HARMONIC ANALYSIS OPERATORS ASSOCIATED WITH LAGUERRE POLYNOMIAL EXPANSIONS ON VARIABLE LEBESGUE SPACES

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Dedicated to the memory of our beloved Eleonor Harboure.

ABSTRACT. In this paper we give sufficient conditions on a measurable function $p:(0,\infty)^n \to [1,\infty)$ in order that harmonic analysis operators (maximal operators, Riesz transforms, Littlewood–Paley functions and multipliers) associated with α-Laguerre polynomial expansions are bounded on the variable Lebesgue space $L^{p(\cdot)}((0,\infty)^n,\mu_\alpha)$, where $d\mu_\alpha(x) = 2^n \prod_{j=1}^n \frac{x_j^{2\alpha_j+1}e^{-x_j^2}}{\Gamma(\alpha_j+1)}dx$, being $\alpha=(\alpha_1,\dots,\alpha_n)\in[0,\infty)^n$ and $x=(x_1,\dots,x_n)\in(0,\infty)^n$.

1. Introduction and main results

In this article we establish $L^{p(\cdot)}-$boundedness properties of harmonic analysis operators appearing in the context of Laguerre polynomials.

For every $\alpha > -1$ and $k \in \mathbb{N} := \{0,1,2,\dots\}$, the normalized Laguerre polynomial of type α and degree k is defined by the formula (c.f. [24], [43])

$$L_k^\alpha(x) = \sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)k!}}e^x x^{-\alpha}\frac{d^k}{dx^k}\left(e^{-x}x^{k+\alpha}\right), \quad x\in(0,\infty).$$

Let $\alpha = (\alpha_1,\dots,\alpha_n) \in (-1,\infty)^n$. For every $k=(k_1,\dots,k_n)\in\mathbb{N}^n$, the k–th Laguerre polynomial of type α and degree $\widehat{k} := k_1+\cdots+k_n$ is defined by

$$L_k^\alpha(x) = \prod_{i=1}^n L_{k_i}^{\alpha_i}(x_i), \quad x=(x_1,\dots,x_n)\in(0,\infty)^n := \mathbb{R}_+^n.$$

The sequence of polynomials $\{L_k^\alpha\}_{k\in\mathbb{N}^n}$ is an orthonormal basis for $L^2(\mathbb{R}_+^n,\nu_\alpha)$ being $d\nu_\alpha(x) = \prod_{j=1}^n \frac{x_j^{\alpha_j}e^{-x_j}}{\Gamma(\alpha_j+1)}dx$ a non-doubling measure defined on $\mathbb{R}_+^n$, see [24, §4.21] for the orthonormality of the family.

We define, for each $k\in\mathbb{N}^n$, $\mathcal{L}_k^\alpha(x) = \prod_{i=1}^n L_{k_i}^{\alpha_i}(x_i^2)$, $x=(x_1,\dots,x_n)\in\mathbb{R}_+^n$. The sequence $\{\mathcal{L}_k^\alpha\}_{k\in\mathbb{N}^n}$ is an orthonormal basis for $L^2(\mathbb{R}_+^n,\mu_\alpha)$ where

$$d\mu_\alpha(x) = 2^n\prod_{j=1}^n \frac{x_j^{2\alpha_j+1}e^{-x_j^2}}{\Gamma(\alpha_j+1)}dx$$

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is the pull-back measure from $d\nu_\alpha$ on $\mathbb{R}^n_+$ through the one-to-one and onto change of variables $\Psi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined as $\Psi(x) = x^2 := (x_1^2, \cdots, x_n^2)$, for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$.

We consider the differential Laguerre operator defined on $\mathbb{R}^n_+$ as follows

$$\mathbb{\Delta}_\alpha = -\frac{1}{4}\sum_{j=1}^n \left(\frac{d^2}{dx_j^2} + \left(\frac{2\alpha_j+1}{x_j} - 2x_j \right) \frac{d}{dx_j} \right).$$

It turns out that the polynomials $\mathcal{L}_k^\alpha$ are eigenfunctions of the operator $\mathbb{\Delta}_\alpha$, with $\mathbb{\Delta}_\alpha \mathcal{L}_k^\alpha = \widehat{k}\mathcal{L}_k^\alpha$ for every $k \in \mathbb{N}^n$.

For every $f \in L^2(\mathbb{R}^n_+, \mu_\alpha)$ and $k \in \mathbb{N}^n$, we denote

$$c_k^\alpha(f) = \int_{\mathbb{R}^n_+} \mathcal{L}_k^\alpha(x) f(x) d\mu_\alpha(x).$$

We define the operator Δ_α by

$$\Delta_\alpha f = \sum_{k \in \mathbb{N}^n} \lambda_k c_k^\alpha(f) \mathcal{L}_k^\alpha, \quad f \in D(\Delta_\alpha),$$

where $\lambda_k = \widehat{k}$ for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, and

$$D(\Delta_\alpha) = \left\{ f \in L^2(\mathbb{R}^n_+, \mu_\alpha) : \sum_{k \in \mathbb{N}^n} |\lambda_k c_k^\alpha(f)|^2 < \infty \right\},$$

is the domain of Δ_α on $L^2(\mathbb{R}^n_+, \mu_\alpha)$. Note that $\Delta_\alpha f = \mathbb{\Delta}_\alpha f$ for every $f \in C_c^\infty(\mathbb{R}^n_+)$, the space of smooth and compactly supported functions on $\mathbb{R}^n_+$.

The operator Δ_α is symmetric and positive, and $-\Delta_\alpha$ generates a semigroup of operators $\{W_t^\alpha\}_{t>0}$ in $L^2(\mathbb{R}^n_+, \mu_\alpha)$ where, for every $t > 0$,

$$W_t^\alpha(f) = \sum_{k \in \mathbb{N}^n} e^{-\lambda_k t} c_k^\alpha(f) \mathcal{L}_k^\alpha, \quad f \in L^2(\mathbb{R}^n_+, \mu_\alpha).$$

According to the Hille-Hardy formula ([24, (4.17.6)] with x and y replaced by x^2 and y^2 respectively, and t by e^{-t}), we have that

$$\sum_{k \in \mathbb{N}^n} e^{-\lambda_k t} \mathcal{L}_k^\alpha(x) \mathcal{L}_k^\alpha(y) = \prod_{j=1}^n \frac{\Gamma(\alpha_j+1)}{1-e^{-t}} \left(e^{-t/2} x_j y_j \right)^{-\alpha_j} I_{\alpha_j}\left(\frac{2e^{-t/2} x_j y_j}{1-e^{-t}} \right) \times \exp\left(-\frac{e^{-t}}{1-e^{-t}} (x_j^2 + y_j^2) \right),$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $t > 0$. Here I_ν is the modified Bessel function of the first kind and order $\nu > -1$. We can write, for every $f \in L^2(\mathbb{R}^n_+, \mu_\alpha)$ and $t > 0$,

$$W_t^\alpha(f)(x) = \int_{\mathbb{R}^n_+} W_t^\alpha(x, y) f(y) d\mu_\alpha(y), \quad x \in \mathbb{R}^n_+, \tag{1.1}$$

being

$$W_t^\alpha(x, y) = \prod_{j=1}^n \frac{\Gamma(\alpha_j+1)}{1-e^{-t}} \left(e^{-t/2} x_j y_j \right)^{-\alpha_j} I_{\alpha_j}\left(\frac{2e^{-t/2} x_j y_j}{1-e^{-t}} \right) e^{-\frac{e^{-t}}{1-e^{-t}}(x_j^2+y_j^2)},$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $t > 0$. The integral in (1.1) defines, for every $t > 0$, a contraction on $L^p(\mathbb{R}^n_+, \mu_\alpha)$, for every $1 \leq p \leq \infty$. By defining, for each $t > 0$, W_t^α by (1.1), the family $\{W_t^\alpha\}_{t>0}$ is a symmetric diffusion semigroup in Stein's sense in $(\mathbb{R}^n_+, \mu_\alpha)$ (see [42, p. 65]).

The Poisson semigroup $\{P_t^\alpha\}_{t>0}$ associated with the operators $-\sqrt{\Delta_\alpha}$ is defined by

$$P_t^\alpha(f) = \sum_{k\in\mathbb{N}^n} e^{-t\sqrt{\lambda_k}} c_k^\alpha(f)\mathcal{L}_k^\alpha, \quad f \in L^2(\mathbb{R}_+^n, \mu_\alpha), t > 0.$$

By using the subordination formula, we have that

$$P_t^\alpha(f) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W_u^\alpha(f)du, \quad f \in L^2(\mathbb{R}_+^n, \mu_\alpha), t > 0. \tag{1.2}$$

We can write, for every $t > 0$ and $f \in L^2(\mathbb{R}_+^n, \mu_\alpha)$,

$$P_t^\alpha(f)(x) = \int_{\mathbb{R}_+^n} P_t^\alpha(x,y)f(y)d\mu_\alpha(y), \quad x \in \mathbb{R}_+^n, \tag{1.3}$$

where

$$P_t^\alpha(x,y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W_u^\alpha(x,y)du, \quad x, y \in \mathbb{R}_+^n, t > 0.$$

For each $t > 0$, the integral in (1.3) defines a contraction on $L^p(\mathbb{R}_+^n, \mu_\alpha)$ when $1 \le p \le \infty$. By defining P_t^α as in (1.2), $\{P_t^\alpha\}_{t>0}$ is a Stein symmetric diffusion semigroup in $(\mathbb{R}_+^n, \mu_\alpha)$.

The study of harmonic analysis in the Laguerre setting was initiated by Muckenhoupt ([29, 30]). Muckenhoupt's context is transferred to ours by applying the transform mapping Ψ mentioned above (see, for instance, [41]).

The maximal operators W_*^α and P_*^α are defined by

$$W_*^\alpha(f) = \sup_{t>0} |W_t^\alpha(f)|, \quad P_*^\alpha(f) = \sup_{t>0} |P_t^\alpha(f)|.$$

From [42, p. 73], it follows that both W_*^α and P_*^α are bounded on $L^p(\mathbb{R}_+^n, \mu_\alpha)$ for every $1 < p \le \infty$. Muckenhoupt ([29]) proved that W_*^α is bounded from $L^1(\mathbb{R}_+, \mu_\alpha)$ into $L^{1,\infty}(\mathbb{R}_+, \mu_\alpha)$. He considered the one-dimensional case. This result was extended to higher dimensions by Dinger ([20]). Note that the subordination formula (1.2) allows us to deduce the L^p-boundedness properties for P_*^α from the corresponding ones of W_*^α. The holomorphic Laguerre semigroups and the maximal operators associated with them where studied in [40].

Taking into account the spectral decomposition of Δ_α and [33, §7.2] we define the first order Riesz-Laguerre transform associated to Δ_α as

$$R_\alpha^i f = \sum_{k\in\mathbb{N}^n\setminus\{(0,\dots,0)\}} \frac{1}{\sqrt{\lambda_k}} \partial_{x_i}\mathcal{L}_k^\alpha(x)c_k^\alpha(f), \qquad f \in L^2(\mathbb{R}_+^n, \mu_\alpha).$$

Thus the operator R_α^i turns out to be bounded on $L^2(\mathbb{R}_+^n, \mu_\alpha)$.

Moreover, we can also define the higher order Riesz-Laguerre transforms as an extension of the first order ones in the following way

$$R_\alpha^\beta f = \sum_{k\in\mathbb{N}^n\setminus\{(0,\dots,0)\}} \frac{1}{\lambda_k^{\hat{\beta}/2}} D_x^\beta \mathcal{L}_k^\alpha(x)c_k^\alpha(f),$$

with $\beta \in \mathbb{N}^n \setminus \{(0,\dots,0)\}$ and $D_x^\beta = \frac{\partial^{\hat{\beta}}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}$. They are also bounded on $L^2(\mathbb{R}_+^n, \mu_\alpha)$, see [33]. Let us remark that $R_\alpha^i f = R_\alpha^{e_i} f$ with e_i the i-th unit vector and $f \in L^2(\mathbb{R}_+^n, \mu_\alpha)$.

For every $b > 0$ we define the fractional integral Δ_α^{-b} as the $-b$ power of Δ_α, given, for every $f \in L^2(\mathbb{R}_+^n, \mu_\alpha)$, by

$$\Delta_\alpha^{-b} f = \sum_{k\in\mathbb{N}^n\setminus\{0\}} \lambda_k^{-b} c_k^\alpha(f)\mathcal{L}_k^\alpha.$$

Let us notice that for any number $b > 0$ in [41] it was proved that the integral kernel for Δ_α^{-b} is given by

$$K_b(x,y) = \frac{1}{\Gamma(b)} \int_0^\infty t^{b-1} \left(W_t^\alpha(x,y) - 1\right) dt$$
$$= \frac{1}{\Gamma(b)} \int_0^1 (-\log r)^{b-1} \left(W_{-\log r}^\alpha(x,y) - 1\right) \frac{dr}{r}.$$

If $f \in C_c^\infty(\mathbb{R}_+^n)$ we have that, for every $\beta \in \mathbb{N}^n \setminus \{(0,\dots,0)\}$,

$$R_\alpha^\beta f = D^\alpha \Delta_\alpha^{-\widehat{\beta}/2} f.$$

According to what was done in [41] we can also conclude that the operator R_α^β, off the diagonal, is given by the smooth kernel $D_x^\beta K_{\frac{\widehat{\beta}}{2}}(x,y)$, i.e.

$$R_\alpha^\beta f(x) = \int_{\mathbb{R}_+^n} D_x^\beta K_{\frac{\widehat{\beta}}{2}}(x,y) f(y)\, d\mu_\alpha(y),$$

for all $x \notin \operatorname{supp}(f)$ when $f \in C_c^\infty(\mathbb{R}_+^n)$.

From [8, Theorem 1.1], [41, Theorem 1.1] and [32, Theorem 13], we deduce that R_α^β can be extended from $L^2(\mathbb{R}_+^n, \mu_\alpha) \cap L^p(\mathbb{R}_+^n, \mu_\alpha)$ to $L^p(\mathbb{R}_+^n, \mu_\alpha)$ as a bounded operator on $L^p(\mathbb{R}_+^n, \mu_\alpha)$ when $1 < p < \infty$. It can also be extended from $L^1(\mathbb{R}_+^n, \mu_\alpha)$ into $L^{1,\infty}(\mathbb{R}_+^n, \mu_\alpha)$ for $\widehat{\beta} \leq 2$ and from $L^1(\mathbb{R}_+^n, w\mu_\alpha)$ into $L^{1,\infty}(\mathbb{R}_+^n, \mu_\alpha)$, with $w(y) = (1+\sqrt{|y|})^{\widehat{\beta}-2}$, for $\widehat{\beta} > 2$ (see [21]). We continue denoting by R_α^β to those extensions. Furthermore, there exists a constant c_β such that, for every $f \in L^p(\mathbb{R}_+^n, \mu_\alpha)$, $1 \leq p < \infty$,

$$R_\alpha^\beta(f)(x) = c_\beta f(x) + \lim_{\epsilon \to 0^+} \int_{y \in \mathbb{R}_+^n,\, |x-y|>\epsilon} R_\alpha^\beta(x,y) f(y) d\mu_\alpha(y), \qquad \text{a.e. } x \in \mathbb{R}_+^n,$$

where

$$(1.4) \qquad R_\alpha^\beta(x,y) = \frac{1}{\Gamma\left(\frac{\widehat{\beta}}{2}\right)} \int_0^\infty t^{\frac{\widehat{\beta}}{2}-1} D_x^\beta W_t^\alpha(x,y) dt,$$
$$= \frac{1}{\Gamma\left(\frac{\widehat{\beta}}{2}\right)} \int_0^1 (-\log r)^{\frac{\widehat{\beta}}{2}-1} D_x^\beta W_{-\log r}^\alpha(x,y)\, \frac{dr}{r},$$

for $x, y \in \mathbb{R}_+^n$, $x \neq y$.

We consider the Littlewood–Paley functions $g_\alpha^{\beta,k}$ defined for Poisson semigroups $\{P_t^\alpha\}_{t>0}$ for $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$ such that $k + \widehat{\beta} > 0$, as follows

$$g_\alpha^{\beta,k}(f)(x) = \left(\int_0^\infty \left| t^{k+\widehat{\beta}} \partial_t^k D_x^\beta P_t^\alpha(f)(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \; x \in \mathbb{R}_+^n.$$

For simplicity, when $\beta = \mathbf{0} = (0,\dots,0)$, we shall write $g_\alpha^k = g_\alpha^{\mathbf{0},k}$. According to [42, Corollary 1], g_α^k is bounded on $L^p(\mathbb{R}_+^n, \mu_\alpha)$ for every $k \in \mathbb{N}$, $k \geq 1$ and $1 < p < \infty$. In [9, Theorem 1.2] it was recently proved that g_α^k is bounded from $L^1(\mathbb{R}_+, \mu_\alpha)$ into $L^{1,\infty}(\mathbb{R}_+, \mu_\alpha)$. Nowak in [32, Theorems 6 and 7] proved L^p-boundedness properties for $1 < p < \infty$ for Littlewood-Paley functions associated with Laguerre polynomial expansions in the ν_α-context including one spatial derivative.

