The Hochschild Cohomology of the Algebra of Differential Operators Tangent to a Central Arrangement of Lines

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Abstract. Given a central arrangement of lines $\mathcal{A}$ in a 2-dimensional vector space V over a field of characteristic zero, we study the algebra $\mathscr{D}(\mathcal{A})$ of differential operators on V which are logarithmic along $\mathcal{A}$. Among other things we determine the Hochschild cohomology of $\mathscr{D}(\mathcal{A})$ as a Gerstenhaber algebra, establish a connection between that cohomology and the de Rham cohomology of the complement $M(\mathcal{A})$ of the arrangement, determine the isomorphism group of $\mathscr{D}(\mathcal{A})$ and classify the algebras of that form up to isomorphism.

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Contents

1 Introduction

Let us fix a ground field $\Bbbk$ of characteristic zero, a finite-dimensional vector space V and a central arrangement of hyperplanes $\mathcal{A}$ in V. We let S be the algebra of polynomial functions of V, fix a defining polynomial $Q \in S$ for $\mathcal{A}$, and consider, following K. Saito [15], the Lie algebra

$$\operatorname{Der}(\mathcal{A}) = \{\delta \in \operatorname{Der}(S) : \delta(Q) \in QS\}$$

of derivations of S logarithmic with respect to $\mathcal{A}$, which is, geometrically speaking, the Lie algebra of vector fields on V which are tangent to the hyperplanes of $\mathcal{A}$. This Lie algebra is a very interesting invariant of the arrangement and has been the subject of a lot of work — we refer to the book of P. Orlik and H. Terao [12] and the one by A. Dimca [5] for surveys on this subject. In particular, using this Lie algebra we can define an important class of arrangements: we say that an arrangement $\mathcal{A}$ is *free* if $\operatorname{Der}(\mathcal{A})$ is free as a left S-module. For example, central arrangements of lines in the plane are free, as are, according to a beautiful result of Terao [18], the arrangements of reflecting hyperplanes of a finite group generated by pseudo-reflections.

Now, along with $\operatorname{Der}(\mathcal{A})$ we can consider also the associative algebra $\mathscr{D}(\mathcal{A})$ generated inside the algebra $\operatorname{End}_{\Bbbk}(S)$ of linear endomorphisms of the vector space S by $\operatorname{Der}(\mathcal{A})$ and the set of maps given by left multiplication by elements of S: we call it the *algebra of differential operators tangent* to the arrangement $\mathcal{A}$ — in the literature its elements are also called *logarithmic differential operators*, but we prefer the more geometric term. When $\mathcal{A}$ is free, it coincides with the algebra of differential operators on S which preserve the ideal QS of S and all its powers, studied for example by F. J. Calderón-Moreno [3] or by the second author in [17].

The purpose of this paper is to study, from the point of view of noncommutative algebra and homological algebra, this algebra $\mathscr{D}(\mathcal{A})$ in the simplest case of a free arrangement, that of central line arrangements.

Let us describe briefly our results. We thus assume in what follows that $\mathcal{A}$ is a central arrangement of $r+2$ lines in a 2-dimensional vector space V, and for simplicity we suppose that $\mathcal{A}$ has at least five lines, so that $r \geq 3$. We let $Q \in S$ be a defining polynomial for $\mathcal{A}$, that is, a square-free product of linear forms on V with the union of the hyperplanes of $\mathcal{A}$ as zero locus. As S is a subalgebra of $\mathscr{D}(A)$, we view Q as an element of the latter.

Theorem A. *The algebra $\mathscr{D}(\mathcal{A})$ is a noetherian domain, it has global dimension 4 and projective dimension as a bimodule over itself also equal to 4. The Hochschild cohomology $\mathrm{HH}^\bullet(\mathscr{D}(\mathcal{A}))$ of $\mathscr{D}(\mathcal{A})$ has Hilbert series*

$$\sum_{i\geq 0} \dim \mathrm{HH}^i(\mathscr{D}(\mathcal{A}))\cdot t^i = 1 + (r+2)t + (2r+3)t^2 + (r+2)t^3.$$

The algebra $\mathscr{D}(\mathcal{A})$ has Hochschild homology and cyclic homology isomorphic to those of a polynomial algebra $\Bbbk[X]$, and periodic cyclic homology and K-theory isomorphic to that of the ground field $\Bbbk$. It is a twisted Calabi–Yau algebra of dimension 4, the element Q of $\mathscr{D}(A)$ is normal, and the modular automorphism $\sigma : \mathscr{D}(\mathcal{A}) \to \mathscr{D}(\mathcal{A})$ of $\mathscr{D}(\mathcal{A})$ is the unique one such that for all $a \in \mathscr{D}(\mathcal{A})$ one has

$$Qa = \sigma(a)Q.$$

These claims are contained in Propositions 4.7, 6.1, 7.2 and 8.10. In Propositions 5.3 and 5.6 we describe completely the cup product and the Gerstenhaber Lie structure on $\mathrm{HH}^\bullet(\mathscr{D}(\mathcal{A}))$ — we refer to their statements for the precise details, which are technical. The calculations needed in order to do these computations are annoyingly involved.

We obtain a very concrete description of $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ in Proposition 5.2, along with one of its Lie algebra structure in Proposition 5.6:

Theorem B. *Let $Q = \alpha_1 \cdots \alpha_{r+2}$ be a factorization of the defining polynomial as a product of linear factors, so that $\alpha_1, \ldots, \alpha_{r+2}$ are linear polynomials on V whose zero loci are the hyperplanes of $\mathcal{A}$.*

(*i*) *For each $i \in \{1, \ldots, r+2\}$ there is a unique derivation $\partial_i : \mathscr{D}(\mathcal{A}) \to \mathscr{D}(\mathcal{A})$ such that $\partial_i(f) = 0$ for all $f \in S$ and $\partial_i(\delta) = \delta(\alpha_i)/\alpha_i$ for all $\delta \in \mathrm{Der}(\mathcal{A})$.*

(*ii*) *The set of classes of $\partial_1, \ldots, \partial_{r+2}$ in $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$, which we view as the space of outer derivations of the algebra $\mathscr{D}(\mathcal{A})$, is a basis.*

(*iii*) *The Lie algebra $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ is abelian.*

The elements $\partial_1, \ldots, \partial_{r+2}$ are canonically determined and in a natural bijection with the set of hyperplanes. We do not have a description along the same lines of the rest of the cohomology. In Proposition 5.4, though, we do obtain the following piece of information:

Theorem C. *The subalgebra $\mathscr{H}$ of $\mathrm{HH}^\bullet(\mathscr{D}(\mathcal{A}))$ generated by the component $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ of degree 1 is isomorphic to the de Rham cohomology of the complement $M(\mathcal{A})$ of the arrangement. It is generated as a graded-commutative algebra by the $r+2$ elements $\partial_1, \ldots, \partial_{r+2}$ of $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ subject to the relations*

$$\partial_i \smile \partial_j + \partial_j \smile \partial_k + \partial_k \smile \partial_i = 0,$$

one for each choice of three pairwise distinct elements i, j, k of $\{1, \ldots, r+2\}$.

