# MAXIMAL FUNCTION CHARACTERIZATION OF HARDY SPACES RELATED TO LAGUERRE POLYNOMIAL EXPANSIONS 

JORGE J. BETANCOR, ESTEFANÍA DALMASSO, PABLO QUIJANO, AND ROBERTO SCOTTO


#### Abstract

In this paper we introduce the atomic Hardy space $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ associated with the non-doubling probability measure $d \gamma_{\alpha}(x)=\frac{2 x^{2 \alpha+1}}{\Gamma(\alpha+1)} e^{-x^{2}} d x$ on $(0, \infty)$, for $\alpha>-\frac{1}{2}$. We obtain characterizations of $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ by using two local maximal functions. We also prove that the truncated maximal function defined through the heat semigroup generated by the Laguerre differential operator is bounded from $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$.


## 1. Introduction and main results

We consider, for $\alpha>-\frac{1}{2}$, the probability measure $d \gamma_{\alpha}(x)=\frac{2 x^{2 \alpha+1}}{\Gamma(\alpha+1)} e^{-x^{2}} d x$ on $(0, \infty)$. The measure $\gamma_{\alpha}$ is not doubling (not even locally doubling) with respect to the usual metric in $(0, \infty)$. We could define Hardy spaces as in [24], since it is clear that $\sup _{r, x \in(0, \infty)} \gamma_{\alpha}(I(x, r)) / r<\infty$, where for every $r, x \in(0, \infty)$, we denote $I(x, r):=(x-r, x+r) \cap(0, \infty)$. However, Tolsa's definition is not satisfactory for our purposes because the harmonic analysis operators associated with Laguerre polynomial expansions are not Calderón-Zygmund operators in this setting. We will define the Hardy space related to the measure $\gamma_{\alpha}$ in $(0, \infty)$ following the ideas developed in [17] (see also [4]) for the Ornstein-Uhlenbeck setting and the non-standard Gaussian measure $d \gamma(x)=\pi^{-n / 2} e^{-|x|^{2}} d x$ in $\mathbb{R}^{n}$.

In order to select a family of intervals over which $\gamma_{\alpha}$ is indeed doubling, we consider the admissibility function $m(x)=\min \left\{1, \frac{1}{x}\right\}$, for $x \in(0, \infty)$. Given $a>0$, we say that an interval $I(x, r)$ with $0<r \leq x$ is $a$-admissible if $r \leq a m(x)$. The class of such intervals will be denoted by $\mathcal{B}_{a}$. To simplify notation, we shall write $\mathcal{B}:=\mathcal{B}_{1}$. For every $a>0$, it is easy to see that the measure $\gamma_{\alpha}$ is doubling on $\mathcal{B}_{a}$, that is, there exists $C_{\alpha, a}>0$ such that

$$
\gamma_{\alpha}(I(x, 2 r)) \leq C_{\alpha, a} \gamma_{\alpha}(I(x, r)), \quad I(x, r) \in \mathcal{B}_{a}
$$

The atoms will be defined over these families of intervals as follows. Given $1<q \leq \infty$ and $a>0$, a measurable function $b$ defined on $(0, \infty)$ is said to be an $(a, q, \alpha)$-atom when $b(x)=1$ for every $x \in(0, \infty)$, or there exist $0<r \leq x$ such that $I(x, r) \in \mathcal{B}_{a}$ and satisfying that
(i) $\operatorname{supp}(b) \subset I(x, r)$;
(ii) $\|b\|_{L^{q}\left((0, \infty), \gamma_{\alpha}\right)} \leq \gamma_{\alpha}(I(x, r))^{\frac{1}{q}-1}$, where $\frac{1}{q}=0$ when $q=\infty$;
(iii) $\int_{0}^{\infty} b(y) d \gamma_{\alpha}(y)=0$.

Date: August 16, 2022.
2020 Mathematics Subject Classification. 42A50, 42B20, 42B25.
Key words and phrases. Hardy spaces, atoms, maximal functions, Laguerre polynomials.
The first author is partially supported by PID2019-106093GB-I00 (Ministerio de Ciencia e Innovación, Spain). The second and fourth authors are partially supported by grants PICT-2019-2019-00389 (ANPCyT), PIP-11220200101916CO (CONICET) and CAI+D 2019-015 (UNL).

For $1<q \leq \infty$ and $a>0$, the atomic Hardy space $\mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ consists of all of those $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ such that $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$ where, for every $j \in \mathbb{N}, b_{j}$ is an $(a, q, \alpha)$-atom and $\lambda_{j} \in \mathbb{C}$ with $\sum_{j=0}^{\infty}\left|\lambda_{j}\right|<\infty$. Note that, under the above conditions, the series $\sum_{j=0}^{\infty} \lambda_{j} b_{j}$ converges in $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. We define, for every $f \in \mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$,

$$
\|f\|_{\mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)}=\inf \sum_{j=0}^{\infty}\left|\lambda_{j}\right|
$$

where the infimum is taken over all the sequences $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}} \in \mathbb{C}$ with $\sum_{j=0}^{\infty}\left|\lambda_{j}\right|<\infty$ and $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$, where $b_{j}$ is an $(a, q, \alpha)$-atom, for every $j \in \mathbb{N}$.

Actually, the space $\mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ does not depend on $a$ and $q$. The following result can be proved by proceeding as in [17, Section 2] and [18, Theorem 2.2].

Theorem 1.1. Let $\alpha>-\frac{1}{2}, a>0$ and $1<q \leq \infty$. Then, $\mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ and $\mathcal{H}_{1}^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$ coincide, algebraic and topologically.

In view of the above result, we will simply write $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ to refer to the Hardy space $\mathcal{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$, for any $a>0$ and $1<q \leq \infty$.

We now introduce a maximal function we shall use in order to characterize $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Our definition is inspired in the one given in [16, p. 273].

We consider the measure $\mathfrak{m}_{\alpha}$ on $(0, \infty)$ by $d \mathfrak{m}_{\alpha}(x)=x^{2 \alpha+1} d x$. By $C_{c}^{1}(0, \infty)$ we denote the space of continuously differentiable functions with compact support in $(0, \infty)$.

We define, for every $x \in(0, \infty)$, the sets $\mathcal{A}_{x}^{\alpha}$ and $\mathcal{A}_{x, \text { loc }}^{\alpha}$ of functions as follows. Given $x \in(0, \infty)$, a function $\phi \in C_{c}^{1}(0, \infty)$ is said to be in $\mathcal{A}_{x}^{\alpha}$ when there exists $0<r \leq x$ such that $\operatorname{supp}(\phi) \subset I(x, r)$ and

$$
\mathfrak{m}_{\alpha}(I(x, r))\|\phi\|_{\infty} \leq 1, \quad r \mathfrak{m}_{\alpha}(I(x, r))\left\|\phi^{\prime}\right\|_{\infty} \leq 1
$$

On the other hand, when all of the above hold with some $0<r \leq \min \{x, m(x)\}$, $\phi$ is said to be in $\mathcal{A}_{x, \text { loc }}^{\alpha}$. In other words, $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ when $\phi \in \mathcal{A}_{x}^{\alpha}$ with $I(x, r) \in \mathcal{B}$. Here, $\|g\|_{\infty}$ denotes the essential supremum of $g$ in $(0, \infty)$ with respect to the Lebesgue measure (equivalently, to $\mathfrak{m}_{\alpha}$ or $\gamma_{\alpha}$ ).

We are now in position to define the maximal functions we shall be dealing with. Given $f \in L^{1}\left((0, \delta), \mathfrak{m}_{\alpha}\right)$ for every $\delta>0$, we set

$$
\mathcal{M}_{\alpha}(f)(x)=\sup _{\phi \in \mathcal{A}_{x}^{\alpha}}\left|\int_{0}^{\infty} f(y) \phi(y) d \mathfrak{m}_{\alpha}(y)\right|, \quad x \in(0, \infty)
$$

and

$$
\mathcal{M}_{\alpha, \text { loc }}(f)(x)=\sup _{\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}}\left|\int_{0}^{\infty} f(y) \phi(y) d \mathfrak{m}_{\alpha}(y)\right|, \quad x \in(0, \infty)
$$

Theorem 1.2. Given $\alpha>-\frac{1}{2}$. A function $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ is in $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ if and only if $\mathcal{M}_{\alpha, \operatorname{loc}}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $\mathcal{E}_{\alpha}(f):=\int_{0}^{\infty} y\left|\int_{y}^{\infty} f d \gamma_{\alpha}\right| d y<\infty$. Furthermore, for every $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$,

$$
\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)} \simeq\left\|\mathcal{M}_{\alpha, \operatorname{loc}}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathcal{E}_{\alpha}(f)<\infty .
$$

In [8] Dziubański defined Hardy spaces associated with Laguerre function sequences that are basis in $L^{2}((0, \infty), d x)$. He introduced the admissibility function $w(x)=\frac{1}{8} \min \left\{x, \frac{1}{x}\right\}, x \in(0, \infty)$ in order to define the atoms. Later, Cha and Li ([5] and [6]) used the function $w$ to define some BMO-type spaces that can be identified with the duals of the Hardy spaces introduced in [8].

Hardy spaces $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ can also be defined using the admissibility function $w$ instead of the function $m$. The denominator 8 in the function $w$ is not important for us but we prefer to keep the notation given in [5], [6] and [8].

If $a \in(0,8)$, we say that an interval $I(x, r)$ is in $\mathbb{B}_{a}$ when $x \in(0, \infty)$ and $0<r \leq a w(x)$. We write $\mathbb{B}:=\mathbb{B}_{1}$.

Let $1<q \leq \infty$ and $a \in(0,8)$. A measurable function $b$ in $(0, \infty)$ is said to be an $(a, q, \alpha)_{w^{-}}$atom when $b(x)=1$ for every $x \in(0, \infty)$, or there exist $x, r \in(0, \infty)$ such that $I(x, r) \in \mathbb{B}_{a}$ and the following properties hold
(i) $\operatorname{supp}(b) \subset I(x, r)$;
(ii) $\|b\|_{L^{q}\left((0, \infty), \gamma_{\alpha}\right)} \leq \gamma_{\alpha}(I(x, r))^{\frac{1}{q}-1}$;
(iii) $\int_{0}^{\infty} b(x) d \gamma_{\alpha}(x)=0$.

We now define the atomic Hardy space $\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ as follows. We say that a measurable function $f$ defined on $(0, \infty)$ belongs to $\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ whenever $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$, where, for every $j \in \mathbb{N}, b_{j}$ is an $(a, q, \alpha)_{w^{-}}$-atom and $\lambda_{j} \in \mathbb{C}$ being $\sum_{j=0}^{\infty}\left|\lambda_{j}\right|<\infty$. For every $f \in \mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$, we define

$$
\|f\|_{\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)}=\inf \sum_{j=0}^{\infty}\left|\lambda_{j}\right|
$$

where the infimum is taken over all the sequences $\left\{\lambda_{j}\right\}_{j \in \mathbb{N}}$ of complex numbers for which $\sum_{j=0}^{\infty}\left|\lambda_{j}\right|<\infty$ and $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$ with $(a, q, \alpha)_{w}$-atoms $b_{j}$, for every $j \in \mathbb{N}$.

In order to characterize this space by a local maximal function, we shall consider the class of functions $\mathbb{A}$ defined by

$$
\mathbb{A}=\left\{\phi \in C_{c}^{1}(-1,1):\|\phi\|_{\infty} \leq 1,\left\|\phi^{\prime}\right\|_{\infty} \leq 1\right\}
$$

For $\phi \in \mathbb{A}$, we write $\phi_{t}(x)=\frac{1}{t} \phi\left(\frac{x}{t}\right)$, for $x \in \mathbb{R}$ and $t \in(0, \infty)$. Note that if $\phi \in \mathbb{A}$, then $\operatorname{supp}\left(\phi_{t}(x-\cdot)\right) \subset I(x, t) \subset(0, \infty)$ provided that $0<t \leq x<\infty$.

For every $a \in(0,8)$, we define the local maximal function $\mathbb{M}_{a, \text { loc }}$ by

$$
\mathbb{M}_{a, \text { loc }}(f)(x)=\sup \left\{\left|\phi_{t} * f(x)\right|: \phi \in \mathbb{A}, 0<t<a w(x)\right\}, \quad x \in(0, \infty),
$$

for every $f \in L_{\text {loc }}^{1}((0, \infty), d x)$, where by $*$ we denote the usual convolution in $\mathbb{R}$.
We also introduce, for every $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$, the following quantity

$$
\mathbb{E}_{\alpha}(f)=\int_{0}^{1} \frac{1}{y}\left|\int_{0}^{y} f(x) d \gamma_{\alpha}(x)\right| d y+\int_{1}^{\infty} y\left|\int_{y}^{\infty} f(x) d \gamma_{\alpha}(x)\right| d y
$$

We will prove that $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ can also be characterized by means of $\mathbb{M}_{a, \text { loc }}$ and $\mathbb{E}_{\alpha}$. Moreover, the atomic spaces $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ coincide for any $a \in(0,1]$ and $1<q \leq \infty$.
Theorem 1.3. Let $\alpha>-\frac{1}{2}, a \in(0,1]$ and $1<q \leq \infty$. For each $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$, the following statements are equivalent.
(a) $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$;
(b) $f \in \mathbb{H}_{2 a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$;
(c) $\mathbb{M}_{a, \operatorname{loc}}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $\mathbb{E}_{\alpha}(f)<\infty$.

Moreover, the quantities $\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)},\|f\|_{\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)}$ and

$$
\left\|\mathbb{M}_{a, \operatorname{loc}}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathbb{E}_{\alpha}(f)
$$

are equivalent.
Remark 1.4. From the proof of this theorem (see Section 3) we shall deduce that the equivalence of properties (b) and (c) still holds for every $\alpha>-1$.
Remark 1.5. The independence of the parameters $a \in(0,1]$ and $1<q \leq \infty$ in the space $\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)$ follows from Theorem 1.3.

We now give some definitions and basic properties about Laguerre polynomial expansions and the heat semigroup generated by the Laguerre operator.

Let $\alpha>-\frac{1}{2}$. For every $k \in \mathbb{N}$, the Laguerre polynomial $L_{k}^{\alpha}$ of order $\alpha$ and degree $k$ is defined (see [14]) by

$$
L_{k}^{\alpha}(x)=\sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1) k!}} e^{x} x^{-\alpha} \frac{d^{k}}{d x^{k}}\left(e^{-x} x^{k+\alpha}\right), \quad x \in(0, \infty)
$$

We consider the Laguerre operator $\widetilde{\Delta}_{\alpha}$ given by

$$
\widetilde{\Delta}_{\alpha}(f)(x)=-\frac{1}{4}\left(\frac{d^{2}}{d x^{2}}+\left(\frac{2 \alpha+1}{x}-2 x\right) \frac{d}{d x}\right) f(x), \quad f \in C^{2}(0, \infty)
$$

By setting, for every $k \in \mathbb{N}, \mathcal{L}_{k}^{\alpha}(x)=L_{k}^{\alpha}\left(x^{2}\right), x \in(0, \infty)$, the sequence $\left\{\mathcal{L}_{k}^{\alpha}\right\}_{k \in \mathbb{N}}$ is an orthonormal basis in $L^{2}\left((0, \infty), \gamma_{\alpha}\right)$. Furthermore, $\widetilde{\Delta}_{\alpha} \mathcal{L}_{k}^{\alpha}=k \mathcal{L}_{k}^{\alpha}, k \in \mathbb{N}$.