We say that a function m is of Laplace transform type when

$$m(x) = x \int_0^\infty \phi(y) e^{-xy} dy, \quad x \in \mathbb{R}_+,$$

being $\phi \in L^\infty(\mathbb{R}_+)$. Given m of Laplace transform type, we define the spectral multiplier for Δ_α, T_m^α, associated with m by

$$T_m^\alpha(f) = \sum_{k\in\mathbb{N}^n} m(\lambda_k) c_k^\alpha(f) \mathcal{L}_\alpha^k, \quad f \in L^2(\mathbb{R}_+^n, \mu_\alpha).$$

Since m is bounded, T_m^α is bounded on $L^2(\mathbb{R}_+^n, \mu_\alpha)$. According to [42, Corollary 3, p. 121], T_m^α can be extended from $L^2(\mathbb{R}_+^n, \mu_\alpha) \cap L^p(\mathbb{R}_+^n, \mu_\alpha)$ to $L^p(\mathbb{R}_+^n, \mu_\alpha)$ as a bounded operator on $L^p(\mathbb{R}_+^n, \mu_\alpha)$ when $1 < p < \infty$. In [38] it was established that T_m^α can be extended from $L^2(\mathbb{R}_+^n, \mu_\alpha) \cap L^1(\mathbb{R}_+^n, \mu_\alpha)$ to $L^1(\mathbb{R}_+^n, \mu_\alpha)$ as a bounded operator from $L^1(\mathbb{R}_+^n, \mu_\alpha)$ into $L^{1,\infty}(\mathbb{R}_+^n, \mu_\alpha)$. From a higher dimension version of [6, Theorem 1.1] we deduce that, for every $f \in L^p(\mathbb{R}_+^n, \mu_\alpha)$ with $1 \le p < \infty$,

$$T_m^\alpha(f)(x) = \lim_{\epsilon\to 0^+} \left(\Lambda(\epsilon) f(x) + \int_{\substack{|x-y|>\epsilon,\\ y\in\mathbb{R}_+^n}} K_\phi^\alpha(x,y) f(y) d\mu_\alpha(y) \right), \quad \text{a.e. } x \in \mathbb{R}_+^n,$$

where $\Lambda \in L^\infty(\mathbb{R}_+)$ and

$$K_\phi^\alpha(x,y) = \int_0^\infty \phi(t) \left(-\frac{\partial}{\partial t}\right) W_t^\alpha(x,y) dt, \quad x, y \in \mathbb{R}_+^n, x \ne y.$$

A special case of multiplier of Laplace transform type is the imaginary power $\Delta_\alpha^{i\beta}$ of Δ_α that appears when $m(x) = x^{i\beta}$, for $x \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$.

Our objective is to give conditions on a function $p : \mathbb{R}_+^n \to [1,\infty)$ in order that the operators we have just defined (maximal operators, Riesz transforms, Littlewood–Paley functions and multipliers of Laplace transform type) are bounded on $L^{p(\cdot)}(\mathbb{R}_+^n, \mu_\alpha)$.

Exhaustive studies about Lebesgue spaces with variable exponent (also called generalized Lebesgue spaces or variable Lebesgue spaces) can be found in the monographs [14] and [18].

Assume that $p : \mathbb{R}_+^n \to [1,\infty)$ is measurable. We say that a measurable function f on $\mathbb{R}_+^n$ belongs to $L^{p(\cdot)}(\mathbb{R}_+^n, \mu_\alpha)$ if the modular $\varrho_{p(\cdot),\mu_\alpha}(f/\lambda)$ is finite for some $\lambda > 0$, where

$$\varrho_{p(\cdot),\mu_\alpha}(g) = \int_{\mathbb{R}_+^n} |g(x)|^{p(x)} d\mu_\alpha(x).$$

We define on $L^{p(\cdot)}(\mathbb{R}_+^n, \mu_\alpha)$ the Luxemburg norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)}$ associated with $\varrho_{p(\cdot),\mu_\alpha}$, that is,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot),\mu_\alpha}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

The space $\left(L^{p(\cdot)}(\mathbb{R}_+^n, \mu_\alpha), \|\cdot\|_{L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)}\right)$ is a Banach function space. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}_+^n) := L^{p(\cdot)}(\mathbb{R}_+^n, dx)$ and its norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}_+^n)} := \|\cdot\|_{L^{p(\cdot)}(\mathbb{R}_+^n,dx)}$ are defined in the obvious way.

Lebesgue spaces with variable exponents appear associated to physics problems, image processing and modeling of electrorheological fluids (see, for instance, [1], [10] and [37]).

As it is well-known, the Hardy–Littlewood maximal function $M_{\rm HL}$ plays a central role in the study of L^p-boundedness properties of harmonic analysis operators. The following conditions on the exponent $p(\cdot)$ arise related with the boundedness of $M_{\rm HL}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ ([13] and [17]):

(a) Local log-Hölder condition: a measurable function $p : \Omega \subset \mathbb{R}^n \to [1, \infty)$ is said to be in $\mathrm{LH}_0(\Omega)$ if there exists $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, 0 < |x-y| < \tfrac{1}{2}.$$

(b) Decay log-Hölder condition: a measurable function $p : \Omega \subset \mathbb{R}^n \to [1, \infty)$ is said to be in $\mathrm{LH}_\infty(\Omega)$ when there exists $C > 0$ and $p_\infty \geq 1$ such that

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \Omega.$$

We define $\mathrm{LH}(\Omega) = \mathrm{LH}_0(\Omega) \cap \mathrm{LH}_\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$.

If $p : \Omega \subset \mathbb{R}^n \to [1, \infty)$ is measurable, we denote by $p^- = \operatorname{ess\,inf}_\Omega p$ and $p^+ = \operatorname{ess\,sup}_\Omega p$ the essential infimum and supremum of p on Ω, respectively.

If $1 < p^- \leq p^+ < \infty$ and $p \in \mathrm{LH}(\mathbb{R}^n)$, then the Hardy–Littlewood maximal function is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ ([13]). However, $p \in \mathrm{LH}(\mathbb{R}^n)$ is not necessary for this boundedness ([14, Examples 4.1 and 4.43]). The same conditions on p, $1 < p^- \leq p^+ < \infty$ and $p \in \mathrm{LH}(\mathbb{R}^n)$, assure that the Calderón–Zygmund singular integrals are bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ ([14, Theorem 5.39]).

In [15], Dalmasso and Scotto studied Riesz transforms in the Gaussian setting on variable Lebesgue spaces. In order to do this, they introduced a new class of exponents which is contained in $\mathrm{LH}_\infty(\mathbb{R}^n)$. A measurable function $p : \Omega \subset \mathbb{R}^n \to [1, \infty)$ is said to be in $\mathcal{P}_e^\infty(\Omega)$ when there exists $C > 0$ and $p_\infty \geq 1$ such that

$$|p(x) - p_\infty| \leq \frac{C}{|x|^2}, \quad x \in \Omega \setminus \{(0, \dots, 0)\}.$$

If $p_\infty \geq 1$, $A > 0$ and $q \geq 2$ are given, the functions $p(x) = p_\infty + \frac{A}{(e+|x|)^q}$, for $x \in \mathbb{R}^n$, are in $\mathcal{P}_e^\infty(\mathbb{R}^n)$. Main properties of the functions in $\mathcal{P}_e^\infty(\mathbb{R}^n)$ were established in [15]. Maximal operators defined by the heat semigroup ([28]) and Riesz type singular integrals ([16] and [31]) associated with the Ornstein-Uhlenbeck differential operator were studied on $L^{p(\cdot)}(\mathbb{R}^n, \gamma_n)$ with $p \in \mathrm{LH}_0(\mathbb{R}^n) \cap \mathcal{P}_e^\infty(\mathbb{R}^n)$, where $d\gamma_n$ denotes the Gaussian measure.

We now state the main results of this article concerning $L^{p(\cdot)}$-boundedness properties of harmonic analysis operators in the Laguerre setting.

Theorem 1.1. *Let $\alpha \in [0, \infty)^n$. Assume that $p \in \mathrm{LH}_0(\mathbb{R}^n_+) \cap \mathcal{P}_e^\infty(\mathbb{R}^n_+)$ with $1 < p^- \leq p^+ < \infty$. We denote by T_α one of the following operators:*

(a) The maximal operators W_*^α and P_*^α;
(b) The Laguerre-Riesz transformation R_α^β, $\beta \in \mathbb{N}^n \setminus \{(0, \dots, 0)\}$;
(c) The Littlewood–Paley functions $g_\alpha^{\beta,k}$ associated with the Poisson semigroup $\{P_t^\alpha\}_{t>0}$, where $\beta \in \mathbb{N}^n$ and $k \in \mathbb{N}$, such that $k + \widehat{\beta} > 0$;
(d) The Laguerre spectral multipliers T_m^α, where m is a Laplace transform type function.

Then, T_α is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$.

Hereinafter, we prove Theorem 1.1. In Section 2, we explain the method we develop in order to prove that the operators given in (a)–(d) are bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$. In Section 3, we introduce a global operator that will be a key ingredient for proving our main theorem. In the following sections, we establish the $L^{p(\cdot)}$-boundedness for each class of operators. Our method exploits the decomposition of the operators into a local part and a global part, which is usual in the study of harmonic analysis in the Laguerre setting, but we need a careful adaptation to the variable exponent context.

Throughout this paper, C and c will always denote positive constants that may change in each occurrence.

2. The method for proving our results

In this section we describe the method we apply to prove the boundedness results.

The polynomial measure $\mathfrak{m}_\alpha$ on $\mathbb{R}^n_+$ defined by $d\mathfrak{m}_\alpha(x) = \prod_{i=1}^n x_i^{2\alpha_i+1}dx_i$ is doubling on $\mathbb{R}^n_+$. Thus, the triple $(\mathbb{R}^n_+, |\cdot|, \mathfrak{m}_\alpha)$ is a homogeneous space in the sense of Coifman and Weiss ([11]).

Let X be a Banach space. Suppose that $K : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus D \to X$ is a strongly measurable function, where $D = \{(x, x) : x \in \mathbb{R}^n_+\}$, satisfying the following two conditions:

(i) *Size condition:* there exists $C > 0$ such that

$$\|K(x, y)\|_X \leq \frac{C}{\mathfrak{m}_\alpha(B(x, |x-y|))}, \quad x, y \in \mathbb{R}^n_+, x \neq y;$$

(ii) *Regularity condition:* there exists $C > 0$ such that

$$\|K(x, y) - K(z, y)\|_X \leq \frac{C|x-z|}{|x-y|\ \mathfrak{m}_\alpha(B(x, |x-y|))}$$

and

$$\|K(x, y) - K(x, z)\|_X \leq \frac{C|y-z|}{|x-y|\ \mathfrak{m}_\alpha(B(x, |x-y|))}$$

for every $x, y, z \in \mathbb{R}^n_+$ with $|x-z| \leq \frac{1}{2}|x-y|$.

When the function K verifies (i) and (ii), we say that K is an X-valued Calderón–Zygmund kernel with respect to the homogeneous space $(\mathbb{R}^n_+, |\cdot|, \mathfrak{m}_\alpha)$ in the Banach space X.

For every exponent $q : \mathbb{R}^n_+ \to [1, \infty)$, we denote by $L^{q(\cdot)}_X(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ the X-Bochner Lebesgue space with variable exponent q, defined in the natural way.

Assume T is a bounded operator from $L^2(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^2_X(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$. We say that T is an X-valued Calderón–Zygmund operator associated with the Calderón–Zygmund kernel K when, for every $f \in C^\infty_c(\mathbb{R}^n_+)$,

$$Tf(x) = \int_{\mathbb{R}^n_+} K(x, y)f(y)d\mathfrak{m}_\alpha(y), \qquad \text{a.e. } x \notin \operatorname{supp}(f).$$

Here, the integral is understood in the X-Bochner sense.

According to [23, Theorem 1.1] (see also [36]), if T is an X-valued Calderón–Zygmund operator on $(\mathbb{R}^n_+, |\cdot|, \mathfrak{m}_\alpha)$, T can be extended, for every $1 \leq p < \infty$, from $L^2(\mathbb{R}^n_+, \mathfrak{m}_\alpha) \cap L^p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ to $L^p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ as a bounded operator from $L^p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^p_X(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ when $1 < p < \infty$, and from $L^1(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^{1,\infty}_X(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ when $p = 1$.

Any non-negative measurable function w on $\mathbb{R}^n_+$ is named a weight. For every $1 < p < \infty$, we say that a weight w on $\mathbb{R}^n_+$ is in the Muckenhoupt class $A_p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ when

$$\sup_B \left(\frac{1}{\mathfrak{m}_\alpha(B)} \int_B w(x)d\mathfrak{m}_\alpha(x)\right) \left(\frac{1}{\mathfrak{m}_\alpha(B)} \int_B w(x)^{-\frac{1}{p-1}}d\mathfrak{m}_\alpha(x)\right)^{p-1} < \infty,$$

where the supremum is taken over all the balls B in $\mathbb{R}^n_+$.

A weight w on $\mathbb{R}^n_+$ is said to be in the Muckenhoupt class $A_1(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ when there exists $C > 0$ such that, for every ball $B \subset \mathbb{R}^n_+$,

$$\frac{1}{\mathfrak{m}_\alpha(B)} \int_B w(x)d\mathfrak{m}_\alpha(x) \leq C \operatorname*{ess\,inf}_{y \in B} w(y).$$

We also define $A_\infty(\mathbb{R}^n_+, \mathfrak{m}_\alpha) = \bigcup_{p\geq 1} A_p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$.

If T is an X-valued Calderón–Zygmund operator on $(\mathbb{R}^n_+, |\cdot|, \mathfrak{m}_\alpha)$, for every $w \in A_p(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ and $1 < p < \infty$, the operator T can be extended from $L^2(\mathbb{R}^n_+, \mathfrak{m}_\alpha) \cap$

$L^p(\mathbb{R}^n_+, w, \mathfrak{m}_\alpha)$ to $L^p(\mathbb{R}^n_+, w, \mathfrak{m}_\alpha)$ as a bounded operator from $L^p(\mathbb{R}^n_+, w, \mathfrak{m}_\alpha)$ into $L^p_X(\mathbb{R}^n_+, w, \mathfrak{m}_\alpha)$ (see, for instance, [25, Theorem 1.1]).

Rubio de Francia's extrapolation theorem works for spaces of homogeneous type ([3, Theorem 3.5]). The arguments in the proof of [12, Theorem 1.3] allow us to deduce that if T is an X-valued Calderón–Zygmund operator on $(\mathbb{R}^n_+, |\cdot|, \mathfrak{m}_\alpha)$, T defines a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^{p(\cdot)}_X(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$, provided that $1 < p^- \leq p^+ < \infty$ and the $\mathfrak{m}_\alpha$-Hardy–Littlewood maximal function is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ (see also [19, Theorem 4.8]). We recall that according to [2, Theorems 1.4 and 1.7], the Hardy–Littlewood maximal operator defined by the measure $\mathfrak{m}_\alpha$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ provided $1 < p^- \leq p^+ < \infty$ and $p \in \mathrm{LH}(\mathbb{R}^n_+)$ (see also [16, Theorem 5.2]). We also notice that T is well-defined for $f \in L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ thanks to the embedding $L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha) \hookrightarrow L^{p^-}(\mathbb{R}^n_+, \mathfrak{m}_\alpha) + L^{p^+}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ ([18, Theorem 3.3.11]).

The maximal operators and the Littlewood–Paley function can be studied by using Banach valued operators. Indeed, we can write

$$P^\alpha_*(f) = \|P^\alpha_t(f)\|_{L^\infty(\mathbb{R}_+)}, \qquad W^\alpha_*(f) = \|W^\alpha_t(f)\|_{L^\infty(\mathbb{R}_+)}$$

and

$$g^{\beta,k}_\alpha(f) = \left\| t^{k+\widehat{\beta}} \partial^k_t D^\beta_x P^\alpha_t(f) \right\|_{L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)}.$$

We define

$$q_\pm(x, y, s) = \sum_{i=1}^n (x_i^2 + y_i^2 \pm 2x_i y_i s_i),$$

with $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ and $s = (s_1, \dots, s_n) \in (-1,1)^n$. We split $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (-1,1)^n$ into two parts. Let $\tau > 0$ and let us fix $C_0 > 0$ whose exact value will be specified later. The local part L_τ is defined by

$$L_\tau = \left\{ (x, y, s) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (-1,1)^n : \sqrt{q_-(x, y, s)} \leq \frac{C_0 \tau}{1 + |x| + |y|} \right\}$$

and the global part G_τ is given by

$$G_\tau = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (-1,1)^n \setminus L_\tau.$$

By taking into account the integral representation for the modified Bessel function I_ν, $\nu > -\frac{1}{2}$ ([24, (5.10.22)]), for every $t > 0$, the integral kernel of W^α_t can be written as

$$W^\alpha_t(x, y) = \frac{1}{(1 - e^{-t})^{n + \widehat{\alpha}}} \int_{(-1,1)^n} \exp\left(-\frac{q_-\left(e^{-t/2}x, y, s\right)}{1 - e^{-t}} + |y|^2 \right) \Pi_\alpha(s) ds,$$

for $x, y \in \mathbb{R}^n_+$, where $\widehat{\alpha} = \sum_{i=1}^n \alpha_i$ and $\Pi_\alpha(s) = \prod_{i=1}^n \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + 1/2)\sqrt{\pi}} (1 - s_i^2)^{\alpha_i - 1/2}$ for $s = (s_1, \dots, s_n) \in (-1,1)^n$.

As in [41], we consider a smooth function φ on $\mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (-1,1)^n$ such that $0 \leq \varphi \leq 1$,

$$\varphi(x, y, s) = \begin{cases} 1, & (x, y, s) \in L_1, \\ 0, & (x, y, s) \notin L_2, \end{cases}$$

and

$$|\nabla_x \varphi(x, y, s)| + |\nabla_y \varphi(x, y, s)| \leq \frac{C}{q_-(x, y, s)^{1/2}}, \quad x, y \in \mathbb{R}^n_+, s \in (-1,1)^n.$$

We also define, for $x, y \in \mathbb{R}^n_+$ and $t > 0$,

$$W^\alpha_{t,\mathrm{loc}}(x, y) = \int_{(-1,1)^n} \frac{\exp\left(-\frac{q_-\left(e^{-t/2}x, y, s\right)}{1 - e^{-t}} + |y|^2 \right)}{(1 - e^{-t})^{n + \widehat{\alpha}}} \Pi_\alpha(s) \varphi(x, y, s) ds$$

and

$$W^\alpha_{t,\mathrm{glob}}(x,y) = W^\alpha_t(x,y) - W^\alpha_{t,\mathrm{loc}}(x,y).$$

Suppose that T_α is one of the operators considered in Theorem 1.1. This operator is defined by using the heat integral kernel $W^\alpha_t(x,y)$. We decompose the operator T_α as

$$|T_\alpha| \leq |T_{\alpha,\mathrm{loc}}| + |T_{\alpha,\mathrm{glob}}|,$$

where $T_{\alpha,\mathrm{loc}}$ is defined as T_α but replacing $W^\alpha_t(x,y)$ by $W^\alpha_{t,\mathrm{loc}}(x,y)$, and in $T_{\alpha,\mathrm{glob}}$ the kernel $W^\alpha_t(x,y)$ is replaced by $W^\alpha_{t,\mathrm{glob}}(x,y)$.

