\,,\,\rangle_{\mathfrak{v}}$. Assume that j is injective. Then if the metric is naturally reductive, the set $\{j(z)\}$ with $z \in \mathfrak{z}$ builds a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$, Lie subalgebra which admits an ad-invariant metric (Theorems 3.2 and 3.3). The converse also holds and it is consequence of the same theorems.

We also give a naturally reductive homogeneous structure for these cases. Recall that the notion of *homogeneous structure* was introduced by Ambrose and Singer [1] to characterize connected simply connected and complete homogeneous Riemannian manifolds, revalidated to the pseudo-Riemannian case by Gadea and Oubiña [12].

Our results here improve the understanding of some geometrical features such as the isometry group -Proposition 3.5- and it sets up the construction of new examples; in particular we get naturally reductive metrics in the Heisenberg Lie group H_{2n+1} for $n \geq 1$. We also bring into focus the underlying geometry with a study of 2-step nilpotent pseudo-Riemannian Lie groups with nondegenerate center, study which completes the work in the paper.

We note that a construction of naturally reductive pseudo-Riemannian Lie groups was proposed by the author in [25]. In particular by this method one obtains naturally reductive k-step nilpotent Lie groups with $k \geq 3$, which are not symmetric. A next step in our investigations is to generalize the results of the present paper to those cases.

2. Preliminaries

In this section we set the basic notions for the study of the geometry of a 2-step nilpotent Lie group equipped with a left-invariant pseudo-Riemannian metric. Here and along the paper left-invariant means invariant under a left action. We focus on those metrics for which the center is nondegenerate. At the end of the section we give the definition of natural reductiveness.

A *metric* on a real vector space $\mathfrak{v}$ is a nondegenerate symmetric bilinear form $\langle\,,\,\rangle : \mathfrak{v}\times\mathfrak{v} \to \mathbb{R}$. Whenever $\mathfrak{v}$ is the Lie algebra of a given Lie group G, by identifying $\mathfrak{v}$ with the set of left-invariant vector fields on G, the metric induces by mean of the left translations, a pseudo-Riemannian metric tensor on the corresponding Lie group. Conversely a pseudo-Riemannian metric on G invariant under left translations is completely determined by its value at the identity tangent space T_eG.

Let $\mathfrak{n}$ denote a 2-step nilpotent Lie algebra furnished with a metric $\langle\,,\,\rangle$ for which the center is nondegenerate. Then $\mathfrak{n}$ can be decomposed into a orthogonal direct sum

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \qquad \text{with } \mathfrak{v} = \mathfrak{z}^{\perp}$$

where as usual $\mathfrak{z}$ denotes the center of $\mathfrak{n}$. The Lie bracket on $\mathfrak{n}$ induces for $x \in \mathfrak{z}$ the skew-adjoint linear map $j(x) : \mathfrak{v} \to \mathfrak{v}$ given by

$$\langle [u,v], x\rangle = \langle j(x)u, v\rangle \qquad \text{for } x \in \mathfrak{z}, u, v \in \mathfrak{v}. \tag{2.1}$$

Conversely let $(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}})$ and $(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ denote vector spaces endowed with (not necessarily definite) metrics. Let $\mathfrak{n}$ denote the direct sum as vector spaces $\mathfrak{n} = \mathfrak{z}\oplus\mathfrak{v}$ and let $\langle\,,\,\rangle$ denote the metric given by

$$\langle\,,\,\rangle_{|\mathfrak{z}\times\mathfrak{z}} = \langle\,,\,\rangle_{\mathfrak{z}} \qquad \langle\,,\,\rangle_{|\mathfrak{v}\times\mathfrak{v}} = \langle\,,\,\rangle_{\mathfrak{v}} \qquad \langle \mathfrak{z}, \mathfrak{v}\rangle = 0. \tag{2.2}$$

Let $j : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ be a linear map such that $j(z)$ is skew-adjoint with respect to $\langle\,,\,\rangle_{\mathfrak{v}}$ for all $z \in \mathfrak{z}$. Then $\mathfrak{n}$ becomes a 2-step nilpotent Lie algebra if one defines a Lie bracket by the relation in (2.1) and so that $\mathfrak{z}$ is contained in the center of $\mathfrak{n}$. By using left translations the corresponding connected Lie group N is endowed with a pseudo-Riemannian metric.

In the situation where the metric is definite, the inner product $\langle\,,\,\rangle_+$ produces a decomposition of the center of the Lie algebra $\mathfrak{n}$ as a orthogonal direct sum as vector spaces

$$\mathfrak{z} = \ker j \oplus C(\mathfrak{n}),$$

where $C(\mathfrak{n})$ denotes the commutator and moreover j is injective if and only if there is no Euclidean factor in the De Rahm decomposition of the simply connected

Lie group $(N, \langle\,,\,\rangle_+)$ (see [14]). This does not necessarily hold in the pseudo-Riemannian case (see the next example where $ker(j) = [\mathfrak{n}, \mathfrak{n}]$).

Example 2.1. Let $\mathbb{R} \times \mathfrak{h}_3$ be the 2-step nilpotent Lie algebra spanned by the vectors e_1, e_2, e_3, e_4 with the Lie bracket $[e_1, e_2] = e_3$. Define a metric $\langle\,,\,\rangle$ where the nontrivial relations are

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_4\rangle = 1.$$

After (2.1) one can verify that $j(e_3) \equiv 0$, while

$$j(e_4) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that $e_4 \notin C(\mathbb{R} \times \mathfrak{h}_3)$ and $kerj = \mathbb{R}e_3 = C(\mathbb{R} \times \mathfrak{h}_3)$, that is $kerj = C(\mathfrak{n})$.

Let $\mathsf{O}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ denote the group of linear maps on $\mathfrak{v}$ which are isometries for $\langle\,,\,\rangle_{\mathfrak{v}}$ and whose Lie algebra $\mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ is the set of linear maps on $\mathfrak{v}$ that are skew-adjoint with respect to $\langle\,,\,\rangle_{\mathfrak{v}}$. To describe the group of isometries we shall make use of the next result.

Proposition 2.2. [6] *Let N denote a 2-step nilpotent Lie group endowed with a left-invariant pseudo-Riemannian metric with respect to which the center is nondegenerate. Then the group of isometries fixing the identity coincides with the group of orthogonal automorphisms of N.*

Denote by H the group of orthogonal automorphisms and by N also the subgroup of isometries consisting of ℓ_n: the left translation by the element $n \in N$. Consider the isometries of the form $h\ell_n$ where $h \in H$ and $n \in N$, and denote it by $I_a(N)$. Then N is a normal subgroup of $I_a(N)$, $N \cap H = \{e\}$ and therefore for $I(N)$ the isometry group in our situation one has

$$I(N) = I_a(N) = H \ltimes N.$$

Whenever $(N, \langle\,,\,\rangle)$ is simply connected, we do not distinguish between the group of automorphisms of N and of $\mathfrak{n}$. Thus one obtains that the group H is given by

$$H = \{(\phi, T) \in \mathsf{O}(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}}) \times \mathsf{O}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}}) : Tj(x)T^{-1} = j(\phi x), \quad x \in \mathfrak{z}\} \tag{2.3}$$

while its Lie algebra $\mathfrak{h} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle\,,\,\rangle)$ is

$$\mathfrak{h} = \{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}}) : [B, j(x)] = j(Ax), \quad x \in \mathfrak{z}\}. \tag{2.4}$$

In fact, let ψ denote a orthogonal automorphism of $(\mathfrak{n}, \langle\,,\,\rangle)$. Thus $\psi(\mathfrak{z}) \subseteq \mathfrak{z}$ and since $\mathfrak{v} = \mathfrak{z}^{\perp}$ then $\psi(\mathfrak{v}) \subseteq \mathfrak{v}$. Set $\phi := \psi_{|_{\mathfrak{z}}}$ and $T := \psi_{|_{\mathfrak{v}}}$, thus $(\phi, T) \in \mathsf{O}(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}}) \times \mathsf{O}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ such that

$$\begin{array}{rcll} \langle \phi^{-1}[u,v], x\rangle &=& \langle [Tu, Tv], j(x)\rangle & \text{if and only if} \\ \langle j(\phi x)u, v\rangle &=& \langle j(x)Tu, Tv\rangle & \end{array}$$

which implies (2.3). By derivating (2.3) one gets (2.4).