For every $f \in L^{2}\left((0, \infty), \gamma_{\alpha}\right)$, we define

$$
c_{k}^{\alpha}(f)=\int_{0}^{\infty} f(x) \mathcal{L}_{k}^{\alpha}(x) d \gamma_{\alpha}(x), \quad k \in \mathbb{N}
$$

We consider the operator $\Delta_{\alpha}$ given by

$$
\Delta_{\alpha}(f)=\sum_{k=0}^{\infty} k c_{k}^{\alpha}(f) \mathcal{L}_{k}^{\alpha}, \quad f \in D\left(\Delta_{\alpha}\right)
$$

where

$$
D\left(\Delta_{\alpha}\right)=\left\{f \in L^{2}\left((0, \infty), \gamma_{\alpha}\right): \sum_{k=0}^{\infty}\left|k c_{k}^{\alpha}(f)\right|^{2}<\infty\right\}
$$

Thus $\Delta_{\alpha}$ is an extension on $L^{2}\left((0, \infty), \gamma_{\alpha}\right)$ of $\widetilde{\Delta}_{\alpha}$ firstly defined on $C_{c}^{\infty}(0, \infty)$ (the space of smooth functions with compact support in $(0, \infty)$ ). The operator $\Delta_{\alpha}$ is self-adjoint and positive and, moreover, $-\Delta_{\alpha}$ generates a semigroup of operators $\left\{W_{t}^{\alpha}\right\}_{t>0}$ in $L^{2}\left((0, \infty), \gamma_{\alpha}\right)$ where, for every $t>0$, that

$$
W_{t}^{\alpha}(f)=\sum_{k=0}^{\infty} e^{-k t} c_{k}^{\alpha}(f) \mathcal{L}_{k}^{\alpha}, \quad f \in L^{2}\left((0, \infty), \gamma_{\alpha}\right)
$$

By using the Hille-Hardy formula ([14, (4.17.6)]) we can write, for every $x, y, t>0$,

$$
\begin{aligned}
\sum_{k=0}^{\infty} e^{-k t} \mathcal{L}_{k}^{\alpha}(x) \mathcal{L}_{k}^{\alpha}(y)= & \frac{\Gamma(\alpha+1)}{1-e^{-t}}\left(e^{-t / 2} x y\right)^{-\alpha} I_{\alpha}\left(\frac{2 e^{-t / 2} x y}{1-e^{-t}}\right) \\
& \times \exp \left(-\frac{e^{-t}}{1-e^{-t}}\left(x^{2}+y^{2}\right)\right)
\end{aligned}
$$

where $I_{\alpha}$ denotes the modified Bessel function of the first kind and order $\alpha$.
We get, for every $f \in L^{2}\left((0, \infty), \gamma_{\alpha}\right)$ and $t>0$,

$$
\begin{equation*}
W_{t}^{\alpha}(f)(x)=\int_{0}^{\infty} W_{t}^{\alpha}(x, y) f(y) d \gamma_{\alpha}(y), \quad x \in(0, \infty) \tag{1.1}
\end{equation*}
$$

being, for every $x, y, t>0$,

$$
W_{t}^{\alpha}(x, y)=\frac{\Gamma(\alpha+1)}{1-e^{-t}}\left(e^{-t / 2} x y\right)^{-\alpha} I_{\alpha}\left(\frac{2 e^{-t / 2} x y}{1-e^{-t}}\right) \exp \left(-\frac{e^{-t}}{1-e^{-t}}\left(x^{2}+y^{2}\right)\right)
$$

The integral in (1.1) is absolutely convergent for every $f \in L^{p}\left((0, \infty), \gamma_{\alpha}\right)$ and every $1 \leq p<\infty$. Moreover, by defining, for every $t>0, W_{t}^{\alpha}$ on $L^{p}\left((0, \infty), \gamma_{\alpha}\right)$ with $1 \leq p<\infty$, by (1.1) $\left\{W_{t}^{\alpha}\right\}_{t>0}$ is a symmetric diffusion semigroup in the sense of Stein ([23]).

The maximal operator defined by $\left\{W_{t}^{\alpha}\right\}_{t>0}$ is given by

$$
W_{*}^{\alpha}(f)=\sup _{t>0}\left|W_{t}^{\alpha}(f)\right| .
$$

It is known that $W_{*}^{\alpha}$ is bounded on $L^{p}\left((0, \infty), \gamma_{\alpha}\right)$, for every $1<p<\infty$ (see [23, p. 73]). Furthermore, from the Muckenhoupt's results ([19]) it can be deduced that $W_{*}^{\alpha}$ is bounded from $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$.

Suggested by the results in [15] and [20] we consider the truncated maximal operator $\mathbb{W}_{*}^{\alpha}$ defined by

$$
\mathbb{W}_{*}^{\alpha}(f)=\sup _{0<t<m(x)^{2}}\left|W_{t}^{\alpha}(f)\right| .
$$

Since $\mathbb{W}_{*}^{\alpha}(f) \leq W_{*}^{\alpha}(f), \mathbb{W}_{*}^{\alpha}$ is also bounded on $L^{p}\left((0, \infty), \gamma_{\alpha}\right)$, for every $1<p<\infty$, and from $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$. Furthermore, we can prove the following endpoint estimate for $\mathbb{W}_{*}^{\alpha}$.

Theorem 1.6. For any $\alpha \geq 0$, the operator $\mathbb{W}_{*}^{\alpha}$ is bounded from $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$.

Two open questions related to our results are worth noting:
(1) Is the maximal operator $W_{*}^{\alpha}$ bounded from the Hardy space $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ ?
(2) Can the Hardy space $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ be characterized by using the maximal operator $\mathbb{W}_{*}^{\alpha}$ ?

As far as we know, these two questions have not been resolved even in the OrnsteinUhlenbeck setting (see [15] and [20]).

We will now present the proofs for our Theorems 1.2, 1.3 and 1.6 in the following sections.

Throughout this paper, by $C$ and $c$ we will always denote positive constants that may change in each occurrence. We will use many times the following properties (see, for instance, [25, Section 1.2]) without mentioning them: for every $a>0$ there exists $C_{a}>1$ such that $\frac{1}{C_{a}} m(x) \leq m(y) \leq C_{a} m(x)$ and $\frac{1}{C_{a}} \gamma(x) \leq \gamma(y) \leq C_{a} \gamma(x)$ provided that $|x-y| \leq a m(x)$, where $\gamma(x)=e^{-|x|^{2}}$ for $x \in(0, \infty)$.

## 2. Proof of Theorem 1.2.

We now establish some properties that we will use in the proof of Theorem 1.2.
Lemma 2.1. Let $\alpha>-\frac{1}{2}, x_{0}>0$ and $0<r_{0} \leq \min \left\{x_{0}, m\left(x_{0}\right)\right\}$.
Suppose that $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ with $\operatorname{supp}(f) \subset I\left(x_{0}, r_{0}\right)$. Then, the support of $\mathcal{M}_{\alpha, \text { loc }}(f)$ is contained in $I\left(x_{0}, h m\left(x_{0}\right)\right)=\left(x_{0}-h m\left(x_{0}\right), x_{0}+h m\left(x_{0}\right)\right) \cap(0, \infty)$, for a certain $h>1$ which does not depend on $f, x_{0}$ or $r_{0}$.

Proof. We fix $x>0$ and take $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ with $\operatorname{supp}(\phi) \subset I(x, r)$, for some $0<r \leq \min \{x, m(x)\}$. Clearly, we can assume $\operatorname{supp}(\phi) \cap I\left(x_{0}, r_{0}\right) \neq \emptyset$ since, otherwise, $\int_{0}^{\infty} \phi(y) f(y) y^{2 \alpha+1} d y=0$. Choosing $z \in \operatorname{supp}(\phi) \cap I\left(x_{0}, r_{0}\right)$, it follows that $m(x) \sim m(z) \sim m\left(x_{0}\right)$, from which yields that $x \in I\left(x_{0}, h m\left(x_{0}\right)\right)$, for some $h>1$. Thus, we have proved that $\operatorname{supp}\left(\mathcal{M}_{\alpha, \operatorname{loc}}(f)\right) \subset I\left(x_{0}, h m\left(x_{0}\right)\right)$.

We consider, for every $j \in \mathbb{N} \backslash\{0\}, I_{j}=(\sqrt{j-1}, \sqrt{j+1})$ and $I_{0}=(0,1)$. The center and the radius of $I_{j}$ are $c_{j}=\frac{1}{2}(\sqrt{j+1}+\sqrt{j-1})$ and $r_{j}=1 /\left(2 c_{j}\right)$, respectively, for every $j \in \mathbb{N} \backslash\{0\}$, and the center and radius of $I_{0}$ are $c_{0}=1 / 4$ and $r_{0}=1 / 4$, respectively. Then, $I_{j} \in \mathcal{B}$ for each $j \in \mathbb{N}$. We also define $\mathcal{I}_{j}=I_{j}$, $\mathcal{I}_{-j}=(-\sqrt{j+1},-\sqrt{j-1}), j \in \mathbb{N} \backslash\{0\}$, and $\mathcal{I}_{0}=(-1,1)$. We consider a partition of unity $\left\{\hat{\eta}_{j}\right\}_{j \in \mathbb{N}}$ associated with $\left\{\mathcal{I}_{j}\right\}_{j \in \mathbb{Z}}$ (see [18, p. 1685]). We define $\eta_{j}=\hat{\eta}_{j}$, $j \in \mathbb{N}, j \geq 1$, and $\eta_{0}=\left.\left(\hat{\eta}_{0}\right)\right|_{I_{0}}$.

Lemma 2.2. Let $\alpha>-\frac{1}{2}$. There exist $h, C>0$ such that, for every $x \in(0, \infty)$ and $j \in \mathbb{N}$, and any $g \in L^{1}\left((0, \delta), \gamma_{\alpha}\right)$, for each $\delta>0$,

$$
\mathcal{M}_{\alpha, \operatorname{loc}}\left(g \eta_{j} \gamma\right)(x) \leq C \gamma\left(c_{j}\right) \mathcal{M}_{\alpha, \operatorname{loc}}(g)(x) \chi_{h I_{j}}(x)
$$

Proof. Let $g \in L^{1}\left((0, \delta), \gamma_{\alpha}\right)$, for each $\delta>0, x \in(0, \infty)$ and $j \in \mathbb{N}$. Assume that $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ and define $\psi=\phi \eta_{j} \gamma / \gamma\left(c_{j}\right)$. We have that $\psi \in C_{c}^{1}(0, \infty)$ and
(i) $\operatorname{supp}(\psi) \subset \operatorname{supp}(\phi) \subset I(x, r)$, for some $0<r \leq \min \{x, m(x)\}$;
(ii) $\mathfrak{m}_{\alpha}(I(x, r))\|\psi\|_{\infty} \leq C \mathfrak{m}_{\alpha}(I(x, r))\|\phi\|_{\infty} \leq C$;
(iii) since $|\eta| \leq 1, r_{j}\left|\eta_{j}^{\prime}\right| \leq C$, and $0<r \leq 1$, we have that

$$
\begin{aligned}
r \mathfrak{m}_{\alpha}(I(x, r)) \psi^{\prime}(y) \leq & \frac{r \mathfrak{m}_{\alpha}(I(x, r))}{\gamma\left(c_{j}\right)}\left(\left|\phi^{\prime}(y)\right| \eta_{j}(y) \mid \gamma(y)\right. \\
& \left.+|\phi(y)|\left|\eta_{j}^{\prime}(y)\right| \gamma(y)+2|\phi(y)|\left|\eta_{j}(y)\right| y \gamma(y)\right) \\
\leq & C,
\end{aligned}
$$

for any $y \in(0, \infty)$. Indeed, assume that $I(x, r) \cap I_{j} \neq \emptyset$. By taking $z \in I(x, r) \cap I_{j}$ we deduce that $m(x) \sim m(z) \sim m\left(c_{j}\right)$. Furthermore, if $y \in I_{j}$, then

$$
\begin{aligned}
r y & =r\left(y-c_{j}+c_{j}\right) \leq r\left(r_{j}+c_{j}\right) \\
& \leq m(x)\left(r_{j}+c_{j}\right) \leq C m\left(c_{j}\right)\left(r_{j}+c_{j}\right) \leq C .
\end{aligned}
$$

Also, according to Lemma 2.1, $\operatorname{supp}\left(\mathcal{M}_{\alpha, \text { loc }}\left(g \eta_{j} \gamma\right)\right) \subset h I_{j}$ for some $h \geq 1$. With this, we obtain the desired result.

Lemma 2.3. Let $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $j \in \mathbb{N}$. We define

$$
b_{j}=\frac{\int f \eta_{j} d \gamma_{\alpha}}{\int \eta_{j} d \gamma_{\alpha}} .
$$

Suppose that $\mathcal{M}_{\alpha, \operatorname{loc}}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Then, $\mathcal{M}_{\alpha}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right) \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$ with

$$
\left\|\mathcal{M}_{\alpha}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right)\right\|_{L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)} \leq C \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f) d \gamma_{\alpha}
$$

where $C, h>0$ are are independent of $f$ and $j$.
Proof. Since we can split

$$
\begin{aligned}
& \mathcal{M}_{\alpha}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right)(x) \\
& \quad=\mathcal{M}_{\alpha, \text { loc }}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right)(x)+\sup _{\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}}\left|\int_{0}^{\infty}\left(f-b_{j}\right) \eta_{j} \gamma(y) \phi(y) d \mathfrak{m}_{\alpha}(y)\right|,
\end{aligned}
$$

for any $x \in(0, \infty)$, we will estimate their norms in $L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$ separately.
For the first term, according to Lemma 2.2, applied twice, we get

$$
\begin{aligned}
& \int_{0}^{\infty} \mathcal{M}_{\alpha, \mathrm{loc}}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right)(x) d \mathfrak{m}_{\alpha}(x) \\
& \leq \int_{0}^{\infty} \mathcal{M}_{\alpha, \mathrm{loc}}\left(f \eta_{j} \gamma\right)(x) d \mathfrak{m}_{\alpha}(x)+\left|b_{j}\right| \int_{0}^{\infty} \mathcal{M}_{\alpha, \mathrm{loc}}\left(\eta_{j} \gamma\right)(x) d \mathfrak{m}_{\alpha}(x) \\
& \leq C\left(\int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x) d \mathfrak{m}_{\alpha}(x) \gamma\left(c_{j}\right)+\left|b_{j}\right| \int_{h I_{j}} \mathcal{M}_{\alpha, \mathrm{loc}}(1)(x) d \mathfrak{m}_{\alpha}(x) \gamma\left(c_{j}\right)\right) \\
& \quad \leq C\left(\int_{h I_{j}} \mathcal{M}_{\left.\alpha, \operatorname{loc}(f)(x) d \gamma_{\alpha}(x)+\left|b_{j}\right| \gamma_{\alpha}\left(h I_{j}\right)\right)}\right.
\end{aligned}
$$

Since $|f| \leq \mathcal{M}_{\alpha, \text { loc }}(f)$ and

$$
\left|b_{j}\right|=\left|\frac{\int f \eta_{j} d \gamma_{\alpha}}{\int \eta_{j} d \gamma_{\alpha}}\right| \leq \frac{1}{\gamma_{\alpha}\left(\frac{1}{2} I_{j}\right)} \int_{I_{j}}|f| d \gamma_{\alpha} \leq C \frac{1}{\gamma_{\alpha}\left(I_{j}\right)} \int_{I_{j}}|f| d \gamma_{\alpha}
$$

we obtain

$$
\begin{aligned}
& \int_{0}^{\infty} \mathcal{M}_{\alpha, \text { loc }}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right)(x) d \mathfrak{m}_{\alpha}(x) \\
& \leq C\left(\int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)+\frac{\gamma_{\alpha}\left(h I_{j}\right)}{\gamma_{\alpha}\left(I_{j}\right)} \int_{I_{j}}|f| d \gamma_{\alpha}(x)\right) \\
& \quad \leq C \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)
\end{aligned}
$$

Suppose now that $\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}$, with $x \in h I_{j}$. By recalling the notation $I(x, m(x)):=(x-m(x), x+m(x)) \cap(0, \infty)$, we have that $\operatorname{supp}(\phi) \subset I(x, r)$ for some $0<r \leq x$ with $r>m(x)$, and $\mathfrak{m}_{\alpha}(I(x, m(x)))\|\phi\|_{\infty} \leq 1$. Hence,

$$
\begin{aligned}
\left|\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha}\right| & \leq \frac{C}{\mathfrak{m}_{\alpha}(I(x, m(x)))} \int_{I_{j}}\left|f-b_{j}\right| d \gamma_{\alpha} \\
& \leq \frac{C}{\mathfrak{m}_{\alpha}(I(x, m(x)))}\left(\int_{I_{j}}|f| d \gamma_{\alpha}+\left|b_{j}\right| \gamma_{\alpha}\left(I_{j}\right)\right) \\
& \leq \frac{C}{\mathfrak{m}_{\alpha}(I(x, m(x)))} \int_{I_{j}}|f| d \gamma_{\alpha}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_{h I_{j}} \sup _{\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}} \mid & \int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha} \mid d \mathfrak{m}_{\alpha}(x) \\
& \leq C \int_{I_{j}}|f| d \gamma_{\alpha} \int_{h I_{j}} \frac{d \mathfrak{m}_{\alpha}(x)}{\mathfrak{m}_{\alpha}(I(x, m(x)))}
\end{aligned}
$$