We shall prove that both $T_{\alpha,\mathrm{loc}}$ and $T_{\alpha,\mathrm{glob}}$ are bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$ provided that p satisfies the hypotheses imposed on Theorem 1.1.

In order to prove the $L^{p(\cdot)}$-boundedness of $T_{\alpha,\mathrm{glob}}$, we introduce, for every $\varepsilon \in [0,1)$, a positive measurable function $H_{\alpha,\varepsilon}$ defined on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ verifying that the operator $\mathcal{H}_{\alpha,\varepsilon}$ given by

$$\mathcal{H}_{\alpha,\varepsilon}(f)(x) = \int_{\mathbb{R}^n_+} H_{\alpha,\varepsilon}(x,y) f(y) d\mu_\alpha(y), \quad x \in \mathbb{R}^n_+$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$. Then, we prove that there exists $\varepsilon \in [0,1)$ for which

$$|T_{\alpha,\mathrm{glob}} f(x)| \leq \mathcal{H}_{\alpha,\varepsilon}(|f|)(x), \quad x \in \mathbb{R}^n_+.$$

Secondly, we prove that $T_{\alpha,\mathrm{loc}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$. We consider the following Banach spaces

$$X(W^\alpha_*) = X(P^\alpha_*) = L^\infty(\mathbb{R}_+),$$

for every $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$ such that $k + \widehat{\beta} > 0$,

$$X\left(g^{\beta,k}_\alpha\right) = L^2\left(\mathbb{R}_+, \frac{dt}{t}\right),$$

and for every $\beta \in \mathbb{N} \setminus \{0\}$ and every multiplier m of Laplace transform type,

$$X\left(R^\beta_\alpha\right) = X(T_m) = \mathbb{C}.$$

We can write

$$|T_{\alpha,\mathrm{loc}}(f)| = \|\mathbb{T}_\alpha(f)\|_{X(T_\alpha)}$$

where, for $x \in \mathbb{R}^n_+$,

$$\mathbb{T}_\alpha(f)(x) = \int_{\mathbb{R}^n_+} \int_{(-1,1)^n} \mathcal{M}_\alpha(x,y,s)\varphi(x,y,s)\Pi_\alpha(s)ds\, f(y) d\mathfrak{m}_\alpha(y).$$

Here, the function $\mathcal{M}_\alpha : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times (-1,1)^n \to X(T_\alpha)$ is strongly measurable and the integral is understood in the $X(T_\alpha)$-Bochner sense. We write

$$\mathbb{M}_\alpha(x,y) = \int_{(-1,1)^n} \mathcal{M}_\alpha(x,y,s)\varphi(x,y,s)\Pi_\alpha(s)ds, \quad x,y \in \mathbb{R}^n_+.$$

Thus, $\mathbb{M}_\alpha : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus D \to X(T_\alpha)$ is strongly measurable.

The operator $\mathbb{T}_\alpha$ is bounded from $L^2(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^2_{X(T_\alpha)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$. We prove that $\mathbb{T}_\alpha$ is an $X(T_\alpha)$-valued Calderón–Zygmund operator associated with $\mathbb{M}_\alpha$. Then, according to the above-mentioned arguments, $\mathbb{T}_\alpha$ defines a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$ into $L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+, \mathfrak{m}_\alpha)$. We are going to see that $\mathbb{T}_\alpha$ is also bounded from $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$ into $L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+, \mu_\alpha)$. Note that the measure μ_α is not doubling on $(\mathbb{R}^n_+, |\cdot|)$.

As stated in [39, Lemma 4], there exists a sequence $\{x(\ell)\}_{\ell \in \mathbb{N}} \subset \mathbb{R}^n_+$ such that, if we set

$$B_\ell = \left\{ x \in \mathbb{R}^n_+ : |x - x(\ell)| \leq \frac{1}{2(1 + |x(\ell)|)} \right\}, \quad \ell \in \mathbb{N},$$

the following properties hold

(i) $\mathbb{R}^n_+ = \bigcup_{\ell\in\mathbb{N}} B_\ell$;

(ii) for every $\delta > 1$, the family $\{\delta B_\ell\}_{\ell\in\mathbb{N}}$ has bounded overlap;

(iii) there exists $C > 1$ such that, for every $\ell \in \mathbb{N}$ and every measurable subset E of B_ℓ,

$$\frac{1}{C} e^{-|x(\ell)|^2} \mathfrak{m}_\alpha(E) \leq \mu_\alpha(E) \leq C e^{-|x(\ell)|^2} \mathfrak{m}_\alpha(E).$$

Furthermore, for every $\eta > 0$, there exists $\delta > 1$ such that, if $\ell \in \mathbb{N}$, $x \in B_\ell$ and $y \notin \delta B_\ell$, then $(x, y, s) \notin L_\eta$ for each $s \in (-1, 1)^n$ (see [39, Remark 5]).

We have that

$$\|\mathbb{T}_\alpha f\|_{L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+,\mu_\alpha)} = \left\| \|\mathbb{T}_\alpha f\|_{X(T_\alpha)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)},$$

so, according to [18, Corollary 3.2.14],

$$\|\mathbb{T}_\alpha f\|_{L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+,\mu_\alpha)} \leq 2 \sup_{\|F\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)} \leq 1} \int_{\mathbb{R}^n_+} \|\mathbb{T}_\alpha f(x)\|_{X(T_\alpha)} |F(x)| d\mu_\alpha(x).$$

Here, p' denotes the Hölder conjugate exponent of p, i.e., $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for every $x \in \mathbb{R}^n_+$.

Fix $F \in L^{p'(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$ with $\|F\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)} \leq 1$. By virtue of the properties (i), (ii) and (iii), for certain $\delta > 1$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \|\mathbb{T}_\alpha f(x)\|_{X(T_\alpha)} |F(x)| d\mu_\alpha(x) \\
&\quad \leq \sum_{\ell\in\mathbb{N}} \int_{B_\ell} \|\mathbb{T}_\alpha f(x)\|_{X(T_\alpha)} |F(x)| d\mu_\alpha(x) \\
&\quad = \sum_{\ell\in\mathbb{N}} \int_{B_\ell} \|\mathbb{T}_\alpha (f\chi_{\delta B_\ell})(x)\|_{X(T_\alpha)} |F(x)| d\mu_\alpha(x) \\
&\quad \leq C \sum_{\ell\in\mathbb{N}} e^{-|x(\ell)|^2} \int_{B_\ell} \|\mathbb{T}_\alpha (f\chi_{\delta B_\ell})(x)\|_{X(T_\alpha)} |F(x)| d\mathfrak{m}_\alpha(x) \\
&\quad \leq C \sum_{\ell\in\mathbb{N}} e^{-|x(\ell)|^2} \|\mathbb{T}_\alpha(f\chi_{\delta B_\ell})\|_{L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)} \|F\chi_{B_\ell}\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)} \\
&\quad \leq C \sum_{\ell\in\mathbb{N}} e^{-|x(\ell)|^2} \|f\chi_{\delta B_\ell}\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)} \|F\chi_{B_\ell}\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)}.
\end{aligned}$$

We have used Hölder's inequality with variable exponents (see, for instance, [18, Lemma 3.2.20]).

Since $p \in \mathcal{P}^\infty_e(\mathbb{R}^n_+)$ and $1 < p^- \leq p^+ < \infty$, we also have $p' \in \mathcal{P}^\infty_e(\mathbb{R}^n_+)$ with $1 < (p')^- \leq (p')^+ < \infty$. From [15, Lemma 2.5], by proceeding as in [15, (3.12)] and the following lines, we get

$$e^{-|x(\ell)|^2/p_\infty} \|f\chi_{\delta B_\ell}\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)} \leq \|f\chi_{\delta B_\ell}\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}$$

and

$$e^{-|x(\ell)|^2/p'_\infty} \|F\chi_{B_\ell}\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mathfrak{m}_\alpha)} \leq \|F\chi_{B_\ell}\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)},$$

where p'_∞ is the conjugate exponent of p_∞.

By means of [15, Corollary 2.8], we obtain

$$\int_{\mathbb{R}^n_+} \|\mathbb{T}_\alpha f(x)\|_{X(T_\alpha)} |F(x)| d\mu_\alpha(x) \leq C \sum_{\ell\in\mathbb{N}} \|f\chi_{\delta B_\ell}\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)} \|F\chi_{B_\ell}\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}$$

$$\leq C\sum_{\ell\in\mathbb{N}}\left\|f\chi_{\delta B_\ell}e^{-|\cdot|^2/p(\cdot)}\prod_{i=1}^n\frac{x_i^{(2\alpha_i+1)/p(\cdot)}}{\Gamma(\alpha_i+1/2)^{1/p(\cdot)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n_+)}$$
$$\times\left\|F\chi_{B_\ell}e^{-|\cdot|^2/p'(\cdot)}\prod_{i=1}^n\frac{x_i^{(2\alpha_i+1)/p'(\cdot)}}{\Gamma(\alpha_i+1/2)^{1/p'(\cdot)}}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n_+)}$$
$$\leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}\|F\|_{L^{p'(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}.$$

Hence, we conclude that

$$\|\mathbb{T}_\alpha f\|_{L^{p(\cdot)}_{X(T_\alpha)}(\mathbb{R}^n_+,\mu_\alpha)}\leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}.$$

We have thus proved that the operator $T_{\alpha,\mathrm{loc}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ provided that the exponent function p satisfies the conditions of Theorem 1.1.

3. An auxiliary result

In this section we establish a result that will be useful to prove $L^{p(\cdot)}$-boundedness for the global parts of the operators considered in Theorem 1.1.

Given $\alpha\in[0,\infty)^n$ and $\varepsilon\in[0,1)$, we define the global operator

$$\mathcal{H}_{\alpha,\varepsilon}(f)(x)=\int_{\mathbb{R}^n_+}H_{\alpha,\varepsilon}(x,y)f(y)d\mathfrak{m}_\alpha(y),\quad x\in\mathbb{R}^n_+,$$

where

$$H_{\alpha,\epsilon}(x,y)=\int_{(-1,1)^n}H_{\alpha,\varepsilon}(x,y,s)(1-\varphi(x,y,s))\Pi_\alpha(s)ds$$

and

(3.1)

$$H_{\alpha,\varepsilon}(x,y,s)=\begin{cases}e^{-(1-\varepsilon)|y|^2}, & \sum_{i=1}^n x_iy_is_i\leq 0,\\ q_+(x,y,s)^{n+\widehat{\alpha}}e^{-\frac{(1-\varepsilon)}{2}\left(|y|^2-|x|^2+\sqrt{q_+(x,y,s)q_-(x,y,s)}\right)}, & \sum_{i=1}^n x_iy_is_i>0.\end{cases}$$

Proposition 3.1. *Let $\alpha\in[0,\infty)^n$. Suppose that $p\in\mathrm{LH}_0(\mathbb{R}^n_+)\cap\mathcal{P}^\infty_e(\mathbb{R}^n_+)$ with $1<p^-\leq p^+<\infty$ and let $0<\varepsilon<\frac{1}{(p^-)'}\wedge\frac{1}{n+\widehat{\alpha}}$. Then, the operator $\mathcal{H}_{\alpha,\varepsilon}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$.*

Proof. We decompose $\mathcal{H}_{\alpha,\varepsilon}(f)=\mathcal{H}^{(1)}_{\alpha,\varepsilon}(f)+\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)$, where

$$\mathcal{H}^{(1)}_{\alpha,\varepsilon}(f)(x)=\int_{E_x}H_{\alpha,\varepsilon}(x,y,s)(1-\varphi(x,y,s))\Pi_\alpha(s)dsf(y)d\mathfrak{m}_\alpha(y),$$

and

$$\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)(x)=\int_{F_x}H_{\alpha,\varepsilon}(x,y,s)(1-\varphi(x,y,s))\Pi_\alpha(s)dsf(y)d\mathfrak{m}_\alpha(y),$$

being

$$E_x=\left\{(y,s)\in\mathbb{R}^n_+\times(-1,1)^n:\sum_{i=1}^n x_iy_is_i\leq 0\right\},$$
$$F_x=\left\{(y,s)\in\mathbb{R}^n_+\times(-1,1)^n:\sum_{i=1}^n x_iy_is_i>0\right\}.$$

Let $f\in L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ be given such that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}\leq 1$. For $x\in\mathbb{R}^n_+$, we have that

$$\left|\mathcal{H}^{(1)}_{\alpha,\varepsilon}(f)(x)\right|\leq\int_{\mathbb{R}^n_+}e^{-(1-\varepsilon)|y|^2}|f(y)|\int_{(-1,1)^n}|1-\varphi(x,y,s)|\Pi_\alpha(s)dsd\mathfrak{m}_\alpha(y)$$

$$\leq C \int_{\mathbb{R}^n_+} e^{-(1-\varepsilon)|y|^2} |f(y)| d\mathfrak{m}_\alpha(y).$$

Since $\varepsilon < 1/(p^-)'$, we can write $1-\varepsilon = \tilde{\varepsilon} + 1/p^-$ with $\tilde{\varepsilon} > 0$. Thus, by Hölder's inequality with $p^- > 1$ we have

$$\begin{aligned}
&\left|\mathcal{H}^{(1)}_{\alpha,\varepsilon}(f)(x)\right| \\
&\leq C \int_{\mathbb{R}^n_+} e^{-\left(\tilde{\varepsilon}+\frac{1}{p^-}\right)|y|^2} |f(y)| d\mathfrak{m}_\alpha(y) \\
&\leq C \left(\int_{\mathbb{R}^n_+} e^{-|y|^2} |f(y)|^{p^-} d\mathfrak{m}_\alpha(y)\right)^{1/p^-} \left(\int_{\mathbb{R}^n_+} e^{-\tilde{\varepsilon}(p^-)'|y|^2} d\mathfrak{m}_\alpha(y)\right)^{1/(p^-)'} \\
&\leq C \left(\int_{\mathbb{R}^n_+\cap\{|f|>1\}} |f(y)|^{p(y)} d\mu_\alpha(y) + \int_{\mathbb{R}^n_+\cap\{|f|\leq 1\}} d\mu_\alpha(y)\right)^{1/p^-} \leq C,
\end{aligned}$$

since $\int_{\mathbb{R}^n_+} |f(y)|^{p(y)} d\mu_\alpha(y) \leq 1$ and μ_α is a probability measure on $\mathbb{R}^n_+$.

Therefore, by the homogeneity of the norm,

$$\left\|\mathcal{H}^{(1)}_{\alpha,\varepsilon}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}.$$

for any $f \in L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$.

We now study $\mathcal{H}^{(2)}_{\alpha,\varepsilon}$. We have that

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} |\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)(x)|^{p(x)} d\mu_\alpha(x) \leq C \int_{\mathbb{R}^n_+} \left(\int_{F_x} |f(y)| e^{\frac{-|y|^2}{p(y)}} e^{\frac{|y|^2}{p(y)} - \frac{|x|^2}{p(x)}}\right. \\
&\times \left. q_+(x,y,s)^{n+\widehat{\alpha}} e^{-\frac{(1-\varepsilon)}{2}(|y|^2-|x|^2+\sqrt{q_+(x,y,s)q_-(x,y,s)})} \Pi_\alpha(s) ds d\mathfrak{m}_\alpha(y)\right)^{p(x)} d\mathfrak{m}_\alpha(x).
\end{aligned}$$

Note that we can write

$$\begin{aligned}
&q_+(x,y,s)q_-(x,y,s) \\
&\qquad= \left(|x|^2+|y|^2+2\sum_{i=1}^n x_i y_i s_i\right)\left(|x|^2+|y|^2-2\sum_{i=1}^n x_i y_i s_i\right) \\
&\qquad= \left(|x|^2+|y|^2\right)^2 - 4\left(\sum_{i=1}^n x_i y_i s_i\right)^2 \\
&\qquad= |x|^4+|y|^4+2|x|^2|y|^2 - 4\left(\sum_{i=1}^n x_i y_i s_i\right)^2 \\
&\qquad= \left(|x|^2-|y|^2\right)^2 + 4\left(|x|^2|y|^2 - \left(\sum_{i=1}^n x_i y_i s_i\right)^2\right) \\
&\qquad\geq \left(|x|^2-|y|^2\right)^2 + 4\left(|x|^2|y|^2 - |\langle (x_1,\dots,x_n),(s_1y_1,\dots,s_ny_n)\rangle|^2\right) \\
&\qquad\geq \left(|x|^2-|y|^2\right)^2 + 4\left(|x|^2|y|^2 - |x|^2|(s_1y_1,\dots,s_ny_n)|^2\right) \\
&\qquad\geq \left(|x|^2-|y|^2\right)^2,
\end{aligned}$$

for each $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n_+$ and $s=(s_1,\dots,s_n)\in(-1,1)^n$.