Proposition 2.3. *Let N denote a simply connected 2-step nilpotent Lie group endowed with a left-invariant pseudo-Riemannian metric with respect to which the center is nondegenerate. Then the group of isometries is*

$$I(N) = H \ltimes N.$$

where N denotes the set of left translations by elements of N and the isotropy subgroup H is given by the isometric automorphisms (2.3) with Lie algebra as in (2.4).

Example 2.4. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra equipped with an inner product denoted by $\langle\,,\,\rangle_+$. Let $J_z \in \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_+)$ denote the maps in (2.1).

We shall consider an indefinite metric $\langle\,,\,\rangle$ on $\mathfrak{n}$ by changing the sign of the metric on the center $\mathfrak{z}$; thus the metric on $\mathfrak{v}$ remains invariant for each of these two metrics and we take

$$\langle z_i, z_j\rangle = -\langle z_i, z_j\rangle_+ \qquad \text{for } z_i, z_j \in \mathfrak{z} \qquad \text{and} \qquad \langle \mathfrak{z}, \mathfrak{v}\rangle = 0.$$

By (2.1) the maps $j(z)$ for the metric $\langle\,,\,\rangle$ on $\mathfrak{n}$ are

$$-\langle z, [u,v]\rangle_+ = -\langle J(z)u, v\rangle_+ = \langle j(z)u, v\rangle = \langle z, [u,v]\rangle, \qquad \text{for } z \in \mathfrak{z}$$

that is $j(z) = -J(z)$ for every $z \in \mathfrak{z}$.

We work out an example on the Heisenberg Lie group H_3. This is the simply connected Lie group whose Lie algebra is $\mathfrak{h}_3$ which is spanned by the vectors e_1, e_2, e_3, with the nontrivial Lie bracket relation $[e_1, e_2] = e_3$. The canonical left-invariant metric $\langle\,,\,\rangle_+$ is that one obtained by declaring the basis $\{e_1, e_2, e_3\}$ to be orthogonal and the map $J(e_3)$ for $\langle\,,\,\rangle_+$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A Lorentzian metric $\langle\,,\,\rangle$ is obtained on H_3 by changing the sign of the canonical metric on the center. Kaplan showed that $(H_3, \langle\,,\,\rangle_+)$ is naturally reductive [18]. In the next sections we shall see that $(H_3, \langle\,,\,\rangle)$ and generalizations of it, are also naturally reductive in the pseudo-Riemannian context.

By (2.3) the group of isometries for any of these both metrics is $(\mathbb{R}\times O(2))\ltimes H_3$, where the action of the isotropy group is given by $(\lambda, A)\cdot(z+v) = \lambda z + Av$ for $z \in \mathfrak{z}$ and $v \in \mathfrak{v} = span\{e_1, e_2\}$, $\lambda \in \mathbb{R}$ and $A \in O(2)$.

Definition 1. A homogeneous manifold M is said to be *naturally reductive* if there is a transitive Lie group of isometries G with Lie algebra $\mathfrak{g}$ and there exists a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ complementary to $\mathfrak{h}$ in $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebra of the isotropy group H, such that

$$\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m} \quad \text{and} \quad \langle [x,y]_{\mathfrak{m}}, z\rangle + \langle y, [x,z]_{\mathfrak{m}}\rangle = 0 \qquad \text{for all } x,y,z \in \mathfrak{m}.$$

Frequently we will say that a metric on a homogeneous space M is naturally reductive even though it is not naturally reductive with respect to a particular transitive group of isometries (see Lemma 2.3 in [14]).

For naturally reductive metrics the geodesics passing through $m \in M$ are of the form

$$\gamma(t) = \exp(tx)\cdot m \qquad \text{for some } x \in \mathfrak{m}.$$

Indeed pseudo-Riemannian symmetric spaces are naturally reductive. In particular 2-step nilpotent Lie groups equipped with a bi-invariant metric provide examples of naturally reductive pseudo-Riemannian 2-step nilpotent Lie groups where the center is degenerate (see for instance [24]).

3. Naturally reductive metrics with nondegenerate center: a characterization

In this section we achieve a characterization of naturally reductive pseudo-Riemannian simply connected 2-step nilpotent Lie groups with nondegenerate center. We also find the corresponding naturally reductive homogeneous structure.

Lemma 3.1. *Let $(\mathfrak{n}, \langle\, ,\, \rangle)$ denote a 2-step nilpotent Lie algebra equipped with a metric for which its center $\mathfrak{z}$ is nondegenerate and assume j is injective. Let $\mathfrak{h} = \mathfrak{so}(\mathfrak{n}, \langle\, ,\, \rangle) \cap \mathrm{Der}(\mathfrak{n})$ denote the Lie subalgebra of the group of isometries fixing the identity element in the corresponding simply connected Lie group N. Then*

i) *$\mathfrak{h}$ leaves each of $\mathfrak{z}$ and $\mathfrak{v}$ invariant.*

ii) *For $\phi \in \mathfrak{h}$,*

$$\phi_{|_{\mathfrak{z}}} = j^{-1} \circ \mathrm{ad}_{\mathfrak{so}(\mathfrak{v})}\, \phi_{|_{\mathfrak{v}}} \circ j;$$

in particular $\phi \to \phi_{|_{\mathfrak{v}}}$ is an isomorphism of $\mathfrak{h}$ onto a subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}})$.

iii) *Let $\phi \in \mathfrak{so}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}})$. Then ϕ extends to an element of $\mathfrak{h}$ if and only if $[\phi, j(\mathfrak{z})] \subseteq j(\mathfrak{z})$ and $j^{-1} \circ \mathrm{ad}_{\mathfrak{so}(\mathfrak{v})}\, \phi_{|_{\mathfrak{v}}} \circ j \in \mathfrak{so}(\mathfrak{z}, \langle\, ,\, \rangle_{\mathfrak{z}})$.*

PROOF. i) is easy to prove. We shall show (ii) and (iii). Let $A \in \mathfrak{so}(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}})$ and $B \in \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$, the linear map ϕ which agrees with $(A, B) \in \mathfrak{z} \oplus \mathfrak{v}$ lies in $\mathfrak{h}$ if and only if

$$\langle j(Ax)u, v\rangle = \langle (Bj(x) - j(x)B)u, v\rangle \qquad \text{for } x \in \mathfrak{z},\ u, v \in \mathfrak{v}$$

which is equivalent to $j(A(x)) = [B, j(x)]$ the last $[\,,\,]$ denotes the Lie bracket in $\mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ and since j was assumed injective one gets $A = j^{-1} \circ \mathrm{ad}_{\mathfrak{so}(\mathfrak{v})}(B) \circ j$. □

The proof of the next theorem coincides with that one given by C. Gordon in [14]. For the sake of completeness we include it here. However the consequences are quite different from the Riemannian situation.