Using our precise description of $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ and the techniques of J. Alev and M. Chamarie [1], we arrive in Section 8 at a description of the automorphism group of the algebra $\mathscr{D}(\mathcal{A})$, stated there as Theorem 8.7. Since the arrangement $\mathcal{A}$ is central, the Lie algebra $\operatorname{Der}(\mathcal{A})$ is a *graded* S-module, and that grading turns $\mathscr{D}(\mathcal{A})$ into a graded algebra: we will use this structure in the following result.

Theorem D. *Let G be the subgroup of $\mathrm{GL}(V)$ of maps which preserve the arrangement $\mathcal{A}$.*

- (*i*) *There is an action of G on a vector space W of dimension $r+2$ such that the semidirect product $G \ltimes W$ is isomorphic to the group $\operatorname{Aut}_0(\mathscr{D}(\mathcal{A}))$ of algebra automorphisms of $\mathscr{D}(\mathcal{A})$ which respect the grading.*
- (*ii*) *An element of $\mathscr{D}(\mathcal{A})$ is locally* ad*-nilpotent if and only if it belongs to S. The set $\operatorname{Exp}(\mathcal{A}) = \{\exp \operatorname{ad}(f) : f \in S\}$ of the automorphisms of $\mathscr{D}(\mathcal{A})$ obtained as exponentials of locally* ad*-nilpotent elements is a subgroup of the full group of automorphisms $\operatorname{Aut}(\mathscr{D}(\mathcal{A}))$.*
- (*iii*) *There is an action of $\operatorname{Aut}_0(\mathscr{D}(\mathcal{A}))$ on $\operatorname{Exp}(\mathcal{A})$ such that there is an isomorphism of groups $\operatorname{Aut}(\mathscr{D}(\mathcal{A})) = \operatorname{Aut}_0(\mathscr{D}(\mathcal{A})) \ltimes \operatorname{Exp}(\mathcal{A})$.*

This knowledge of the automorphism group of $\mathscr{D}(\mathcal{A})$ allows us to describe the set of normal elements of the algebra and its birational class: putting together our Propositions 8.9 and 8.13 we obtain the following result.

Theorem E. *The set of normal elements of $\mathscr{D}(\mathcal{A})$ is the saturated multiplicatively closed subset generated by Q. The maximal normal localization of $\mathscr{D}(\mathcal{A})$, which is therefore $\mathscr{D}(\mathcal{A})[\frac{1}{Q}]$, is isomorphic to the localization of the Weyl algebra $\mathscr{D}(S)[\frac{1}{Q}]$.*

Finally, using —as it is often done— normal elements, we are able to classify in Proposition 8.11 the algebras under study up to isomorphism:

Theorem F. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two central arrangements of lines in V. The algebras $\mathscr{D}(\mathcal{A})$ and $\mathscr{D}(\mathcal{A}')$ are isomorphic if and only if the arrangements $\mathcal{A}$ and $\mathcal{A}'$ themselves are linearly isomorphic.*

This means, essentially, that we can reconstruct the arrangement from the algebra $\mathscr{D}(\mathcal{A})$ of its differential operators.

We expect most of the above results to hold in the general case of a free arrangement of hyperplanes of arbitrary rank. As our computations here make clear, some technology is needed in order to deal with more complicated cases. In future work, we will show how to organize this computation using the language of Lie–Rinehart pairs [14] and their cohomology theory. On the other hand, one can interpret the second cohomology space $\mathrm{HH}^2(\mathscr{D}(\mathcal{A}))$ as classifying infinitesimal deformations of the algebra $\mathscr{D}(\mathcal{A})$ and use $\mathrm{HH}^3(\mathscr{D}(\mathcal{A}))$ and our description of the Gerstenhaber bracket to study the deformation theory of $\mathscr{D}(\mathcal{A})$. This produces a somewhat concrete interpretation of the second cohomology space in geometrical terms. As this involves quite a bit of calculation, we defer the exposition of these results to a later paper.

The paper is organized as follows. We start in Section 2 by giving a concrete realization of the algebra $\mathscr{D}(\mathcal{A})$ as an iterated Ore extension of a polynomial ring and proving some useful lemmas. In Section 3 we construct a resolution for $\mathscr{D}(\mathcal{A})$ and in Sections 4 and 5 we present the computation of the Hochschild cohomology $\mathrm{HH}^\bullet(\mathscr{D}(\mathcal{A}))$ and its Gerstenhaber algebra structure. Section 6 gives the much easier determination of the Hochschild homology, cyclic homology, periodic cyclic homology and K-theory of our algebra, followed by the proof, in Section 7, of the twisted Calabi–Yau property. Finally, in the last section we determine the automorphism group of $\mathscr{D}(\mathcal{A})$ and classify the algebras of this form up to isomorphism.

Some notations. We will use the symbols $\triangleright$ and $\triangleleft$ to denote the left and right actions of an algebra on a bimodule whenever this improves clarity. We will have a ground field $\Bbbk$ of characteristic zero. All vector spaces and algebras are implicitly defined over $\Bbbk$, and unadorned $\otimes$ and Hom are taken with respect to $\Bbbk$. If M is a vector space, we will often denote by

$$\boxed{M}$$

an element of M about which we do not need to be specific. We will use the vertical bar $|$ as a synomym for $\otimes$ whenever horizontal space is scarce.
We refer to the book [12] for a general reference about hyperplane arrangements and their derivations, and to C. Weibel's book [19] for generalities about homological algebra and, in particular, Hochschild, cyclic and periodic theories. Finally, whenever we mention K-theory we refer to both K_0 and higher K-theory.

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2 The algebra of differential operators associated to a central arrangement of lines

2.1. We fix once and for all a ground field $\Bbbk$ of characteristic zero and put $S = \Bbbk[x, y]$. We view S as a graded algebra as usual, with both x and y of degree 1, and for each $p \geq 0$ we write S_p the homogeneous component of S of degree p.
We write $\mathrm{Der}(S)$ for the Lie algebra of $\Bbbk$-linear derivations of S, which is a free left graded S-module, freely generated by the usual partial derivatives ∂_x, $\partial_y : S \to S$, which are homogeneous elements of $\mathrm{Der}(S)$ of degree -1. On the other hand, we write $\mathscr{D}(S)$ the associative algebra of regular differential

operators on S, as defined, for example, in [11, §15.5]. As this is by definition a subalgebra of $\operatorname{End}_{\Bbbk}(S)$, there is a tautological structure of left $\mathscr{D}(S)$-module on S.