Since $\mathfrak{m}_{\alpha}$ is doubling and $m(x) \sim m\left(c_{j}\right)$ provided that $x \in h I_{j}$, where the equivalence constant is not depending on $j$, we obtain

$$
\begin{aligned}
\int_{h I_{j}} \sup _{\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}} & \left|\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha}\right| d \mathfrak{m}_{\alpha}(x) \\
& \leq C \int_{I_{j}}|f| d \gamma_{\alpha} \int_{h I_{j}} \frac{d \mathfrak{m}_{\alpha}(x)}{\mathfrak{m}_{\alpha}\left(I\left(x, C m\left(c_{j}\right)\right)\right)} \\
& \leq C \int_{I_{j}}|f| d \gamma_{\alpha} \int_{h I_{j}} \frac{d \mathfrak{m}_{\alpha}(x)}{\mathfrak{m}_{\alpha}\left(I_{j}\right)} \leq C \int_{I_{j}}|f| d \gamma_{\alpha}
\end{aligned}
$$

We now study

$$
\int_{\left(h I_{j}\right)^{c}} \sup _{\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}}\left|\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha}\right| d \mathfrak{m}_{\alpha}(x)
$$

Let $x \in\left(h I_{j}\right)^{c}$ and $\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}$. Assume that $\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha} \neq 0$. Then, $\operatorname{supp}^{\infty}(\phi) \cap I_{j} \neq \emptyset$ which yields that $\operatorname{supp}(\phi) \subset I(x, r)$, with $r>\operatorname{dist}\left(x, I_{j}\right)$. Since $\int_{0}^{\infty}\left(f-b_{j}\right) \eta_{j} d \gamma_{\alpha}=0$ we can write

$$
\begin{aligned}
& \left|\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha}\right| \\
& =C\left|\int_{0}^{\infty}\left(\phi(y)-\phi\left(c_{j}\right)\right)\left(f(y)-b_{j}\right) \eta_{j}(y) \gamma(y) d \mathfrak{m}_{\alpha}(y)\right|
\end{aligned}
$$

$$
\begin{aligned}
& \leq C \int_{I_{j}}\left|\phi(y)-\phi\left(c_{j}\right)\right|\left|f(y)-b_{j}\right|\left|\eta_{j}(y)\right| d \gamma_{\alpha}(y) \\
& \leq C \frac{r_{j}}{r \mathfrak{m}_{\alpha}(I(x, r))} \int_{I_{j}}\left|f(y)-b_{j}\right| d \gamma_{\alpha}(y) \\
& \leq C \frac{r_{j}}{\operatorname{dist}\left(x, I_{j}\right) \mathfrak{m}_{\alpha}\left(I\left(x, \operatorname{dist}\left(x, I_{j}\right)\right)\right)} \int_{I_{j}}|f(y)| d \gamma_{\alpha}(y)
\end{aligned}
$$

We choose $k_{0} \in \mathbb{N}$ such that $2^{k_{0}} \leq h<2^{k_{0}+1}$. Thus, calling $R_{k}\left(c_{j}, r_{j}\right)=$ $I\left(c_{j}, 2^{k+1} r_{j}\right) \backslash I\left(c_{j}, 2^{k} r_{j}\right)$,

$$
\begin{aligned}
\int_{\left(h I_{j}\right)^{c}} & \frac{1}{\operatorname{dist}\left(x, I_{j}\right) \mathfrak{m}_{\alpha}\left(I\left(x, \operatorname{dist}\left(x, I_{j}\right)\right)\right)} d \mathfrak{m}_{\alpha}(x) \\
& \leq C \sum_{k=k_{0}}^{\infty} \int_{R_{k}\left(c_{j}, r_{j}\right)} \frac{x^{2 \alpha+1}}{\operatorname{dist}\left(x, I_{j}\right) \mathfrak{m}_{\alpha}\left(I\left(x, \operatorname{dist}\left(x, I_{j}\right)\right)\right)} d x \\
& \leq C \sum_{k=k_{0}}^{\infty} \int_{R_{k}\left(c_{j}, r_{j}\right)} \frac{1}{\operatorname{dist}\left(x, I_{j}\right)^{2}} d x \\
& \leq C \sum_{k=0}^{\infty} \int_{2^{k} r_{j}+c_{j}}^{2^{k+1} r_{j}+c_{j}} \frac{d x}{\left(2^{k} r_{j}\right)^{2}} \\
& \leq \frac{C}{r_{j}} \sum_{k=0}^{\infty} 2^{-k}=\frac{C}{r_{j}}
\end{aligned}
$$

where, in the first inequality, we have used that $\mathfrak{m}_{\alpha}(I(z, r)) \geq C z^{2 \alpha+1} r$, for every $z, r>0$ (see $[26,(1.4)])$.

Therefore

$$
\int_{\left(h I_{j}\right)^{c}} \sup _{\phi \in \mathcal{A}_{x}^{\alpha} \backslash \mathcal{A}_{x, \text { loc }}^{\alpha}}\left|\int_{0}^{\infty}\left(f-b_{j}\right) \phi \eta_{j} \gamma d \mathfrak{m}_{\alpha}\right| d \mathfrak{m}_{\alpha}(x) \leq C \int_{I_{j}}|f| d \gamma_{\alpha} .
$$

and we can conclude that

$$
\int_{0}^{\infty} \mathcal{M}_{\alpha}\left(\left(f-b_{j}\right) \eta_{j} \gamma\right) d \mathfrak{m}_{\alpha} \leq C \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f) d \gamma_{\alpha}
$$

Suppose that $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ with $\int_{0}^{\infty} f d \gamma_{\alpha}=0, \mathcal{M}_{\alpha, \operatorname{loc}}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $\mathcal{E}_{\alpha}(f)<\infty$. If $\int_{0}^{\infty} f d \gamma_{\alpha} \neq 0$ we apply our result to $f-\int_{0}^{\infty} f d \gamma_{\alpha}$. To prove the "if" implication of Theorem 1.2 we are going to see that, for some $a>0$, $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$, where, for every $j \in \mathbb{N}, b_{j}$ is an $(a, \infty, \alpha)$-atom, and $\lambda_{j} \in \mathbb{C}$ being

$$
\sum_{j=0}^{\infty}\left|\lambda_{j}\right| \leq C\left(\int_{0}^{\infty}\left|\mathcal{M}_{\alpha, \operatorname{loc}} f(y)\right| d \gamma_{\alpha}(y)+\mathcal{E}_{\alpha}(f)\right)
$$

To achieve this goal, we are going to consider the Hardy space $H_{\alpha}^{1}(0, \infty)$ associated with the Bessel operator

$$
S_{\alpha}=-\frac{d^{2}}{d x^{2}}-\frac{2 \alpha+1}{x} \frac{d}{d x}
$$

that is, the Hardy space defined by the heat semigroup generated by $S_{\alpha}$ (see [2] and [26]). We shall also take into account the following atoms: we say that a measurable function $b$ is a $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atom when there exist $x_{0}, r_{0} \in(0, \infty)$ such that
(a) $\operatorname{supp}(b) \subset I\left(x_{0}, r_{0}\right)$;
(b) $\|b\|_{\infty} \leq \mathfrak{m}_{\alpha}\left(I\left(x_{0}, r_{0}\right)\right)^{-1}$;
(c) $\int_{0}^{\infty} b(y) d \mathfrak{m}_{\alpha}(y)=0$.

As it was proved in [2], the space $H_{\alpha}^{1}(0, \infty)$ coincides with the atomic Hardy space $\left.H^{1}\left((0, \infty),|\cdot|, \mathfrak{m}_{\alpha}\right)\right)$ defined through these ( $\left.\mathfrak{m}_{\alpha}, \infty\right)$-atoms in $\left((0, \infty),|\cdot|, \mathfrak{m}_{\alpha}\right)$, in the sense of Coifman and Weiss ([7]). Clearly, the triple $\left((0, \infty),|\cdot|, \mathfrak{m}_{\alpha}\right)$ is a space of homogeneous type. According to [2, p. 201] the space $H_{\alpha}^{1}((0, \infty)$ also coincides with the Coifman-Weiss atomic Hardy space $H^{1}\left((0, \infty), d_{\alpha}, \mathfrak{m}_{\alpha}\right)$ associated with the normal homogeneous space $\left((0, \infty), d_{\alpha}, \mathfrak{m}_{\alpha}\right)$. Here $d_{\alpha}$ represents the metric defined on $(0, \infty)$ by $d_{\alpha}(x, y)=\frac{1}{2 \alpha+2}\left|x^{2 \alpha+2}-y^{2 \alpha+2}\right|, x, y \in(0, \infty)$.

The following lemma gives us a decomposition for any function in $H_{\alpha}^{1}(0, \infty)$ supported on an interval $I$ into $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atoms with controlled support.
Lemma 2.4. Let $\alpha>-\frac{1}{2}$. There exists $C>0$ such that, for every interval $I \subset(0, \infty)$ and $g \in H_{\alpha}^{1}(0, \infty)$ such that $\operatorname{supp}(g) \subset I$, we can write $g=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$ in $L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$, where, for every $j \in \mathbb{N}$, $b_{j}$ is a $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atom having its support contained in $2 I$, and $\lambda_{j} \in \mathbb{C}$ being $\sum_{j=0}^{\infty}\left|\lambda_{j}\right| \leq C\|g\|_{H_{\alpha}^{1}(0, \infty)}$.
Proof. If $I=(0, \infty)$ the property is clear from [2, Theorem 1.7]. Let $I$ be a bounded interval of $(0, \infty)$ and suppose that $g \in H_{\alpha}^{1}(0, \infty)$ with $\operatorname{supp}(g) \subset I$. Then, according to [16, Theorem 4.13] by introducing a cut off function with respect to the intervals $I$ and $2 I$ in the maximal functions we deduce that $g \in H^{1}\left(2 I, d_{\alpha}, \mathfrak{m}_{\alpha}\right)$ and

$$
\begin{equation*}
\|g\|_{H^{1}\left(2 I, d_{\alpha}, \mathfrak{m}_{\alpha}\right)} \leq C\|g\|_{H^{1}\left((0, \infty), d_{\alpha}, \mathfrak{m}_{\alpha}\right)} \leq C\|g\|_{H_{\alpha}^{1}(0, \infty)} . \tag{2.1}
\end{equation*}
$$

Therefore, we can write $g=\sum_{j=1}^{\infty} \lambda_{j} b_{j}+\lambda_{0} b_{0}$ in $L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$, where $b_{j}$ is an $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atom such that $\operatorname{supp}\left(b_{j}\right) \subset 2 I$, for every $j \in \mathbb{N} \backslash\{0\}, b_{0}(x)=1$ for every $x \in 2 I$, and $\lambda_{j} \in \mathbb{C}$, for every $j \in \mathbb{N}$, being

$$
\begin{equation*}
\sum_{j=0}^{\infty}\left|\lambda_{j}\right| \leq C\|g\|_{H_{\alpha}^{1}(0, \infty)} \tag{2.2}
\end{equation*}
$$

Since $\int_{0}^{\infty} g(y) d \mathfrak{m}_{\alpha}(y)=\int_{0}^{\infty} b_{j}(y) d \mathfrak{m}_{\alpha}(y)=0$, for every $j \in \mathbb{N} \backslash\{0\}$, and the series $\sum_{j=1}^{\infty} \lambda_{j} b_{j}+\lambda_{0} b_{0}$ converges in $L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$, it follows that $\lambda_{0}=0$. Note that the constants $C>0$ in (2.1) and (2.2) do not depend on $g$ or $I$.

The following result will allow us to obtain an atomic decomposition of a function $g$ into $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atoms whenever $\mathcal{M}_{\alpha}(g) \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$.
Lemma 2.5. Let $\alpha>-\frac{1}{2}$. There exists $C>0$ such that if $g \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$, then $g \in H_{\alpha}^{1}(0, \infty)$ provided that $\mathcal{M}_{\alpha}(g) \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$ being

$$
\|g\|_{H_{\alpha}^{1}(0, \infty)} \leq C\left\|\mathcal{M}_{\alpha}(g)\right\|_{L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)}
$$

Proof. Assume that $0 \leq \phi \in C_{c}^{\infty}(0, \infty)$ and $\operatorname{supp}(\phi) \subset(0,1)$. We define, for every $t>0, \phi_{t}(x)=t^{-2 \alpha-2} \phi(x / t), x \in(0, \infty)$, and, for every $t, x \in(0, \infty)$, $\Phi_{x, t}={ }_{\alpha} \tau_{x}\left(\phi_{t}\right)$. Here ${ }_{\alpha} \tau_{x}$ denotes the $x$-translation associated with the operator $S_{\alpha}$, also called Hankel $x$-translation (see [9], [11] and [13]). We can write

$$
\Phi_{x, t}(y)=\int_{|x-y|}^{x+y} \phi_{t}(z) D_{\alpha}(x, y, z) z^{2 \alpha+1} d z, \quad x, y, t \in(0, \infty)
$$

where, for every $x, y, z \in(0, \infty)$,

$$
D_{\alpha}(x, y, z)=\frac{2^{2 \alpha-1} \Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right) \sqrt{\pi}}(x y z)^{-2 \alpha}(\Upsilon(x, y, z))^{2 \alpha-1}, \quad x, y, z \in(0, \infty)
$$

being $\Upsilon(x, y, z)$ the area of a triangle with side lengths $x, y, z$ when this triangle exists and $\Upsilon(x, y, z)=0$ in other cases. It follows that $\Phi_{x, t}(y)=0$ for every $|x-y|>t$ and $t>0$.

We also have that, for any $x, y, t \in(0, \infty)$,

$$
\begin{equation*}
\Phi_{x, t}(y)=\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right) \sqrt{\pi}} \int_{0}^{\pi} \phi_{t}\left(\sqrt{x^{2}+y^{2}-2 x y \cos \theta}\right)(\sin \theta)^{2 \alpha} d \theta \tag{2.3}
\end{equation*}
$$

Notice that, since $\phi \in C_{c}^{\infty}(0, \infty)$, for every $x \in(0, \infty)$ it satisfies the following properties
(a) $0 \leq \phi(x) \leq C\left(1+x^{2}\right)^{-\alpha-3 / 2}$;
(b) $\left|\phi^{\prime}(x)\right| \leq C x\left(1+x^{2}\right)^{-\alpha-5 / 2}$;
(c) $\left|\phi^{\prime \prime}(x)\right| \leq C\left(1+x^{2}\right)^{-\alpha-5 / 2}$.

Then, according to [2, Theorem 2.7] and [26, Theorem 1.1], $g$ is in $H_{\alpha}^{1}(0, \infty)$ provided that $g \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$ and

$$
\sup _{t>0}\left|\int_{0}^{\infty} \Phi_{x, t}(y) g(y) d \mathfrak{m}_{\alpha}(y)\right| \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)
$$

Since $\mathcal{M}_{\alpha}(g) \in L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$, it is enough to show that for every $g \in L_{\mathrm{loc}}^{1}(0, \infty)$,

$$
\begin{equation*}
\sup _{t>0}\left|\int_{0}^{\infty} \Phi_{x, t}(y) g(y) d \mathfrak{m}_{\alpha}(y)\right| \leq \mathcal{M}_{\alpha}(g)(x), \quad x \in(0, \infty) . \tag{2.4}
\end{equation*}
$$

We will do so by showing that $\Phi_{x, t} \in \mathcal{A}_{x}^{\alpha}$ for every $x, t \in(0, \infty)$.
First, we are going to see that

$$
\begin{equation*}
\left|\Phi_{x, t}(y)\right| \leq \frac{C}{\mathfrak{m}_{\alpha}(I(x, t))}, \quad x, y, t \in(0, \infty) \tag{2.5}
\end{equation*}
$$

By taking $0<\epsilon<2 \alpha+1$, and using that $2(1-\cos \theta) \sim \theta^{2}$ for every $\theta \in[0, \pi]$, we have that

$$
\begin{aligned}
\left|\Phi_{x, t}(y)\right| & \leq \frac{C}{t^{2 \alpha+2}} \int_{0}^{\pi}\left|\phi\left(\frac{\sqrt{x^{2}+y^{2}-2 x y \cos \theta}}{t}\right)\right|(\sin \theta)^{2 \alpha} d \theta \\
& \leq \frac{C}{t^{2 \alpha+2}}\left(\int_{0}^{t / \sqrt{x y}} \theta^{2 \alpha}\left(\frac{t}{\sqrt{(x-y)^{2}+x y \theta^{2}}}\right)^{2 \alpha+1-\epsilon} d \theta\right. \\
& \left.+\int_{t / \sqrt{x y}}^{\pi} \theta^{2 \alpha}\left(\frac{t}{\sqrt{(x-y)^{2}+x y \theta^{2}}}\right)^{2 \alpha+1+\epsilon} d \theta\right) \\
& \leq C\left(\frac{1}{t^{1+\varepsilon}} \int_{0}^{t / \sqrt{x y}} \theta^{\epsilon-1} d \theta(x y)^{-\alpha-(1-\epsilon) / 2}\right. \\
& \left.+\frac{1}{t^{1-\varepsilon}} \int_{t / \sqrt{x y}}^{\infty} \theta^{-\epsilon-1} d \theta(x y)^{-\alpha-(1+\epsilon) / 2}\right) \\
& \leq C \frac{1}{t(x y)^{\alpha+1 / 2}}, \quad x, y, t \in(0, \infty) .
\end{aligned}
$$