Theorem 3.2. *Let $(N, \langle\,,\,\rangle)$ denote a 2-step simply connected Lie group equipped with a left-invariant pseudo-Riemannian metric such that the center is nondegenerate and assume j is injective. Then the metric is naturally reductive with respect to $G = H \ltimes N$ being H the group of orthogonal automorphisms, if and only if*

(i) *$j(\mathfrak{z})$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ and*

(ii) *$[j(x), j(y)] = j(\tau_x y)$ where $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle\,,\,\rangle_{\mathfrak{z}})$ for any $x \in \mathfrak{z}$.*

PROOF. Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ be the Lie algebra of $G = H \ltimes N$ and assume N is naturally reductive with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Set $\pi : \mathfrak{n} \to \mathfrak{h}$ so that

$$\mathfrak{m} = \{x + \pi(x) : x \in \mathfrak{n}\}.$$

The condition for natural reductiveness says

$$\langle [x + \pi(x), y + \pi(y)]_{\mathfrak{m}}, z + \pi(z)\rangle_{\mathfrak{m}} = -\langle y + \pi(y), [x + \pi(x), z + \pi(z)]_{\mathfrak{m}}\rangle_{\mathfrak{m}}$$

where $\langle\,,\,\rangle$ is the pseudo-Riemannian metric on $\mathfrak{m}$, so that the previous equality can be interpreted on $\mathfrak{n}$ as

$$\langle [x, y] + \pi(x)y - \pi(y)x, z\rangle = -\langle y, [x, z] + \pi(x)z - \pi(z)x\rangle. \tag{3.1}$$

where $\pi(x)$ is view as a linear operator on $\mathfrak{n}$ and one writes $\pi(x)y = [x, y]$ when $x, y \in \mathfrak{n}$. Since $\pi(x) \in \mathfrak{so}(\mathfrak{n}, \langle\,,\,\rangle)$ the terms involving $\pi(x)$ cancel and (3.1) yields

$$\mathrm{ad}(y)^* z + \mathrm{ad}(z)^* y = \pi(y)z + \pi(z)y \qquad \text{for all } y, z \in \mathfrak{n}. \tag{3.2}$$

Since $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, one has

$$[\pi(x), y + \pi(y)] = \pi(x)y + [\pi(x), \pi(y)] \in \mathfrak{m}$$

and therefore

$$\pi(\pi(x)y) = [\pi(x), \pi(y)] \qquad \text{for all } x, y \in \mathfrak{n}. \tag{3.3}$$

If $z \in \mathfrak{z}$ and $y \in \mathfrak{v}$, $\mathrm{ad}(z)^* y = 0$ and (3.2) says

$$j(z)y = \pi(y)z + \pi(z)y. \tag{3.4}$$

But $\pi(y)z \in \mathfrak{z}$ and $\pi(z)y \in \mathfrak{v}$, so (3.4) implies

$$\pi(z)_{|\mathfrak{v}} = j(z) \in \mathfrak{so}(\mathfrak{v}, \langle\, , \rangle_{\mathfrak{v}}) \quad \text{for every } z \in \mathfrak{z}.$$

It then follows that

$$[j(x), j(\mathfrak{z})] \subset j(\mathfrak{z}) \text{ and } [j(x), j(y)] = j(\tau_x y) \quad \text{for } \tau_x \in \mathfrak{so}(\mathfrak{z}, \langle\, , \rangle_{\mathfrak{z}}), \quad x, y \in \mathfrak{z}.$$

Conversely if (i) and (ii) hold, extend $j(x)$ to an element $\pi(x)$ of $\mathfrak{h}$ such that the restriction of $\pi(x)$ to $\mathfrak{z}$ is given by the left-hand side of (ii). Extend ρ as a linear map of $\mathfrak{n}$ by declaring $\pi_{|\mathfrak{v}} \equiv 0$. We claim (3.3) holds for all $x, y \in \mathfrak{n}$. In fact it is easy to verify it if at least one of $x, y \in \mathfrak{v}$. Assume $x, y \in \mathfrak{z}$, then

$$\pi(\pi(x)y)_{|\mathfrak{v}} = j(j^{-1}[j(x), j(y)]) = [j(x), j(y)]$$

and therefore (3.3) is true after (3.1) ii). Define

$$\mathfrak{l} = \pi(\mathfrak{n}), \quad \mathfrak{m} = \{x + \pi(x) : x \in \mathfrak{n}\}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}.$$

By (3.3) $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{h}$ and $[\mathfrak{l}, \mathfrak{m}] \subseteq \mathfrak{m}$ and since $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{n}$, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$.

We assert that (3.2) is valid. This can be easily checked whenever at least one of $x, y \in \mathfrak{v}$. If both $x, y \in \mathfrak{z}$ the left-hand side of (3.2) is zero. The right-hand side lies in $\mathfrak{z} \cap ker(\pi)$, but $\ker(\pi) = \ker(j)$ and since j is injective one has $\mathfrak{z} \cap ker(\pi) = \{0\}$, which proves (3.2). By following the argument preceding (3.2) backwards, one can see that M is naturally reductive with respect to $\mathfrak{k}$. □

In the conditions of Theorem (3.2) it follows that if $(N, \langle\, , \rangle)$ is naturally reductive then the bilinear map τ defines a Lie algebra structure on $\mathfrak{z}$ and the map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v}, \langle\, , \rangle_{\mathfrak{v}})$ becomes a real representation of the Lie algebra $(\mathfrak{z}, \tau)$. Furthermore the metric on $\mathfrak{v}$ is $j(\mathfrak{z})$-invariant and since $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle\, , \rangle_{\mathfrak{z}})$ the metric on $\mathfrak{z}$ is $\mathrm{ad}(\mathfrak{z})$-invariant, where ad denotes the adjoint representation of $(\mathfrak{z}, \tau)$.

Conversely let $\mathfrak{g}$ be a real Lie algebra endowed with an $\mathrm{ad}(\mathfrak{g})$-invariant metric $\langle\, , \rangle_{\mathfrak{g}}$ and let $(\pi, \mathfrak{v})$ be a faithful representation of $\mathfrak{g}$ endowed with a $\pi(\mathfrak{g})$-invariant metric $\langle\, , \rangle_{\mathfrak{v}}$ and without trivial subrepresentations, that is, $\bigcap_{x \in \mathfrak{g}} ker\pi(x) = \{0\}$. Define a 2-step nilpotent Lie algebra structure on the vector space underlying $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ by the following bracket

$$\begin{aligned} &[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{n}} = [\mathfrak{g}, \mathfrak{v}]_{\mathfrak{n}} = 0 \qquad [\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{g} \\ &\langle [u, v], x \rangle_{\mathfrak{g}} = \langle \pi(x)u, v \rangle_{\mathfrak{v}} \quad \forall x \in \mathfrak{g}, u, v \in \mathfrak{v} \end{aligned} \tag{3.5}$$

and equip $\mathfrak{n}$ with the metric $\langle\, ,\, \rangle$ obtained as the product metric

$$\langle\, ,\, \rangle_{|\mathfrak{g}\times\mathfrak{g}} = \langle\, ,\, \rangle_{\mathfrak{g}} \qquad \langle\, ,\, \rangle_{|\mathfrak{v}\times\mathfrak{v}} = \langle\, ,\, \rangle_{\mathfrak{v}} \qquad \langle \mathfrak{g}, \mathfrak{v}\rangle = 0. \tag{3.6}$$

Take N the simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ and endow it with the left-invariant metric determined by $\langle\, ,\, \rangle$.

Since $(\pi, \mathfrak{v})$ has no trivial subrepresentations, the center of $\mathfrak{n}$ coincides with $\mathfrak{g}$. Moreover $\mathfrak{v}$ is its orthogonal complement and the transformation $j(x)$ defined as in (2.1) is precisely $\pi(x)$ for every $x \in \mathfrak{g}$. Since $(\pi, \mathfrak{v})$ is faithful, the commutator of $\mathfrak{n}$ is $\mathfrak{g}$: $C(\mathfrak{n}) = \mathfrak{g}$. Since the set $\{\pi(x)\}_{x\in\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}})$ we conclude that $(N, \langle\, ,\, \rangle)$ is naturally reductive.