There is an injective morphism of algebras $\phi : S \to \mathscr{D}(S)$ such that $\phi(s)(a) = as$ for all choices of s and a in S which we will view as an identification; elements in its image are the differential operators of order zero. Since S is a regular algebra, the algebra $\mathscr{D}(S)$ is generated as a subalgebra of $\operatorname{End}_{\Bbbk}(S)$ by S and $\operatorname{Der}(S)$; see [11, Corollary 15.5.6]. A consequence of this is that $\mathscr{D}(S)$ is generated as an algebra by x, y, ∂_x and ∂_y, and in fact these elements generate it freely subject to the relations

$$[x, y] = [\partial_x, y] = [\partial_y, x] = [\partial_x, \partial_y] = 0, \qquad [\partial_x, x] = [\partial_y, y] = 1.$$

It follows easily from this that $\mathscr{D}(S)$ has a $\mathbb{Z}$-grading with x and y in degree 1 and ∂_x and ∂_y in degree -1, and that with respect to this grading S is a graded $\mathscr{D}(S)$-module.

2.2. We fix an integer $r \geq -1$ and consider a central arrangement $\mathcal{A}$ of $r + 2$ lines in the plane $\mathbb{A}^2$. Up to a change of coordinates, we may assume that the line with equation $x = 0$ is one of the lines in $\mathcal{A}$, so that the defining polynomial Q of the arrangement is of the form xF for some square-free homogeneous polynomial $F \in S$ of degree $r + 1$ which does not have x as a factor. Up to multiplying by a scalar, which does not change anything substantial, we may assume that $F = x\bar{F} + y^{r+1}$ for some $\bar{F} \in S_r$.

We let $\operatorname{Der}(\mathcal{A})$ be the Lie algebra of derivations of S that preserve the arrangement, as in [12, §4.1], so that

$$\operatorname{Der}(\mathcal{A}) = \{\delta \in \operatorname{Der}(S) : \delta(Q) \in QS\}.$$

This a graded Lie subalgebra of $\operatorname{Der}(S)$. The two derivations

$$E = x\partial_x + y\partial_y, \qquad D = F\partial_y$$

are elements of $\operatorname{Der}(\mathcal{A})$ of degrees 0 and r, and it follows immediately from Saito's criterion [12, Theorem 4.19] that the set $\{E, D\}$ is a basis of $\operatorname{Der}(\mathcal{A})$ as a graded S-module; this is the content of Example 4.20 in that book.

The *algebra of differential operators tangent to the arrangement* $\mathcal{A}$ is the subalgebra $\mathscr{D}(\mathcal{A})$ of $\mathscr{D}(S)$ generated by S and $\operatorname{Der}(\mathcal{A})$. It follows immediately from the remarks above that $\mathscr{D}(\mathcal{A})$ is generated by x, y, E and D, and a computation shows that the following commutation relations hold in $\mathscr{D}(\mathcal{A})$:

$$\begin{aligned}
&[y, x] = 0, \\
&[D, x] = 0, && [D, y] = F, \\
&[E, x] = x, && [E, y] = y, && [E, D] = rD.
\end{aligned} \tag{1}$$

Since these generators are homogeneous elements in $\mathscr{D}(S)$ —with E of degree 0, x and y of degree 1 and D of degree r— we see that the algebra $\mathscr{D}(\mathcal{A})$ is a

graded subalgebra of $\mathscr{D}(S)$ and, by restricting the structure from $\mathscr{D}(S)$, that S is a graded $\mathscr{D}(\mathcal{A})$-module.
The set of commutation relations given above is in fact a presentation of the algebra $\mathscr{D}(\mathcal{A})$. More precisely, we have:

Lemma. *The algebra $\mathscr{D}(\mathcal{A})$ is isomorphic to the iterated Ore extension $S[D][E]$. It is a noetherian domain and the set $\{x^i y^j D^k E^l : i, j, k, l \geq 0\}$ is a $\Bbbk$-basis for $\mathscr{D}(\mathcal{A})$.*

Here we view D as a derivation of S, so that we way construct the Ore extension $S[D]$, and view E as a derivation of this last algebra, so as to be able extend once more to obtain $S[D][E]$.

Proof. It is clear at this point that the obvious map $\pi : S[D][E] \to \mathscr{D}(\mathcal{A})$ is a surjective morphism of algebras, so we need only prove that it is injective. To do that, let us suppose that there exists a non-zero element L in $S[D][E]$ whose image under the map π is zero, and suppose that $L = \sum_{i,j\geq 0} f_{i,j} D^i E^j$, with coefficients $f_{i,j} \in S$ for all i, $j \geq 0$, almost all of which are zero. As L is non-zero, we may consider the number $m = \max\{i + j : f_{i,j} \neq 0\}$.
Let us now fix a point $p = (a, b) \in \mathbb{A}^2$ which is not on any line of the arrangement $\mathcal{A}$, so that $aF(a, b) \neq 0$, and let $\mathscr{O}_p$ be the completion of S at the ideal $(x - a, y - b)$ or, more concretely, the algebra of formal series in $x - a$ and $y - b$. We view $\mathscr{O}_p$ as a left module over $\mathscr{D}(S)$ in the tautological way and, by restriction, as a left $\mathscr{D}(\mathcal{A})$-module. There exist formal series ϕ and ψ in $\mathscr{O}_p$ such that

$$E \cdot \phi = 1, \qquad D \cdot \phi = 0, \qquad E \cdot \psi = 0, \qquad D \cdot \psi = x^r.$$

Indeed, we may choose $\phi = \ln x$ to satisfy the first two conditions, and the last two ones are equivalent to the equations

$$\partial_x \psi = -\frac{x^{r-1} y}{F}, \qquad \partial_y \psi = \frac{x^r}{F},$$

which can be solved for ψ, as the usual well-known sufficient integrability condition from elementary calculus holds. If now s, $t \in \mathbb{N}_0$ are such that $s+t = m$, a straightforward computation shows that $L \cdot \phi^s \psi^t = s!t!x^{rt} f_{s,t}$ in $\mathscr{O}_p$, and this implies that $f_{s,t} = 0$. This contradicts the choice of m and this contradiction proves what we want. $\square$

2.3. We will use the following two simple lemmas a few times:

Lemma. *Suppose that $r \geq 2$. If α, $\beta \in S_1$ are such that $\alpha F_x + \beta F_y = 0$, then $\alpha = \beta = 0$.*

The conclusion of this lemma is false if $r < 2$.