On the other hand, according to [15, Lemma 2.5], since $p \in \mathcal{P}^\infty_e(\mathbb{R}^n_+)$ then

$$e^{\frac{|y|^2}{p(y)} - \frac{|x|^2}{p(x)}} \sim e^{\frac{|y|^2-|x|^2}{p_\infty}}, \quad x,y \in \mathbb{R}^n_+.$$

Here $p_\infty > 1$. Whence, it follows that

$$
\begin{aligned}
&q_+(x,y,s)^{n+\widehat{\alpha}} \exp\left(-\tfrac{(1-\varepsilon)}{2}(|y|^2-|x|^2+\sqrt{q_+(x,y,s)q_-(x,y,s)}\right) e^{\frac{|y|^2}{p(y)}-\frac{|x|^2}{p(x)}} \\
&\le Cq_+(x,y,s)^{n+\widehat{\alpha}} \exp\left(\left(\tfrac{1}{p_\infty}-\tfrac{1-\varepsilon}{2}\right)\left(|y|^2-|x|^2\right)-\tfrac{(1-\varepsilon)}{2}\sqrt{q_+(x,y,s)q_-(x,y,s)}\right) \\
&\le C\left(q_+(x,y,s)\right)^{n+\widehat{\alpha}} \exp\left(-a_\varepsilon\sqrt{q_+(x,y,s)q_-(x,y,s)}\right),
\end{aligned}
$$

for every $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n_+$ such that $(x,y,s)\in G_1$ and $\sum_{i=1}^n x_iy_is_i \ge 0$. We recall that

$$
G_1=\left\{(x,y,s)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\times(-1,1)^n : \sqrt{q_-(x,y,s)} \ge \frac{C_0}{1+|x|+|y|}\right\}.
$$

Above we have set $a_\varepsilon = \frac{1-\varepsilon}{2} - \left|\frac{1}{p_\infty}-\frac{1-\varepsilon}{2}\right|$. Note that $a_\varepsilon>0$ because $\varepsilon < 1/(p^-)'$ and $(p^-)'=(p')^+ \ge p'_\infty$.

We get

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}\left|\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)(x)\right|^{p(x)} d\mu_\alpha(x) \\
&\quad\le C\int_{\mathbb{R}^n_+}\left(\int_{F_x}|f(y)|e^{\frac{-|y|^2}{p(y)}}|1-\varphi(x,y,s)|q_+(x,y,s)^{n+\widehat{\alpha}}\right. \\
&\qquad\left.\times\exp\left(-a_\varepsilon\sqrt{q_+(x,y,s)q_-(x,y,s)}\right)\Pi_\alpha(s)ds\,d\mathfrak{m}_\alpha(y)\right)^{p(x)} d\mathfrak{m}_\alpha(x).
\end{aligned}
$$

In order to complete the study of $\mathcal{H}^{(2)}_{\alpha,\varepsilon}$ we use Stein complex interpolation. We consider firstly $n=1$. For every $z\in\mathbb{C}$ with $\mathrm{Re}(z)>-\frac{1}{2}$, we define the operator $\mathbb{H}^{(2)}_{z,\varepsilon}$ by

$$
\begin{aligned}
\mathbb{H}^{(2)}_{z,\varepsilon}(h)(x) &= \int_0^\infty K^{(2)}_{z,\varepsilon}(x,y)h(y)y^{2z+1}dy\; x^{\frac{2z+1}{p(x)}} \\
&= \widehat{\mathcal{H}}^{(2)}_{z,\varepsilon}(h)(x)\; x^{\frac{2z+1}{p(x)}}, \quad x\in\mathbb{R}_+,
\end{aligned}
$$

where

$$
\begin{aligned}
K^{(2)}_{z,\varepsilon}(x,y) = \int_{-1}^1 &\chi_{F_x}(y,s)(1-\varphi(x,y,s))(q_+(x,y,s))^{z+1} \\
&\times\exp\left(-a_\varepsilon\sqrt{q_+(x,y,s)q_-(x,y,s)}\right)(1-s^2)^{z-\frac{1}{2}}ds, \quad x,y\in\mathbb{R}_+,
\end{aligned}
$$

and a_ε is as above.

For every $z\in\mathbb{C}$ with $\mathrm{Re}(z)>-\frac{1}{2}$ and every simple function h defined on $(\mathbb{R}_+,dx)$, $\mathbb{H}^{(2)}_{z,\varepsilon}(h)$ is a measurable function on $(\mathbb{R}_+,dx)$.

Assume that r, y, c_1, $c_2>0$, b_1, b_2, m_1 and m_2 are positive bounded measurable functions on $\mathbb{R}_+$, and A_1 and A_2 are two measurable subsets of $\mathbb{R}_+$ with finite Lebesgue measure. We define

$$
F_{y,r}(z) = \int_{B(y,r)} \mathbb{H}^{(2)}_{z,\varepsilon}\left(c_1^{m_1(\cdot)z+b_1(\cdot)}\chi_{A_1}(\cdot)\right)(x)c_2^{m_2(x)z+b_2(x)}\chi_{A_2}(x)dx,
$$

for $z\in\mathbb{C}$, $\mathrm{Re}(z)>-\frac{1}{2}$. The function $F_{y,r}$ is analytic on $\Omega=\left\{z\in\mathbb{C} : \mathrm{Re}(z)>-\frac{1}{2}\right\}$. Furthermore, for every $-\frac{1}{2}<c<d<\infty$,

$$
\sup_{c\le\mathrm{Re}(z)\le d}|F_{y,r}(z)|<\infty.
$$

Thus, the family $\left\{\mathbb{H}^{(2)}_{z,\varepsilon}\right\}_{z\in\Omega}$ is an analytic family of admissible growth in every strip $\{z\in\mathbb{C} : c<\mathrm{Re}(z)<d\}$, with $-\frac{1}{2}<c<d<\infty$ (see [27, §3]).

Let $k \in \mathbb{N}$, $k > 1$. We take $\alpha = \frac{k}{2} - 1$. For every $\overline{x} \in \mathbb{R}^k$ we write $x = |\overline{x}|$. If $\overline{x}$, $\overline{y} \in \mathbb{R}^k$ and θ is the angle between $\overline{x}$ and $\overline{y}$, we have that

$$|\overline{x} \pm \overline{y}|^2 = q_\pm(x, y, \cos(\theta)),$$

and also that $(x, y, \cos(\theta)) \in L_1$ if and only if $|\overline{x} - \overline{y}| < C_0/(1+x+y)$. By integrating in spherical coordinates on $\mathbb{R}^k$ and by performing the change of variable $s = \cos(\theta)$ we obtain

$$\left|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)(x)\right| \leq C x^{\frac{k-1}{p(x)}} \int_{|\overline{x}-\overline{y}|>\frac{C_0}{1+x+y}} |\overline{x}+\overline{y}|^k e^{-a_\varepsilon|\overline{x}-\overline{y}||\overline{x}+\overline{y}|} |h(y)| d\overline{y},$$

for $x = |\overline{x}| \in \mathbb{R}_+$. We consider the operators

$$T_1(h)(\overline{x}) = \int_{\substack{|\overline{x}-\overline{y}|>\frac{C_0}{1+x+y} \\ 2|\overline{x}-\overline{y}|\geq|\overline{x}+\overline{y}|}} |\overline{x}+\overline{y}|^k e^{-a_\varepsilon|\overline{x}-\overline{y}||\overline{x}+\overline{y}|} h(\overline{y}) d\overline{y},$$

and

$$T_2(h)(\overline{x}) = \int_{\substack{|\overline{x}-\overline{y}|>\frac{C_0}{1+x+y} \\ 2|\overline{x}-\overline{y}|<|\overline{x}+\overline{y}|}} |\overline{x}+\overline{y}|^k e^{-a_\varepsilon|\overline{x}-\overline{y}||\overline{x}+\overline{y}|} h(\overline{y}) d\overline{y},$$

for $\overline{x} \in \mathbb{R}^k$. We are going to see that T_1 and T_2 are bounded on $L^{\overline{p}(\cdot)}(\mathbb{R}^k, dx)$, where $\overline{p}(\overline{x}) = p(|\overline{x}|)$, $\overline{x} \in \mathbb{R}^k$.

Note firstly that

$$\begin{aligned}
|T_1(h)(\overline{x})| &\leq C\left(\int_{B(-\overline{x},1)} |h(\overline{y})| d\overline{y} + \sum_{\ell=1}^{\infty} \int_{\ell\leq|\overline{x}+\overline{y}|<\ell+1} e^{-c|\overline{x}+\overline{y}|^2} |h(\overline{y})| d\overline{y}\right) \\
&\leq C \sum_{\ell=0}^{\infty} e^{-c\ell^2} \int_{B(-\overline{x},\ell+1)} |h(\overline{y})| d\overline{y} \\
&\leq C M_{\mathrm{HL}}(h)(-\overline{x}), \quad \overline{x} \in \mathbb{R}^k.
\end{aligned}$$

Here, M_{HL} represents the Hardy–Littlewood maximal function in $\mathbb{R}^k$.

On the other hand, according to [22, (16) and (17)], if $2|\overline{x} - \overline{y}| < |\overline{x} + \overline{y}|$, then $|\overline{y}| \leq 3|\overline{x}|$ and $\frac{4}{3}|\overline{x}| \leq |\overline{x} + \overline{y}| \leq 4|\overline{x}|$. We obtain

$$\begin{aligned}
|T_2(h)(\overline{x})| &\leq C \int_{|\overline{x}-\overline{y}|>\frac{C_0}{1+4x}} |\overline{x}|^k e^{-c|\overline{x}||\overline{x}-\overline{y}|} |h(\overline{y})| d\overline{y} \\
&\leq \begin{cases} \int_{|\overline{x}-\overline{y}|\leq 4} |h(\overline{y})| d\overline{y} \leq C M_{\mathrm{HL}}(h)(\overline{x}) & \text{if } |\overline{x}| \leq 1, \\ \int_{|\overline{x}-\overline{y}|>C_0/(5|\overline{x}|)} |\overline{x}|^k e^{-c|\overline{x}||\overline{x}-\overline{y}|} |h(\overline{y})| d\overline{y} & \text{if } |\overline{x}| > 1. \end{cases}
\end{aligned}$$

Since $\overline{p}(\overline{x}) = \overline{p}(-\overline{x})$, and under the imposed conditions for $p(\cdot)$, M_{HL} is bounded on $L^{\overline{p}(\cdot)}(\mathbb{R}^k, dx)$ (see Lemma A.2 for $n = 1$), the arguments developed in [15, pp. 417 and 418] allow us to conclude that T_1 and T_2 are bounded on $L^{\overline{p}(\cdot)}(\mathbb{R}^k, dx)$.

We have, therefore, that the operator $T := T_1 + T_2$ is bounded on $L^{\overline{p}(\cdot)}(\mathbb{R}^k, dx)$.

Since

$$\left|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)(x)\right| \leq C x^{\frac{k-1}{p(x)}} T(|\widetilde{h}|)(\overline{x}), \quad x = |\overline{x}|, \quad x \in \mathbb{R}^k,$$

we get

$$\int_0^\infty \left|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)(x)\right|^{p(x)} dx = \int_0^\infty \left|\widehat{\mathcal{H}}^{(2)}_{\alpha,\varepsilon}(h)(x)\right|^{p(x)} x^{k-1} dx \leq C \int_{\mathbb{R}^k} \left|T(|\widetilde{h}|)(|\overline{x}|)\right|^{\overline{p}(\overline{x})} d\overline{x},$$

where $\widetilde{h}(\overline{y}) = h(|\overline{y}|)$, $\overline{y} \in \mathbb{R}^k$. Hence

$$\left\|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq C \left\|T\left(|\widetilde{h}|\right)\right\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^k,dx)} \leq C \left\|\widetilde{h}\right\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^k,dx)}.$$

Naming $h_k(u) = h(u)u^{\frac{k-1}{p(u)}}$, $u \in \mathbb{R}_+$, we also have

$$\int_{\mathbb{R}^k} \left|\widetilde{h}(\overline{x})\right|^{\overline{p}(x)} d\overline{x} = \int_{\mathbb{R}^k} |h(|\overline{x}|)|^{p(|\overline{x}|)} d\overline{x} = C\int_0^\infty |h(x)|^{p(x)} x^{k-1} dx$$
$$= C\int_0^\infty \left|h(x)x^{\frac{k-1}{p(x)}}\right|^{p(x)} dx = C\int_0^\infty |h_k(x)|^{p(x)} dx,$$

which yields $\|\widetilde{h}\|_{L^{\overline{p}(\cdot)}(\mathbb{R}^k,dx)} \leq C\|h_k\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}$. We conclude that

$$\left\|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq C\left\|h_k\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}.$$

We now consider, for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > -\frac{1}{2}$,

$$\mathcal{C}_{z,\varepsilon}(f)(x) = \mathcal{H}^{(2)}_{z,\varepsilon}(f_z)(x), \quad x \in \mathbb{R}_+,$$

where $f_z(y) = f(y)y^{-\frac{2z+1}{p(y)}}$, $y \in \mathbb{R}_+$.

The family $\{\mathcal{C}_{z,\varepsilon}\}_{\mathrm{Re}(z)>-\frac{1}{2}}$ is an analytic family of admissible growth in every strip $\{z \in \mathbb{C} : c < \mathrm{Re}(z) < d\}$ with $-\frac{1}{2} < c < d < \infty$ ([27, §3]). For every $k \in \mathbb{N}$, $k > 1$, we have that

$$\left\|\mathcal{C}_{\frac{k}{2}-1,\varepsilon}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq C_0\left\|f\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}$$

and, for each $t \in \mathbb{R}$,

$$\left\|\mathcal{C}_{\frac{k}{2}-1+it,\varepsilon}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq \left\|\mathcal{C}_{\frac{k}{2}-1,\varepsilon}(|f|)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq C_0\left\|f\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}.$$

According to [27, Theorem 1], for every $\alpha \geq 0$,

$$\left\|\mathcal{C}_{\alpha,\varepsilon}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)} \leq C_\alpha\left\|f\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}.$$

It follows that, for every $\alpha \geq 0$,

$$\int_0^\infty \left|\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)(x)\right|^{p(x)} d\mu_\alpha(x) \leq C\int_0^\infty \left|\mathbb{H}^{(2)}_{\alpha,\varepsilon}\left(f(\cdot)e^{-\frac{|\cdot|^2}{p(\cdot)}}\right)(x)\right|^{p(x)} dx$$
$$= C\int_0^\infty \left|\mathcal{C}_{\alpha,\varepsilon}\left(f(\cdot)e^{-\frac{|\cdot|^2}{p(\cdot)}}(\cdot)^{\frac{2\alpha+1}{p(y)}}\right)(x)\right|^{p(x)} dx.$$

Then

$$\left\|\mathcal{H}^{(2)}_{\alpha,\varepsilon}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,\mu_\alpha)} \leq C\left\|\mathcal{C}_{\alpha,\varepsilon}\left(f(\cdot)e^{-\frac{|\cdot|^2}{p(\cdot)}}(\cdot)^{\frac{2\alpha+1}{p(\cdot)}}\right)\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}$$
$$\leq C\left\|f(\cdot)e^{-\frac{|\cdot|^2}{p(\cdot)}}(\cdot)^{\frac{2\alpha+1}{p(\cdot)}}\right\|_{L^{p(\cdot)}(\mathbb{R}_+,dx)}$$
$$\leq C\left\|f\right\|_{L^{p(\cdot)}(\mathbb{R}_+,\mu_\alpha)}.$$

We conclude that the operator $\mathcal{H}_{\alpha,\varepsilon}$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+,\mu_\alpha)$.

We now prove that $\mathcal{H}_{\alpha,\varepsilon}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ when the dimension n is greater than one.

Let $n \in \mathbb{N}$, $n > 1$. We define

$$\mathbb{H}^{(2)}_{z,\varepsilon}(h)(x) = \int_{\mathbb{R}^n_+} K^{(2)}_{z,\varepsilon}(x,y)h(y)\prod_{j=1}^n y_j^{2z_j+1} dy \prod_{j=1}^n x_j^{\frac{2z_j+1}{p(x)}},$$

for $x=(x_1,\dots,x_n)\in\mathbb{R}^n_+$ and $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ with $\mathrm{Re}(z_j)>-\frac{1}{2}$ for each $j=1,\dots,n$, where

$$K^{(2)}_{z,\varepsilon}(x,y)=\int_{(-1,1)^n}\chi_{F_x}(y,s)(1-\varphi(x,y,s))q_+(x,y,s)^{n+\widehat{z}}$$
$$\times\exp\left(-a_\varepsilon\sqrt{q_+(x,y,s)q_-(x,y,s)}\right)\prod_{j=1}^n(1-s_j^2)^{z_j-1/2}ds,$$

for $x,y\in\mathbb{R}^n_+$, z and a_ε as before.

Let $k=(k_1,\dots,k_n)\in\mathbb{N}^n$, $k_j>1$, $j=1,\dots,n$. We consider $\alpha_j=k_j/2-1$, $j=1,\dots,n$, and $\alpha=(\alpha_1,\dots,\alpha_n)$. We have that

$$\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)(x)=\int_{\mathbb{R}^n_+}K^{(2)}_{\alpha,\varepsilon}(x,y)h(y)\prod_{j=1}^n y_j^{k_j-1}dy\prod_{j=1}^n x_j^{\frac{k_j-1}{p(x)}},$$

for $x=(x_1,\dots,x_n)\in\mathbb{R}^n_+$, and

$$K^{(2)}_{\alpha,\varepsilon}(x,y)=\int_{(-1,1)^n}\chi_{F_x}(y,s)(1-\varphi(x,y,s))q_+(x,y,s)^{\widehat{k}/2}$$
$$\times\exp\left(-a_\varepsilon\sqrt{q_+(x,y,s)q_-(x,y,s)}\right)\prod_{j=1}^n(1-s_j^2)^{\alpha_j-1/2}ds,$$

for $x,y\in\mathbb{R}^n_+$. We define $\overline{p}(\overline{x_1},\dots,\overline{x_n})=p(x_1,\dots,x_n)$, where $x_j=|\overline{x_j}|$, $\overline{x_j}\in\mathbb{R}^{k_j}_+$, $j=1,\dots,n$. Integrating in multi-radial polar coordinates we have that

$$\left|\mathbb{H}^{(2)}_{\alpha,\varepsilon}(h)(x)\right|\le C\int_{|\overline{x}-\overline{y}|>\frac{C_0}{1+|\overline{x}|+\overline{y}|}}|\overline{x}+\overline{y}|^{\widehat{k}}e^{-a_\varepsilon|\overline{x}-\overline{y}||\overline{x}+\overline{y}|}|h(|\overline{y_1}|,\dots,|\overline{y_n}|)|d\overline{y}\prod_{j=1}^n x_j^{\frac{k_j-1}{p(x)}},$$

for $x=(x_1,\dots,x_n)=(|\overline{x_1}|,\dots,|\overline{x_n}|)\in\mathbb{R}^n_+$ and $\overline{x}=(\overline{x_1},\dots,\overline{x_n})\in\prod_{j=1}^n\mathbb{R}^{k_j}=\mathbb{R}^{\widehat{k}}$.