Theorem 3.3. *Let N denote a naturally reductive simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ and metric $\langle\, ,\, \rangle$. Assume j is injective. Then $\tau_x y = j^{-1}[j(x), j(y)]$ defines a Lie algebra structure on $\mathfrak{z}$ and $\langle\, ,\, \rangle_{\mathfrak{z}\times\mathfrak{z}}$ is ad-invariant on $(\mathfrak{z}, \tau)$.*

Conversely let $\mathfrak{g}$ denote a Lie algebra equipped with an ad-invariant metric $\langle\, ,\, \rangle_{\mathfrak{g}}$ and let $(\pi, \mathfrak{v})$ be a real faithful representation of $\mathfrak{g}$ without trivial subrepresentations and endowed with a $\pi(\mathfrak{g})$-invariant metric $\langle\, ,\, \rangle_{\mathfrak{v}}$. Let $\mathfrak{n}$ denote the direct sum of vector spaces

$$\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$$

endowed with the Lie bracket given by (2.1) and furnished with the metric $\langle\, ,\, \rangle$ as in (3.6). Then the corresponding simply connected 2-step nilpotent Lie group $(N, \langle\, ,\, \rangle)$ is a naturally reductive pseudo-Riemannian space.

Remark 1. Suppose the representation $(\pi, \mathfrak{v})$ of $\mathfrak{g}$ is not faithful. Thus

$$z \in ker\pi \Longleftrightarrow \langle z, [u, v]\rangle = 0 \quad \forall u, v \in \mathfrak{v}$$

$\Longrightarrow z \in C(\mathfrak{n})^{\perp}$. Since the metric on the center $\mathfrak{g}$ is indefinite, $ker\pi \cap C(\mathfrak{n})$ could be nontrivial, so that the sum as vector spaces $ker\pi + C(\mathfrak{n})$ is not necessarily direct.

When π has some trivial subrepresentation,

$$u \in \cap_{x\in\mathfrak{g}}\pi(x) \Longleftrightarrow \langle \pi(x)u, v\rangle = 0 \quad \forall v \in \mathfrak{v},$$

$\Longrightarrow \langle x, [u, v]\rangle = 0$ for all $x \in \mathfrak{g}$, thus $[u, v] = 0$ for all $v \in \mathfrak{v}$ which says $u \in \mathfrak{z}(\mathfrak{n})$.

Remark 2. In the Riemannian case, the condition of the metric to be positive definite says that $\mathfrak{g}$ must be compact. In the general case the Lie algebra $\mathfrak{g}$ carries an ad-invariant (definite or not) metric.

Example 3.4. The Killing form on any semisimple Lie algebra is an ad-invariant metric.

Any Lie algebra $\mathfrak{g}$ can be embedded into a Lie algebra which admits an ad-invariant metric. In fact, the cotangent $\mathrm{T}^* \mathfrak{g} = \mathfrak{g} \ltimes_{coad} \mathfrak{g}^*$, being *coad* the coadjoint representation, admits a neutral ad-invariant metric which is given by:

$$\langle (x_1, \varphi_1), (x_2, \varphi_2) \rangle = \varphi_1(x_2) + \varphi_2(x_1) \qquad x_1, x_2 \in \mathfrak{g}, \quad \varphi_1, \varphi_2 \in \mathfrak{g}^*.$$

Notice that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic subspaces in $\mathrm{T}^* \mathfrak{g}$.

A *data set* $(\mathfrak{g}, \mathfrak{v}, \langle\,,\,\rangle)$ consists of

(i) a Lie algebra $\mathfrak{g}$ equipped with an ad-invariant metric $\langle\,,\,\rangle_{\mathfrak{g}}$,

(ii) a real faithful representation of $\mathfrak{g}$, $(\pi, \mathfrak{v})$, without trivial subrepresentations,

(iii) $\langle\,,\,\rangle$ a $\mathfrak{g}$-invariant metric on $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$, i.e. $\langle\,,\,\rangle_{|\mathfrak{g}\times\mathfrak{g}} = \langle\,,\,\rangle_{\mathfrak{g}}$ is $\mathrm{ad}(\mathfrak{g})$-invariant, $\langle\,,\,\rangle_{|\mathfrak{v}\times\mathfrak{v}}$ is $\pi(\mathfrak{g})$-invariant and $\langle \mathfrak{g}, \mathfrak{v} \rangle = 0$.

By (3.3) a data set $(\mathfrak{g}, \mathfrak{v}, \langle\,,\,\rangle)$ determines a naturally reductive simply connected 2-step nilpotent Lie group denoted by $N(\mathfrak{g}, \mathfrak{v})$ whose Lie algebra is the underlying vector space $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ with the Lie bracket defined by (3.5).

We study the isometry group in this case. Let $\mathfrak{h}$ denote the Lie algebra of the isometries fixing the identity element; by (2.4) an element $D \in \mathfrak{h}$ is a skew-adjoint derivation which can be written as $D = (A, B) \in \mathfrak{so}(\mathfrak{g}, \langle\,,\,\rangle_{\mathfrak{g}}) \times \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ such that

$$B\pi(x) - \pi(x)B = \pi(Ax), \quad \forall x \in \mathfrak{g}.$$

Denote by $[\,,\,]_{\mathfrak{n}}$ the Lie bracket on $\mathfrak{n}$ and by $[\,,\,]$ the Lie brackets on $\mathfrak{g}$ and $\mathrm{End}(\mathfrak{v})$. Then

$$\begin{aligned} \pi(A[x,y]) &= B\pi([x,y]) - \pi([x,y])B = B[\pi(x), \pi(y)] - [\pi(x), \pi(y)]B \\ &= [B, [\pi(x), \pi(y)]] = [[B, \pi(x)], \pi(y)] + [\pi(x), [B, \pi(y)]] \\ &= [\pi(Ax), \pi(y)] + [\pi(x), \pi(Ay)] = \pi([Ax, y] + [x, Ay]). \end{aligned}$$

Since π is faithful then

$$A[x,y] = [Ax, y] + [x, Ay] \qquad \text{for all } x, y \in \mathfrak{g},$$

that is, $A \in \mathrm{Der}(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g}, \langle\,,\,\rangle_{\mathfrak{g}})$.

Proposition 3.5. *The group of isometries fixing the identity on a naturally reductive pseudo-Riemannian 2-step nilpotent Lie group $N(\mathfrak{g}, \mathfrak{v})$ as in (3.3) has Lie algebra*

$$\mathfrak{h} = \{(A, B) \in (\mathrm{Der}(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g}, \langle\,,\,\rangle_{\mathfrak{g}})) \times \mathfrak{so}(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}}) \,:\, [\pi(x), B] = \pi(Ax) \quad \forall x \in \mathfrak{g}\}.$$

Whenever $\mathfrak{g}$ is semisimple, the ad-invariant metric on $\mathfrak{g}$ is essentially the Killing form; therefore any skew-adjoint derivation of $\mathfrak{g}$ is of the form $\mathrm{ad}(x)$ for some $x \in \mathfrak{g}$. In this case one can consider $\mathfrak{g} \subset \mathfrak{h}$ where the action is given as

$$x \cdot (z + v) = \mathrm{ad}(x)z + \pi(x)v \qquad x \in \mathfrak{g},\, z + v \in \mathfrak{n}$$

being $\mathrm{ad}(x)$ the adjoint map on the semisimple Lie algebra $\mathfrak{g}$. Thus an element $D = (A, B) \in \mathfrak{h}$ is of the form

$$(A, B) = (\mathrm{ad}(x), \pi(x)) + (0, B') \qquad x \in \mathfrak{g}$$

with $B' = B - \pi(x) \in \mathrm{End}_{\mathfrak{g}}(\mathfrak{v}) \cap \mathfrak{so}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}}) = \mathfrak{e}_{\mathfrak{g}}$, where $\mathrm{End}_{\mathfrak{g}}(\mathfrak{v})$ denotes the set of intertwinning operators of the representation $(\pi, \mathfrak{v})$ of $\mathfrak{g}$. Since $\mathfrak{g}$ and $\mathfrak{e}_{\mathfrak{g}}$ commute, then $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{e}_{\mathfrak{g}}$ is a direct sum of Lie algebras, here we identify $\mathfrak{g}$ with the set $\{(\mathrm{ad}(x), \pi(x)) : x \in \mathfrak{g}\} \subseteq \mathfrak{h}$. This argues the following result.