Then, whenever $0<\frac{x}{2}<y<2 x<\infty$

$$
\left|\Phi_{x, t}(y)\right| \leq \frac{C}{t x^{2 \alpha+1}}, \quad t \in(0, \infty)
$$

On the other hand, if $0<y<\frac{x}{2}<y$ or $2 x<y<\infty$, we get

$$
\begin{aligned}
\left|\Phi_{x, t}(y)\right| & \leq \frac{C}{t^{2 \alpha+2}} \int_{0}^{\pi}\left(\frac{t}{\sqrt{x^{2}+y^{2}-2 x y \cos \theta}}\right)^{2 \alpha+1} \theta^{2 \alpha} d \theta \\
& \leq \frac{C}{t|x-y|^{2 \alpha+1}}
\end{aligned}
$$

$$
\leq \frac{C}{t x^{2 \alpha+1}}, \quad t \in(0, \infty)
$$

Since $\mathfrak{m}_{\alpha}(I(x, t)) \sim t(x+t)^{2 \alpha+1}$, for any $x, t \in(0, \infty)$ and, as it is clear from (2.3),

$$
\left|\Phi_{x, t}(y)\right| \leq \frac{C}{t^{2 \alpha+2}}, \quad x, y, t \in(0, \infty)
$$

we conclude that $(2.5)$ holds.
Now, for every $x, y, t \in(0, \infty)$,

$$
\begin{aligned}
\partial_{y} \Phi_{x, t}(y)=\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right) \sqrt{\pi} t^{2 \alpha+3}} \int_{0}^{\pi} \phi^{\prime} & \left(\frac{\sqrt{x^{2}+y^{2}-2 x y \cos \theta}}{t}\right) \\
& \times \frac{y-x \cos \theta}{\sqrt{x^{2}+y^{2}-2 x y \cos \theta}}(\sin \theta)^{2 \alpha} d \theta
\end{aligned}
$$

Since, for every $x, y \in(0, \infty)$ and $\theta \in(0, \pi)$,

$$
(y-x \cos \theta)^{2}=y^{2}+x^{2}(\cos \theta)^{2}-2 x y \cos \theta=x^{2}+y^{2}-2 x y \cos \theta-\left(1-\cos ^{2} \theta\right) x^{2}
$$

we get

$$
\begin{aligned}
\left|\partial_{y} \Phi_{x, t}(y)\right| & \leq \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right) \sqrt{\pi} t^{2 \alpha+3}} \int_{0}^{\pi} \phi^{\prime}\left(\frac{\sqrt{x^{2}+y^{2}-2 x y \cos \theta}}{t}\right)(\sin \theta)^{2 \alpha} d \theta \\
& =\frac{1}{t}{ }_{\alpha} \tau_{x}\left(\left|\phi^{\prime}\right|\right)(y) \\
& \leq \frac{1}{t \mathfrak{m}_{\alpha}(I(x, t))}, \quad x, y, t \in(0, \infty)
\end{aligned}
$$

In order to see the last inequality we can proceed as above by considering $\left|\phi^{\prime}\right|$ instead of $\phi$. Thus, we have proved (2.4) and this finishes the proof.

Hence, according to Lemmas 2.3, 2.4, and 2.5, for every $j \in \mathbb{N}$ we obtain that $\left(f-b_{j}\right) \eta_{j} \gamma \in H_{\alpha}^{1}(0, \infty)$, and there exist a sequence $\left\{\lambda_{i, j}\right\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence $\left\{a_{i, j}\right\}_{i \in \mathbb{N}}$ of $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atoms such that $\operatorname{supp}\left(a_{i, j}\right) \subset 2 I_{j}$, for every $i \in \mathbb{N}$, and

$$
\left(f-b_{j}\right) \eta_{j} \gamma=\sum_{i=0}^{\infty} \lambda_{i, j} a_{i, j}
$$

being $\sum_{i=0}^{\infty}\left|\lambda_{i, j}\right| \leq C \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f) d \gamma_{\alpha}$.
Moreover, for each $j \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{i=0}^{\infty}\left|\lambda_{i, j}\right|\left\|a_{i, j} / \gamma\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} & =\sum_{j=0}^{\infty}\left|\lambda_{i, j}\right|\left\|a_{i, j}\right\|_{L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)} \\
& \leq C \sum_{i=0}^{\infty}\left|\lambda_{i, j}\right| \\
& \leq C \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f) d \gamma_{\alpha}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty}\left|\lambda_{i, j}\right|\left\|a_{i, j} / \gamma\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} & \leq C \sum_{j=0}^{\infty} \int_{h I_{j}} \mathcal{M}_{\alpha, \operatorname{loc}}(f) d \gamma_{\alpha} \\
& \leq C\left\|\mathcal{M}_{\alpha, \operatorname{loc}(f)}\right\|_{\left.L^{1}(0, \infty), \gamma_{\alpha}\right)}
\end{aligned}
$$

so the double series $\sum_{(i, j) \in \mathbb{N} \times \mathbb{N}}\left|\lambda_{i, j}\right|\left|a_{i, j} / \gamma\right|$ converges in $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Then, since the last series consists of positive functions, the series $\sum_{(i, j) \in \mathbb{N} \times \mathbb{N}}\left|\lambda_{i, j}\right|\left|a_{i, j} / \gamma\right|$ converges almost everywhere in $(0, \infty)$ and we can write

$$
\sum_{(i, j) \in \mathbb{N} \times \mathbb{N}} \lambda_{i, j} a_{i, j} / \gamma=\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \lambda_{i, j} a_{i, j} / \gamma
$$

in $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and almost everywhere in $(0, \infty)$.
Furthermore, for every $i, j \in \mathbb{N}, a_{i, j} / \gamma$ is a $(1, \infty, \alpha)$-atom as $\operatorname{supp}\left(a_{i, j}\right) \subset 2 I_{j}$. We have then just proved that

$$
\sum_{j=0}^{\infty}\left(f-b_{j}\right) \eta_{j}=\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \lambda_{i, j} a_{i, j} / \gamma
$$

Our next objective is to study $\sum_{j=0}^{\infty} b_{j} \eta_{j}$. Since

$$
\left|b_{j}\right| \leq \frac{C}{\gamma_{\alpha}\left(I_{j}\right)} \int_{I_{j}}|f| d \gamma_{\alpha}, \quad j \in \mathbb{N}
$$

we get

$$
\left\|\sum_{j=0}^{\infty}\left|b_{j}\right|\left|\eta_{j}\right|\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C \sum_{j=0}^{\infty} \int_{I_{j}}|f| d \gamma_{\alpha} \leq C\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}
$$

Then, there exists an increasing function $\psi: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$
\lim _{k \rightarrow \infty} \sum_{j=0}^{\psi(k)}\left|b_{j}(x)\right|\left|\eta_{j}(x)\right|
$$

exists for almost every $x \in(0, \infty)$. We conclude that the series $\sum_{j=0}^{\infty}\left|b_{j}\right|\left|\eta_{j}\right|$ converges almost everywhere in $(0, \infty)$.

For $j \in \mathbb{N}$, we define $\widetilde{\eta}_{j}=\eta_{j} / \int_{0}^{\infty} \eta_{j} d \gamma_{\alpha}$, so we can write $b_{j} \eta_{j}=\widetilde{\eta}_{j} \int_{0}^{\infty} f \eta_{j} d \gamma_{\alpha}$. We also consider $\mu_{k}=\sum_{j=k}^{\infty} \eta_{j}$, for $k \in \mathbb{N}$. Since

$$
\int_{0}^{\infty}|f(y)| \sum_{j=k}^{\infty} \eta_{j}(y) d \gamma_{\alpha}(y) \leq C \int_{0}^{\infty}|f(y)| d \gamma_{\alpha}(y), \quad k \in \mathbb{N}
$$

it follows that

$$
\sum_{j=k}^{\infty} \int_{0}^{\infty} f(y) \eta_{j}(y) d \gamma_{\alpha}=\int_{0}^{\infty} f(y) \mu_{k}(y) d \gamma_{\alpha}(y), \quad k \in \mathbb{N}
$$

We get

$$
\sum_{j=0^{\infty}} b_{j} \eta_{j}=\left|\int_{0}^{\infty} f(y) \mu_{k}(y) d \gamma_{\alpha}(y)\right| \leq \sum_{j=k}^{\infty} \int_{I_{j}}|f(y)| d \gamma_{\alpha}(y) \leq C \int_{\sqrt{k-1}}^{\infty}|f(y)| d \gamma_{\alpha}(y)
$$

where the last integral tends to zero as $k \rightarrow \infty$. By summation by parts, since $\lim _{k \rightarrow \infty} \eta_{k}(x)=0, x \in(0, \infty)$, we deduce

$$
\sum_{j=0}^{\infty} b_{j} \eta_{j}=\sum_{j=0}^{\infty} \int_{0}^{\infty} f(y) \eta_{j}(y) d \gamma_{\alpha}(y) \widetilde{\eta}_{j}=\sum_{k=0}^{\infty} \int_{0}^{\infty} f(y) \mu_{k+1}(y) d \gamma_{\alpha}(y)\left(\widetilde{\eta}_{k+1}-\widetilde{\eta}_{k}\right)
$$

We claim there exist $C>0$ such that $C\left(\widetilde{\eta}_{k+1}-\widetilde{\eta}_{k}\right)$ is a $(2, \infty, \alpha)$-atom, for every $k \in \mathbb{N}$. Indeed
(i) $\int_{0}^{\infty}\left(\widetilde{\eta}_{k+1}-\widetilde{\eta}_{k}\right) d \gamma_{\alpha}=\int_{0}^{\infty} \widetilde{\eta}_{k+1} d \gamma_{\alpha}-\int_{0}^{\infty} \widetilde{\eta}_{k} d \gamma_{\alpha}=0$ for each $k \in \mathbb{N}$;
(ii) $\operatorname{supp}\left(\widetilde{\eta}_{k+1}-\widetilde{\eta}_{k}\right) \subset[\sqrt{k-1}, \sqrt{k+2}]$, for $k \in \mathbb{N} \backslash\{0\}$, while $\widetilde{\eta}_{1}-\widetilde{\eta}_{0}$ is supported on $[0, \sqrt{2}]$;
(iii) When $k \in \mathbb{N} \backslash\{0\},[\sqrt{k-1}, \sqrt{k+2}] \subset D_{k}$, where

$$
D_{k}:=\left[\frac{\sqrt{k-1}+\sqrt{k+2}}{2}-\frac{2}{\sqrt{k-1}+\sqrt{k+2}}, \frac{\sqrt{k-1}+\sqrt{k+2}}{2}+\frac{2}{\sqrt{k-1}+\sqrt{k+2}}\right],
$$

and

$$
\left\|\widetilde{\eta}_{k+1}-\widetilde{\eta}_{k}\right\|_{\infty} \leq \frac{1}{\gamma_{\alpha}\left(\frac{1}{2} I_{k}\right)}+\frac{1}{\gamma_{\alpha}\left(\frac{1}{2} I_{k+1}\right)} \leq C \frac{1}{\gamma_{\alpha}\left(D_{k}\right)}
$$

While

$$
\left\|\widetilde{\eta}_{1}-\widetilde{\eta}_{0}\right\|_{\infty} \leq \frac{1}{\gamma_{\alpha}(0, \sqrt{2})}+\frac{1}{\gamma_{\alpha}\left(0, \frac{1}{2}\right)} \leq C \frac{1}{\gamma_{\alpha}(0, \sqrt{2})}
$$

Now, we are going to see that

$$
\sum_{k=0}^{\infty}\left|\int_{0}^{\infty} f(y) \mu_{k}(y) d \gamma_{\alpha}(y)\right| \leq C\left(\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathcal{E}_{\alpha}(f)\right)
$$

We can write, for every $k \in \mathbb{N}$,

$$
\begin{aligned}
\int_{0}^{\infty} f(x) \mu_{k}(x) d \gamma_{\alpha}(x) & =\int_{0}^{\infty}\left(\int_{0}^{x} \mu_{k}^{\prime}(y) d y+\mu_{k}\left(0^{+}\right)\right) f(x) d \gamma_{\alpha}(x) \\
& =\int_{0}^{\infty} \mu_{k}^{\prime}(y) \int_{y}^{\infty} f(x) d \gamma_{\alpha}(x) d y+\mu_{k}\left(0^{+}\right) \int_{0}^{\infty} f(x) d \gamma_{\alpha}(x)
\end{aligned}
$$

Here $\phi\left(0^{+}\right)$stands for $\lim _{x \rightarrow 0^{+}} \phi(x)$. We have that $\mu_{k}\left(0^{+}\right)=\sum_{j=k}^{\infty} \eta_{j}\left(0^{+}\right)=0$, for each $k \in \mathbb{N}, k \geq 1$, and $\mu_{0}\left(0^{+}\right)=1$. As in [18, p. 1687] we get that $\operatorname{supp}\left(\mu_{k}^{\prime}\right) \subset I_{k}$ and

$$
\left|\mu_{k}^{\prime}(y)\right| \leq \frac{C}{r_{k}} \leq C\left(1+c_{k}\right), \quad k \in \mathbb{N}
$$

where $C$ is independent of $k$. Hence,

$$
\begin{aligned}
\sum_{k=0}^{\infty}\left|\int_{0}^{\infty} f(y) \mu_{k}(y) d \gamma_{\alpha}(y)\right| \leq & C \sum_{k=0}^{\infty} \int_{I_{k}}\left(1+c_{k}\right)\left|\int_{y}^{\infty} f(x) d \gamma_{\alpha}(x)\right| d y \\
& +\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \\
\leq & C \int_{0}^{\infty}(1+y)\left|\int_{y}^{\infty} f(x) d \gamma_{\alpha}(x)\right| d y \\
& +\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \\
\leq & C\left(\mathcal{E}_{\alpha}(f)+\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}\right)
\end{aligned}
$$

Our purpose is established and the proof of the "if" implication of Theorem 1.2 is finished.