Proof. Suppose that F_1, F_2 and F_3 are three distinct linear factors of F (here is where we need the hypothesis that the arrangement has at least 4 lines, so that

$r \geq 2$) so that $F = F_1F_2F_3F'$ for some $F' \in S_{r-2}$; as F has degree at least 3, this is possible. We have $F_x \equiv F_{1x}F_2F_3F'$ and $F_y \equiv F_{1y}F_2F_3F'$ modulo F_1, so that $(\alpha F_{1x} + \beta F_{1y})F_2F_3F' \equiv 0 \mod F_1$. Since F is square free, this tells us that F_1 divides $\alpha F_{1x} + \beta F_{1y}$ and, since both polynomials have the same degree and $F_1 \neq 0$, that there exists a scalar λ such that $\alpha F_{1x} + \beta F_{1y} = \lambda F_1$. Of course, we can do the same with the other two factors F_2 and F_3. We can state this by saying that the matrix $\left(\begin{smallmatrix} \alpha_x & \beta_x \\ \alpha_y & \beta_y \end{smallmatrix}\right)$ has the three vectors $\left(\begin{smallmatrix} F_{1x} \\ F_{1y} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F_{2x} \\ F_{2y} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F_{3x} \\ F_{3y} \end{smallmatrix}\right)$ as eigenvectors. Since no two of these are linearly dependent, because F is square-free, this implies that the matrix is in fact a scalar multiple of the identity, and there is a $\mu \in \Bbbk$ such that $\alpha = \mu x$ and $\beta = \mu y$. The hypothesis is then that $\mu(r+1)F = \mu(xF_x + yF_y) = 0$, so that $\mu = 0$. This proves the claim. □

2.4. Lemma. *If $\alpha_1, \ldots, \alpha_{r+1} \in S_1$ are such that $F = \prod_{i=1}^{r+1} \alpha_i$, then the set of quotients $\{\frac{F}{\alpha_1}, \ldots, \frac{F}{\alpha_{r+1}}\}$ is a basis for S_r.*

Proof. Suppose $c_1, \ldots, c_{r+1} \in \Bbbk$ are scalars such that $\sum_{i=1}^{r+1} c_i \frac{F}{\alpha_i} = 0$. If $j \in \{1, \ldots, r+1\}$, we then have $c_j \frac{F}{\alpha_j} \equiv 0$ modulo α_j and, since F is square-free, this implies that in fact $c_j = 0$. The set $\{\frac{F}{\alpha_1}, \ldots, \frac{F}{\alpha_{r+1}}\}$ is therefore linearly independent. Since $\dim S_r = r+1$, this completes the proof. □

3 A projective resolution

3.1. We keep the situation of the previous section, and write from now on A instead of $\mathscr{D}(\mathcal{A})$. Our immediate objective is to construct a projective resolution of A as an A-bimodule, and we do this by looking at A as a deformation of a commutative polynomial algebra, which suggests that it should have a resolution resembling the usual Koszul complex.

3.2. If U is a vector space and $u \in U$, there are derivations

$$\nabla_x^u, \nabla_y^u : S \to S \otimes U \otimes S$$

of S into the S-bimodule $S \otimes U \otimes S$ uniquely determined by the condition that

$$\nabla_x^u(x) = 1 \otimes u \otimes 1, \qquad \nabla_x^u(y) = 0, \qquad \nabla_y^u(x) = 0, \qquad \nabla_y^u(y) = 1 \otimes u \otimes 1,$$

and in fact we have, for every i, $j \geq 0$, that

$$\nabla_x^u(x^iy^j) = \sum_{s+t+1=i} x^s \otimes u \otimes x^ty^j, \qquad \nabla_y^u(x^iy^j) = \sum_{s+t+1=j} x^iy^s \otimes u \otimes y^s.$$

We consider also the derivation

$$\nabla \coloneqq \nabla_x^x + \nabla_y^y : S \to S \otimes S_1 \otimes S.$$

It is the unique derivation such that $\nabla(\alpha) = 1 \otimes \alpha \otimes 1$ for all $\alpha \in S_1$. There is, on the other hand, a unique morphism of S-bimodules

$$d : S \otimes S_1 \otimes S \to S \otimes S$$

such that $d(1 \otimes \alpha \otimes 1) = \alpha \otimes 1 - 1 \otimes \alpha$ for all $\alpha \in S_1$, and we have

$$d(\nabla(f)) = f \otimes 1 - 1 \otimes f \tag{2}$$

whenever f is in S. To check this last equality, it is enough to notice that the composition $d \circ \nabla : S \to S \otimes S$ is a derivation and, since S_1 generates S as an algebra, that the equality holds when $f \in S_1$.

3.3. Let V be the subspace of A spanned by x, y, D and E. This is a graded subspace and its grading induces on the exterior algebra $\Lambda^\bullet(V)$ an internal grading. If ω is an element of an exterior power $\Lambda^p(V)$ of V, we write $(-) \wedge \omega$ for the map of A-bimodules

$$A \otimes S_1 \otimes A \to A \otimes \Lambda^{p+1} V \otimes A$$

such that $(1 \otimes \alpha \otimes 1) \wedge \omega = 1 \otimes \alpha \wedge \omega \otimes 1$ for all $\alpha \in S_1$.

3.4. Recall that we are using the bar $|$ as a synomym for the tensor product $\otimes$ over the ground field. There is a chain complex $\mathbf{P}$ of free graded A-bimodules of the form

$$0 \longrightarrow A|\Lambda^4 V|A \xrightarrow{d_4} A|\Lambda^3 V|A \xrightarrow{d_3} A|\Lambda^2 V|A \xrightarrow{d_2} A|V|A \xrightarrow{d_1} A|A \tag{3}$$

with A^e-linear maps homogeneous of degree zero and such that

$$d_1(1|v|1) = [v, 1|1], \qquad \forall v \in V;$$

$$\begin{aligned}
&d_2(1|x \wedge y|1) = [x, 1|y|1] - [y, 1|x|1];\\
&d_2(1|x \wedge E|1) = [x, 1|E|1] - [E, 1|x|1] + 1|x|1;\\
&d_2(1|y \wedge E|1) = [y, 1|E|1] - [E, 1|y|1] + 1|y|1;\\
&d_2(1|x \wedge D|1) = [x, 1|D|1] - [D, 1|x|1];\\
&d_2(1|y \wedge D|1) = [y, 1|D|1] - [D, 1|y|1] + \nabla(F);\\
&d_2(1|D \wedge E|1) = [D, 1|E|1] - [E, 1|D|1] + r|D|1;
\end{aligned}$$

$$\begin{aligned}
d_3(1|x \wedge y \wedge D|1) &= [x, 1|y \wedge D|1] - [y, 1|x \wedge D|1] + [D, 1|x \wedge y|1]\\
&\qquad + \nabla(F) \wedge x;\\
d_3(1|x \wedge y \wedge E|1) &= [x, 1|y \wedge E|1] - [y, 1|x \wedge E|1] + [E, 1|x \wedge y|1]\\
&\qquad - 2|x \wedge y|1;\\
d_3(1|x \wedge D \wedge E|1) &= [x, 1|D \wedge E|1] - [D, 1|x \wedge E|1] + [E, 1|x \wedge D|1]\\
&\qquad - (r + 1)|x \wedge D|1;\\
d_3(1|y \wedge D \wedge E|1) &= [y, 1|D \wedge E|1] - [D, 1|y \wedge E|1] + [E, 1|y \wedge D|1]\\
&\qquad + \nabla(F) \wedge E - (r + 1)|y \wedge D|1;
\end{aligned}$$

$$\begin{aligned}
d_4(1|x \wedge y \wedge D \wedge E|1) &= [x, 1|y \wedge D \wedge E|1] - [y, 1|x \wedge D \wedge E|1]\\
&\quad + [D, 1|x \wedge y \wedge E|1] - [E, 1|x \wedge y \wedge D|1]\\
&\quad + \nabla(F) \wedge x \wedge E + (r + 2)|x \wedge y \wedge D|1.
\end{aligned}$$

That $\mathbf{P}$ is indeed a complex follows from a direct calculation that we omit. Let us only show how the element $\nabla(F)$ first comes into the picture: we have

$$\begin{aligned}
d_1(d_2(1|y \wedge D|1)) &= d_1\Big([y, 1|D|1] - [D, 1|y|1] + \nabla(F)\Big)\\
&= [y, D]|1 - 1|[y, D] + d(\nabla(F))
\end{aligned}$$

and this is zero precisely because of the equality (2). The appearence of $\nabla(F)$ in d_3 and d_4 provides for similar cancellations.
More interestingly, this complex is exact:

Lemma. *The complex* $\mathbf{P}$ *is a projective resolution of* A *as an* A*-bimodule, with augmentation* $d_0 : A|A \to A$ *such that* $d_0(1|1) = 1$.