Corollary 3.6. *In the conditions of (3.5) with data set* $(\mathfrak{g}, \mathfrak{v}, \langle\, ,\, \rangle)$ *for* $\mathfrak{g}$ *semisimple, the group of isometries fixing the identity element is*

$$H = G \times U \qquad\qquad U = \mathrm{End}_{\mathfrak{g}}(\mathfrak{v}) \cap \mathsf{O}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}}).$$

Proof. By (2.3) we have that

$$H = \{(\phi, T) \in \mathsf{O}(\mathfrak{g}, \langle\, ,\, \rangle_{\mathfrak{g}}) \times \mathsf{O}(\mathfrak{v}, \langle\, ,\, \rangle_{\mathfrak{v}}) : T\pi(x)T^{-1} = \pi(\phi x), \quad x \in \mathfrak{g}\}.$$

Hence $\phi = \pi^{-1} \circ \mathrm{Ad}(T) \circ \pi \in \mathrm{Aut}(\mathfrak{g})$. Since $\mathfrak{g}$ is semisimple any automorphism of $\mathfrak{g}$ is an inner automorphism, thus there exist $g \in G$ such that $\phi = \mathrm{Ad}(g)$. By the paragraph above, $(Ad(g), \pi(g)) \in H$ and therefore $\pi(g)^{-1}T \in U$. Hence

$$(\phi, T) = (\mathrm{Ad}(g), \pi(g)) \cdot (I, \pi(g)^{-1}T),$$

which says $H = G \times U$. □

Remark 3. Compare with the Riemannian case [20].

3.1. **A naturally reductive homogeneous structure.** Ambrose and Singer [1] achieved an infinitesimal characterization of connected simply connected and complete homogeneous Riemannian manifolds in terms of a (1,2) tensor. This condition was generalized to the pseudo-Riemannian case in the work given by Gadea and Oubiña [12]. For naturally reductive Riemannian spaces a more restricted condition was found by Tricerri and Vanhecke [28] (see also [29]) and generalized to the pseudo-Riemannian case by Gadea and Oubiña (Proposition 4.1 in [13]). While Tricerri and Vanhecke [28] achieved the classification of homogeneous Riemannian structures, in the pseudo-Riemannian case, a complete classification is still a pending item. There are studies in low dimensions as shown in [3, 4, 5].

Theorem 3.7. *A connected complete and simply connected pseudo-Riemannian manifold* (M, g) *is naturally reductive if and only if there exists a tensor field* T *of type (1,2) on* M *such that*

(i) $g(T_x y, z) + g(y, T_x z) = 0$

(ii) $(\nabla_x R)(y, z) = [T_x, R(y, z)] - R(T_x y, z) - R(y, T_x z)$

(iii) $(\nabla_x)T_y = [T_x, T_y] - T_{T_x y}$

(iv) $T_x x = 0$ *for all* $x \in \xi(M)$.

for $x, y, z \in \xi(M)$, *where* ∇ *denotes the Levi Civita connection of* (M, g) *and* R *the corresponding curvature tensor.*

Let $\tilde{\nabla}$ define on (M, g) by $\tilde{\nabla} := T - \nabla$, then the conditions (i)-(iii) above can be rewritten in the following way:

(i)' $\tilde{\nabla} g = 0$

(ii)' $\tilde{\nabla} R = 0$

(iii)' $\tilde{\nabla} T = 0$.

The aim is to give a naturally reductive homogeneous structure for the simply connected pseudo-Riemannian 2-step nilpotent Lie groups $(N, \langle\,,\,\rangle)$ constructed with the data of Theorem (3.3). Since the metric $\langle\,,\,\rangle$ is left-invariant, it suffices to define such a T for left-invariant vector fields, that is, on $\mathfrak{n}$.

Theorem 3.8. *Let* N *be the simply connected Lie group endowed with the left-invariant pseudo-Riemannian naturally reductive metric induced by* $\langle\,,\,\rangle$ *on its Lie algebra* $\mathfrak{n}$ *(3.3). The following tensor* T *defines a naturally reductive structure:*

$$T_{x_1+v_1} x_2 + v_2 = \tau_{x_1} x_2 + \frac{1}{2}([v_1, v_2] + \pi(x_1)v_2 - \pi(x_2)v_1) \tag{3.7}$$

Proof. Since τ_x and $\pi(x)$ are skew-adjoint on $(\mathfrak{g}, \langle\,,\,\rangle_{\mathfrak{g}})$ and $(\mathfrak{v}, \langle\,,\,\rangle_{\mathfrak{v}})$ respectively for every $x \in \mathfrak{g}$, by computing one gets $\tilde{\nabla}\langle\,,\,\rangle = 0$. By using the expression of the curvature tensor in (5.6) one can compute and verify that $\tilde{\nabla} R = 0$ and also $\tilde{\nabla} T = 0$, thus T is a homogeneous structure. Clearly $T_{x+v} x + v = 0$ so that T is naturally reductive. □

Compare with [21].

4. Examples and applications

Below we expose examples of naturally reductive metrics on 2-step nilpotent Lie groups. This is achieved by translating the data at the Lie algebra level to the corresponding simply connected Lie group by following the key results provided in (3.3). We shall make use of Euclidean and semisimple Lie algebras in order to obtained ad-invariant metrics. For further details on Lie algebras with ad-invariant metrics see for instance [10, 22].

(i) *Riemannian examples.* Naturally reductive Riemannian nilmanifolds arise by considering a data set with $\mathfrak{g}$ compact. Recall that if $\mathfrak{g}$ is compact then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{c}$ where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is a compact semisimple Lie algebra and $\mathfrak{c}$ is the center (see [31]). In [20] they were extended studied.

In the Riemannian case the converse of (*) above holds [32].

(ii) *Modified Riemannian.* Take any of the data sets corresponding to the positive definite case and follow the ideas in (2.4). Clearly all requirements in (3.3) apply and so one can produce naturally reductive pseudo-Riemannian metrics of signature $(\dim \mathfrak{g}, \dim \mathfrak{v})$.

Let $N(\mathfrak{g}, \mathfrak{v})$ denote a Riemannian naturally reductive nilmanifold obtained from a data set $(\mathfrak{g}, \mathfrak{v}, \langle\, ,\, \rangle)$ (where $\mathfrak{g}$ is compact). Let $\tilde{N}(\mathfrak{g}, \mathfrak{v})$ denote the pseudo-Riemannian 2-step nilpotent Lie group obtained by changing the sign of the metric on $\mathfrak{g}$. Therefore by [32]

$$N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}') \qquad \Longleftrightarrow \qquad \mathfrak{n}(\mathfrak{g}, \mathfrak{v}) \simeq \mathfrak{n}(\mathfrak{g}, \mathfrak{v}')$$

and this occurs if and only if there exists an isometric isomorphism $\phi : (\mathfrak{g}, \langle\, ,\, \rangle_+) \to (\mathfrak{g}', \langle\, ,\, \rangle'_+)$ and a isometry $T : (\mathfrak{v}, \langle\, ,\, \rangle_+) \to (\mathfrak{v}', \langle, , \rangle_+)$ such that

$$T\pi(x)T^{-1} = \pi'(\phi x) \qquad \text{for all } x \in \mathfrak{g}.$$

Clearly $\phi : (\mathfrak{g}, -\langle\, ,\, \rangle_+) \to (\mathfrak{g}', -\langle\, ,\, \rangle'_+)$ is also a isometric isomorphism, so that the corresponding simply connected Lie groups are isometric. Thus one has what follows.

Proposition 4.1. *If $N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}')$ then $\tilde{N}(\mathfrak{g}, \mathfrak{v}) \simeq \tilde{N}'(\mathfrak{g}', \mathfrak{v}')$.*

In [20] detailed conditions to get the isometries $N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}')$ were obtained.