We will now prove the "only if" implication of Theorem 1.2. We will see that, for every $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$,

$$
\begin{equation*}
\left\|\mathcal{M}_{\alpha, \operatorname{loc}}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathcal{E}_{\alpha}(f) \leq C\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)} \tag{2.6}
\end{equation*}
$$

In order to do this we will first show that there exists $C>0$ such that, for every $(1, \infty, \alpha)$-atom $b$,

$$
\begin{equation*}
\left\|\mathcal{M}_{\alpha, \operatorname{loc}}(b)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathcal{E}_{\alpha}(b) \leq C \tag{2.7}
\end{equation*}
$$

Assume that $b$ is a $(1, \infty, \alpha)$-atom associated with the interval $I=I\left(x_{0}, r_{0}\right)$ where $0<r_{0} \leq \min \left\{x_{0}, m\left(x_{0}\right)\right\}$. According to Lemma 2.1, $\mathcal{M}_{\alpha, \text { loc }}(b)$ is supported on the interval $\widetilde{I}=I\left(x_{0}, h m\left(x_{0}\right)\right)$. Let $x \in(0, \infty)$ and suppose that $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ is associated to the interval $J=I\left(x, r_{1}\right)$. Then,

$$
\left|\int_{0}^{\infty} b(y) \phi(y) d \mathfrak{m}_{\alpha}(y)\right| \leq\|b\|_{L^{\infty}\left((0, \infty), \gamma_{\alpha}\right)} \int_{I \cap J}|\phi(y)| d \mathfrak{m}_{\alpha}(y)
$$

$$
\begin{aligned}
& \leq\|b\|_{L^{\infty}\left((0, \infty), \gamma_{\alpha}\right)} \frac{\mathfrak{m}_{\alpha}(I \cap J)}{\mathfrak{m}_{\alpha}(J)} \\
& \leq \frac{1}{\gamma_{\alpha}(I)}
\end{aligned}
$$

which yields

$$
\mathcal{M}_{\alpha, \operatorname{loc}}(b)(x) \leq \frac{1}{\gamma_{\alpha}(I)}
$$

We deduce that

$$
\int_{2 I} \mathcal{M}_{\alpha, \text { loc }}(b)(x) d \gamma_{\alpha}(x) \leq \frac{\gamma_{\alpha}(2 I)}{\gamma_{\alpha}(I)} \leq C
$$

Suppose now that $x \in \widetilde{I} \backslash(2 I)$ and $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ associated to the interval $J$ given above. We can write

$$
\int_{0}^{\infty} b(y) \phi(y) d \mathfrak{m}_{\alpha}(y)=\int_{0}^{\infty} b(y)\left(\phi(y)-\phi\left(x_{0}\right)\right) d \mathfrak{m}_{\alpha}(y)+\phi\left(x_{0}\right) \int_{0}^{\infty} b(y) d \mathfrak{m}_{\alpha}(y)
$$

and estimate each term separately.
Since $\operatorname{supp}(\phi) \subset J$ and $\operatorname{supp}(b) \subset I$ we can assume that $r_{1}>\operatorname{dist}(x, I)$ (otherwise, if $0<r_{1} \leq \operatorname{dist}(x, I)$, then $J \subset I^{c}$ and $\left.\int_{0}^{\infty} b \phi d \mathfrak{m}_{\alpha}=0\right)$. Since $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$, by [26, (1.4)] we have that

$$
\begin{aligned}
\left|\int_{0}^{\infty} b(y)\left(\phi(y)-\phi\left(x_{0}\right)\right) d \mathfrak{m}_{\alpha}(y)\right| & \leq \int_{0}^{\infty}|b(y)|\left|\phi(y)-\phi\left(x_{0}\right)\right| d \mathfrak{m}_{\alpha}(y) \\
& \leq \frac{1}{r_{1} \mathfrak{m}_{\alpha}(J)} \int_{I}\left|y-x_{0}\right||b(y)| d \mathfrak{m}_{\alpha}(y) \\
& \leq C \frac{r_{0}}{r_{1} \mathfrak{m}_{\alpha}(J)} \int_{I}|b(y)| \frac{\gamma(y)}{\gamma\left(x_{0}\right)} d \mathfrak{m}_{\alpha}(y) \\
& \leq C \frac{r_{0}}{r_{1} \mathfrak{m}_{\alpha}(J) \gamma\left(x_{0}\right)} \\
& \leq C \frac{r_{0}}{\operatorname{dist}(x, I) \mathfrak{m}_{\alpha}(I(x, \operatorname{dist}(x, I))) \gamma\left(x_{0}\right)} \\
& \leq C \frac{r_{0}}{d(x, I)^{2} x^{2 \alpha+1} \gamma\left(x_{0}\right)}
\end{aligned}
$$

On the other hand, since $\int_{0}^{\infty} b(y) d \gamma_{\alpha}(y)=0$, we get

$$
\int_{0}^{\infty} b(y) d \mathfrak{m}_{\alpha}(y)=\int_{0}^{\infty} b(y) y^{2 \alpha+1} \frac{\gamma\left(x_{0}\right)-\gamma(y)}{\gamma\left(x_{0}\right)} d y
$$

As in $[17,(3.6)]$ we obtain

$$
\left|\frac{\gamma\left(x_{0}\right)-\gamma(y)}{\gamma\left(x_{0}\right)}\right| \leq C r_{0}\left(1+x_{0}\right), \quad y \in I
$$

Then, using that $\int_{0}^{\infty}|b(y)| d \gamma_{\alpha}(y) \leq C$, we get

$$
\left|\int_{0}^{\infty} b(y) d \mathfrak{m}_{\alpha}(y)\right| \leq C \frac{r_{0}\left(1+x_{0}\right)}{\gamma\left(x_{0}\right)}
$$

By applying again $[26,(3.6)]$, we have that

$$
\begin{aligned}
\left|\phi\left(x_{0}\right) \int_{0}^{\infty} b(y) d \mathfrak{m}_{\alpha}(y)\right| & \leq C\left|\phi\left(x_{0}\right)\right| \frac{r_{0}\left(1+x_{0}\right)}{\gamma\left(x_{0}\right)} \\
& \leq C \frac{r_{0}\left(1+x_{0}\right)}{\gamma\left(x_{0}\right) \mathfrak{m}_{\alpha}(J)} \\
& \leq C \frac{r_{0}\left(1+x_{0}\right)}{\gamma\left(x_{0}\right) x^{2 \alpha+1} r_{1}}
\end{aligned}
$$

$$
\leq C \frac{r_{0}\left(1+x_{0}\right)}{\gamma\left(x_{0}\right) x^{2 \alpha+1} \operatorname{dist}(x, I)} .
$$

By combining the above estimates we conclude that, for every $x \in \widetilde{I} \backslash(2 I)$,

$$
\mathcal{M}_{\alpha, \mathrm{loc}}(b)(x) \leq C\left(\frac{r_{0}}{d(x, I)^{2} x^{2 \alpha+1} \gamma\left(x_{0}\right)}+\frac{r_{0}\left(1+x_{0}\right)}{\operatorname{dist}(x, I) x^{2 \alpha+1} \gamma\left(x_{0}\right)}\right)
$$

We deduce that

$$
\begin{aligned}
\int_{\widetilde{I} \backslash(2 I)} & \mathcal{M}_{\alpha, \operatorname{loc}(b)(x) d \gamma_{\alpha}(x)} \\
& \leq C\left(\int_{\widetilde{I} \backslash(2 I)} \frac{r_{0}}{d(x, I)^{2} \gamma\left(x_{0}\right)} d \gamma(x)+\int_{\widetilde{I} \backslash(2 I)} \frac{r_{0}\left(1+x_{0}\right)}{\operatorname{dist}(x, I) \gamma\left(x_{0}\right)} d \gamma(x)\right) \\
& \leq C\left(r_{0} \int_{(2 I)^{c}} \frac{d x}{\operatorname{dist}(x, I)^{2}}+r_{0}\left(1+x_{0}\right) \int_{\widetilde{I} \backslash(2 I)} \frac{d x}{\operatorname{dist}(x, I)}\right) \\
& \leq C\left(1+r_{0}\left(1+x_{0}\right) \log \left(\frac{m\left(x_{0}\right)}{r_{0}}\right)\right) \\
& \leq C\left(1+\frac{r_{0}}{m\left(x_{0}\right)} \log \left(\frac{m\left(x_{0}\right)}{r_{0}}\right)\right) \leq C .
\end{aligned}
$$

It follows that

$$
\int_{0}^{\infty} \mathcal{M}_{\alpha, \operatorname{loc}}(b)(x) d \gamma_{\alpha}(x) \leq C
$$

where $C>0$ does not depend on $b$.
Clearly, if $b(x)=1$ for every $x \in(0, \infty)$, then $\mathcal{M}_{\alpha, \text { loc }}(b) \leq 1$ and

$$
\int_{0}^{\infty} \mathcal{M}_{\alpha, \operatorname{loc}}(b)(x) d \gamma_{\alpha}(x) \leq 1
$$

for this atom.
We now prove that, for a certain $C>0, \mathcal{E}_{\alpha}(b) \leq C$, for every $(1, \infty, \alpha)$-atom $b$. Suppose that $b$ is a $(1, \infty, \alpha)$-atom associated to the interval $I=I\left(x_{0}, r_{0}\right)$. Since $\int_{0}^{\infty} b d \gamma_{\alpha}=0$ we deduce that $\int_{x}^{\infty} b d \gamma_{\alpha}=0$ for $x \notin I$, and therefore

$$
\left|\int_{x}^{\infty} b(y) d \gamma_{\alpha}(y)\right| \leq \int_{I}|b(y)| d \gamma_{\alpha}(y) \chi_{I}(x) \leq \chi_{I}(x), \quad x \in(0, \infty)
$$

Hence,

$$
\begin{aligned}
\mathcal{E}_{\alpha}(b) & \leq \int_{I} x\left|\int_{x}^{\infty} b(y) d \gamma_{\alpha}(y)\right| d x \\
& \leq \int_{I}|b(y)| \int_{x_{0}-r_{0}}^{y} x d x d \gamma_{\alpha}(y) \\
& \leq \int_{I}|b(y)| d \gamma_{\alpha}(y)\left(\frac{\left(x_{0}+r_{0}\right)^{2}-\left(x_{0}-r_{0}\right)^{2}}{2}\right) \leq 2 .
\end{aligned}
$$

In the last inequality we have taken into account that $x_{0} r_{0} \leq 1$.
On the other hand, if $b(x)=1$ for every $x \in(0, \infty)$, we can write

$$
\mathcal{E}_{\alpha}(1)=\int_{0}^{\infty} x\left|\int_{x}^{\infty} d \gamma_{\alpha}(y)\right| d x \leq \int_{0}^{\infty} \int_{0}^{y} x d x d \gamma_{\alpha}(y) \leq C \int_{0}^{\infty} y^{2 \alpha+3} e^{-y^{2}} d y \leq C
$$

Thus (2.7) is proved.
As it was established by Bownik ([3]) in the Euclidean setting, the property (2.7) is not sufficient to deduce the validity of (2.6).

In order to obtain (2.6), we are going to see that $\mathcal{M}_{\alpha, \text { loc }}$ is bounded from $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$. Let $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $\lambda>0$. We consider the sequence $\left\{I_{i}\right\}_{i \in \mathbb{N}}$ introduced in the first part of this proof. We have that

$$
\left\{x \in(0, \infty): \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x)>\lambda\right\} \subset \bigcup_{i=0}^{\infty}\left\{x \in I_{i}: \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x)>\lambda\right\}
$$

Then,

$$
\begin{aligned}
\gamma_{\alpha}\left(\left\{x \in(0, \infty): \mathcal{M}_{\alpha, \mathrm{loc}}(f)(x)>\lambda\right\}\right) & \leq \sum_{i=0}^{\infty} \gamma_{\alpha}\left(\left\{x \in I_{i}: \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x)>\lambda\right\}\right) \\
& \leq \sum_{i=0}^{\infty} \gamma\left(c_{i}\right) \mathfrak{m}_{\alpha}\left(\left\{x \in I_{i}: \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x)>\lambda\right\}\right)
\end{aligned}
$$

We can write, for every $x \in(0, \infty)$ and $\phi \in \mathcal{A}_{x, \text { loc }}^{\alpha}$,

$$
\left|\int_{0}^{\infty} f(y) \phi(y) d \mathfrak{m}_{\alpha}(y)\right| \leq \frac{1}{\mathfrak{m}_{\alpha}(I(x, r))} \int_{x-r}^{x+r}|f(y)| d \mathfrak{m}_{\alpha}(y)
$$

for some $0<r \leq \min \{x, m(x)\}$. Then, since $m\left(c_{i}\right) \sim m(x)$ for every $x \in I_{i}$ and $y \in I(x, r)$, we get $y \in C I_{i}$ and

$$
\int_{x-r}^{x+r}|f(y)| d \mathfrak{m}_{\alpha}(y)=\int_{x-r}^{x+r}|f(y)| \chi_{C I_{i}}(y) d \mathfrak{m}_{\alpha}(y)
$$

We deduce that, for $i \in \mathbb{N}$,

$$
\mathcal{M}_{\alpha, \operatorname{loc}}(f)(x) \leq M_{\mathfrak{m}_{\alpha}}\left(f \chi_{C I_{i}}\right)(x), \quad x \in I_{i}
$$

where $M_{\mathfrak{m}_{\alpha}}$ denotes the centered Hardy-Littlewood maximal function defined by the measure $\mathfrak{m}_{\alpha}$ on $(0, \infty)$.

It is well-known that $M_{\mathfrak{m}_{\alpha}}$ is bounded from $L^{1}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \mathfrak{m}_{\alpha}\right)$. Thus,

$$
\begin{aligned}
\gamma_{\alpha}(\{x \in(0, \infty) & \left.\left.: \mathcal{M}_{\alpha, \operatorname{loc}}(f)(x)>\lambda\right\}\right) \\
& \leq \sum_{i=0}^{\infty} \gamma\left(c_{i}\right) \mathfrak{m}_{\alpha}\left(\left\{x \in I_{i}: M_{\mathfrak{m}_{\alpha}}\left(f \chi_{C I_{i}}\right)(x)>\lambda\right\}\right) \\
& \leq \frac{C}{\lambda} \sum_{i=0}^{\infty} \gamma\left(c_{i}\right) \int_{C I_{i}}|f(y)| d \mathfrak{m}_{\alpha}(y) \\
& \leq \frac{C}{\lambda} \sum_{i=0}^{\infty} \int_{C I_{i}}|f(y)| d \gamma_{\alpha}(y) \\
& \leq \frac{C}{\lambda} \int_{0}^{\infty}|f(y)| d \gamma_{\alpha}(y)
\end{aligned}
$$

Therefore, we have proved that the local maximal function $\mathcal{M}_{\alpha, \text { loc }}$ is bounded from $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$.

Suppose that $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$, where, for every $j \in \mathbb{N}, b_{j}$ is a $(1, \infty, \alpha)$-atom and $\lambda_{j} \in \mathbb{C}$ being $\sum_{j=0}^{\infty}\left|\lambda_{j}\right|<\infty$. The series defining $f$ converges in $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and from the boundedness proved above for $\mathcal{M}_{\alpha, \text { loc }}$,

$$
\mathcal{M}_{\alpha, \operatorname{loc}}(f)=\lim _{k \rightarrow \infty} \mathcal{M}_{\alpha, \operatorname{loc}}\left(\sum_{j=0}^{k} \lambda_{j} b_{j}\right), \quad \text { in } L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)
$$

Then, there exists an increasing function $\psi: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$
\mathcal{M}_{\alpha, \text { loc }}(f)(x)=\lim _{k \rightarrow \infty} \mathcal{M}_{\alpha, \text { loc }}\left(\sum_{j=0}^{\psi(k)} \lambda_{j} b_{j}\right)(x), \quad \text { a.e. } x \in(0, \infty)
$$

which yields

$$
\begin{aligned}
\mathcal{M}_{\alpha, \text { loc }}(f)(x) & \leq \lim _{k \rightarrow \infty} \sum_{j=0}^{\psi(k)}\left|\lambda_{j}\right| \mathcal{M}_{\alpha, \text { loc }}\left(b_{j}\right)(x) \\
& =\sum_{j=0}^{\infty}\left|\lambda_{j}\right| \mathcal{M}_{\alpha, \text { loc }}\left(b_{j}\right)(x), \quad \text { a.e. } x \in(0, \infty) .
\end{aligned}
$$

By (2.7) it follows that

$$
\left\|\mathcal{M}_{\alpha, \operatorname{loc}(f)}\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq \sum_{j=0}^{\infty}\left|\lambda_{j}\right|\left\|\mathcal{M}_{\alpha, \operatorname{loc}}\left(b_{j}\right)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C \sum_{j=0}^{\infty}\left|\lambda_{j}\right|,
$$

so we can conclude that $\left\|\mathcal{M}_{\alpha, \operatorname{loc}}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)}$.
Since $f=\sum_{j=0}^{\infty} \lambda_{j} b_{j}$ in $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ we obtain

$$
\int_{x}^{\infty} f(y) d \gamma_{\alpha}(y)=\sum_{j=0}^{\infty} \lambda_{j} \int_{x}^{\infty} b_{j}(y) d \gamma_{\alpha}(y), \quad x \in(0, \infty)
$$

Then,

$$
\left|\int_{x}^{\infty} f(y) d \gamma_{\alpha}(y)\right| \leq \sum_{j=0}^{\infty}\left|\lambda_{j}\right|\left|\int_{x}^{\infty} b_{j}(y) d \gamma_{\alpha}(y)\right|, \quad x \in(0, \infty)
$$

By using the monotone convergence theorem and (2.7) we get

$$
\mathcal{E}_{\alpha}(f) \leq \sum_{j=0}^{\infty}\left|\lambda_{j}\right| \mathcal{E}_{\alpha}\left(b_{j}\right) \leq C \sum_{j=0}^{\infty}\left|\lambda_{j}\right| \leq C\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)}
$$

The proof is now finished.

## 3. Proof of Theorem 1.3

(b) $\Rightarrow$ (a) It is sufficient to note that if $b$ is an $(a, q, \alpha)_{w}$-atom, then $b$ is also an ( $a, q, \alpha$ )-atom, when $0<a \leq 1$.
(a) $\Rightarrow$ (c) Let $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Then $\mathcal{M}_{\alpha, \text { loc }}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ by Theorem 1.2. We shall prove that $\mathbb{M}_{1, \text { loc }}(f) \leq C \mathcal{M}_{\alpha, \text { loc }}(f)$.