Proof. For each $p \in \mathbb{N}_0$ we consider the subspace $\mathcal{F}_pA = \langle x^i y^j D^k E^l : k+l \leq p\rangle$ of A. As a consequence of Lemma 2.2, one sees that $\mathcal{F}A = (\mathcal{F}_pA)_{p\geq 0}$ is an exhaustive and increasing algebra filtration on A and that the corresponding associated graded algebra $\operatorname{gr}(A)$ is isomorphic to the usual commutative polynomial ring $\Bbbk[x, y, D, E]$. Since V is a subspace of A, we can restrict the filtration of A to one on V, and the latter induces as usual a filtration on each exterior power $\Lambda^p V$. In this way we obtain a filtration on each component of the complex $\mathbf{P}$, which turns out to be compatible with its differentials, as can be checked by inspection. The complex $\operatorname{gr}(\mathbf{P})$ obtained from $\mathbf{P}$ by passing to associated graded objects in each degree is isomorphic to the Koszul resolution of $\operatorname{gr}(A)$ as a $\operatorname{gr}(A)$-bimodule and it is therefore acyclic over $\operatorname{gr}(A)$. A standard argument using the filtration of $\mathbf{P}$ concludes from this that the complex $\mathbf{P}$ itself acyclic over A. As its components are manifestly free A-bimodules, this proves the lemma. □

3.5. One almost immediate application of having a projective resolution for our algebra as a bimodule over itself is in computing its global dimension.

PROPOSITION. *The global dimension of A is equal to* 4.

Of course, as A is noetherian, there is no need to distinguish between the left and the right global dimensions.

Proof. If $\lambda \in \Bbbk$ let M_λ be the left A-module which as a vector space is freely spanned by an element u_λ and on which the action of A is such that $x \cdot u_\lambda = y \cdot u_\lambda = D \cdot u_\lambda = 0$ and $E \cdot u_\lambda = \lambda u_\lambda$. It is easy to see that all 1-dimensional A-modules are of this form and that $M_\lambda \cong M_\mu$ iff $\lambda = \mu$, but we will not need this.
The complex $\mathbf{P} \otimes_A M_\lambda$ is a projective resolution of M_λ as a left A-module, and therefore the cohomology of $\operatorname{Hom}_A(\mathbf{P} \otimes_A M_\lambda, M_\mu)$ is canonically isomorphic with $\operatorname{Ext}_A^\bullet(M_\lambda, M_\mu)$. Let us identify $\operatorname{Hom}_A(\mathbf{P} \otimes_A M_\lambda, M_\mu)$ with $M_\mu \otimes M_\lambda^* \otimes \Lambda^\bullet V^*$, with V^* the dual space $\operatorname{Hom}(V, \Bbbk)$ and M_λ^* the vector space $\operatorname{Hom}(M_\lambda, \Bbbk)$, and write ξ for a fixed non-zero element in $M_\mu \otimes M_\lambda^*$. Up to that identification, the complex has the form

$$M_\mu \otimes M_\lambda^* \xrightarrow{\delta^0} M_\mu \otimes M_\lambda^* \otimes V^* \xrightarrow{\delta^1} M_\mu \otimes M_\lambda^* \otimes \Lambda^2 V^* \xrightarrow{\delta^2}$$
$$\longrightarrow M_\mu \otimes M_\lambda^* \otimes \Lambda^3 V^* \xrightarrow{\delta^3} M_\mu \otimes M_\lambda^* \otimes \Lambda^4 V^*$$

with differentials given by

$$\begin{aligned}
&\delta^0(\xi) = (\mu - \lambda)\xi \otimes \hat{E}, \\
&\delta^1(a\xi \otimes \hat{x} + b\xi \otimes \hat{y} + c\xi \otimes \hat{D} + d\xi \otimes \hat{D}) \\
&\qquad = (\lambda + 1 - \mu)a\xi \otimes \hat{x} \wedge \hat{E} + (\lambda + 1 - \mu)b\xi \otimes \hat{y} \wedge \hat{E} \\
&\qquad\qquad + (\lambda + r - \mu)c\xi \otimes \hat{D} \wedge \hat{E}, \\
&\delta^2(a\xi \otimes \hat{x} \wedge \hat{y} + b\xi \otimes \hat{x} \wedge \hat{E} + c\xi \otimes \hat{y} \wedge \hat{E} + d\xi \otimes \hat{x} \wedge \hat{D} \\
&\qquad\qquad + e\xi \otimes \hat{y} \wedge \hat{D} + f\xi \otimes \hat{D} \wedge \hat{E}) \\
&\qquad = (\mu - \lambda - 2)a\xi \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + (\mu - \lambda - r - 1)d\xi \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\
&\qquad\qquad + (\mu - \lambda - r - 1)e\xi \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^3(a\xi \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + b\xi \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + c\xi \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + d\xi \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) \\
&\qquad = (\lambda + r + 2 - \mu)a\xi \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}
\end{aligned}$$

for all choices of a, b, c, d, e and f in $\Bbbk$.

An easy computation shows that

$$\dim \operatorname{Ext}^p_A(M_\lambda, M_{\lambda+r+2}) = \begin{cases} 1, & \text{if } p = 3 \text{ or } p = 4; \\ 0, & \text{in any other case.} \end{cases}$$

In particular, $\operatorname{Ext}^4_A(M_\lambda, M_{\lambda+r+2}) \neq 0$ and therefore $\operatorname{gldim} A \geq 4$. On the other hand, we have constructed a projective resolution of A as an A-bimodule of length 4, so that the projective dimension of A as a bimodule is $\operatorname{pdim}_{A^e} A \leq 4$. Since $\operatorname{gldim} A \leq \operatorname{pdim}_{A^e} A$, the proposition follows from this. □