(iii) *Abelian center.* Let $\mathbb{R}^{2n}$ be equipped with a metric B, that is, B is determined by a nonsingular symmetric linear map b such that

$$B(x, y) = \langle bx, y \rangle \qquad \text{being } \langle\, ,\, \rangle \text{ the canonical inner product on } \mathbb{R}^{2n}.$$

Let $t \in \mathfrak{so}(\mathbb{R}^{2n}, B)$, that is t may satisfy $t^* = -btb$ where t^* denotes adjoint with respect to the canonical inner product on $\mathbb{R}^{2n}$.

Any nonsingular $t \in \mathfrak{so}(\mathbb{R}^{2n}, B)$ gives rise to a faithful representation of $\mathbb{R}$ to $(\mathbb{R}^{2n}, B)$ without trivial subrepresentations. Let $\mathfrak{n}$ be the vector space direct sum

$\mathbb{R}z \oplus \mathbb{R}^{2n}$ equipped with a metric $\langle\, ,\, \rangle$ such that

$$\langle z, \mathbb{R}^{2n}\rangle = 0 \qquad \langle z, z\rangle = \lambda \in \mathbb{R} - \{0\} \qquad \langle\, ,\, \rangle_{\mathbb{R}^{2n}} = B.$$

Define a Lie bracket on $\mathfrak{n}$ by

$$[z, y] = 0 \qquad \forall y \in \mathfrak{n} \qquad \text{and} \qquad \langle [u, v], z\rangle = B(tu, v) \quad u, v \in \mathbb{R}^{2n}.$$

According to (3.3) this Lie bracket makes of $\mathfrak{n} = \mathbb{R}z \oplus \mathbb{R}^{2n}$ a 2-step nilpotent Lie algebra and the given metric is naturally reductive whenever the center is nondegenerate. This Lie algebra is isomorphic to the Heisenberg Lie algebra.

Furthermore, the group of isometries fixing the identity element has Lie algebra

$$\mathfrak{h} = \mathcal{Z}_{\mathfrak{so}(\mathbb{R}^{2n}, B)}(t) \tag{4.1}$$

where $\mathcal{Z}_{\mathfrak{so}(\mathbb{R}^{2n}, B)}(t)$ denotes the centralizer of t in $\mathfrak{so}(\mathbb{R}^{2n}, B)$, which can be verified by applying Proposition (3.5).

In this way one gets naturally reductive metrics on the Heisenberg Lie group of dimension 2n+1. The converse also holds.

Proposition 4.2. *Any left-invariant pseudo-Riemannian metric on the Heisenberg Lie group H_{2n+1} for which the center is nondegenerate is naturally reductive.*

The isotropy group has Lie algebra $\mathfrak{h}$ as in (4.1).

Proof. Let $\mathfrak{h}_{2n+1}$ denote the Lie algebra of H_{2n+1} and decompose it as a orthogonal direct sum $\mathfrak{h}_{2n+1} = \mathbb{R}z \oplus \mathfrak{v}$. Then the restriction of the metric to $\mathfrak{v}$ defines a metric B of signature (k, m). The map j defined in (2.1) is indeed skew-adjoint with respect to $B := \langle\, ,\, \rangle_{|\mathfrak{v}\times\mathfrak{v}}$ and it generates a subalgebra of $\mathfrak{so}(\mathfrak{v}, B)$. Thus $z \to j(z)$ defines a faithful representation without trivial subrepresentations since by taking $t := j(z)$ one has

$$tu = 0 \Longleftrightarrow B(tu, v) = 0 \quad \forall v \in \mathfrak{v} \Longleftrightarrow B(z, [u, v]) = 0 \quad \forall v \in \mathfrak{v}.$$

But since the center is nondegenerate then $[u, v] = 0$ for all $v \in \mathfrak{v}$, which implies $j(z)u = 0$ and thus $u = 0$. Indeed any nondegenerate metric on $\mathbb{R}z$ is ad-invariant. Hence the statements of (3.3) are satisfied and the metric on $\mathfrak{h}_{2n+1}$ is naturally reductive. □

Example 4.3. Let $\mathfrak{h}_3$ denote the Heisenberg Lie algebra of dimension three with basis e_1, e_2, e_3 satisfying the Lie bracket $[e_1, e_2] = e_3$. Lorentzian metrics on $\mathfrak{h}_3$ with nondegenerate center can be defined by

$$\begin{array}{rrcccl} (1) & -\langle e_3, e_3\rangle & = & 1 & = & \langle e_1, e_1\rangle = \langle e_2, e_2\rangle \\ (2) & \langle e_3, e_3\rangle & = & 1 & = & -\langle e_1, e_1\rangle = \langle e_2, e_2\rangle \end{array}$$

Thus in the basis e_1, e_2 the map $j_1(e_3)$ for the metric in (1) is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(compare with (2.4)) while $j_2(e_3)$ for the metric (2) one has

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The construction on the Heisenberg Lie algebra, can be extended in the following way. Set B a nondegenerate symmetric bilinear form on $\mathbb{R}^k$ and let $t_1, \dots t_l$ be commuting linear maps in $\mathfrak{so}(\mathbb{R}^k, B)$ and such that $\bigcap_i ker(t_i) = \{0\}$.

Set $\mathfrak{n} = \mathbb{R}^l \oplus \mathbb{R}^k$ direct sum of vector spaces, equipped $\mathbb{R}^l$ with any metric and $\mathfrak{n}$ with the product metric such that $\langle \mathbb{R}^l, \mathbb{R}^k \rangle = 0$.

The triple $(\mathbb{R}^l, \mathbb{R}^k, \langle \, , \, \rangle)$ is a data set which induces a naturally reductive metric on the corresponding simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$.

Semisimple center. Let $\mathbb{R}^{p,q}$ denote the real vector space $\mathbb{R}^{p+q}$ endowed with a metric $\langle \, , \rangle_{p,q}$ of signature (p, q). Let $\mathfrak{so}(p, q)$ denote the set of skew-adjoint transformations for $\langle \, , \rangle_{p,q}$. This a semisimple Lie algebra and the Killing form K is a natural ad-invariant metric on $\mathfrak{so}(p, q)$. Indeed $\mathfrak{so}(p, q)$ acts on $\mathbb{R}^{p,q}$ just by evaluation. Take the direct sum as vector spaces $\mathfrak{n} = \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}$ and equipped with the product metric $\langle \, , \, \rangle_{\mathfrak{n}}$ such that $\langle \, , \, \rangle_{\mathfrak{so}(p,q)\times\mathfrak{so}(p,q)} = K$, $\langle \, , \, \rangle_{\mathbb{R}^{p,q}\times\mathbb{R}^{p,q}} = \langle \, , \, \rangle_{p,q}$ and $\langle \mathfrak{so}(p, q), \mathbb{R}^{p,q} \rangle = 0$. Thus a Lie bracket can be defined on $\mathfrak{n}$ by

$$K([u, v], A) = \langle Au, v \rangle_{p,q} \qquad \text{for all } u, v \in \mathbb{R}^{p,q}, A \in \mathfrak{so}(p, q).$$

The corresponding 2-step nilpotent Lie group equipped with the left-invariant metric induced by the metric above, makes of N a naturally reductive pseudo-Riemannian space -Theorem (3.3)-.

A similar construction can be done by restriction of the evaluating action to a nondegenerate subalgebra of $\mathfrak{so}(p, q)$.

(v) *Modified tangent semisimple.* The Killing form K is an ad-invariant metric on any semisimple Lie algebra $\mathfrak{g}$.

Take the Lie algebra $\mathfrak{g}$ together with the Killing form and let $\mathfrak{v}$ denote the underlying vector space to $\mathfrak{g}$ endowed also with the Killing form metric. To this pair $(\mathfrak{g}, \mathfrak{v})$ attach

- the metric given by $\langle \, , \, \rangle_{\mathfrak{g}} = \langle \, , \, \rangle_{\mathfrak{v}} = K$ and $\langle \mathfrak{g}, \mathfrak{v} \rangle = 0$;
- the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{so}(\mathfrak{v}, K)$.