Let $\phi \in \mathbb{A}$ and $t, x \in(0, \infty)$ such that $t \leq w(x)$. We define

$$
\varphi_{x, t}(y)=\frac{1}{t y^{2 \alpha+1}} \phi\left(\frac{x-y}{t}\right), y \in(0, \infty) .
$$

We have that $\operatorname{supp}\left(\varphi_{x, t}\right) \subset(x-t, x+t) \subset\left(\frac{19}{20} x, \frac{21}{20} x\right)$ and $I(x, t) \in \mathcal{B}$. Then, by $[26,(1.4)]$, for every $y \in(0, \infty)$,

$$
\left|\varphi_{x, t}(y)\right| \leq \frac{1}{t y^{2 \alpha+1}} \leq \frac{C}{t x^{2 \alpha+1}} \leq \frac{C}{t(x+t)^{2 \alpha+1}} \leq \frac{C}{\mathfrak{m}_{\alpha}((I(x, t))}
$$

On the other hand, we get

$$
\partial_{y} \varphi_{x, t}(y)=-\frac{2 \alpha+1}{t y^{2 \alpha+2}} \phi\left(\frac{x-y}{t}\right)-\frac{1}{t^{2} y^{2 \alpha+1}} \phi^{\prime}\left(\frac{x-y}{t}\right), \quad y \in(0, \infty) .
$$

Then, for each $y \in(0, \infty)$,

$$
\left|\partial_{y} \varphi_{x, t}(y)\right| \leq C\left(\frac{1}{t y^{2 \alpha+2}}+\frac{1}{t^{2} y^{2 \alpha+1}}\right) \leq \frac{C}{t^{2} x^{2 \alpha+1}} \leq \frac{C}{t \mathfrak{m}_{\alpha}(I(x, t))}
$$

so it follows that, for a certain $C>0, C \varphi_{x, t} \in \mathcal{A}_{x, \text { loc }}^{\alpha}$ and, therefore, $\mathbb{M}_{1, \text { loc }}(f) \leq$ $C \mathcal{M}_{\alpha, \text { loc }}(f)$ as claimed.

Moreover, since $\mathbb{M}_{a, \text { loc }}(f) \leq \mathbb{M}_{1, \text { loc }}(f)$ for any $0<a \leq 1$, we also have that $\mathbb{M}_{a, \text { loc }}(f) \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ for any $0<a \leq 1$ provided that $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$.

By proceeding as in the proof of $\mathcal{E}_{\alpha}(f)<\infty$ for $f \in \mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ (see Section 2, p. 15), in order to see that $\mathbb{E}_{\alpha}(f)<\infty$ for such $f$, it is sufficient to prove that there exists $C>0$ such that, for every $(1, \infty, \alpha)$-atom $b$,

$$
\mathbb{E}_{\alpha, 1}(b):=\int_{0}^{1} \frac{1}{y}\left|\int_{0}^{y} b(x) d \gamma_{\alpha}(x)\right| d y \leq C .
$$

If $b(x)=1$ for every $x \in(0, \infty)$, then

$$
\begin{aligned}
\mathbb{E}_{\alpha, 1}(b) & =\int_{0}^{1} \frac{1}{y} \int_{0}^{y} x^{2 \alpha+1} e^{-x^{2}} d x d y \\
& \leq \int_{0}^{1} \frac{1}{y} \int_{0}^{y} x^{2 \alpha+1} d x d y \\
& \leq \frac{1}{2 \alpha+2} \int_{0}^{1} y^{2 \alpha+1} d y=(2 \alpha+2)^{-2}
\end{aligned}
$$

Suppose now that $b$ is a $(1, \infty, \alpha)$-atom associated to an interval $I\left(x_{0}, r_{0}\right)$ with $0<r_{0} \leq x_{0}$ and $r_{0} \leq m\left(x_{0}\right)$.

Since $\int_{0}^{\infty} b(y) d \gamma_{\alpha}(y)=0$, we have $\int_{0}^{y} b(x) d \gamma_{\alpha}(x)=0$, provided that $\left|y-x_{0}\right| \geq r_{0}$. Then,

$$
\begin{aligned}
\mathbb{E}_{\alpha, 1}(b) & =\int_{\left(x_{0}-r_{0}, x_{0}+r_{0}\right) \cap(0,1)} \frac{1}{y}\left|\int_{0}^{y} b(x) d \gamma_{\alpha}(x)\right| d y \\
& \leq C \frac{\gamma\left(x_{0}\right)}{\gamma_{\alpha}\left(\left(x_{0}-r_{0}, x_{0}+r_{0}\right)\right)} \int_{x_{0}-r_{0}}^{x_{0}+r_{0}} \frac{1}{y} \int_{0}^{y} x^{2 \alpha+1} d x d y \\
& \leq C \frac{\gamma\left(x_{0}\right)}{\gamma_{\alpha}\left(\left(x_{0}-r_{0}, x_{0}+r_{0}\right)\right)} \int_{x_{0}-r_{0}}^{x_{0}+r_{0}} y^{2 \alpha+1} d y \\
& \leq \frac{C}{\gamma_{\alpha}\left(\left(x_{0}-r_{0}, x_{0}+r_{0}\right)\right)} \int_{x_{0}-r_{0}}^{x_{0}+r_{0}} d \gamma_{\alpha}(y) \leq C
\end{aligned}
$$

Thus the proof of $(\mathrm{a}) \Rightarrow(\mathrm{c})$ is finished.
(c) $\Rightarrow$ (b) Let $a \leq 1$. We will see that if $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$, then $f \in$ $\mathbb{H}_{2 a}^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$ provided that (c) holds for this $a$. In order to prove this property we proceed as in the proof of the corresponding property in Theorem 1.2 (see also the proof of [18, Theorem 3.3]). We sketch the main steps.

By using [5, Lemma 2.1] (see also [8, p. 276]) we define the sequence of positive numbers $\left\{c_{j}\right\}_{j \in \mathbb{Z}}$ given by $c_{0}=1, c_{j}=c_{j-1}+a w\left(c_{j-1}\right)$ for $j>0$ and $c_{j}=$ $c_{j+1}-a w\left(c_{j+1}\right)$ for $j<0$. Then, for every $j \in \mathbb{Z}$, the interval $\mathbb{I}_{j}=I\left(c_{j}, r_{j}\right)$ where $r_{j}=a w\left(c_{j}\right)$, verify the following properties
(i) $(0, \infty)=\bigcup_{j \in \mathbb{Z}} \mathbb{I}_{j}$;
(ii) For every $k \in \mathbb{Z}, \mathbb{I}_{k} \cap \mathbb{I}_{j}=\emptyset$ provided that $j \notin\{k-1, k, k+1\}$;

We choose a partition of unity $\left\{\eta_{j}\right\}_{j \in \mathbb{Z}}$ such that $\left|\eta_{j}^{\prime}\right| \leq C \frac{1}{r_{j}}$, for every $j \in \mathbb{Z}$.
Assume that $0<\beta<8$ and $|x-y| \leq \beta w(x)$, with $x, y \in(0, \infty)$. We shall see that $w(x) / h(\beta) \leq w(y) \leq h(\beta) w(x)$ where $h$ is a positive, increasing and continuous function on $(0,8)$. An explicit form for the function $h$ can be obtained. We first consider $x \geq 1$, so $w(x)=\frac{1}{8 x}$. Then, $x-\frac{\beta}{8 x} \leq y \leq x+\frac{\beta}{8 x}$. If $1 \leq y \leq x+\frac{\beta}{8 x}=\frac{8 x^{2}+\beta}{8 x}$,
since $y>x-\beta w(x)=\frac{8 x^{2}-\beta}{8 x}$, we get

$$
w(y)=\frac{1}{8 y} \leq \frac{x}{8 x^{2}-\beta} \leq \frac{x}{(8+\beta) x^{2}}=\frac{1}{(8-\beta) x}=\frac{8}{8-\beta} w(x)
$$

Also we have that

$$
w(y)=\frac{1}{8 y} \geq \frac{x}{8 x^{2}+\beta} \geq \frac{x}{(8+\beta) x^{2}}=\frac{1}{(8+\beta) x}=\frac{8}{8+\beta} w(x)
$$

Suppose now that $x-\beta w(x)<y<1$. Then, $w(y)=\frac{y}{8}$. Since $x-\frac{\beta}{8 x}<1$, it follows that $x<z:=\frac{2+\sqrt{4+2 \beta}}{4}$. We obtain

$$
w(y)<\frac{1}{8}<\frac{z}{8 x}=z w(x)
$$

Since

$$
\frac{8-\beta}{8}=1-\frac{\beta}{8} \leq x-\frac{\beta}{8 x} \leq y
$$

we get

$$
w(x) \leq \frac{1}{8} \leq \frac{1}{8-\beta} y \leq \frac{8}{8-\beta} w(y)
$$

Assume now that $x<1$. We can proceed in a similar way. We have that $w(x)=\frac{x}{8}$ and $\left(1-\frac{\beta}{8}\right) x<y<\left(1+\frac{\beta}{8}\right) x$. If $y<1$, then

$$
\frac{8-\beta}{8} w(x)<w(y)=\frac{y}{8}<\frac{8+\beta}{8} \frac{x}{8}=\frac{8+\beta}{8} w(x) .
$$

Suppose that $y \geq 1$. Since $x>\frac{8}{8+\beta} y$, we obtain

$$
w(x)=\frac{x}{8}=\frac{x^{2}}{8 x} \leq \frac{1}{8 x}<\frac{8+\beta}{8} \frac{1}{8 y}=\frac{8+\beta}{8} w(y)
$$

and

$$
w(y)=\frac{1}{8 y} \leq \frac{x}{8 x}<\frac{8+\beta}{8} \frac{x}{8}=\frac{8+\beta}{8} w(x) .
$$

Suppose now that $g \in L_{\mathrm{loc}}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ and $j \in \mathbb{Z}$. Let $\phi \in \mathbb{A}, x \in(0, \infty)$ and $0<t \leq a w(x)$. Then, since $\left|x-c_{j}\right| \leq|x-y|+\left|y-c_{j}\right| \leq a\left(w(x)+w\left(c_{j}\right) \leq 3 a w\left(c_{j}\right)\right.$ provided that $y \in \mathbb{I}_{j}$ and $|x-y|<t$, it follows that

$$
\left(\phi_{t} *\left(g \eta_{j} \gamma_{\alpha}\right)\right)(x)=0, \quad x \notin 3 \mathbb{I}_{j} .
$$

Therefore $\operatorname{supp}\left(\mathbb{M}_{a, \text { loc }}\left(g \eta_{j} \gamma_{\alpha}\right)\right) \subset 3 \mathbb{I}_{j}$.
We shall now see that

$$
\begin{equation*}
\mathbb{M}_{a, \operatorname{loc}}\left(g \eta_{j} \gamma_{\alpha}\right)(x) \leq C \mathbb{M}_{a, \operatorname{loc}}(g)(x) \gamma_{\alpha}\left(c_{j}\right) \chi_{3 \mathbb{I}_{j}}(x), \quad x \in(0, \infty) \tag{3.1}
\end{equation*}
$$

by showing that $C \varphi_{x, t} \in \mathbb{A}$, for every $x \in 3 \mathbb{I}_{j}$ and $0<t \leq a w(x)$ and some constant $C>0$ that does not depend on $x$ and $t$, being

$$
\varphi_{x, t}(z)=\phi(z) \eta_{j}(x-t z) \gamma_{\alpha}(x-t z) / \gamma_{\alpha}\left(c_{j}\right), \quad z \in(0, \infty)
$$

Let $x \in 3 \mathbb{I}_{j}$ and $0<t \leq a w(x)$. Since there exists $C>0$ such that

$$
\begin{equation*}
\frac{1}{C} \gamma_{\alpha}(y) \leq \gamma_{\alpha}\left(c_{j}\right) \leq C \gamma_{\alpha}(y), \quad y \in 5 \mathbb{I}_{j} \tag{3.2}
\end{equation*}
$$

and

$$
\left|x-t z-c_{j}\right| \leq\left|x-c_{j}\right|+t|z| \leq 3 a w\left(c_{j}\right)+a w(x) \leq 5 a w\left(c_{j}\right), \quad|z| \leq 1
$$

we obtain that $\left|\varphi_{x, t}(z)\right| \leq C$ for any $|z| \leq 1$.
On the other hand,

$$
\begin{aligned}
\partial_{z} \varphi_{x, t}(z)=\frac{1}{\gamma_{\alpha}\left(c_{j}\right)}( & \phi^{\prime}(z) \eta_{j}(x-t z) \gamma_{\alpha}(x-t z)-t \phi(z) \eta_{j}^{\prime}(x-t z) \gamma_{\alpha}(x-t z) \\
& \left.-t \phi(z) \eta_{j}(x-t z) \gamma_{\alpha}^{\prime}(x-t z)\right), \quad|z| \leq 1
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\left|\partial_{z} \varphi_{x, t}(z)\right| \leq & \frac{C}{\gamma_{\alpha}\left(c_{j}\right)}\left(\gamma_{\alpha}(x-t z)+\frac{a w(x) \gamma_{\alpha}(x-t z)}{r_{j}}\right. \\
& \left.+a w(x) \gamma_{\alpha}(x-t z)\left(\frac{1}{x}+x\right)\right) \leq C, \quad|z| \leq 1
\end{aligned}
$$

Thus, for a certain $C>0, C \varphi_{x, t} \in \mathbb{A}$ and we can write, for every $x \in(0, \infty)$ and $0<t \leq a w(x)$,

$$
\left(\phi_{t} *\left(g \eta_{j} \gamma_{\alpha}\right)\right)(x)=\left(\left(\varphi_{x, t}\right)_{t} * g\right)(x) \gamma_{\alpha}\left(c_{j}\right) \chi_{3 \mathbb{I}_{j}}(x)
$$

so (3.1) follows.
Let $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ such that $\int_{0}^{\infty} f d \gamma_{\alpha}=0$ and $j \in \mathbb{Z}$. We define

$$
\begin{equation*}
b_{j}=\frac{\int_{0}^{\infty} f \eta_{j} d \gamma_{\alpha}}{\int_{0}^{\infty} \eta_{j} d \gamma_{\alpha}} \tag{3.3}
\end{equation*}
$$

Our next objective is to see that $\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha} \in H^{1}(\mathbb{R}, d x)$ and

$$
\left\|\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right\|_{H^{1}(\mathbb{R}, d x)} \leq C \int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)
$$

Here $H^{1}(\mathbb{R}, d x)$ denotes the classical Hardy space on $\mathbb{R}$.
Let $\phi \in \mathbb{A} \cap C_{c}^{\infty}(0, \infty)$ such that $\int_{\mathbb{R}} \phi(x) d x \neq 0$. By using (3.1) and (3.3) we have that

$$
\begin{aligned}
\int_{0}^{\infty} \sup _{0<t \leq a w(x)} & \left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| d x \\
& \leq C \gamma_{\alpha}\left(c_{j}\right) \int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}\left(f-b_{j}\right)(x) d x \\
& \leq C \gamma_{\alpha}\left(c_{j}\right)\left(\int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d x+\left|b_{j}\right| \int_{3 \mathbb{I}_{j}} d x\right) \\
& \leq C\left(\int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)+\frac{\gamma_{\alpha}\left(3 \mathbb{I}_{j}\right)}{\gamma_{\alpha}\left(\mathbb{I}_{j}\right)} \int_{\mathbb{I}_{j}}|f(x)| d \gamma_{\alpha}(x)\right) \\
& \leq C \int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x),
\end{aligned}
$$

where we have used that, since $\eta_{j}(z)=1$ for $z \in \frac{1}{2} \mathbb{I}_{j}, \int_{0}^{\infty} \eta_{j} d \gamma_{\alpha} \geq C \gamma_{\alpha}\left(\mathbb{I}_{j}\right)$. We are going to see that

$$
\int_{0}^{\infty} \sup _{t>a w(x)}\left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| d x \leq C \int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)
$$

We firstly note that for every $x \in 3 \mathbb{I}_{j}$ and $t \geq a w(x)$, (3.3) leads to

$$
\begin{aligned}
\left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| & \leq \frac{C}{t} \int_{\mathbb{I}_{j}}\left|f(y)-b_{j}\right| d \gamma_{\alpha}(y) \\
& \leq \frac{C}{w(x)} \int_{\mathbb{I}_{j}}\left(|f(y)|+\left|b_{j}\right|\right) d \gamma_{\alpha}(y) \\
& \leq \frac{C}{w\left(c_{j}\right)} \int_{\mathbb{I}_{j}}|f(y)| d \gamma_{\alpha}(y)
\end{aligned}
$$

Then,

$$
\int_{3 \mathbb{I}_{j}} \sup _{t>a w(x)}\left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| d x \leq C \frac{\left|\mathbb{I}_{j}\right|}{w\left(c_{j}\right)} \int_{\mathbb{I}_{j}}|f(y)| d \gamma_{\alpha}(y)
$$

$$
\leq C \int_{\mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(y) d \gamma_{\alpha}(y)
$$