We now proceed as in the above one-dimensional case. In order to do this, notice that if we define $\overline{p}$ by $\overline{p}(\overline{x})=p(|\overline{x_1}|,\dots,|\overline{x_n}|)$, for $\overline{x}=(\overline{x_1},\dots,\overline{x_n})\in\mathbb{R}^{\widehat{k}}_+$, then $\overline{p}$ belongs to $\mathrm{LH}(\mathbb{R}^{\widehat{k}}_+)$, with $1<\overline{p}^-\le\overline{p}^+<\infty$, by virtue of Lemma A.2. Hence, the Hardy–Littlewood maximal operator M_{HL} on $\mathbb{R}^{\widehat{k}}_+$ is bounded on $L^{\overline{p}(\cdot)}\left(\mathbb{R}^{\widehat{k}}_+\right)$.

We consider, for every $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ such that $\mathrm{Re}(z_j)>-\frac{1}{2}$, for each $j=1,\dots,n$, the operator

$$\mathcal{C}_{z,\varepsilon}(f)(x)=\mathbb{H}^{(2)}_{z,\varepsilon}(f_z)(x),\quad x\in\mathbb{R}^n_+,$$

where $f_z(y)=f(y)\prod_{j=1}^n y_j^{-\frac{2z_j+1}{p(y)}}$ for $y=(y_1,\dots,y_n)\in\mathbb{R}^n_+$. The proof can be concluded as in the one-dimensional case by using an n-dimensional version of the Stein complex interpolation with variable exponent. This result can be proved by proceeding as in the proof of [27, Theorem 1] and by using an n-dimensional version of the Three Lines Theorem (see Theorem A.1 and [4, Proposition 21]). □

4. Proof of Theorem 1.1 for maximal operators

According to the subordination formula (1.2), since $\frac{t}{2\sqrt{\pi}}\int_0^\infty\frac{e^{-t^2/(4u)}}{u^{3/2}}du=1$ for each $t>0$, we deduce that

$$P^\alpha_*(f)(x)\le W^\alpha_*(f)(x),\quad x\in\mathbb{R}^n_+.$$

Hence, it suffices to see that W^α_* is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$.

We firstly study its global part $W^\alpha_{*,\mathrm{glob}}$ given, for $x\in\mathbb{R}^n_+$, by

$W^\alpha_{*,\mathrm{glob}}(f)(x)$

$$= \sup_{t>0} \left| \int_{\mathbb{R}^n_+} \int_{(-1,1)^n} \frac{e^{-\frac{q_-\left(e^{-t/2}x,y,s\right)}{1-e^{-t}}+|y|^2}}{(1-e^{-t})^{n+\widehat{\alpha}}} (1-\varphi(x,y,s))\Pi_\alpha(s)ds f(y) d\mu_\alpha(y) \right|.$$

By performing the change of variables $1-e^{-t}=u$, $t>0$, and then replacing u by t, we can write

$$W^\alpha_{*,\mathrm{glob}}(f)(x)$$
$$= \sup_{0<t<1} \left| \int_{\mathbb{R}^n_+} \int_{(-1,1)^n} \frac{e^{-\frac{q_-(\sqrt{1-t}x,y,s)}{t}+|y|^2}}{t^{n+\widehat{\alpha}}} (1-\varphi(x,y,s))\Pi_\alpha(s)ds f(y) d\mu_\alpha(y) \right|.$$

Let $(x,y,s) \in G_1$ (recall the definition on page 8). We consider

$$u(t) = \frac{(1-t)|x|^2 + |y|^2 - 2\sum_{i=1}^n x_i y_i s_i \sqrt{1-t}}{t}, \quad t \in (0,1).$$

Setting $a = |x|^2 + |y|^2$ and $b = 2\sum_{i=1}^n x_i y_i s_i$, we have

$$u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2, \quad t \in (0,1). \tag{4.1}$$

We also define

$$v(t) = \frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}}, \quad t \in (0,1).$$

We are going to study the supremum of $v(t)$, for $t \in (0,1)$, by proceeding as in the proof of [26, Proposition 2.1]. The derivative of v is

$$v'(t) = -\frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}} \left(u'(t) + \frac{n+\widehat{\alpha}}{t} \right), \quad t \in (0,1),$$

where

$$u'(t) = -\frac{a}{t^2} + b\left(\frac{1}{2t\sqrt{1-t}} + \frac{\sqrt{1-t}}{t^2} \right) = \frac{-2a\sqrt{1-t} + bt + 2b(1-t)}{2t^2\sqrt{1-t}}, \quad t \in (0,1).$$

Thus,

$$\begin{aligned} v'(t) &= -\frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}} \left(\frac{-2a\sqrt{1-t} - bt + 2b}{2t^2\sqrt{1-t}} + \frac{n+\widehat{\alpha}}{t} \right) \\ &= -\frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}} \cdot \frac{2\sqrt{1-t}(t(n+\widehat{\alpha}) - a) + b(2-t)}{2t^2\sqrt{1-t}}, \quad t \in (0,1). \end{aligned}$$

By choosing $C_0 > 1$ large enough, we can prove $a > n + \widehat{\alpha}$ for any $(x,y,s) \in G_1$. Indeed, let us remark that

$$|b| \le 2\sum_{i=1}^n |x_i||y_i| \le |x|^2 + |y|^2 = a.$$

Besides,

$$a = \frac{a-b+a+b}{2} \ge \frac{q_-(x,y,s) + a - |b|}{2} \ge \frac{1}{2} q_-(x,y,s).$$

Also,

$$\sqrt{a} \ge \frac{1}{\sqrt{2}}(|x|+|y|).$$

Fix $(x,y,s) \in G_1$. If $|x|+|y|<1$ then

$$a \ge \frac{1}{2} q_-(x,y,s) > \frac{1}{2} \frac{C_0^2}{(1+|x|+|y|)^2} \ge \frac{C_0^2}{8} > \frac{C_0}{8}$$

since we shall take $C_0 > 1$. And, if $|x| + |y| \geq 1$, then

$$a = \sqrt{a}\sqrt{a} \geq \frac{1}{\sqrt{2}}(|x|+|y|)\frac{1}{\sqrt{2}}\sqrt{q_-(x,y,s)} > \frac{C_0}{2}\frac{|x|+|y|}{1+|x|+|y|} \geq \frac{C_0}{2}\frac{1}{2} = \frac{C_0}{4} > \frac{C_0}{8}.$$

Therefore, taking $C_0 > 8(n+\widehat{\alpha})$ we get that $a > n+\widehat{\alpha}$ on G_1 as claimed.

Then, if $b \leq 0$, $v'(t) > 0$ for each $t \in (0,1)$, so

$$\sup_{0<t<1} v(t) \leq v(1) = e^{-|y|^2}.$$

On the other hand, if $b > 0$, from the property $a > n+\widehat{\alpha}$, the equation

$$2\sqrt{1-t}(a - t(n+\widehat{\alpha})) = b(2-t)$$

has a unique solution t_n. The arguments developed in [26, p. 850] allow us to conclude that

$$\sup_{0<t<1} v(t) \sim v(t_0),$$

where $t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \sim \sqrt{\frac{q_-(x,y,s)}{q_+(x,y,s)}}$.

Then,

$$\sup_{0<t<1} v(t) \leq C\left(\frac{q_+(x,y,s)}{q_-(x,y,s)}\right)^{\frac{n+\widehat{\alpha}}{2}} \exp\left(-\frac{|y|^2-|x|^2}{2} - \frac{\sqrt{q_+(x,y,s)q_-(x,y,s)}}{2}\right),$$

provided that C_0 satisfies the above condition. From now on, C_0 will be fixed such that the stated condition holds.

Since $q_+(x,y,s)q_-(x,y,s) \geq c$ for every $(x,y,s) \in G_1$ (see [21, p. 264]) , we have that

$$\sqrt{\frac{q_+(x,y,s)}{q_-(x,y,s)}} \leq Cq_+(x,y,s).$$

Therefore, for every $(x,y,s) \in G_1$

$$\sup_{0<t<1} v(t) \leq Cq_+(x,y,s)^{n+\widehat{\alpha}} e^{-\frac{|y|^2-|x|^2}{2} - \frac{\sqrt{q_+(x,y,s)q_-(x,y,s)}}{2}} = CH_{\alpha,0}(x,y,s)$$

where $H_{\alpha,0}$ is the function given in (3.1). Hence, $W^\alpha_{*,\text{glob}}$ is pointwise smaller than a multiple of $\mathcal{H}_{\alpha,0}$, which is a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ by Proposition 3.1, so $W^\alpha_{*,\text{glob}}$ is also bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$.

We now study $W^\alpha_{*,\text{loc}}$ defined by

$$W^\alpha_{*,\text{loc}}(f)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n_+}\int_{(-1,1)^n} \frac{\exp\left(\frac{-q_-\left(e^{-t/2}x,y,s\right)}{1-e^{-t}} + |y|^2\right)}{(1-e^{-t})^{n+\widehat{\alpha}}}\varphi(x,y,s)\Pi_\alpha(s)ds f(y)d\mu_\alpha(y)\right|,$$

for $x \in \mathbb{R}^n_+$. Setting $u = 1-e^{-t}$ and then replacing u by t, we can write

$$W^\alpha_{*,\text{loc}}(f)(x) = \sup_{0<t<1}\left|\int_{\mathbb{R}^n_+} K^\alpha_t(x,y)f(y)d\mathfrak{m}_\alpha(y)\right|$$

for $x = (x_1,\dots,x_n) \in \mathbb{R}^n_+$, where

$$K^\alpha_t(x,y) = \int_{(-1,1)^n}\frac{\exp\left(-\frac{(1-t)|x|^2+|y|^2-2\sqrt{1-t}\sum_{i=1}^n x_iy_is_i}{t}\right)}{t^{n+\widehat{\alpha}}}\varphi(x,y,s)\Pi_\alpha(s)ds,$$

for $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}^n_+$ and $t \in (0,1)$.

As it was explained in Section 2, we shall see that $W^\alpha_{*,\rm loc}$ is a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ as a consequence of vector valued Calderón–Zygmund theory.

According to [38, (2.6)] we have that

$$q_-\left(\sqrt{1-t}x,y,s\right)\geq q_-(x,y,s)-C\left(1-\sqrt{1-t}\right)=q_-(x,y,s)-C\frac{t}{1+\sqrt{1-t}},$$

for $x,y\in\mathbb{R}^n_+$, $t\in(0,1)$, $s\in(-1,1)^n$ and $(x,y,s)\in L_2$.

Then,

$$|K^\alpha_t(x,y)|\leq C\int_{(-1,1)^n}\frac{e^{-q_-(x,y,s)/t}}{t^{n+\widehat{\alpha}}}\Pi_\alpha(s)ds\leq C\int_{(-1,1)^n}\frac{\Pi_\alpha(s)}{q_-(x,y,s)^{n+\widehat{\alpha}}}ds$$

for $x,y\in\mathbb{R}^n_+$ and $t\in(0,1)$.

According to [7, Lemma 3.1] (see also [33, Lemma 2.1]), we get

$$\sup_{t>0}|K^\alpha_t(x,y)|\leq\frac{C}{\mathfrak{m}_\alpha(B(x,|y-x|))},\quad x,y\in\mathbb{R}^n_+,\ x\neq y. \tag{4.2}$$

Let $j=1,\dots,n$. We have that,

$$\partial_{x_j}K^\alpha_t(x,y)=\int_{(-1,1)^n}\left(\frac{-2x_j(1-t)+2y_js_j\sqrt{1-t}}{t^{n+1+\widehat{\alpha}}}\varphi(x,y,s)+\frac{\frac{\partial\varphi}{\partial x_j}(x,y,s)}{t^{n+\widehat{\alpha}}}\right)$$
$$\times\exp\left(-\frac{(1-t)|x|^2+|y|^2-2\sum_{i=1}^n x_iy_is_i\sqrt{1-t}}{t}\right)\Pi_\alpha(s)ds,$$

for $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n_+$, and $t>0$.

According to the properties of φ and using again [7, Lemma 3.1], since

$$\begin{aligned}\left|x_j\sqrt{1-t}-y_js_j\right|^2&=x_j^2(1-t)+y_j^2s_j^2-2x_jy_js_j\sqrt{1-t}\\&\leq(1-t)|x|^2+|y|^2-2\sum_{i=1}^n x_iy_is_i\sqrt{1-t},\end{aligned} \tag{4.3}$$

for $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n_+$, $t\in(0,1)$ and $s\in(-1,1)^n$, we get

$$\begin{aligned}|\partial_{x_j}K^\alpha_t(x,y)|&\leq C\int_{(-1,1)^n}\frac{e^{-c\frac{q_-(x,y,s)}{t}}}{t^{n+\frac12+\widehat{\alpha}}}\Pi_\alpha(s)ds\\&\leq C\int_{(-1,1)^n}\frac{\Pi_\alpha(s)}{q_-(x,y,s)^{n+\frac12+\widehat{\alpha}}}ds\\&\leq C\frac{1}{|x-y|\mathfrak{m}_\alpha(B(x,|x-y|))},\end{aligned}$$

for $x,y\in\mathbb{R}^n_+$, $x\neq y$, and $t>0$. Hence,

$$\sup_{t>0}|\partial_{x_j}K^\alpha_t(x,y)|+\sup_{t>0}|\partial_{y_j}K^\alpha_t(x,y)|\leq C\frac{1}{|x-y|\mathfrak{m}_\alpha(B(x,|x-y|))}, \tag{4.4}$$

for $x,y\in\mathbb{R}^n_+$, $x\neq y$.

Let $N\in\mathbb{N}$. We consider the space $C([1/N,N])$ of continuous functions in $[1/N,N]$ with the usual maximum norm. We define, for every $x,y\in\mathbb{R}^n_+$, $x\neq y$,

$$[K^\alpha(x,y)](t)=K^\alpha_t(x,y),\quad t>0.$$

By proceeding as above we can see that, for every $x\in\mathbb{R}^n_+$, the mapping $\Phi_x(y)=K^\alpha(x,y)$, $y\in\mathbb{R}^n_+$, is continuous from $\mathbb{R}^n_+$ into $C([1/N,N])$, and then, Φ_x is weakly measurable. Since $C([1/N,N])$ is separable, we conclude that, for every $x\in\mathbb{R}^n_+$, Φ_x is strongly measurable (see [44, p. 131]). According to (4.2) and (4.4) we deduce that K^α is a $C([1/N,N])$-valued Calderón–Zygmund kernel with respect to $(\mathbb{R}^n_+,|\cdot|,\mathfrak{m}_\alpha)$.

Suppose λ is a complex measure supported in $[1/N,N]$ and $f\in C_c^\infty(\mathbb{R}_+^n)$. By using (4.2) we obtain

$$\int_{[1/N,N]}\int_{\mathbb{R}_+^n}|K_t^\alpha(x,y)||f(y)|d\mathfrak{m}_\alpha(y)d|\lambda|(t)<\infty,\quad x\notin \operatorname{supp}(f),$$

because $|\lambda|([1/N,N])<\infty$. Here $|\lambda|$ denotes the total variation of λ. It follows that

$$(4.5)\qquad \begin{aligned}&\int_{[1/N,N]}\int_{\mathbb{R}_+^n}K_t^\alpha(x,y)f(y)d\mathfrak{m}_\alpha(y)d\lambda(t)\\ &\qquad=\int_{\mathbb{R}_+^n}\int_{[1/N,N]}K_t^\alpha(x,y)f(y)d\lambda(t)d\mathfrak{m}_\alpha(y),\quad x\notin \operatorname{supp}(f).\end{aligned}$$

We define the functional S_λ on $C([1/N,N])$ by

$$S_\lambda(g)=\int_{[1/N,N]}g(t)d\lambda(t),\quad g\in C([1/N,N]).$$

Equality (4.5) says that, by understanding the integral under S_λ in the $C([1/N,N])$-Bochner sense,

$$S_\lambda\left[\int_{\mathbb{R}_+^n}[K^\alpha(x,y)](\cdot)f(y)d\mathfrak{m}_\alpha(y)\right]=\int_{[1/N,N]}W_{t,\mathrm{loc}}^\alpha(f)(x)d\lambda(t),\quad x\notin \operatorname{supp}(f).$$

Since the dual of $C([1/N,N])$ is the space $\mathcal{M}([1/N,N])$ of complex measures supported on $[1/N,N]$ we conclude that, for every $x\notin\operatorname{supp}(f)$

$$W_{t,\mathrm{loc}}^\alpha(f)(x)=\left[\int_{\mathbb{R}_+^n}[K^\alpha(x,y)](\cdot)f(y)d\mathfrak{m}_\alpha(y)\right](t),\quad t\in[1/N,N].$$

According to [42, p. 73], the maximal operator W_*^α is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$. Also, $W_{*,\mathrm{glob}}^\alpha$ is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$ (see the first part of this proof). Then, $W_{*,\mathrm{loc}}^\alpha$ is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$. Hence, there exists $C>0$ such that, for every $N\in\mathbb{N}$,

$$(4.6)\qquad \Big\|\big\|W_{t,\mathrm{loc}}^\alpha(f)\big\|_{C([1/N,N])}\Big\|_{L^2(\mathbb{R}_+^n,\mu_\alpha)}\le C\|f\|_{L^2(\mathbb{R}_+^n,\mu_\alpha)},$$

for $f\in L^2(\mathbb{R}_+^n,\mu_\alpha)$. By using (4.2), (4.4) and (4.6) as it was explained in Section 2 we get

$$\left\|\sup_{t\in[1/N,N]}|W_{t,\mathrm{loc}}^\alpha(f)(x)|\right\|_{L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)}$$

for $f\in L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)$ and $C>0$ independent of $N\in\mathbb{N}$.