The adjoint representation on $\mathfrak{g}$ is faithful and there is no trivial subrepresentations, so that $(\mathfrak{g}, \mathfrak{v}, K + K)$ constitutes a data set for a 2-step nilpotent Lie group $N(\mathfrak{g}, \mathfrak{v})$ and by (3.3) it is naturally reductive pseudo-Riemannian. Clearly the signature of this metric is twice as much the signature of B and the isometry group can be computed with (3.6).

Remark 4. Bi-invariant metrics give rise to examples with degenerate center. In [26] the author produces an example of a naturally reductive 2-step nilpotent Lie group with degenerate center but where the metric is not bi-invariant.

5. On the geometry of pseudo-Riemannian 2-step nilpotent Lie groups

The aim of this section is to write explicitly some geometric features of pseudo-Riemannian 2-step nilpotent Lie groups.

Recall that a 2-step nilpotent Lie algebra $\mathfrak{n}$ is said to be *nonsingular* if $\mathrm{ad}(x)$ maps $\mathfrak{n}$ onto $\mathfrak{z}$ for every $x \in \mathfrak{n} - \mathfrak{z}$. Suppose $\mathfrak{n}$ is equipped with a metric as in (1.1) then $\mathfrak{n}$ is nonsingular if and only if $j(x)$ is nonsingular for every $x \in \mathfrak{z}$. We shall say that a Lie group is nonsingular if its corresponding Lie algebra is nonsingular.

Whenever N is simply connected 2-step nilpotent the exponential map $\exp : \mathfrak{n} \to N$ produces global coordinates. In terms of this map the product for $z_1, z_2 \in \mathfrak{z}$, $v_1, v_2 \in \mathfrak{v}$ can be obtained by making use of the next equality

$$\exp(z_1 + v_1)\exp(z_2 + v_2) = \exp(z_1 + z_2 + \frac{1}{2}[v_1, v_2] + v_1 + v_2).$$

We shall study the geometry of 2-step nilpotent Lie groups when they are endowed with a left-invariant (pseudo-Riemannian) metric $\langle\, ,\, \rangle$ with respect to which the center is nondegenerate. In the Riemannian case see the work of P. Eberlein [9].

The covariant derivative ∇ is left-invariant, hence one can see ∇ as a bilinear form on $\mathfrak{n}$ getting the formula

$$\nabla_x y = \frac{1}{2}([x, y] - \mathrm{ad}(x)^* y - \mathrm{ad}(y)^* x) \qquad \text{for } x, y \in \mathfrak{n}, \tag{5.1}$$

where $\mathrm{ad}(x)^*$ denotes the adjoint of $\mathrm{ad}(x)$. By writing this explicitly one obtains

$$\begin{array}{rcll} \nabla_x y & = & \frac{1}{2}[x, y] & \text{for all } x, y \in \mathfrak{v} \\ \nabla_x y = \nabla_y x & = & -\frac{1}{2} j(y) x & \text{for all } x \in \mathfrak{v}, y \in \mathfrak{z} \\ \nabla_x y & = & 0 & \text{for all } x, y \in \mathfrak{z} \end{array} \tag{5.2}$$

Since translations on the left are isometries, to describe the geodesics of $(N, \langle\, ,\, \rangle)$ it suffices to describe those geodesics that begin at the identity $e \in N$. Let $\gamma(t)$

be a curve with $\gamma(0) = e$, and let $\gamma'(0) = z_0 + v_0 \in \mathfrak{n}$, where $z_0 \in \mathfrak{z}$ and $v_0 \in \mathfrak{v}$. In exponential coordinates we write

$$\gamma(t) = \exp(z(t) + v(t)), \qquad \text{where } z(t) \in \mathfrak{z},\ v(t) \in \mathfrak{v} \qquad \text{for all } t$$

and ask $z'(0) = z_0$, $v'(0) = v_0$.

The curve $\gamma(t)$ is a geodesic if and only if the following equations are satisfied:

$$v''(t) = j(z_0)v'(t) \text{ for all } t \in \mathbb{R} \tag{5.3}$$

$$z_0 \equiv z'(t) + \frac{1}{2}[v'(t), v(t)] \text{ for all } t \in \mathbb{R} \tag{5.4}$$

These equations were derived by A. Kaplan in [18] to study 2-step nilpotent groups N of Heisenberg type, but the proof is valid in general for 2-step nilpotent Lie groups equipped with a left-invariant pseudo-Riemannian metric where the center is nondegenerate as noted in [11] and [2].

Let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$. Write $\gamma'(0) = z_0 + v_0$, where $z_0 \in \mathfrak{z}$ and $v_0 \in \mathfrak{v}$ and identify $\mathfrak{n} = T_eN$. Then

$$\gamma'(t) = dL_{\gamma(t)}(e^{tj(z_0)}v_0 + z_0) \qquad \text{for all } t \in \mathbb{R} \tag{5.5}$$

where $e^{tj(z_0)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} j(z_0)^n$. In fact, write $\gamma(t) = exp(z(t) + v(t))$, where $z(t)$ and $v(t)$ lie in $\mathfrak{z}$ and $\mathfrak{v}$ respectively for all $t \in \mathbb{R}$. By using the previous equations (5.3) one has

$$\begin{aligned} \gamma'(t) &= d\exp_{z(t)+v(t)}(z'(t) + v'(t))_{z(t)+v(t)} \\ &= dL_{\gamma(t)}(z'(t) + \tfrac{1}{2}[v'(t), v(t)] + v') \\ &= dL_{\gamma(t)}(z_0 + v'). \end{aligned}$$

Now by integrating the first equation of (5.3) one gets $v'(t) = e^{tj(z_0)}v_0$ which proves (5.5).

For x, y elements in $\mathfrak{n}$ the curvature tensor is defined by

$$R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}.$$

Using (5.2) one gets

$$R(x,y)z = \begin{cases} \frac{1}{2}j([x,y])z - \frac{1}{4}j([y,z])x + \frac{1}{4}j([x,z])y & \text{for } x,y,z \in \mathfrak{v}, \\ \frac{1}{4}[j(z)x,y] - [x,j(z)y] & \text{for } x,y \in \mathfrak{v},\ z \in \mathfrak{z}, \\ -\frac{1}{4}[x,j(y)z] & \text{for } x,z \in \mathfrak{v},\ y \in \mathfrak{z}, \\ -\frac{1}{4}j(y)j(z)x & \text{for } x \in \mathfrak{v}, y,z \in \mathfrak{z}, \\ \frac{1}{4}[j(x),j(y)]z & \text{for } x,y \in \mathfrak{z}, z \in \mathfrak{v}, \\ 0 & \text{for } x,y,z \in \mathfrak{z}. \end{cases} \tag{5.6}$$

Let $\Pi \subseteq \mathfrak{n}$ denote a nondegenerate plane and let Q be given by

$$Q(x,y) = \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2.$$

The nondegeneracy property is equivalent to ask $Q(v,w) \neq 0$ for one -hence every- basis $\{v,w\} \in \Pi$ [23]. The sectional curvature of Π is the number $K(x,y) := \langle R(x,y)y,x\rangle/Q(x,y)$, which is independent of the choice of the basis. Now take a orthonormal basis for Π, that is a linearly independent set $\{x,y\}$ such that $\langle x,y\rangle = 0$ and $\langle x,x\rangle = \pm 1$ and $\langle y,y\rangle = \pm 1$.

After (5.6) one obtains

$$K(x,y) = \begin{cases} -\frac{3\varepsilon_1\varepsilon_2}{4}\langle [x,y],[x,y]\rangle & \text{for } x,y \in \mathfrak{v} \\ -\frac{\varepsilon_1\varepsilon_2}{4}\langle j(y)x, j(y)x\rangle & \text{for } x \in \mathfrak{v}, y \in \mathfrak{z}, \\ 0 & \text{for } x,y \in \mathfrak{z} \end{cases} \tag{5.7}$$

being $\varepsilon_1 := \langle x,x\rangle$ and $\varepsilon_2 := \langle y,y\rangle$.