Here, $|A|$ denotes the Lebesgue measure of $A$, for every measurable set $A \subset \mathbb{R}$.
On the other hand, by proceeding as in the corresponding estimation in Section 2 we can deduce that

$$
\int_{\left(3 \mathbb{I}_{j}\right)^{c}} \sup _{t>\operatorname{aw}(x)}\left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| d x \leq C \int_{\mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(y) d \gamma_{\alpha}(y)
$$

Thus we prove that

$$
\int_{0}^{\infty} \sup _{t>a w(x)}\left|\phi_{t} *\left(\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right)(x)\right| d x \leq C \int_{\mathbb{I}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(y) d \gamma_{\alpha}(y)
$$

Then,

$$
\left\|\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}\right\|_{H^{1}(\mathbb{R}, d x)} \leq C \int_{3 \mathbb{I}_{j}} \mathbb{M}_{a, \text { loc }}(f)(y) d \gamma_{\alpha}(y)
$$

By using now [18, Lemma 2.1] we can see that

$$
\left(f-b_{j}\right) \eta_{j} \gamma_{\alpha}=\sum_{i=0}^{\infty} \lambda_{i, j} g_{i, j}, \quad \text { in } L^{1}((0, \infty), d x)
$$

where, for every $i \in \mathbb{N}, g_{i, j}$ is a $(1, \infty)$-classical atom supported on $2 \mathbb{I}_{j}$ and $\lambda_{i, j} \in \mathbb{C}$ being

$$
\sum_{i=0}^{\infty}\left|\lambda_{i, j}\right| \leq C \int_{3 \mathbb{K}_{j}} \mathbb{M}_{a, \operatorname{loc}}(f)(y) d \gamma_{\alpha}(y)
$$

According to (3.2), there exists $C>0$ that is not depending on $j$ such that $C g_{i, j} / \gamma_{\alpha}$ is a $(2 a, \infty, \alpha)_{w}$-atom, for every $i \in \mathbb{N}$, and

$$
\left(f-b_{j}\right) \eta_{j}=\sum_{i=0}^{\infty} \lambda_{i j} g_{i, j} / \gamma_{\alpha}, \quad \text { in } L^{1}\left((0, \infty), \gamma_{\alpha}\right) .
$$

As in Section 2 we can prove that

$$
\sum_{j \in \mathbb{Z}}\left(f-b_{j}\right) \eta_{j}=\sum_{j \in \mathbb{Z}} \sum_{i=0}^{\infty} \lambda_{i, j} g_{i, j} / \gamma_{\alpha}, \quad \text { in } L^{1}\left((0, \infty), \gamma_{\alpha}\right),
$$

and

$$
\sum_{j \in \mathbb{Z}} \sum_{i=0}^{\infty}\left|\lambda_{i, j}\right| \leq C \int_{0}^{\infty} \mathbb{M}_{a, \text { loc }}(f)(y) d \gamma_{\alpha}(y)
$$

As in [18, p. 1687] we define $\mu_{k}=\sum_{j=k}^{\infty} \eta_{j}, k \in \mathbb{Z}$, and we also consider $\widetilde{\eta}_{j}=$ $\eta_{j} / \int_{0}^{\infty} \eta_{j} d \gamma_{\alpha}, j \in \mathbb{Z}$. By using summation by parts we have that

$$
\sum_{j \in \mathbb{Z}} b_{j} \widetilde{\eta}_{j}=\sum_{k=-\infty}^{+\infty}\left(\widetilde{\eta}_{k}-\widetilde{\eta}_{k-1}\right) \int_{0}^{\infty} f \mu_{k} d \gamma_{\alpha}
$$

There exists $C>0$ such that $C\left(\widetilde{\eta}_{k}-\widetilde{\eta}_{k-1}\right)$ is a $(2 a, \infty, \alpha)_{w}$-atom, for every $k \in \mathbb{Z}$. On the other hand recalling that $\int f d \gamma_{\alpha}=0$, we get

$$
\begin{aligned}
\int_{0}^{\infty} f(x) \mu_{k}(x) d \gamma_{\alpha}(x)= & \int_{0}^{1} f(x)\left(-\int_{x}^{1} \mu_{k}^{\prime}(y) d y+\mu_{k}(1)\right) d \gamma_{\alpha}(x) \\
& +\int_{1}^{\infty} f(x)\left(\int_{1}^{x} \mu_{k}^{\prime}(y) d y+\mu_{k}(1)\right) d \gamma_{\alpha}(x) \\
= & -\int_{0}^{1} \mu_{k}^{\prime}(y) \int_{0}^{y} f(x) d \gamma_{\alpha}(x) d y
\end{aligned}
$$

$$
+\int_{1}^{\infty} \mu_{k}^{\prime}(y) \int_{y}^{\infty} f(x) d \gamma_{\alpha}(x) d y, \quad k \in \mathbb{Z}
$$

Since, for every $k \in \mathbb{Z}, \operatorname{supp}\left(\mu_{k}^{\prime}\right) \subset \mathbb{I}_{k}$ and $\left|\mu_{k}^{\prime}\right| \leq C / w\left(c_{k}\right)$, we obtain

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}}\left|\int_{0}^{\infty} f(x) \mu_{k}(x) d \gamma_{\alpha}(x)\right| \leq & C\left(\sum_{k \leq 0} \int_{\mathbb{I}_{k}} \frac{1}{c_{k}}\left|\int_{0}^{y} f(x) d \gamma_{\alpha}(x)\right| d y\right. \\
& \left.+\sum_{k>0} \int_{\mathbb{I}_{k}} c_{k}\left|\int_{y}^{\infty} f(x) d \gamma_{\alpha}(x)\right| d y\right) \\
\leq & C\left(\sum_{k \leq 0} \int_{\mathbb{I}_{k}} \frac{1}{y}\left|\int_{0}^{y} f(x) d \gamma_{\alpha}(x)\right| d y\right. \\
& \left.+\sum_{k>0} \int_{\mathbb{I}_{k}} y\left|\int_{y}^{\infty} f(x) d \gamma_{\alpha}(x)\right| d y\right) \\
\leq & C \mathbb{E}_{\alpha}(f) .
\end{aligned}
$$

We write $f=\sum_{j \in \mathbb{Z}}\left(f-b_{j}\right) \eta_{j}+\sum_{j \in \mathbb{Z}} b_{j} \eta_{j}$ and conclude that $f \in \mathbb{H}_{2 a}^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)$ with

$$
\|f\|_{\mathbb{H}_{2 a}^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\left(\int_{0}^{\infty} \mathbb{M}_{a, \operatorname{loc}}(f)(x) d \gamma_{\alpha}(x)+\mathbb{E}_{\alpha}(f)\right)
$$

where the constant $C>0$ does not depend on $f$.
The equivalences of the quantities $\|f\|_{\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)},\|f\|_{\mathbb{H}_{a}^{1, q}\left((0, \infty), \gamma_{\alpha}\right)}$ and

$$
\left\|\mathbb{M}_{a, \operatorname{loc}}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}+\mathbb{E}_{\alpha}(f)
$$

were established during the proofs of $(\mathrm{b}) \Rightarrow(\mathrm{a}) \Rightarrow(\mathrm{c}) \Rightarrow(\mathrm{b})$.

## 4. Proof of Theorem 1.6.

By using an integral representation for the modified Bessel function $I_{\nu}$ ([14, (5.10.22)] we deduce that, for every $t, x, y \in(0, \infty)$

$$
W_{t}^{\alpha}(x, y)=\frac{1}{\left(1-e^{-t}\right)^{\alpha+1}} \int_{-1}^{1} \exp \left(-\frac{q\left(e^{-t / 2} x, y, s\right)}{1-e^{-t}}+y^{2}\right) \Pi_{\alpha}(s) d s
$$

where $\Pi_{\alpha}(s)=\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right) \sqrt{\pi}}\left(1-s^{2}\right)^{\alpha-1 / 2}, s \in(-1,1)$, and $q(x, y, s)=x^{2}+y^{2}-2 x y s$, $x, y \in(0, \infty)$ and $s \in(-1,1)$.

Let $\delta>0$. As in [21] we split $(0, \infty) \times(0, \infty) \times(-1,1)$ in two parts. The first part is named the $\delta$-local part and it is defined by

$$
L_{\delta}=\left\{(x, y, s) \in(0, \infty) \times(0, \infty) \times(-1,1): \sqrt{q(x, y, s)} \leq \frac{\delta}{1+x+y}\right\}
$$

The second part is $G_{\delta}=((0, \infty) \times(0, \infty) \times(-1,1)) \backslash L_{\delta}$ and it is named the $\delta$-global part.

We now decompose, for every $t>0$, the operator $W_{t}^{\alpha}$ in the global and local parts as follows. We choose an smooth function $\psi$ defined in $(0, \infty) \times(0, \infty) \times(-1,1)$ such that $0 \leq \psi \leq 1, \psi(x, y, s)=1,(x, y, s) \in L_{1}, \psi(x, y, s)=0,(x, y, s) \in G_{2}$, and $\left|\partial_{x} \psi(x, y, s)\right|+\left|\partial_{y} \psi(x, y, s)\right| \leq \frac{C}{\sqrt{q(x, y, s)}}, \quad(x, y, s) \in(0, \infty) \times(0, \infty) \times(-1,1)$.
For every $t, x, y>0$, we define the local part $W_{t, \mathrm{loc}}^{\alpha}(x, y)$ of $W_{t}^{\alpha}(x, y)$ by

$$
W_{t, \mathrm{loc}}^{\alpha}(x, y)=\frac{1}{\left(1-e^{-t}\right)^{\alpha+1}} \int_{-1}^{1} \exp \left(-\frac{q\left(e^{-t / 2} x, y, s\right)}{1-e^{-t}}+y^{2}\right) \psi(x, y, s) \Pi_{\alpha}(s) d s
$$

and the global part $W_{t, \text { glob }}^{\alpha}(x, y)$ of $W_{t}^{\alpha}(x, y)$ by

$$
W_{t, \mathrm{glob}}^{\alpha}(x, y)=W_{t}^{\alpha}(x, y)-W_{t, \mathrm{loc}}^{\alpha}(x, y)
$$

We define, for every $t>0$,

$$
W_{t, \mathrm{loc}}^{\alpha}(f)(x)=\int_{0}^{\infty} W_{t, \mathrm{loc}}^{\alpha}(x, y) f(y) d \gamma_{\alpha}(y), \quad x \in(0, \infty)
$$

and

$$
W_{t, \text { glob }}^{\alpha}(f)(x)=\int_{0}^{\infty} W_{t, \text { glob }}^{\alpha}(x, y) f(y) d \gamma_{\alpha}(y), \quad x \in(0, \infty)
$$

Finally we consider the local and global maximal operators defined by

$$
\mathbb{W}_{*, \text { loc }}^{\alpha}(f)=\sup _{0<t<m(x)^{2}}\left|W_{t, \text { loc }}^{\alpha}(f)\right| \quad \text { and } \quad \mathbb{W}_{*, \text { glob }}^{\alpha}(f)=\sup _{0<t<m(x)^{2}}\left|W_{t, \text { glob }}^{\alpha}(f)\right| .
$$

It is clear that $\mathbb{W}_{*}^{\alpha}(f) \leq \mathbb{W}_{*, \text { loc }}^{\alpha}(f)+\mathbb{W}_{*, \text { glob }}^{\alpha}(f)$.
We are going to prove first that the operator $\mathbb{W}_{*, \text { loc }}^{\alpha}$ is bounded from the Hardy space $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Since $\mathbb{W}_{*}^{\alpha}$ is bounded from $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1, \infty}\left((0, \infty), \gamma_{\alpha}\right), \mathbb{W}_{*, \text { loc }}^{\alpha}$ has the same property. Then, in order to prove our objective it is sufficient to see that there exists $C>0$ such that

$$
\begin{equation*}
\left\|\mathbb{W}_{*, \mathrm{loc}}^{\alpha}(b)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C \tag{4.1}
\end{equation*}
$$

for every $(1, \infty, \alpha)$-atom $b$.
If $b(x)=1$, for every $x \in(0, \infty)$, then $\mathbb{W}_{*, \text { loc }}^{\alpha}(b)(x) \leq 1$, for any $x \in(0, \infty)$, and $\left\|\mathbb{W}_{*, \text { loc }}^{\alpha}(b)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq 1$.

On the other hand, suppose that $b$ is a $(1, \infty, \alpha)$-atom associated with an interval $I=\left(c_{I}-r_{I}, c_{I}+r_{I}\right) \in \mathcal{B}$ where $0<r_{I} \leq c_{I}$. We define $a=b \gamma$. We can write

$$
\begin{aligned}
\|a\|_{L^{\infty}\left((0, \infty), \mathfrak{m}_{\alpha}\right)} & \leq C \gamma\left(c_{I}\right)\|b\|_{L^{\infty}\left((0, \infty) ; \gamma_{\alpha}\right)} \\
& \leq C \gamma\left(c_{I}\right)\left(\gamma_{\alpha}(I)\right)^{-1} \\
& \leq C \mathfrak{m}_{\alpha}(I)
\end{aligned}
$$

where $C>0$ does not depend on $b$. Then, $a / C$ is a $\left(\mathfrak{m}_{\alpha}, \infty\right)$-atom.
For every $t, x \in(0, \infty)$,

$$
\begin{aligned}
W_{t, \mathrm{loc}}^{\alpha}(b)(x) & =\int_{0}^{\infty} b(y) \int_{-1}^{1} \frac{\exp \left(-\frac{q\left(e^{-t / 2} x, y, s\right)}{1-e^{-t}}+y^{2}\right)}{\left(1-e^{-t}\right)^{\alpha+1}} \psi(x, y, s) \Pi_{\alpha}(s) d s d \gamma_{\alpha}(y) \\
& =e^{x^{2}} \int_{0}^{\infty} a(y) \int_{-1}^{1} \frac{\exp \left(-\frac{q\left(e^{-t / 2} y, x, s\right)}{1-e^{-t}}\right.}{\left(1-e^{-t}\right)^{\alpha+1}} \psi(x, y, s) \Pi_{\alpha}(s) d s d \mathfrak{m}_{\alpha}(y)
\end{aligned}
$$

We define

$$
K_{t}^{\alpha}(x, y)=\int_{-1}^{1} \frac{\exp \left(-\frac{q\left(e^{-t / 2} y, x, s\right)}{1-e^{-t}}\right)}{\left(1-e^{-t}\right)^{\alpha+1}} \psi(x, y, s) \Pi_{\alpha}(s) d s, \quad t, x, y \in(0, \infty)
$$

According to $[1, \S 4]$ we can deduce

$$
\begin{equation*}
\sup _{0<t \leq m(x)^{2}}\left|\partial_{y} K_{t}(x, y)\right| \leq C \frac{1}{|x-y| \mathfrak{m}_{\alpha}(I(x,|x-y|))}, \quad x, y \in(0, \infty) \tag{4.2}
\end{equation*}
$$

Since $\mathbb{W}_{*}^{\alpha}$ is bounded on $L^{2}\left((0, \infty), \gamma_{\alpha}\right)$, we deduce that

$$
\begin{aligned}
\int_{0}^{\infty}\left|\mathbb{W}_{*, \mathrm{loc}}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x)= & \int_{2 I}\left|\mathbb{W}_{*, \mathrm{loc}}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x) \\
& +\int_{(2 I)^{c}}\left|\mathbb{W}_{*, \text { loc }}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x)
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left(\int_{0}^{\infty}\left|\mathbb{W}_{*}^{\alpha}(b)(x)\right|^{2} d \gamma_{\alpha}(x)\right)^{1 / 2}\left(\gamma_{\alpha}(2 I)\right)^{1 / 2} \\
&+\int_{(2 I)^{c}}\left|\mathbb{W}_{*, \text { loc }}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x) \\
& \leq C\|b\|_{\left.L^{2}\left((0, \infty), \gamma_{\alpha}\right)\right)\left(\gamma_{\alpha}(2 I)\right)^{1 / 2}} \\
&+\int_{(2 I)^{c}}\left|\mathbb{W}_{*, \text { loc }}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x) \\
& \leq C+\int_{(2 I)^{c}}\left|\mathbb{W}_{*, \text { loc }}^{\alpha}(b)(x)\right| d \gamma_{\alpha}(x)
\end{aligned}
$$

On the other hand, we can write

$$
\begin{aligned}
W_{t, \mathrm{loc}}^{\alpha}(b)(x) & =e^{x^{2}} \int_{0}^{\infty} a(y) K_{t}^{\alpha}(x, y) d \mathfrak{m}_{\alpha}(y) \\
& =e^{x^{2}} \int_{0}^{\infty} a(y)\left(K_{t}^{\alpha}(x, y)-K_{t}^{\alpha}\left(x, c_{I}\right)\right) d \mathfrak{m}_{\alpha}(y), \quad x \in(0, \infty)
\end{aligned}
$$

By using (4.2) and [26, (1.4)] we get

$$
\begin{aligned}
\mathbb{W}_{*, \mathrm{loc}}^{\alpha}(b)(x) & \leq C e^{x^{2}} \int_{I}|a(y)| \frac{\left|y-c_{I}\right|}{|x-y| \mathfrak{m}_{\alpha}(I(x,|x-y|))} d \mathfrak{m}_{\alpha}(y) \\
& \leq C \frac{e^{x^{2}} r_{I}}{\mathfrak{m}_{\alpha}(I)} \int_{I} \frac{1}{|x-y|^{2} x^{2 \alpha+1}} d \mathfrak{m}_{\alpha}(y), \quad x \in(0, \infty)
\end{aligned}
$$

It follows that

$$
\int_{(2 I)^{c}} \mathbb{W}_{*, \operatorname{loc}}^{\alpha}(b)(x) d \gamma_{\alpha}(x) \leq C \frac{r_{I}}{\mathfrak{m}_{\alpha}(I)} \int_{I} \int_{(2 I)^{c}} \frac{d x}{|x-y|^{2}} d \mathfrak{m}_{\alpha}(y) \leq C
$$

We conclude that (4.1) holds with constant independent of the atom $b$.
We now prove that the operator $\mathbb{W}_{*, \text { glob }}^{\alpha}$ is bounded from $\mathcal{H}^{1}\left((0, \infty), \gamma_{\alpha}\right)$ into $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. Actually we will see that $\mathbb{W}_{*, \text { glob }}^{\alpha}$ is bounded on $L^{1}\left((0, \infty), \gamma_{\alpha}\right)$.