4 The Hochschild cohomology of $\mathscr{D}(\mathcal{A})$

4.1. We want to compute the Hochschild cohomology of the algebra A. Applying the functor $\operatorname{Hom}_{A^e}(-, A)$ to the resolution $\mathbf{P}$ of 3.4 we get, after standard identifications, the cochain complex

$$A \underset{s^1}{\overset{d^0}{\rightleftarrows}} A \otimes V^* \underset{s^2}{\overset{d^1}{\rightleftarrows}} A \otimes \Lambda^2 V^* \underset{s^3}{\overset{d^1}{\rightleftarrows}} A \otimes \Lambda^3 V^* \underset{s^4}{\overset{d^2}{\rightleftarrows}} A \otimes \Lambda^4 V^* \longrightarrow 0$$

which we denote simply by $A \otimes \Lambda V^*$, with differentials such that

$$\begin{aligned} d^0(a) &= [x,a] \otimes \hat{x} + [y,a] \otimes \hat{y} + [D,a] \otimes \hat{D} + [E,a] \otimes \hat{E}; \\ d^1(a \otimes \hat{x}) &= -[y,a] \otimes \hat{x} \wedge \hat{y} + (a - [E,a]) \otimes \hat{x} \wedge \hat{E} - [D,a] \otimes \hat{x} \wedge \hat{D} \\ &\qquad + \nabla^a_x(F) \otimes \hat{y} \wedge \hat{D}; \end{aligned}$$

$$\begin{aligned} d^1(a \otimes \hat{y}) &= [x,a] \otimes \hat{x} \wedge \hat{y} + (a - [E,a]) \otimes \hat{y} \wedge \hat{E} \\ &\qquad + (\nabla^a_y(F) - [D,a]) \otimes \hat{y} \wedge \hat{D}; \\ d^1(a \otimes \hat{D}) &= [x,a] \otimes \hat{x} \wedge \hat{D} + [y,a] \otimes \hat{y} \wedge \hat{D} + (ra - [E,a]) \otimes \hat{D} \wedge \hat{E}; \\ d^1(a \otimes \hat{E}) &= [x,a] \otimes \hat{x} \wedge \hat{E} + [y,a] \otimes \hat{y} \wedge \hat{E} + [D,a] \otimes \hat{D} \wedge \hat{E}; \end{aligned}$$

$$
\begin{aligned}
d^2(a\otimes\hat{x}\wedge\hat{y}) &= ([D,a]-\nabla_y^a(F))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\\
&\qquad\qquad\qquad\qquad + ([E,a]-2a)\otimes\hat{x}\wedge\hat{y}\wedge\hat{E};\\
d^2(a\otimes\hat{x}\wedge\hat{E}) &= -[y,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}-[D,a]\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}\\
&\qquad\qquad\qquad\qquad + \nabla_x^a(F)\otimes\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^2(a\otimes\hat{y}\wedge\hat{E}) &= [x,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}+(\nabla_y^a(F)-[D,a])\otimes\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^2(a\otimes\hat{x}\wedge\hat{D}) &= -[y,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}+([E,a]-(r+1)a)\otimes\hat{x}\wedge\hat{D}\wedge\hat{E};\\
d^2(a\otimes\hat{y}\wedge\hat{D}) &= [x,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}+([E,a]-(r+1)a)\otimes\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^2(a\otimes\hat{D}\wedge\hat{E}) &= [x,a]\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}+[y,a]\otimes\hat{y}\wedge\hat{D}\wedge\hat{E};
\end{aligned}
$$

$$
\begin{aligned}
d^3(a\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}) &= (-[E,a]+(r+2)a)\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^3(a\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}) &= ([D,a]-\nabla_y^a(F))\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^3(a\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}) &= -[y,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E};\\
d^3(a\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}) &= [x,a]\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}.
\end{aligned}
$$

These differentials are homogeneous with respect to the natural internal grading on the complex $A\otimes\Lambda V^*$ coming from the grading of A. We denote $\gamma : A\otimes\Lambda V^*\to A\otimes\Lambda V^*$ the $\Bbbk$-linear map whose restriction to each homogeneous component of the complex $A\otimes\Lambda V^*$ is simply the multiplication by the degree. There is a homotopy, drawn in the diagram (3) with dashed arrows, with

$$
\begin{aligned}
&s^1(a\otimes\hat{x}+b\otimes\hat{y}+c\otimes\hat{D}+d\otimes\hat{E}) = d,\\
&s^2(a\otimes\hat{x}\wedge\hat{y}+b\otimes\hat{x}\wedge\hat{E}+c\otimes\hat{y}\wedge\hat{E}+d\otimes\hat{x}\wedge\hat{D}\\
&\qquad\qquad + e\otimes\hat{y}\wedge\hat{D}+f\otimes\hat{D}\wedge\hat{E}) = -b\otimes\hat{x}-c\otimes\hat{y}-f\otimes\hat{D},\\
&s^3(a\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}+b\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}+c\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}+d\otimes\hat{y}\wedge\hat{D}\wedge\hat{E})\\
&\qquad\qquad\qquad\qquad = b\otimes\hat{x}\wedge\hat{y}+c\otimes\hat{x}\wedge\hat{D}+d\otimes\hat{y}\wedge\hat{D},\\
&s^4(a\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}) = -a\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}
\end{aligned}
$$

and such that $d\circ s+s\circ d=\gamma$: this tells us that γ induces the zero map on cohomology. Since our ground field $\Bbbk$ has characteristic zero, this implies that the inclusion $(A\otimes\Lambda V^*)_0\to A\otimes\Lambda V^*$ of the component of degree zero of our complex $A\otimes\Lambda V^*$ is a quasi-isomorphism.

4.2. From now on and until the end of this section, we will assume that $r\geq 3$. Let us write the complex $(A\otimes\Lambda V^*)_0$ simply $\mathfrak{X}$ and let us put $T=\Bbbk[E]$, which coincides with A_0. The complex $\mathfrak{X}$ has components

$$
\begin{aligned}
\mathfrak{X}^0 &= A_0,\\
\mathfrak{X}^1 &= A_1\otimes(\Bbbk\hat{x}\oplus\Bbbk\hat{y})\oplus A_r\otimes\Bbbk\hat{D}\oplus A_0\otimes\Bbbk\hat{E},\\
\mathfrak{X}^2 &= A_2\otimes\Bbbk\hat{x}\wedge\hat{y}\oplus A_1\otimes(\Bbbk\hat{x}\wedge\hat{E}\oplus\Bbbk\hat{y}\wedge\hat{E})\oplus A_r\otimes\Bbbk\hat{D}\wedge\hat{E}\\
&\qquad\qquad\qquad\qquad \oplus A_{r+1}\otimes(\Bbbk\hat{x}\wedge\hat{D}\oplus\Bbbk\hat{y}\wedge\hat{D}),
\end{aligned}
$$

$$\begin{aligned}\mathfrak{X}^3 &= A_2 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \oplus A_{r+1} \otimes (\Bbbk\hat{x} \wedge \hat{D} \wedge \hat{E} \oplus \Bbbk\hat{y} \wedge \hat{D} \wedge \hat{E}) \\ &\qquad \oplus A_{r+2} \otimes \Bbbk\hat{x} \wedge \hat{y} \wedge \hat{D}, \\ \mathfrak{X}^4 &= A_{r+2} \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}\end{aligned}$$

and, since $r > 2$, we have that

$$\begin{aligned} &A_0 = T, && A_1 = S_1T, \\ &A_2 = S_2T, && A_r = (S_r \oplus \Bbbk D)T, \\ &A_{r+1} = (S_{r+1} \oplus S_1D)T, && A_{r+2} = (S_{r+2} \oplus S_2D)T. \end{aligned}$$