By using now the monotone convergence theorem (see [18, p. 75]), we conclude that $W_{*,\mathrm{loc}}^\alpha$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)$. Thus, the proof of Theorem 1.1 for W_*^α is finished.

5. Proof of Theorem 1.1 for Riesz transforms

The proof of Theorem 1.1 for Riesz transforms R_α^β of order $\beta\in\mathbb{N}^n\setminus\{(0,\dots,0)\}$, follows the same steps done in the proof of the results in Section 4 by using some results developed in [21] and [41]. We now sketch the proof.

Let $\beta\in\mathbb{N}^n\setminus\{(0,\dots,0)\}$ be given. For every $f\in L^2(\mathbb{R}_+^n,\mu_\alpha)$, we have that

$$R_\alpha^\beta(f)(x)=c_\beta f(x)+\text{p.v.}\int_{\mathbb{R}_+^n}R_\alpha^\beta(x,y)f(y)d\mathfrak{m}_\alpha(y),\quad \text{a.e. } x\in\mathbb{R}_+^n$$

where $c_\beta \in \mathbb{R}$ and

$$R_\alpha^\beta(x,y) = \frac{1}{\Gamma\left(\frac{\widehat{\beta}}{2}\right)} \int_{(-1,1)^n} K_\alpha^\beta(x,y,s)\Pi_\alpha(s)\,ds, \quad x,y \in \mathbb{R}_+^n,\ x \neq y,$$

with

$$\begin{aligned} K_\alpha^\beta(x,y,s) &= \int_0^1 r^{\frac{\widehat{\beta}-2}{2}} \left(\frac{-\log r}{1-r}\right)^{\frac{\widehat{\beta}-2}{2}} \prod_{i=1}^n H_{\beta_i}\left(\frac{\sqrt{r}x_i - y_i s_i}{\sqrt{1-r}}\right) \frac{e^{-\frac{q_-(\sqrt{r}x,y,s)}{1-r}}}{(1-r)^{n+\widehat{\alpha}+1}}\,dr \\ &= \int_0^1 (1-t)^{\frac{\widehat{\beta}-1}{2}} \left(\frac{-\log(1-t)}{t}\right)^{\frac{\widehat{\beta}-2}{2}} \prod_{i=1}^n H_{\beta_i}\left(\frac{\sqrt{1-t}x_i - y_i s_i}{\sqrt{t}}\right) \\ &\qquad \times \frac{e^{-\frac{q_-(\sqrt{1-t}x,y,s)}{t}}}{t^{n+\widehat{\alpha}+1}} \frac{dt}{\sqrt{1-t}}, \end{aligned}$$

being H_{β_i} the one-dimensional Hermite polynomial of degree β_i, $i = 1,\dots,n$, and for the second equality we have made the change of variables $t = 1-r$. In order to establish that R_α^β is bounded on $L^{p(\cdot)}(\mathbb{R}_+^n, \mu_\alpha)$ we can assume that $c_\beta = 0$.

We define $R_{\alpha,\mathrm{loc}}^\beta$ and $R_{\alpha,\mathrm{glob}}^\beta$ in the usual way. Firstly, we shall prove the $L^{p(\cdot)}$-boundedness of the global part.

Taking into account that $|\sqrt{1-t}x_i - y_i s_i| \leq q_-^{\frac{1}{2}}(\sqrt{1-t}x,y,s)$ from (4.3), we get, for every $\varepsilon > 0$,

$$\left|\prod_{i=1}^n H_{\beta_i}\left(\frac{\sqrt{1-t}x_i - y_i s_i}{\sqrt{t}}\right)\right| \leq C \sum_{k=0}^{\widehat{\beta}} \left(\frac{q_-^{\frac{1}{2}}(\sqrt{1-t}x,y,s)}{\sqrt{t}}\right)^k \leq C e^{\varepsilon \frac{q_-(\sqrt{1-t}x,y,x)}{t}}.$$

Also, since the function $t \mapsto (1-t)^{\frac{\widehat{\beta}-1}{2}} \left(-\frac{\log(1-t)}{t}\right)^{\frac{\widehat{\beta}-2}{2}}$ is bounded on $[0,1]$, we have

$$\left|R_{\alpha,\mathrm{glob}}^\beta f(x)\right| \leq C|f(x) + C \int_{\mathbb{R}_+^n} |f(y)| \int_{(-1,1)^n} K_\alpha(x,y,s)\,\Pi_\alpha(s)\,ds\,d\mathfrak{m}_\alpha(y),$$

for $x \in \mathbb{R}_+^n$, being

$$K_\alpha(x,y,s) = \int_0^1 \frac{e^{-(1-\varepsilon)\frac{q_-(\sqrt{1-t}x,y,s)}{t}}}{t^{n+\widehat{\alpha}+1}} \frac{dt}{\sqrt{1-t}}(1 - \varphi(x,y,s))$$

for $y \in \mathbb{R}_+^n$ and $s \in (-1,1)^n$.

We can see that the above kernel is, in turn, bounded by the kernel $H_{\alpha,\varepsilon}(x,y,s)$ given in (3.1) provided that $\varepsilon < \frac{1}{n+\widehat{\alpha}}$. When $\sum_{i=1}^n x_i y_i s_i > 0$ we follow closely the estimates obtained by S. Pérez in [35], taking into account that in this case, for $0 < \varepsilon < \frac{1}{n+\widehat{\alpha}}$,

$$K_\alpha(x,y,s) \leq C_\varepsilon \frac{e^{-(1-\varepsilon)u_0}}{t_0^{n+\widehat{\alpha}}}$$

with $u_0 = \frac{|y|^2 - |x|^2 + \sqrt{q_+(x,y,s)q_-(x,y,s)}}{2}$ and $t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$, being $a = |x|^2 + |y|^2$ and $b = 2\sum_{i=1}^n x_i y_i s_i$.

Indeed, by calling $u(t) = \frac{q_-(\sqrt{1-t}x,y,s)}{t}$, notice that u is the one given in (4.1) at the previous section. We have already proved that, for $b > 0$,

$$\sup_{0<t<1} \frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}} \sim \frac{e^{-u_0}}{t_0^{n+\widehat{\alpha}}}.$$

Thus, for $\nu = \frac{1}{n+\widehat{\alpha}} - \varepsilon > 0$ we have

$$\begin{aligned} K_\alpha(x,y,s) &= \int_0^1 e^{\varepsilon u(t)} \left(\frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}}\right)^{\frac{n+\widehat{\alpha}-1}{n+\widehat{\alpha}}} \left(\frac{e^{-u(t)}}{t^{n+\widehat{\alpha}}}\right)^{\frac{1}{n+\widehat{\alpha}}} \frac{dt}{t\sqrt{1-t}} \\ &\le C \left(\frac{e^{-u_0}}{t_0^{n+\widehat{\alpha}}}\right)^{1-\frac{1}{n+\widehat{\alpha}}} \int_0^1 e^{-\nu u(t)} \frac{dt}{t^2\sqrt{1-t}}. \end{aligned}$$

By performing the change of variable $s = u(t) - u_0$ and following the calculations made in [35, p. 499], the latter expression is bounded by

$$\frac{e^{-(1-\frac{1}{n+\widehat{\alpha}})u_0} e^{-\nu u_0}}{t_0^{n+\widehat{\alpha}-1}} \frac{1}{t_0 \sqrt[4]{(a-b)(a+b)}} \int_0^\infty e^{-\nu s}\left(1 + \frac{1}{\sqrt{s}}\right) ds.$$

Moreover, recalling that $(a-b)(a+b) = q_-(x,y,s)q_+(x,y,s) \ge c$ when $b > 0$ (see [21, p. 264]) we get the estimate claimed above.

For the case $b \le 0$, we have that $\frac{a}{t} - |x|^2 \le u(t) = \frac{q_-(\sqrt{1-t}x,y,s)}{t}$ like in [35, p. 500]. After making the change of variables $a\left(\frac{1}{t}-1\right) = s$ and performing the integration taking into account that on the global part $a \ge c$, we get $K_\alpha(x,y,s) \le Ce^{-(1-\varepsilon)|y|^2}$.

Therefore, $K_\alpha(x,y,s) \le CH_{\alpha,\varepsilon}(x,y,s)$ for $0 < \varepsilon < \frac{1}{n+\widehat{\alpha}}$. From Proposition 3.1 we deduce that the operator $R^\beta_{\alpha,\text{glob}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$ by choosing $0 < \varepsilon < \frac{1}{n+\widehat{\alpha}} \wedge \frac{1}{(p^-)'}$.

According to [33, p. 699] R^β_α is bounded on $L^2(\mathbb{R}^n_+, \mu_\alpha)$. Since, as we have just proved $R^\beta_{\alpha,\text{glob}}$ is bounded on $L^2(\mathbb{R}^n_+, \mu_\alpha)$, $R^\beta_{\alpha,\text{loc}}$ is also bounded on $L^2(\mathbb{R}^n_+, \mu_\alpha)$. By proceeding as in [41, Lemma 3.3] and [7, Lemma 3.1] (see also [21, Proposition 6 and Lemma 7]) we can see that the integral kernel of $R^\beta_{\alpha,\text{loc}}$ is a Calderón–Zygmund kernel with respect to $\mathfrak{m}_\alpha$. The procedure developed in Section 2 leads to see that $R^\beta_{\alpha,\text{loc}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$ and with this we finish the proof of this result.

6. Proof of Theorem 1.1 for Littlewood–Paley functions

In this section we prove Theorem 1.1 for Littlewood–Paley functions $g^{\beta,k}_\alpha$, with $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$ such that $k + \widehat{\beta} > 0$.

Let $k \in \mathbb{N}$, $k \ge 1$. We recall that $g^k_\alpha = g^{\mathbf{0},k}_\alpha$, i.e.

$$g^k_\alpha(f)(x) = \left(\int_0^\infty \left|t^k \partial^k_t P^\alpha_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n_+,$$

where

$$P^\alpha_t(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W^\alpha_u(f)(x) du, \quad x \in \mathbb{R}^n_+, t > 0.$$

We define

$$P^\alpha_{t,\text{loc}}(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W^\alpha_{u,\text{loc}}(f)(x) du, \quad x \in \mathbb{R}^n_+, t > 0.$$

and

$$P^\alpha_{t,\text{glob}}(f)(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W^\alpha_{u,\text{glob}}(f)(x) du, \quad x \in \mathbb{R}^n_+, t > 0.$$

and consider

$$g^k_{\alpha,\text{loc}}(f)(x) = \left(\int_0^\infty \left|t^k \partial^k_t P^\alpha_{t,\text{loc}}(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n_+,$$

and

$$g^k_{\alpha,\text{glob}}(f)(x) = \left(\int_0^\infty \left| t^k \partial_t^k P^\alpha_{t,\text{glob}}(f)(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n_+.$$

We firstly prove that $g^k_{\alpha,\text{glob}}$ defines a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n_+, \mu_\alpha)$.

By using Minkowski inequality we get

$$g^k_{\alpha,\text{glob}}(f)(x) \leq \int_{\mathbb{R}^n_+} |f(y)| \left(\int_0^\infty \left| t^k \partial_t^k P^\alpha_{t,\text{glob}}(x,y) \right|^2 \frac{dt}{t} \right)^{1/2} d\mu_\alpha(y),$$

for $x \in \mathbb{R}^n_+$, where

$$P^\alpha_{t,\text{glob}}(x,y) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}} W^\alpha_{u,\text{glob}}(x,y) du \quad x, y \in \mathbb{R}^n_+, t > 0.$$

We have that

$$\begin{aligned}
t^k \partial_t^k P^\alpha_{t,\text{glob}}(x,y) &= t^k \partial_t^k \left[\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-v}}{\sqrt{v}} W^\alpha_{\frac{t^2}{4v},\text{glob}}(x,y) dv \right] \\
&= t^k \partial_t^{k-1} \left[\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-v}}{\sqrt{v}} \partial_t W^\alpha_{\frac{t^2}{4v},\text{glob}}(x,y) dv \right] \\
&= t^k \partial_t^{k-1} \left[\frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-v}}{v^{3/2}} \left[\partial_z W^\alpha_{z,\text{glob}}(x,y) \right]_{z=\frac{t^2}{4v}} dv \right] \\
&= t^k \partial_t^{k-1} \left[\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4z}}}{\sqrt{z}} \partial_z W^\alpha_{z,\text{glob}}(x,y) dz \right] \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty t^k \partial_t^{k-1} \left[e^{-\frac{t^2}{4z}} \right] \partial_z W^\alpha_{z,\text{glob}}(x,y) \frac{dz}{\sqrt{z}},
\end{aligned}$$

for $x, y \in \mathbb{R}^n_+$ and $t > 0$.

By using Minkowski inequality and [5, Lemma 3] we get

$$\begin{aligned}
&\| t^k \partial_t^k P^\alpha_{t,\text{glob}}(x,y) \|_{L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)} \\
&\quad \leq \frac{1}{\sqrt{\pi}} \int_0^\infty |\partial_z W^\alpha_{z,\text{glob}}(x,y)| \left(\int_0^\infty \left| t^k \partial_t^{k-1} \left[e^{-\frac{t^2}{4z}} \right] \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \frac{dz}{\sqrt{z}} \\
&\quad \leq C \int_0^\infty |\partial_z W^\alpha_{z,\text{glob}}(x,y)| \left(\int_0^\infty \frac{e^{-c\frac{t^2}{z}}}{z^{k-1}} t^{2k-1} dt \right)^{\frac{1}{2}} \frac{dz}{\sqrt{z}} \\
&\quad \leq C \int_0^\infty |\partial_z W^\alpha_{z,\text{glob}}(x,y)| dz \quad x,\ y \in \mathbb{R}^n_+.
\end{aligned}$$

We recall that

$$W^\alpha_{z,\text{glob}}(x,y) = \frac{1}{(1-e^{-z})^{\widehat{\alpha}+n}} \int_{(-1,1)^n} e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}} + |y|^2} (1 - \varphi(x,y,s)) \Pi_\alpha(s) ds,$$

for $x, y \in \mathbb{R}^n_+$ and $z > 0$. Then,

$$\partial_z W^\alpha_{z,\text{glob}}(x,y) = e^{|y|^2} \int_{(-1,1)^n} \partial_z \left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}} \right] (1 - \varphi(x,y,s)) \Pi_\alpha(s) ds,$$

for $x, y \in \mathbb{R}^n_+$ and $z > 0$.

We obtain

$$\begin{aligned}&\left\|t^k\partial_t^k P^\alpha_{t,\mathrm{glob}}(x,y)\right\|_{L^2\left(\mathbb{R}_+,\frac{dt}{t}\right)}\\&\quad\leq Ce^{|y|^2}\int_{(-1,1)^n}\int_0^\infty\left|\partial_z\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dz(1-\varphi(x,y,s))\Pi_\alpha(s)ds,\end{aligned}$$

We have that

$$\partial_z\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]=\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}P_{x,y,s}\left(e^{-z/2}\right),$$

for $x,y\in\mathbb{R}^n_+$ and $s\in(-1,1)^n$, where, for every $x,y\in\mathbb{R}^n_+$ and $s\in(-1,1)^n$, $P_{x,y,s}$ is a polynomial whose degree is at most 4. Then, for every $x,y\in\mathbb{R}^n_+$ and $s\in(-1,1)$, the sign of $P_{x,y,s}$ changes at most four times. We obtain

$$\int_0^\infty\left|\partial_z\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dz\leq C\sup_{z\in\mathbb{R}_+}\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}=\sup_{0<t<1}\frac{e^{-\frac{q_-(\sqrt{1-t}x,y,s)}{t}}}{t^{n+\widehat{\alpha}}},$$

for $x,y\in\mathbb{R}^n_+$ and $s\in(-1,1)^n$.

This estimate allows us to reduce the analysis of the global operator $g^k_{\alpha,\mathrm{glob}}$ to the operator considered when we studied the operator $W^\alpha_{*,\mathrm{glob}}$ in Section 4. Thus, we conclude that the operator $g^k_{\alpha,\mathrm{glob}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$.