We take $\alpha=\frac{k}{2}-1$, with $k \in \mathbb{N}, k \geq 2$ and for every $\bar{x} \in \mathbb{R}^{k}$ we write $x=|\bar{x}|$. For every $\bar{x}, \bar{y} \in \mathbb{R}^{k}$ we have that $|\bar{x}-\bar{y}|^{2}=q(x, y, \cos (\theta))$, where $\theta$ is the angle between $\bar{x}$ and $\bar{y}$. Then, $(x, y, \cos (\theta)) \in L_{1}$ if and only if $|\bar{x}-\bar{y}| \leq \frac{1}{1+x+y}$. We now integrate on $\mathbb{R}^{k}$ using spherical coordinates by performing the change of variables $s=\cos (\theta)$ to obtain

$$
\begin{aligned}
\left|W_{t, \text { glob }}^{\alpha}(f)(x)\right| & \leq C \int_{|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y}} \frac{\exp \left(-\frac{\left|e^{-t / 2} \bar{x}-\bar{y}\right|^{2}}{1-e^{-t}}\right)}{\left(1-e^{-t}\right)^{k / 2}}|f(y)| d \bar{y} \\
& \leq C \int_{|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y}} W_{t}^{O U}(\bar{x}, \bar{y})|f(y)| d \bar{y}, \quad x \in \mathbb{R}^{k}, t \in(0, \infty)
\end{aligned}
$$

Here, for every $\bar{x}, \bar{y} \in \mathbb{R}^{k}$ and $t \in(0, \infty)$, $W_{t}^{O U}(\bar{x}, \bar{y})$ represents the integral kernel of the Ornstein-Uhlenbeck semigroup in $\mathbb{R}^{k}$ (see [12]).

We are going to show that $|\bar{x}-\bar{y}| \geq m(x)$ provided that $|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y}$. In order to do so, we shall consider three cases.
(a) Assume that $y \geq 4$ and $y \leq 2 x$. Then, $x \geq 2$ and $m(x)=\frac{1}{x}$. It follows that

$$
|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y} \geq \frac{1}{1+3 x}=\frac{1}{x} \frac{x}{1+3 x} \geq C \frac{1}{x}
$$

(b) If $y \geq 4$ and $y \geq 2 x$, then

$$
|\bar{x}-\bar{y}| \geq y-x \geq \frac{y}{2} \geq 2 \geq m(x)
$$

(c) Assume that $y \leq 4$. We deduce that, for $x \leq 1$,

$$
|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y} \geq \frac{1}{5+x} \geq \frac{1}{6}=\frac{1}{6} m(x)
$$

while, for $x>1$,

$$
|\bar{x}-\bar{y}| \geq \frac{1}{1+x+y} \geq \frac{1}{5+x} \geq \frac{1}{x} \frac{x}{5+x} \geq C m(x)
$$

Thus our objective is established.
We get

$$
\left|W_{t, \text { glob }}^{\alpha}(f)(x)\right| \leq C \int_{|\bar{x}-\bar{y}| \geq m(x)} W_{t}^{O U}(\bar{x}, \bar{y}) \mid f(y) d \bar{y}, \quad x \in \mathbb{R}^{k}, t \in(0, \infty)
$$

Let $\omega$ be a measurable function in $(0, \infty)$ such that $\omega(x) \in(0,1), x \in(0, \infty)$. We consider, for every $z \in \mathbb{C}$ with $\operatorname{Re}(z)>-\frac{1}{2}$, the operator

$$
S_{\omega}^{z}(f)(x)=e^{-x^{2}} x^{2 z+1} W_{\omega(x), \text { glob }}^{z}\left(f(y) e^{y^{2}} y^{-1-2 z}\right)(x) \chi_{\left(0, m(x)^{2}\right)}(\omega(x)), x \in(0, \infty)
$$

We have that
(a) For every $z \in \mathbb{C}, \operatorname{Re}(z)>-\frac{1}{2}, S_{\omega}^{z}$ is a measurable function on $(0, \infty)$ provided that $f$ is a simple function on $(0, \infty)$.
(b) Suppose that $f, g$ are simple functions on $(0, \infty)$. The function

$$
F(z)=\int_{0}^{\infty} S_{\omega}^{z}(f)(x) g(x) d x, \quad z \in \mathbb{C}, \quad \operatorname{Re}(z)>-\frac{1}{2}
$$

is holomorphic. Furthermore, for every $-1 / 2<c<d<\infty$, there exists $C>0$ such that $|F(x+i y)| \leq C, c \leq x \leq d$ and $y \in \mathbb{R}$.
We have that, for every $\bar{x} \in \mathbb{R}^{k}$ and $\sigma \in \mathbb{R}$,

$$
\left|S_{\omega}^{\frac{k}{2}-1+i \sigma}(f)(x)\right| \leq C e^{-x^{2}} x^{k-1} \sup _{0<t<m(x)^{2}} \int_{|\bar{x}-\bar{y}| \geq m(x)} W_{t}^{O U}(\bar{x}, \bar{y})|f(y)| y^{-k+1} e^{y^{2}} d \bar{y} .
$$

According to [20, Proposition 2.4 (i)] (see also [10]) we obtain

$$
\left\|S_{\omega}^{\frac{k}{2}-1+i \sigma}(f)\right\|_{L^{1}((0, \infty), d x)} \leq C\|f\|_{L^{1}((0, \infty), d x)}, \quad f \in L^{1}((0, \infty), d x), \sigma \in \mathbb{R}
$$

where $C>0$ does not depend on $\sigma \in \mathbb{R}$ and the function $\omega$.
By using [22, Theorem 1] we deduce that, for every $z \in \mathbb{C}, \operatorname{Re}(z) \geq 0$, there exists $C>0$ such that

$$
\left\|S_{\omega}^{z}(f)\right\|_{L^{1}((0, \infty), d x)} \leq C\|f\|_{L^{1}((0, \infty), d x)}, \quad f \in L^{1}((0, \infty), d x)
$$

where $C>0$ does not depend on the function $\omega$.
Then, for every $\alpha>0$ there exists $C>0$ independent of $\omega$ such that

$$
\left\|\mathbb{S}_{\omega}^{\alpha}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}, \quad f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)
$$

where

$$
\mathbb{S}_{\omega}^{\alpha}(f)(x)=W_{\omega(x), \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}(\omega(x)), \quad x \in(0, \infty)
$$

Let $f \in L^{1}\left((0, \infty), \gamma_{\alpha}\right)$. We choose $\left\{t_{j}\right\}_{j=1}^{\ell} \in \mathbb{Q} \cap(0,1)$. For every $x \in(0, \infty)$, we define

$$
\begin{aligned}
j(x)=\min \left\{j=1, \ldots, \ell: \mid W_{t_{j}, \mathrm{glob}}^{\alpha}\right. & (f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{j}\right) \mid \\
& \left.=\max _{k=1, \ldots, \ell}\left|W_{t_{k}, \mathrm{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{k}\right)\right|\right\}
\end{aligned}
$$

and $\omega(x)=t_{j(x)}$.
We also consider, for every $j=1, \ldots, \ell$, the set

$$
A_{j}=\left\{x \in(0, \infty): \omega(x)=t_{j}\right\}
$$

Note that, for every $j=1, \ldots, \ell, x \in A_{j}$ if and only if

$$
\left|W_{t_{j}, \operatorname{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{j}\right)\right|=\max _{k=1, \ldots, \ell}\left|W_{t_{k}, \mathrm{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{k}\right)\right|,
$$

and, when $j>1$, for every $i=1, \ldots, j-1$,

$$
\left|W_{t_{i}, \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{i}\right)\right|<\max _{k=1, \ldots, \ell}\left|W_{t_{k}, \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{k}\right)\right|
$$

It follows that $A_{j}$ is measurable, for every $j=1, \ldots, \ell$.
We can write

$$
\omega=\sum_{j=1}^{\ell} t_{j} \chi_{A_{j}} .
$$

Hence $\omega$ is a measurable function and

$$
\left|\mathbb{S}_{\omega}^{\alpha}(f)(x)\right|=\max _{j=1, \ldots, \ell}\left|W_{t_{j}, \mathrm{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{j}\right)\right|, \quad x \in(0, \infty)
$$

Then

$$
\left\|\max _{j=1, \ldots, \ell}\left|W_{t_{j}, \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{j}\right)\right|\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}
$$

Here $C>0$ does not depend on $f$ and $\left\{t_{j}\right\}_{j=1}^{\ell}$.
We write $\mathbb{Q} \cap(0,1)=\left\{t_{j}\right\}_{j=1}^{\infty}$. We have that, for any $x \in(0, \infty)$

$$
\sup _{t \in \mathbb{Q} \cap(0,1)}\left|W_{t, \mathrm{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}(t)\right|=\lim _{k \rightarrow \infty} \sup _{j=1, \ldots, k}\left|W_{t_{j}, \mathrm{glob}}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}\left(t_{j}\right)\right| .
$$

The monotone convergence theorem leads to

$$
\left\|\sup _{t \in \mathbb{Q} \cap(0,1)}\left|W_{t, \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}(t)\right|\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}
$$

For every $x \in(0, \infty)$ the function $W_{t, \text { glob }}^{\alpha}(f)(x)$ is continuous in $t \in(0, \infty)$. Then, for every $x \in(0, \infty)$,

$$
\sup _{\mathbb{Q} \cap(0, \infty)}\left|W_{t, \text { glob }}^{\alpha}(f)(x) \chi_{\left(0, m(x)^{2}\right)}(t)\right|=\sup _{0<t<m(x)^{2}}\left|W_{t, \text { loc }}^{\alpha}(f)(x)\right| .
$$

We conclude

$$
\left\|\mathbb{W}_{*, \text { glob }}^{\alpha}(f)\right\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)} \leq C\|f\|_{L^{1}\left((0, \infty), \gamma_{\alpha}\right)}
$$

Thus the proof is finished.
Conflict of interests. The authors declare that there is no conflict of interest.

## References

[1] Betancor, J. J., Dalmasso, E., Quijano, P., and Scotto, R. Harmonic analysis operators associated with laguerre polynomial expansions on variable Lebesgue spaces, 2022. arXiv:2202.11137.
[2] Betancor, J. J., Dziubański, J., and Torrea, J. L. On Hardy spaces associated with Bessel operators. J. Anal. Math. 107 (2009), 195-219.
[3] Bownik, M. Boundedness of operators on Hardy spaces via atomic decompositions. Proc. Amer. Math. Soc. 133, 12 (2005), 3535-3542.
[4] Carbonaro, A., Mauceri, G., and Meda, S. $H^{1}$ and BMO for certain locally doubling metric measure spaces of finite measure. Colloq. Math. 118, 1 (2010), 13-41.
[5] Cha, L., and Liu, H. BMO-boundedness of maximal operators and $g$-functions associated with Laguerre expansions. J. Funct. Spaces Appl. (2012), Art. ID 923874, 21.
[6] Cha, L., and Liu, H. BMO spaces for Laguerre expansions. Taiwanese J. Math. 16, 6 (2012), 2153-2186.
[7] Coifman, R. R., and Weiss, G. "Analyse harmonique non-commutative sur certains espaces homogènes". Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. Etude de certaines intégrales singulières.
[8] Dziubański, J. Hardy spaces for Laguerre expansions. Constr. Approx. 27, 3 (2008), 269-287.
[9] Gosselin, J., and Stempak, K. A weak-type estimate for Fourier-Bessel multipliers. Proc. Amer. Math. Soc. 106, 3 (1989), 655-662.
[10] Gutiérrez, C. E., Incognito, A., and Torrea, J. L. Riesz transforms, $g$-functions, and multipliers for the Laguerre semigroup. Houston J. Math. 27, 3 (2001), 579-592.
[11] Haimo, D. T. Integral equations associated with Hankel convolutions. Trans. Amer. Math. Soc. 116 (1965), 330-375.
[12] Harboure, E., Torrea, J. L., and Viviani, B. On the search for weighted inequalities for operators related to the Ornstein-Uhlenbeck semigroup. Math. Ann. 318, 2 (2000), 341-353.
[13] Hirschman, Jr., I. I. Variation diminishing Hankel transforms. J. Analyse Math. 8 (1960/61), 307-336.
[14] Lebedev, N. N. "Special functions and their applications". Dover Publications, Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.
[15] Maas, J., van Neerven, J., and Portal, P. Conical square functions and non-tangential maximal functions with respect to the Gaussian measure. Publ. Mat. 55, 2 (2011), 313-341.
[16] Macías, R. A., and Segovia, C. A decomposition into atoms of distributions on spaces of homogeneous type. Adv. in Math. 33, 3 (1979), 271-309.
[17] Mauceri, G., and Meda, S. BMO and $H^{1}$ for the Ornstein-Uhlenbeck operator. J. Funct. Anal. 252, 1 (2007), 278-313.
[18] Mauceri, G., Meda, S., and Sjögren, P. A maximal function characterization of the Hardy space for the Gauss measure. Proc. Amer. Math. Soc. 141, 5 (2013), 1679-1692.
[19] Muckenhoupt, B. Conjugate functions for Laguerre expansions. Trans. Amer. Math. Soc. 147 (1970), 403-418.
[20] Portal, P. Maximal and quadratic Gaussian Hardy spaces. Rev. Mat. Iberoam. 30, 1 (2014), 79-108.
[21] SASSO, E. Spectral multipliers of Laplace transform type for the Laguerre operator. Bull. Austral. Math. Soc. 69, 2 (2004), 255-266.
[22] Stein, E. M. Interpolation of linear operators. Trans. Amer. Math. Soc. 83 (1956), 482-492.
[23] Stein, E. M. "Topics in harmonic analysis related to the Littlewood-Paley theory". Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.
[24] Tolsa, X. BMO, $H^{1}$, and Calderón-Zygmund operators for non doubling measures. Math. Ann. 319, 1 (2001), 89-149.
[25] Urbina-Romero, W. "Gaussian harmonic analysis". Springer Monographs in Mathematics. Springer, Cham, 2019. With a foreword by Sundaram Thangavelu.
[26] Yang, D., And Yang, D. Real-variable characterizations of Hardy spaces associated with Bessel operators. Anal. Appl. (Singap.) 9, 3 (2011), 345-368.

Jorge J. Betancor
Departamento de Análisis Matemático, Universidad de La Laguna,
Campus de Anchieta, Avda. Astrofísico Sánchez, s/n,
38721 La Laguna (Sta. Cruz de Tenerife), Spain
Email address: jbetanco@ull.es
Estefanía Dalmasso, Pablo Quijano
Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ.
Colectora Ruta Nac. No 168, Paraje El Pozo,
S3007ABA, Santa Fe, Argentina
Email address: edalmasso@santafe-conicet.gov.ar, pabloquijanoar@gmail.com

Roberto Scotto
Universidad Nacional del Litoral, FIQ.
Santiago del Estero 2829,
S3000AOM, Santa Fe, Argentina
Email address: roberto.scotto@gmail.com