In fact, this is where our assumption that $r \geq 3$ intervenes: if $r \leq 2$, then these subspaces of A have different descriptions.
The differentials in $\mathfrak{X}$ can be computed to be given by

$$\begin{aligned}
&\delta^0(a) = x\tau_1(a) \otimes \hat{x} + y\tau_1(a) \otimes \hat{y} + D\tau_r(a) \otimes \hat{D}, \\
&\delta^1(\phi a \otimes \hat{x}) = -\phi y\tau_1(a) \otimes \hat{x} \wedge \hat{y} - (F\phi_y a + \phi D\tau_r(a)) \otimes \hat{x} \wedge \hat{D} \\
&\qquad\qquad + \nabla_x^{\phi a}(F) \otimes \hat{y} \wedge \hat{D}, \\
&\delta^1(\phi a \otimes \hat{y}) = \phi x\tau_1(a) \otimes \hat{x} \wedge \hat{y} + (\nabla_y^{\phi a}(F) - F\phi_y a - \phi D\tau_r(a)) \otimes \hat{y} \wedge \hat{D}, \\
&\delta^1((\phi + \lambda D)a \otimes \hat{D}) = (\phi x\tau_1(a) + \lambda xD\tau_1(a)) \otimes \hat{x} \wedge \hat{D} \\
&\qquad\qquad + (\phi y\tau_1(a) + \lambda F(\tau_1(a) - a) + \lambda yD\tau_1(a)) \otimes \hat{y} \wedge \hat{D}, \\
&\delta^1(a \otimes \hat{E}) = x\tau_1(a) \otimes \hat{x} \wedge \hat{E} + y\tau_1(a) \otimes \hat{y} \wedge \hat{E} + D\tau_r(a) \otimes \hat{D} \wedge \hat{E}, \\
\\
&\delta^2(\phi a \otimes \hat{x} \wedge \hat{y}) = (F\phi_y a + \phi D\tau_r(a) - \nabla_y^{\phi a}(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}, \\
&\delta^2(\phi a \otimes \hat{x} \wedge \hat{E}) = -\phi y\tau_1(a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \\
&\qquad\qquad - (F\phi_y a + \phi D\tau_r(a)) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \nabla_x^{\phi a}(F) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^2(\phi a \otimes \hat{y} \wedge \hat{E}) = \phi x\tau_1(a) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \\
&\qquad\qquad + (\nabla_y^{\phi a}(F) - F\phi_y a - \phi D\tau_r(a)) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^2((\phi + \psi D)a \otimes \hat{x} \wedge \hat{D}) \\
&\qquad\qquad = (-\phi y\tau_1(a) - \psi F(\tau_1(a) - a) - \psi yD\tau_1(a)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}, \\
&\delta^2((\phi + \psi D)a \otimes \hat{y} \wedge \hat{D}) = (\phi x\tau_1(a) + \psi xD\tau_1(a)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}, \\
&\delta^2((\phi + \lambda D)a \otimes \hat{D} \wedge \hat{E}) = (\phi x\tau_1(a) + \lambda xD\tau_1(a)) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\
&\qquad\qquad + (\phi y\tau_1(a) + \lambda yD\tau_1(a) + \lambda F(\tau_1(a) - a)) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^3((\phi + \psi D)a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}) = 0, \\
&\delta^3(\phi a \otimes \hat{x} \wedge \hat{y} \wedge \hat{E}) = (F\phi_y a + \phi D\tau_r(a) - \nabla_y^{\phi a}(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^3((\phi + \psi D)a \otimes \hat{x} \wedge \hat{D} \wedge \hat{E}) \\
&\qquad\qquad = -(\phi y\tau_1(a) + \psi yD\tau_1(a) + \psi F(\tau_1(a) - a)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta^3((\phi + \psi D)a \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) = (\phi x\tau_1(a) + \psi xD\tau_1(a)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned}$$

Here and below $\tau_t : T \to T$ is the $\Bbbk$-linear map such that $\tau_t(E^n) = E^n - (E+t)^n$ for all $n \in \mathbb{N}_0$, and ϕ and ψ denote homogeneous elements of A of appropriate degrees and λ a scalar.

4.3. We proceed to compute the cohomology of the complex $\mathfrak{X}$, starting with degrees zero and four, for which the computation is almost immediate. Indeed, since the kernel of τ_1 and of τ_r is $\Bbbk \subseteq T$, it is clear that $H^0(\mathfrak{X}) = \ker \delta^0 = \Bbbk$. On the other hand, if $\psi \in S_2$ and $a \in T$, we can write $\psi = \psi_1 x + \psi_2 y$ for some ψ_1, $\psi_2 \in S_1$ and there is a $b \in T$ such that $\tau_1(b) = a$, so that

$$\delta^3(-\psi_2 Db \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \psi_1 Db \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) = (\psi Da + \boxed{S_{r+2}T}) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}.$$

Similarly, we have

$$\delta^3(S_{r+1}T \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + S_{r+1}T \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) = S_{r+2}T \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}.$$

These two facts imply that the map δ^3 is surjective, so that $H^4(\mathfrak{X}) = 0$.

4.4. Let $\omega \in \mathfrak{X}^1$ be a 1-cocycle in $\mathfrak{X}$. There are then a, b, c, d, e, $f \in T$, $k \in \mathbb{N}_0$ and $\phi_0, \ldots, \phi_k \in S_r$ such that either $k = 0$ or $\phi_k \neq 0$, and

$$\omega = (xa + yb) \otimes \hat{x} + (xc + yd) \otimes \hat{y} + \left(\sum_{i=0}^{k} \phi_i E^i + De \right) \otimes \hat{D} + f \otimes \hat{E}.$$

If $\bar{e} \in T$ is such that $\tau_r(\bar{e}) = e$, then by replacing ω by $\omega - \delta^0(\bar{e})$, which does not change the cohomology class of ω, we can assume that $e = 0$. The formula for δ^0 then shows that ω is a coboundary iff it is equal to zero. The coefficient of $\hat{x} \wedge \hat{y}$ in $\delta^1(\omega)$ is

$$x^2 \tau_1(c) + xy(\tau_1(d) - \tau_1(a)) - y^2 \tau_1(b) = 0.$$

We therefore have b, c, $d - a \in \Bbbk$. The coefficient of $\hat{D} \wedge \hat{E}$, on the other hand, is $D\tau_r(f) = 0$, so that also $f \in \Bbbk$; exactly the same information comes from the vanishing of the coefficients of $\hat{x} \wedge \hat{E}$ and of $\hat{y} \wedge \hat{E}$. Since $b \in \Bbbk$, the coefficient of $\hat{x} \wedge \hat{D}$ is

$$-Fb - xD\tau_r(a) + \sum_{i=0}^{k} \phi_i x \tau_1(E^i) = 0.$$

We see that $\tau_r(a) = 0$, so that $a \in \Bbbk$, and that $\sum_{i=0}^{k} \phi_i x \tau_1(E^i) = Fb$. This implies that $k \leq 1$, that $-\phi_1 x = Fb$ and therefore, since x is not a factor of F by hypothesis, that $\phi_1 = 0$ and $b = 0$.