The Ricci tensor is given by $Ric(x,y) = \text{trace}(z \to R(z,x)y), z \in \mathfrak{n}$ for arbitrary elements $x,y \in \mathfrak{n}$.

Proposition 5.1. *Let $\{z_i\}$ denote a orthonormal basis of $\mathfrak{z}$ and $\{v_j\}$ a orthonormal basis of $\mathfrak{v}$. It holds*

$$Ric(x,y) = \begin{cases} 0 & \textit{for } x \in \mathfrak{v}, y \in \mathfrak{z} \\ \frac{1}{2}\sum_i \varepsilon_i \langle j(z_i)^2 x, y\rangle & \textit{for } x,y \in \mathfrak{v},\ \varepsilon_i = \langle z_i, z_i\rangle \\ -\frac{1}{4}\sum_j \varepsilon_j \langle j(x)j(y)v_j, v_j\rangle & \textit{for } x,y \in \mathfrak{z},\ \varepsilon_j = \langle v_j, v_j\rangle. \end{cases}$$

Due to symmetries of the curvature tensor, the Ricci tensor is a symmetric bilinear form on $\mathfrak{n}$ and hence there exists a symmetric linear transformation $T : \mathfrak{n} \to \mathfrak{n}$ such that $Ric(x,y) = \langle Tx, y\rangle$ for all $x, y \in \mathfrak{n}$. T is called the Ricci transformation. Let $\{e_k\}$ denote a orthonormal basis of $\mathfrak{n}$; it holds

$$Ric(x,y) = \sum_k \varepsilon_k \langle R(e_k,x)y, e_k\rangle = \langle -\sum_k \varepsilon_k R(e_k,x)e_k, y\rangle$$

which implies

$$T(x) = -\sum_k \varepsilon_k R(e_k,x)e_k, \qquad \text{being } \varepsilon_k = \langle e_k, e_k\rangle. \tag{5.8}$$

According to the results in (5.1) we have that $\mathfrak{z}$ and $\mathfrak{v}$ are T-invariant subspaces and

$$T(x) = \begin{cases} \frac{1}{2}\sum_i \varepsilon_i j(z_i)^2 x & x \in \mathfrak{v}, \quad \varepsilon_i = \langle z_i, z_i\rangle \\ \\ \frac{1}{4}\sum_j \varepsilon_j [v_j, j(x)v_j] & x \in \mathfrak{z} \quad\quad \varepsilon_j = \langle v_j, v_j\rangle. \end{cases}$$

where $\{z_i\}$ and $\{v_j\}$ are orthonormal basis of $\mathfrak{z}$ and $\mathfrak{v}$ respectively.

Remark 5. The formulas above were used in [6] to prove (2.2).

Acknowledgements. The author expresses her deep gratitude to Victor Bangert, for his support during the stay of the author at the Universität Freiburg, where this work was done.

References

[1] W. Ambrose, I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647 - 669.

[2] M. Boucetta, *Ricci flat left-invariant pseudo-Riemannian metrics on 2-step nilpotent Lie groups*, arXiv:0910.2563.

[3] G. Calvaruso, *Homogeneous structures on three dimensional Lorentzian Lie manifolds*, J. Geom. Phys. **57** (2007), 1279 - 1291.

[4] G. Calvaruso, R. A. Marinosci, *Homogeneous geodesics of three dimensional unimodular Lorentzian Lie groups*, Mediterr. J. Math. **3** (2006), 467 - 481.

[5] G. Calvaruso, R. A. Marinosci, *Homogeneous geodesics of non unimodular Lorentzian Lie groups and naturally Lorentzian spaces in dimension three*, Adv. Geom. **8** (2008), 473 - 489.

[6] L. Cordero, P. Parker, *Isometry groups of pseudoriemannian 2-step nilpotent Lie groups*, Houston J. Math. **35** (1) (2009), 49 - 72.

[7] L. Cordero, P. Parker, *Lattices and Periodic Geodesics in Pseudoriemannian 2-step Nilpotent Lie Groups*, Int. J. Geom. Methods Mod. Phys. **5** (2008), 79 - 99.

[8] J. E. D'Atri, W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc. **18** no. 215 (1979).

[9] P. Eberlein, *Geometry of 2-step nilpotent Lie groups with a left-invariant metric*, Ann. sci. Éc. Norm. Sup. **27** (1994), 611 - 660.

[10] G. Favre, L. J. Santharoubane, *Symmetric, invariant, nondegenerate bilinear form on a Lie algebra*, J. Algebra **105** (2) (1987), 451 - 464.

[11] M. Guediri, *Lorentz geometry of 2-step nilpotent Lie groups*, Geom. Dedicata **100** (2003), 11 - 51.

[12] P. M. Gadea, J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (3) (1992), 449 - 465.

[13] P. M. Gadea, J. A. Oubiña, *Reductive homogeneous pseudo-Riemannian manifolds*, Monatsh. Math. **124** (1) (1997), 17 - 34.

[14] C. Gordon, *Naturally reductive homogeneous Riemannian manifolds*, Canad. J. Math. **37** (3) (1985), 467 - 487.

[15] C. Jang, K. Park, T. Lee, *On conjugate points of the group* $H(2,1)$, EAMJ East Asian Math. J. **22** (2) (2006), 249 - 257.

[16] C. Jang, P. E. Parker, *Conjugate loci of pseudo-Riemannian 2-step nilpotent Lie groups with nondegenerate center*, Ann. Global Anal. Geom. **28** (1) (2005), 1 - 18.

[17] C. Jang, P. E. Parker; K. Park, *Pseudo* H*-type 2-step nilpotent Lie groups*, Houston J. Math. **31** (3) (2005), 765 - 786.

[18] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35 - 42.

[19] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Interscience Publ. (1969).

[20] J. Lauret, *Homogeneous nilmanifolds attached to representations of compact Lie groups* Manuscripta math. **99** (1999), 287 - 309.

[21] J. Lauret, *Naturally reductive homogeneous structures on* 2*-step nilpotent Lie groups* Rev. Un. Mat. Argentina **41** (2) (1998), 15 - 23.

[22] A. Medina, P. Revoy, *Algèbres de Lie et produit scalaire invariant* (French) [Lie algebras and invariant scalar products], Ann. Sci. École Norm. Sup. (4) **18** no. 3 (1985), 553 - 561.

[23] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press (1983).

[24] G. Ovando, *Two-step nilpotent Lie algebras with ad-invariant metrics and a special kind of skew-symmetric maps*, J. Algebra Appl. **6** (6) (2007), 897 - 917.

[25] G. Ovando, *Naturally reductive pseudo-Riemannian spaces*, J. Geom. Phys. **61** (1) (2011), 157 - 171 .

[26] G. Ovando, *Examples of naturally reductive pseudo-Riemannian Lie groups*, AIP Conference Proceedings **1360**, XIX International Fall Workshop on Geometry and Physics (2011), 157 - 163.

[27] P. E. Parker, *Pseudoriemannian nilpotent Lie groups with nondegenerate center*, Encyclopedia of Math. Phys., **4** (2006), 94 - 104.

[28] F. Tricerri, L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lect. Notes **83** Cambridge Univ. Press (1983).

[29] F. Tricerri, L. Vanhecke, *Naturally reductive homogeneous spaces and generalized Heisenberg groups*, Composition Math. **52** (1984), 389 - 408.

[30] V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Springer Verlag New York Inc. (1984).

[31] N. Wallach, *Real reductive groups I*, Boston: Academic Press (1988).

[32] E. Wilson, *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), 337 - 346.

Received June 15, 2011

G. Ovando, CONICET and Dept. de Mate., ECEN - FCEIA, Universidad Nacional de Rosario, Pellegrini 250, 2000 Rosario, Argentina.

E-mail address: gabriela@fceia.unr.edu.ar