We now study the operator $g^k_{\alpha,\mathrm{loc}}$. We will use vector valued Calderón–Zygmund theory. In order to have the measurability of the Banach valued functions that appear we are going to consider, for every $N\in\mathbb{N}$, $N\geq 1$, the Banach space $B_N=L^2\left((1/N,N),\frac{dt}{t}\right)$ and in the last step we pass to the limit as N goes to infinity instead of working with the Banach space $L^2(\mathbb{R}_+,\frac{dt}{t})$. Let $N\in\mathbb{N}$, $N\geq 1$. We define the operator

$$G^k_{\alpha,\mathrm{loc}}(f)(x,t)=t^k\partial_t^k P^\alpha_{t,\mathrm{loc}}(f)(x),\quad x\in\mathbb{R}_+,t>0.$$

The integral kernel of $G^k_{\alpha,\mathrm{loc}}$ with respect to $d\mathfrak{m}_\alpha$ is the following

$$M^k_{\alpha,\mathrm{loc}}(x,y,t)=t^k\partial_t^k P^\alpha_{t,\mathrm{loc}}(x,y)e^{-|y|^2},\quad x,y\in\mathbb{R}^n_+,t>0.$$

Since the Poisson semigroup is a Stein symmetric diffusion semigroup, the function g^k_α is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$. In the first part of this proof we establish that $g^k_{\alpha,\mathrm{glob}}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$. Thus, there exists $C>0$ that does not depend on N such that

$$\|G^k_{\alpha,\mathrm{loc}}(f)\|_{L^2_{B_N}(\mathbb{R}^n_+,\mu_\alpha)}\leq C\|f\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)}.$$

By using Minkowski inequality, [5, Lemma 4] and [7, Lemma 3.1] (see also [21, Proposition 6 and Lemma 7]) as above, we get

$$\begin{aligned}
&\|M^k_{\alpha,\mathrm{loc}}(x,y,t)\|_{L^2\left(\mathbb{R}^n_+,\frac{dt}{t}\right)}\\
&\quad\le C\int_{(-1,1)^n}|\varphi(x,y,s)\Pi_\alpha(s)\int_0^\infty\left|\partial_z\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dzds\\
&\quad\le C\int_{(-1,1)^n}|\varphi(x,y,s)\Pi_\alpha(s)\sup_{z\in\mathbb{R}_+}\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}dzds\\
&\quad\le C\int_{(-1,1)^n}\frac{\Pi_\alpha(s)}{q_-(x,y,s)^{\widehat{\alpha}+n}}ds\\
&\quad\le \frac{C}{\mathfrak{m}_\alpha(B(x,|y-x|))},\quad x,y\in\mathbb{R}^n_+,\ x\neq y.
\end{aligned}$$

Let $j=1,\dots,n$. By proceeding in a similar way we can see that

$$\begin{aligned}
&\|\partial_{x_j}M^k_{\alpha,\mathrm{loc}}(x,y,t)\|_{L^2\left(\mathbb{R}^n_+,\frac{dt}{t}\right)}\\
&\quad\le C\int_{(-1,1)^n}|\varphi(x,y,s)\Pi_\alpha(s)\int_0^\infty\left|\partial_z\partial_{x_j}\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dzds\\
&\qquad+C\int_{(-1,1)^n}|\partial_{x_j}\varphi(x,y,s)\Pi_\alpha(s)\int_0^\infty\left|\partial_z\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dzds\\
&\quad\le C\int_{(-1,1)^n}|\varphi(x,y,s)\Pi_\alpha(s)\sup_{z\in\mathbb{R}_+}\left|\partial_{x_j}\left[\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right]\right|dzds\\
&\qquad+C\int_{(-1,1)^n}|\partial_{x_j}\varphi(x,y,s)\Pi_\alpha(s)\sup_{z\in\mathbb{R}_+}\left|\frac{e^{-\frac{q_-(e^{-z/2}x,y,s)}{1-e^{-z}}}}{(1-e^{-z})^{\widehat{\alpha}+n}}\right|dzds\\
&\quad\le C\int_{(-1,1)^n}\frac{\Pi_\alpha(s)}{q_-(x,y,s)^{\widehat{\alpha}+n+1/2}}ds\\
&\quad\le \frac{C}{|x-y|\mathfrak{m}_\alpha(B(x,|y-x|))},\quad x,y\in\mathbb{R}^n_+,\ x\neq y.
\end{aligned}$$

Hence,

$$\|M^k_{\alpha,\mathrm{loc}}(x,y)\|_{B_N}\le\frac{C}{\mathfrak{m}_\alpha(B(x,|x-y|))}\quad x,y\in\mathbb{R}^n_+,\ x\neq y,\tag{6.1}$$

and

$$\begin{aligned}
&\sum_{j=1}^n\left(\|\partial_{x_j}M^k_{\alpha,\mathrm{loc}}(x,y)\|_{B_N}+\|\partial_{y_j}M^k_{\alpha,\mathrm{loc}}(x,y)\|_{B_N}\right)\\
&\qquad\qquad\le\frac{C}{|x-y|\mathfrak{m}_\alpha(B(x,|x-y|))}\quad x,y\in\mathbb{R}^n_+,\ x\neq y,
\end{aligned}$$

where $C>0$ does not depend on N. Suppose that $h\in B_N$ and g is a smooth function with compact support in $\mathbb{R}^n_+$. By using (6.1) we deduce that

$$\begin{aligned}\int_{1/N}^N h(t)G^k_{\alpha,\mathrm{loc}}(g)(x,t)\frac{dt}{t} &= \int_{\mathbb{R}^n_+} g(y)\int_{1/N}^N t^k\partial_t^k P^\alpha_{t,\mathrm{loc}}(x,y)h(t)\frac{dt}{t}d\mathfrak{m}_\alpha(y)\\ &= \int_{1/N}^N h(t)\left[\int_{\mathbb{R}^n_+} K(x,y)g(y)d\mathfrak{m}_\alpha(y)\right](t)\frac{dt}{t},\end{aligned}$$

for $x\notin \operatorname{supp}(f)$, where, for every $x,y\in\mathbb{R}^n_+$, $x\neq y$,

$$[K(x,y)](t)=t^k\partial_t^kP^\alpha_{t,\mathrm{loc}}(x,y),\quad \text{a.e. } t\in(1/N,N),$$

and the integral in the last line is understood in the B_N-Bochner sense. Note that, for every $x\in\mathbb{R}^n_+$, the function Φ_x defined by $\Phi_x(y)=K(x,y)g(y)$, $y\in\mathbb{R}^n_+$, is strongly measurable from $\mathbb{R}^n_+$ into B_N. Indeed, let $x\in\mathbb{R}^n_+$. Since Φ_x is continuous, Φ_x is weakly measurable. By taking into account that B_N is a separable Banach space, Petti's Theorem ([44, p. 131]) allows us to conclude that Φ_x is strongly measurable.

Thus, for every $x\notin\operatorname{supp}(f)$,

$$G^k_{\alpha,\mathrm{loc}}(f)(x,t)=\left[\int_0^\infty K(x,y)f(y)d\mathfrak{m}_\alpha(y)\right](t),$$

in $L^2\left((1/N,N),\frac{dt}{t}\right)$.

The arguments explained in Section 2 allow us to conclude that there exists $C>0$ such that, for every $N\in\mathbb{N}$, $N\geq 1$,

$$\left\|\left\|G^k_{\alpha,\mathrm{loc}}(f)\right\|_{L^2\left((1/N,N),\frac{dt}{t}\right)}\right\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}\leq C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)},$$

for $f\in L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$,

By using the monotone convergence theorem (see [18, p. 75]) we get

$$\left\|g^k_{\alpha,\mathrm{loc}}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)}\leq C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)},$$

for $f\in L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$, and the proof of our result is finished.

Let us consider now the Littlewood–Paley functions including also spatial derivatives. For $\beta\in\mathbb{N}^n\setminus\{(0,\dots,0)\}$ and $k\in\mathbb{N}$, we consider

$$g^{\beta,k}_\alpha(f)(x)=\left(\int_0^\infty\left|t^{k+\widehat{\beta}}\partial_t^kD_x^\beta P^\alpha_t(f)(x)\right|^2\frac{dt}{t}\right)^{1/2},\ x\in\mathbb{R}^n_+.$$

We define the local and global part of $g^{\beta,k}_\alpha$ as follows

$$g^{\beta,k}_{\alpha,\mathrm{loc}}(f)(x)=\left(\int_0^\infty\left|t^{k+\widehat{\beta}}\partial_t^kP^{\alpha,\beta}_{t,\mathrm{loc}}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2},\ x\in\mathbb{R}^n_+.$$

and

$$g^{\beta,k}_{\alpha,\mathrm{glob}}(f)(x)=\left(\int_0^\infty\left|t^{k+\widehat{\beta}}\partial_t^kD_x^\beta P^{\alpha,\beta}_{t,\mathrm{glob}}(f)(x)\right|^2\frac{dt}{t}\right)^{1/2},\ x\in\mathbb{R}^n_+.$$

where

$$P^{\alpha,\beta}_{t,\mathrm{loc}}(f)(x)=\frac{t}{2\sqrt{\pi}}\int_0^\infty\frac{e^{-\frac{t^2}{4u}}}{u^{3/2}}W^{\alpha,\beta}_{u,\mathrm{loc}}(f)(x)dx,\ x\in\mathbb{R}^n_+,t>0,$$

and

$$P^{\alpha,\beta}_{t,\mathrm{glob}}(f)(x)=\frac{t}{2\sqrt{\pi}}\int_0^\infty\frac{e^{-\frac{t^2}{4u}}}{u^{3/2}}W^{\alpha,\beta}_{u,\mathrm{glob}}(f)(x)dx,\ x\in\mathbb{R}^n_+,\ t>0.$$

Here,

$$W_u^{\alpha,\beta}(f)(x) = \int_{\mathbb{R}_+^n} D_x^\beta W_u^\alpha(x,t) f(y) d\mu_\alpha(y),\ x \in \mathbb{R}_+^n, u > 0,$$

and $W_{u,\text{loc}}^{\alpha,\beta}$ and $W_{u,\text{glob}}^{\alpha,\beta}$ are defined in the usual way.

By using Minkowski inequality and [5, Lemma 4] we obtain

$$\begin{aligned}
g_{\alpha,\text{glob}}^{\beta,k}(f)(x) \leq C &\int_{\mathbb{R}_+^n} |f(y)|(1-\varphi(x,y)) \\
&\times \left(\int_0^\infty \left| t^{k+\widehat{\beta}} \partial_t^k \left[\int_0^\infty \frac{te^{-\frac{t^2}{4u}}}{u^{3/2}} D_x^\beta W_u^\alpha(x,y) du \right] \right|^2 \frac{dt}{t} \right)^{1/2} d\mu_\alpha(y) \\
\leq C &\int_{\mathbb{R}_+^n} |f(y)|(1-\varphi(x,y)) \\
&\times \int_0^\infty \left(\int_0^\infty |t^{k+\widehat{\beta}} \partial_t^k (te^{-\frac{t^2}{4u}})| \frac{dt}{t} \right)^{1/2} |D_x^\beta W_u^\alpha(x,y)| \frac{du}{u^{\frac{3}{2}}} d\mu_\alpha(y) \\
\leq C &\int_{\mathbb{R}_+^n} |f(y)|(1-\varphi(x,y)) \int_0^\infty u^{\widehat{\beta}/2-1} |D_x^\beta W_u(x,y)| du d\mu_\alpha(y),
\end{aligned}$$

for $x \in \mathbb{R}_+^n$. From now on we follow the same steps we have done for the higher order Riesz-Laguerre transforms restricted to the global part in order to get the $L^{p(\cdot)}$-boundedness of this operator too, taking into account the representation given in (1.4).

In order to study the local operator $g_{\alpha,\text{loc}}^{\beta,k}$ we use the vector valued Calderón–Zygmund theory. We consider the operator $G_{\alpha,\text{loc}}^{\beta,k}$ defined by

$$G_{\alpha,\text{loc}}^{\beta,k}(f)(x,t) = t^{k+\widehat{\beta}} \partial_t^k P_{t,\text{loc}}^{\alpha,\beta}(f)(x),\ x \in \mathbb{R}_+^n, t > 0.$$

The integral kernel $M_{\alpha,\text{loc}}^{\beta,k}$ of the above operator with respect to $\mathfrak{m}_\alpha$ can be written as follows

$$M_{\alpha,\text{loc}}^{\beta,k}(x,y,t) = \int_{(-1,1)^n} \varphi(x,y) M_\alpha^{\beta,k}(x,y,t,s) \Pi_\alpha(s) ds,\ x,\ y \in \mathbb{R}_+^n, , t > 0,$$

where

$$M_\alpha^{\beta,k}(x,y,t,s) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t^{k+\widehat{\beta}} \partial_t^k \left[te^{-\frac{t^2}{4u}} \right]}{u^{3/2}(1-e^{-u})^{n+\widehat{\alpha}}} D_x^\beta \left[e^{-\frac{q_-(e^{-u/2}x,y,s)}{1-e^{-u}+|y|^2}} \right] du,$$

for $x, y \in \mathbb{R}_+^n$, $t > 0$ and $s \in (-1,1)^n$. By using Minkowski inequality and [5, Lemma 4], according to [21, (2.3)], we deduce that, for every $x, y \in \mathbb{R}_+^n$ and $s \in (-1,1)^n$,

$$\begin{aligned}
&\|M_\alpha^{\beta,k}(x,y,t,s)\|_{L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)} \\
&\leq C \int_0^\infty \frac{\left\| t^{k+\widehat{\beta}} \partial_t^k \left[te^{-\frac{t^2}{4u}} \right] \right\|_{L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)}}{u^{3/2}(1-e^{-u})^{n+\widehat{\alpha}}} \left| D_x^\beta \left[e^{-\frac{q_-(e^{-u/2}x,y,s)}{1-e^{-u}+|y|^2}} \right] \right| du \\
&\leq C \int_0^\infty \frac{u^{\widehat{\beta}/2-1}}{u^{3/2}(1-e^{-u})^{n+\widehat{\alpha}}} \left| D_x^\beta \left[e^{-\frac{q_-(e^{-u/2}x,y,s)}{1-e^{-u}+|y|^2}} \right] \right| du \\
&\leq C \int_0^1 \sqrt{r}^{\widehat{\beta}-2} \left(-\frac{\log(r)}{1-r} \right)^{\frac{\widehat{\beta}-2}{2}} \prod_{i=1}^n \left| H_{\beta_i} \left(\frac{\sqrt{r}x_i - y_i s_i}{\sqrt{1-r}} \right) \right| \frac{e^{-\frac{q_-(rx,y,s)}{1-r}}}{(1-r)^{n+\widehat{\alpha}+1}} dr,
\end{aligned}$$

where we recall that, for every $j \in \mathbb{N}$, H_j denotes the one-dimensional Hermite polynomial of degree j.

As in the Riesz transform $R^{\beta}_{\alpha,\mathrm{loc}}$ case (Section 5), we obtain that

$$\left\| M^{\beta,k}_{\alpha,\mathrm{loc}}(x,y,\cdot) \right\|_{L^2\left(\mathbb{R}_+,\frac{dt}{t}\right)} \leq \frac{C}{\mathfrak{m}_\alpha(B(x,|x-y|))},\ x,\, y \in \mathbb{R}^n_+,\ x \neq y.$$

In a similar way we can see that

$$\sum_{i=1}^{n} \left(\left\| \partial_{x_i} M^{\beta,k}_{\alpha,\mathrm{loc}}(x,y,\cdot) \right\|_{L^2\left(\mathbb{R}_+,\frac{dt}{t}\right)} + \left\| \partial_{y_i} M^{\beta,k}_{\alpha,\mathrm{loc}}(x,y,\cdot) \right\|_{L^2\left(\mathbb{R}_+,\frac{dt}{t}\right)} \right) \leq \frac{C}{|x-y|\mathfrak{m}_\alpha(B(x,|x-y|))},$$

for $x, y \in \mathbb{R}^n_+$, $x \neq y$.

By proceeding as in the first part of the proof when $\beta = 0$, we can prove that the local operator $g^{\beta,k}_{\alpha,\mathrm{loc}}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n_+,\mu_\alpha)$ when we show that $g^{\beta,k}_{\alpha,\mathrm{loc}}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$.

We are going to see that $g^{\beta,k}_{\alpha,\mathrm{loc}}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$ by proving that $g^{\beta,k}_{\alpha}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$. Then, since we have proved that $g^{\beta,k}_{\alpha,\mathrm{glob}}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$, we conclude that $g^{\beta,k}_{\alpha,\mathrm{loc}}$ is bounded on $L^2(\mathbb{R}^n_+,\mu_\alpha)$.

According to [33, p. 699], and by performing a change of variables we obtain that

$$D^{\beta}_x \mathcal{L}^{\alpha}_r(x) = \sum_{(m,\ell)\in\mathcal{A}(\beta)} C^{\beta,\alpha}_{m,\ell}(r) \left(\prod_{i=1}^{n} x_i^{\beta_i - m_i} \right) \mathcal{L}^{\alpha+\beta-m}_{r-\beta+m+\ell}(x),$$

for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n_+$ and $r \in \mathbb{N}^n$, with

$$\mathcal{A}(\beta) = \left\{ (m,\ell) \in \mathbb{N}\times\mathbb{N}^n : 0 \leq m_j \leq \beta_j, 0 \leq \ell_j \leq \frac{\beta_j - m_j}{2}, j = 1,\ldots,n \right\}.$$

Furthermore, for every $(m,\ell) \in \mathcal{A}(\beta)$ and $k \in \mathbb{N}^n$, $C^{\beta,\alpha}_{(m,\ell)} \in \mathbb{R}$ and

$$\left| C^{\beta,\alpha}_{(m,\ell)} \right| \left\| \left(\prod_{i=1}^{n} x_i^{\beta_i - m_i} \right) \mathcal{L}^{\alpha+\beta-m}_{k-\beta+m+\ell} \right\|_{L^2(\mathbb{R}_+,\mu_\alpha)} \leq C_\beta \lambda_k^{\widehat{\beta}/2}. \tag{6.2}$$

Suppose that $f = \sum_{r\in\Lambda} w_r \mathcal{L}^{\alpha}_r$, where Λ is a finite subset of $\mathbb{N}^n$ and $w_r \in \mathbb{C}$ for $r \in \Lambda$. Since for every $(m,\ell) \in \mathcal{A}(\beta)$, the system

$$\left\{ \left(\prod_{i=1}^{n} x_i^{\beta_i - m_i} \right) \mathcal{L}^{\alpha+\beta-m}_{r-\beta+m+\ell} \right\}_{k\in\Lambda_{m,\ell}}$$

is orthogonal with respect to μ_α, where $\Lambda_{m,\ell} = \{k \in \mathbb{N}^n : k_j - \beta_j + m_j + \ell_j \geq 0,\ j = 1,\ldots,n\}$, Bessel inequality leads, by using (6.2), to

$$\begin{aligned} \|g^{\beta,k}_{\alpha}(f)\|^2_{L^2(\mathbb{R}^n_+,\mu_\alpha)} &= \int_0^\infty t^{2(k+\widehat{\beta})-1} \int_{\mathbb{R}^n_+} \left| \sum_{r\in\Lambda} \lambda_r^{k/2} e^{-\sqrt{\lambda_r}t} c_r D^{\beta}_x \mathcal{L}^{\alpha}_r(x) \right|^2 d\mu_\alpha(x)dt \\ &\leq C \int_0^\infty t^{2(k+\widehat{\beta})-1} \sum_{r\in\Lambda} |c_r|^2 e^{-2\sqrt{\lambda_r}t} \lambda_r^{k+\widehat{\beta}} dt \\ &\leq C \sum_{r\in\Lambda} |c_r|^2 = C\|f\|^2_{L^2(\mathbb{R}^n_+,\mu_\alpha)}. \end{aligned}$$

Suppose now that $f \in L^2(\mathbb{R}^n_+,\mu_\alpha)$. For every $m \in \mathbb{N}$, we define

$$f_m = \sum_{\gamma\in\mathbb{N}^n,\widehat{\gamma}\leq m} c^{\alpha}_{\gamma}(f)\mathcal{L}^{\alpha}_{\gamma}.$$

We have that $f_m \to f$, as $m \to \infty$, in $L^2(\mathbb{R}^n_+, \mu_\alpha)$.

It follows that

$$\begin{aligned}|D_x^\beta W_t^\alpha(x,y)| &\leq \frac{C}{(1-e^{-t})^{n+\widehat{\alpha}}} \int_{(-1,1)^n} \left| \partial_x^\beta e^{-\frac{q_-(e^{-t/2}x,y,s)}{1-e^{-t}}+|y|^2} \right| \prod_{i=1}^n (1-s_i^2)^{\alpha_i-1/2} ds \\ &\leq C \frac{e^{-t/2}}{(1-e^{-t})^r} V(|x|,|y|), \quad x,y \in \mathbb{R}^n_+, t>0,\end{aligned}$$

where $r \geq n + \widehat{\alpha}$ and V is a polynomial with positive coefficients. By using [5, Lemma 4] we get

$$\begin{aligned}\left|\partial_t^k D_x^\beta P_t^\alpha(x,y)\right| &\leq C \int_0^\infty \frac{\left|\partial_t^k \left[t e^{-\frac{t^2}{4u}} \right]\right|}{u^{3/2}} |D_x^\beta W_u^\alpha(x,y)| du \\ &\leq C \int_0^\infty \frac{e^{-\frac{t^2}{8u}} e^{-u/2}}{u^{\frac{k+2}{2}}(1-e^{-u})^r} du V(|x|,|y|) \\ &\leq C \left(1 + \int_0^1 \frac{e^{-\frac{t^2}{8u}}}{u^{\frac{k+2}{2}+r}} du \right) V(|x|,|y|) \\ &\leq C \left(1 + t^{-k-1-2r} \right) V(|x|,|y|), \quad x,y \in \mathbb{R}^n_+, t>0.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\partial_t^k D_x^\beta P_t^\alpha(f_m - f)(x)\right| &\leq C \left(1 + t^{-k-1-2r}\right) \int_{\mathbb{R}^n} |f_m(y) - f(y)| V(|x|,|y|) d\mu_\alpha(y) \\ &\leq C \left(1 + t^{-k-1-2r}\right) \left(\int_{\mathbb{R}^n} V^2(|x|,|y|) d\mu_\alpha(y) \right)^{1/2} \\ &\quad \times \|f_m(y) - f(y)\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)},\end{aligned}$$

for $x, y \in \mathbb{R}^n_+$, $t > 0$ and $m \in \mathbb{N}$.

We deduce that

$$\lim_{m\to\infty} t^{k+\widehat{\beta}} \partial_t^k D_x^\beta P_t^\alpha(f_m)(x) = t^{k+\widehat{\beta}} \partial_t^k D_x^\beta P_t^\alpha(f)(x),$$

for $x \in \mathbb{R}^n_+$ and $t > 0$.

By using Fatou's Lemma twice we get

$$\begin{aligned}g_\alpha^{\beta,k}(f)(x) &= \left(\int_0^\infty \left| t^{k+\widehat{\beta}} \partial_t^k D_x^\beta P_t^\alpha(f)(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \lim_{m\to\infty} \left| t^{k+\widehat{\beta}} \partial_t^k D_x^\beta P_t^\alpha(f_m)(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \liminf_{m\to\infty} g_\alpha^{\beta,k}(f_m)(x), \quad x \in \mathbb{R}^n_+,\end{aligned}$$

and then

$$\begin{aligned}\|g_\alpha^{\beta,k}(f)\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)} &\leq \left(\int_{\mathbb{R}^n_+} \liminf_{m\to\infty} |g_\alpha^{\beta,k}(f_m)(x)|^2 d\mu_\alpha(x) \right)^{1/2} \\ &\leq \liminf_{m\to\infty} \|g_\alpha^{\beta,k}(f_m)\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)} \\ &\leq C \lim_{m\to\infty} \|f_m\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)} \\ &\leq C\|f\|_{L^2(\mathbb{R}^n_+,\mu_\alpha)}.\end{aligned}$$

Thus, we have proved that $g_\alpha^{\beta,k}$ is bounded on $L^2(\mathbb{R}^n_+, \mu_\alpha)$.

7. Proof of Theorem 1.1 for Laplace transform type multipliers

We recall that we have

$$T_m^\alpha(f)(x) = \lim_{\epsilon\to 0^+}\left(f(x)\Lambda(\epsilon) + \int_{|x-y|>\epsilon} K_\phi^\alpha(x,y)f(y)d\mu_\alpha(y)\right), \quad \text{a.e. } x\in\mathbb{R}_+^n,$$

where $\Lambda \in L^\infty(\mathbb{R}_+)$ and

$$K_\phi^\alpha(x,y) = \int_0^\infty \phi(t)\left(-\frac{\partial}{\partial t}\right)W_t^\alpha(x,y)dt, \quad x,y\in\mathbb{R}_+^n,\ x\neq y,$$

being $\phi \in L^\infty(\mathbb{R}_+^n)$ and $m(t) = t\int_0^\infty e^{-zt}\phi(z)dz$, $t\in\mathbb{R}_+$.

We define $T_{m,\mathrm{loc}}^\alpha$, $K_{\phi,\mathrm{loc}}^\alpha$, $T_{m,\mathrm{glob}}^\alpha$ and $K_{\phi,\mathrm{glob}}^\alpha$ in the usual way. We firstly observe that

$$\begin{aligned}|K_{\phi,\mathrm{glob}}^\alpha(x,y)| &\\ &\leq C\int_{(-1,1)^n}\int_0^\infty \left|\partial_t\left[\frac{e^{-\frac{q_-(e^{-t/2}x,y,s)}{1-e^{-t}}}}{(1-e^{-t})^{\widehat{\alpha}+n}}\right]\right| |\phi(t)|dt|1-\varphi(x,y,s)|\Pi_\alpha(s)ds\\ &\leq C\int_{(-1,1)^n}\sup_{t>0}\left|\frac{e^{-\frac{q_-(e^{-t/2}x,y,s)}{1-e^{-t}}}}{(1-e^{-t})^{\widehat{\alpha}+n}}\right||1-\varphi(x,y,s)|\Pi_\alpha(s)ds.\end{aligned}$$

By proceeding as in the proof of Section 4 we conclude that $T_{m,\mathrm{glob}}^\alpha$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)$.

We now define $\mathbb{K}_{\phi,\mathrm{loc}}^\alpha(x,y) = e^{-|y|^2}K_{\phi,\mathrm{loc}}^\alpha(x,y)$ for $x,y\in\mathbb{R}_+^n$. By using [38, Lemma 1] and [7, Lemma 3.1] we can see that $\mathbb{K}_{\phi,\mathrm{loc}}^\alpha(x,y)$ is a scalar Calderón–Zygmund kernel with respect to $\mathfrak{m}_\alpha$. According to [42, Corollary 3, p. 121], the Laguerre multiplier T_m^α is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$. Furthermore, as we have just mentioned $T_{m,\mathrm{glob}}^\alpha$ is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$. Then, $T_{m,\mathrm{loc}}^\alpha$ is bounded on $L^2(\mathbb{R}_+^n,\mu_\alpha)$ and on $L^2(\mathbb{R}_+^n,\mathfrak{m}_\alpha)$.

As it was proved in Section 2, we can conclude that $\mathbb{T}_{m,\mathrm{loc}}^\alpha$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+^n,\mu_\alpha)$ and finish the proof of our result.

Appendix A. Auxiliary results

For the sake of completeness we include in this appendix an n-dimensional version of the Three Lines Theorem in the form it was used in Section 3. Although it can be seen as a particular case of [4, Proposition 21], we believe that this simpler form might be enough in many circumstances.

Theorem A.1. *Let $n\in\mathbb{N}$, $n\geq 1$. Assume that, for every $j=1,\dots,n$, a_j, $b_j\in\mathbb{R}$ and $a_j<b_j$. We define $\tau_n = \{z\in\mathbb{C}^n : a_j\leq \mathrm{Re}(z_j)\leq b_j, j=1,\dots,n\}$ and $\mathcal{F}_n = \{z\in\mathbb{C}^n : \mathrm{Re}(z_j)\in\{a_j,b_j\}, j=1,\dots,n\}$. Suppose that U is an open set containing τ_n and $f:U\to\mathbb{C}$ is holomorphic, bounded in τ_n, and such that $|f(z)|\leq K$ for $z\in\mathcal{F}_n$. Then, $|f(z)|\leq K$ for $z\in\tau_n$.*

Proof. We will proceed by induction on the dimension n. The case $n=1$ corresponds to the classical Three Lines Theorem and we refer to [34, Theorem 3.15].

Suppose the result is true for some $n\in\mathbb{N}$, $n\geq 1$. We consider $a_j<b_j$ for $j=1,\dots,n+1$, $\tau_{n+1} = \{z=(z_1,\dots,z_{n+1})\in\mathbb{C}^{n+1} : a_j\leq\mathrm{Re}(z_j)\leq b_j, j=1,\dots,n+1\}$, $\mathcal{F}_{n+1} = \{z=(z_1,\dots,z_{n+1})\in\mathbb{C}^{n+1} : \mathrm{Re}(z_j)\in\{a_j,b_j\}, j=1,\dots,n+1\}$, an open set U containing τ_{n+1}, and a function $f:U\to\mathbb{C}$, holomorphic in U, bounded on τ_{n+1}, and such that $|f(z)|\leq K$ for $z\in\mathcal{F}_{n+1}$.

Let $t \in \mathbb{R}$. We define, $z_{n+1}(t) = a_{n+1} + it$, and $g_t : U_t \to \mathbb{C}$ such that $g_t(z_1, \dots, z_n) = f(z_1, \dots, z_n, z_{n+1}(t))$, where

$$U_t = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : (z_1, \dots, z_n, z_{n+1}(t)) \in U\}.$$

It is clear that U_t is an open set in $\mathbb{C}^n$ that contains τ_n. The function g_t is holomorphic in U_t and bounded on τ_n, and if $z = (z_1, \dots, z_n) \in \mathcal{F}_n$, since $\mathrm{Re}(z_{n+1}) = a_{n+1}$, $|g_t(z_1, \dots, z_n)| = |f(z_1, \dots, z_n, z_{n+1})| \leq K$. Then, using the inductive hypothesis,

$$|g_t(z_1, \dots, z_n)| = |f(z_1, \dots, z_n, z_{n+1}(t))| \leq K$$

for $z = (z_1, \dots, z_n) \in \tau_n$. Thus, we prove that

$$\text{(A.1)} \quad |f(z_1, \dots, z_{n+1})| \leq K \text{ if } \mathrm{Re}(z_j) \in [a_j, b_j],\ j = 1, \dots, n;\ \mathrm{Re}(z_{n+1}) = a_{n+1}.$$

In a similar way, we can see that

$$\text{(A.2)} \quad |f(z_1, \dots, z_{n+1})| \leq K \text{ if } \mathrm{Re}(z_j) \in [a_j, b_j],\ j = 1, \dots, n;\ \mathrm{Re}(z_{n+1}) = b_{n+1}.$$

Let now $c = (c_1, \dots, c_n) \in \prod_{j=1}^n [a_j, b_j]$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. We consider $\tau_0 = \{z \in \mathbb{C} : \mathrm{Re}(z) \in [a_{n+1}, b_{n+1}]\}$, $\mathcal{F}_0 = \{z \in \mathbb{C} : \mathrm{Re}(z) \in \{a_{n+1}, b_{n+1}\}\}$, and $h_t^c : U_0 \to \mathbb{C}$ such that

$$h_t^c(z) = f(c_1 + it_1, \dots, c_n + it_n, z), \quad z \in U_0,$$

where $U_0 = \{z \in \mathbb{C} : (c_1 + it_1, \dots, c_n + it_n, z) \in U\}$. The set U_0 is open in $\mathbb{C}$ and it contains τ_0. The function h_t^c is holomorphic in U_0 and bounded on τ_0. Furthermore, by (A.1) and (A.2), if $z \in \mathcal{F}_0$,

$$|h_t^c(z)| = |f(c_1 + it_1, \dots, c_n + it_n, z)| \leq K.$$

Therefore, by the one-dimensional case, we deduce that

$$|h_t^c(z)| \leq K, \quad z \in \tau_0.$$

Thus we conclude that

$$|f(z)| \leq K, \quad z \in \tau_{n+1}. \qquad \square$$

Lemma A.2. *Let $p : \mathbb{R}_+^n \to [1, \infty)$ be a measurable function such that $p \in \mathrm{LH}(\mathbb{R}_+^n)$ and take $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $k_j \geq 1$ for each $j = 1, \dots, n$. Consider $\overline{x} = (\overline{x_1}, \dots, \overline{x_n}) \in \mathbb{R}^{\widehat{k}}$ with $\overline{x_j} \in \mathbb{R}^{k_j}$, $j = 1, \dots, n$. We define $\overline{p} : \mathbb{R}^{\widehat{k}} \to [1, \infty)$ by $\overline{p}(\overline{x}) = p(|\overline{x_1}|, \dots, |\overline{x_n}|)$. Then, $\overline{p} \in \mathrm{LH}(\mathbb{R}^{\widehat{k}})$. Moreover, if $1 < p^- \leq p^+ < \infty$, also $1 < \overline{p}^- \leq \overline{p}^+ < \infty$.*

Proof. First, we shall see that $\overline{p}$ belongs to $\mathrm{LH}_0(\mathbb{R}^{\widehat{k}})$, so we take $\overline{x} = (\overline{x_1}, \dots, \overline{x_n})$, $\overline{y} = (\overline{y_1}, \dots, \overline{y_n}) \in \mathbb{R}^{\widehat{k}}$, with $\overline{x_j}, \overline{y_j} \in \mathbb{R}^{k_j}$, $j = 1, \dots, n$, and such that $0 < |\overline{x} - \overline{y}| < \frac{1}{2}$. We have that

$$|(|\overline{x_1}| - |\overline{y_1}|, \dots, |\overline{x_n}| - |\overline{y_n}|)| \leq |\overline{x} - \overline{y}|.$$

Indeed, if we write $\overline{x_j} = (x_1^j, \dots, x_{k_j}^j)$, $\overline{y_j} = (y_1^j, \dots, y_{k_j}^j)$, with $j = 1, \dots, n$, this inequality is a consequence of the Cauchy–Schwarz inequality on $\mathbb{R}^{k_j}$, i.e. $|\langle \overline{x_j}, \overline{y_j} \rangle| \leq |\overline{x_j}||\overline{y_j}|$, $j = 1, \dots, n$.

Since $p \in \mathrm{LH}_0(\mathbb{R}_+^n)$ it follows that

$$\begin{aligned}
|\overline{p}(\overline{x}) - \overline{p}(\overline{y})| &= |p(|\overline{x_1}|, \dots, |\overline{x_n}|) - p(|\overline{y_1}|, \dots, |\overline{y_n}|)| \\
&\leq \frac{C}{-\log(|(|\overline{x_1}| - |\overline{y_1}|, \dots, |\overline{x_n}| - |\overline{y_n}|)|)} \\
&\leq \frac{C}{-\log(|\overline{x} - \overline{y}|)}.
\end{aligned}$$

Thus, $\overline{p} \in \mathrm{LH}_0(\mathbb{R}^{\widehat{k}})$.

On the other hand, since $p \in \mathrm{LH}_\infty(\mathbb{R}^n_+)$ and $|(|\overline{x_1}|, \dots, |\overline{x_n}|)| = |\overline{x}|$,

$$|\overline{p}(\overline{x}) - p_\infty| = |p(|\overline{x_1}|, \dots, |\overline{x_n}|) - p_\infty| \leq \frac{C}{\log(e + |(|\overline{x_1}|, \dots, |\overline{x_n}|)|)} = \frac{C}{\log(e + |\overline{x}|)},$$

so $\overline{p} \in \mathrm{LH}_\infty(\mathbb{R}^{\widehat{k}})$ with $\overline{p}_\infty = p_\infty$.

Therefore, we have proved that $\overline{p} \in \mathrm{LH}(\mathbb{R}^{\widehat{k}})$.

Finally, from the definition of $\overline{p}$, it is clear that $\overline{p}^- = p^-$ and $\overline{p}^+ = p^+$, so $1 < p^- \leq p^+ < \infty$ is equivalent to $1 < \overline{p}^- \leq \overline{p}^+ < \infty$. □

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