Finally, using all the information we have so far, we can see that the vanishing of the coefficient of $\hat{y} \wedge \hat{D}$ in $\delta^1(\omega)$ implies that $F_x xa + F_y(xc + yd) = Fd$. Together with Euler's relation $F_x x + F_y y = (r + 1)F$ this tells us that

$$(cx + (d - a)y)F_y = (d - (r + 1)a)F. \tag{4}$$

As F is square-free, it follows[1] from this equality the polynomial $cx + (d - a)y$ is zero so that $c = 0$ and $d = a$ and, finally, that $a = 0$. We conclude in this way that the set of 1-cocycles

$$\phi \otimes \hat{D} + f \otimes \hat{E}, \qquad \phi \in S_r, f \in \Bbbk$$

is a complete, irredundant set of representatives for the elements of $H^1(\mathfrak{X})$.

4.5. Let $\omega \in \mathfrak{X}^3$ be a 3-cocycle, so that

$$\omega = a \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + b \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + c \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + d \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}$$

for some $a \in (S_{r+2} \oplus S_2 D)T$, $b \in S_2 T$, c, $d \in (S_{r+1} \oplus S_1 D)T$ and $\delta^3(\omega) = 0$. For all $\phi \in S_1$ and $e \in T$ we have

$$\begin{aligned}\delta^2(\phi e \otimes \hat{x} \wedge \hat{E}) = -\phi y \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\ + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}\end{aligned}$$

and

$$\delta^2(\phi e \otimes \hat{y} \wedge \hat{E}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

so that by adding to ω an element of $\delta^2(S_1 T \otimes \hat{x} \wedge \hat{E} + S_1 T \otimes \hat{y} \wedge \hat{E})$, which does not change the cohomology class of ω, we can suppose that $b = 0$. Similarly, for all $\phi \in S_2$ and all $e \in T$ we have

$$\delta^2(\phi e \otimes \hat{x} \wedge \hat{y}) = (\boxed{S_{r+2} T} + \phi D \tau_r(e)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D},$$

and for all $\phi \in S_{r+1}$ and all $e \in T$ we have

$$\delta^2(\phi e \otimes \hat{x} \wedge \hat{D}) = -\phi y \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$$

and

$$\delta^2(\phi e \otimes \hat{y} \wedge \hat{D}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}.$$

Using this we see that, up to changing ω by adding to it a 3-coboundary, we can suppose that $a = 0$. Finally, for each $\phi \in S_r$ and all $e \in T$ we have

$$\delta^2(\phi e \otimes \hat{D} \wedge \hat{E}) = \phi x \tau_1(e) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

$$\delta^2(De \otimes \hat{D} \wedge \hat{E}) = x D \tau_1(e) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}$$

and

$$\delta^2(-y \otimes \hat{x} \wedge \hat{E} + \bar{F} E \otimes \hat{D} \wedge \hat{E}) = y^{r+1} \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + \boxed{A_{r+1}} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

[1]Suppose that $u = cx + (d - a)y$ is not zero. Differentiating in (4) with respect to y, we find that $-raF_y = uF_{yy}$. Since x does not divide F, we have $F_{yy} \neq 0$, and then $a \neq 0$ and u divides F_y: from (4) it follows then that u^2 divides F, since the left hand side of that equality is non-zero, and this is absurd because F is square-free.

so we can also suppose that $c \in y^{r+1}ET + yDT$.
There are $l \geq 0$, $\lambda_1, \ldots, \lambda_l$, $\mu_0, \ldots, \mu_l \in \Bbbk$, $\phi_0, \ldots, \phi_l \in S_{r+1}$, $\psi_0, \ldots, \psi_l \in S_1$, $\zeta_0, \ldots, \zeta_l \in S_1$ such that $c = \sum_{i=1}^{l} \lambda_i y^{r+1} E^i + \sum_{i=0}^{l} \mu_i y D E^i$ and $d = \sum_{i=0}^{l} (\phi_i + \psi_i D) E^i$. The vanishing of $\delta^3(\omega)$ means precisely that

$$\sum_{i=0}^{l} \Big(\mu_i y^2 D \tau_1(E^i) - \mu_i y F(E+1)^i - \phi_i x \tau_1(E^i) - \psi_i x D \tau_1(E^i) \Big) + \sum_{i=1}^{l} \lambda_i y^{r+2} \tau_1(E^i) = 0.$$

The left hand side of this equation is an element of $S_{r+2}T \oplus S_2 DT$. The component in S_2DT is $\sum_{i=0}^{l} (\mu_i y^2 - \psi_i x) D \tau_1(E^i) = 0$ and therefore $\mu_i = \psi_i = 0$ for all $i \in \{1, \ldots, l\}$. On the other hand, the component in $S_{r+2}T$ is

$$\sum_{i=1}^{l} \lambda_i y^{r+2} \tau_1(E^i) - \mu_0 y F - \sum_{i=0}^{l} \phi_i x \tau_1(E^i) = 0.$$

This implies that $\lambda_i y^{r+2} - \phi_i x = 0$ if $i \in \{2, \ldots, l\}$, so that $\lambda_i = \phi_i = 0$ for such i, and then the equation reduces to $\lambda_1 y^{r+2} + \mu_0 y F - \phi_1 x = 0$. Recalling from 2.2 that $F = y^{r+1} + x\bar{F}$, we deduce from this that $\lambda_1 = -\mu_0$ and $\phi_1 = \mu_0 y \bar{F}$. We conclude in this way that every 3-cocycle is cohomologous to one of the form

$$(\mu_0 y D - \mu_0 y^{r+1} E) \hat{x} \wedge \hat{D} \wedge \hat{E} + (\phi_0 + \psi_0 D + \mu_0 y \bar{F} E) \hat{y} \wedge \hat{D} \wedge \hat{E} \tag{5}$$

with $\mu_0 \in \Bbbk$, $\phi_0 \in S_{r+1}$ and $\psi_0 \in S_1$, and a direct computation shows that moreover every 3-cochain of this form is a 3-cocycle.
Let now η be a 2-cochain η in $\mathfrak{X}$, so that

$$\eta = \boxed{A_2 \otimes \Bbbk \hat{x} \wedge \hat{y} \oplus A_{r+1} \otimes (\Bbbk \hat{x} \wedge \hat{D} \oplus \Bbbk \hat{y} \wedge \hat{D})} + u \otimes \hat{x} \wedge \hat{E} + v \otimes \hat{y} \wedge \hat{E} + w \otimes \hat{D} \wedge \hat{E}$$

with u, $v \in A_1$ and $w \in A_r$, and let us suppose that its coboundary $\delta^2(\eta)$ is equal to the 3-cocycle written in (5). There are then $l \geq 0$, $\alpha_0, \ldots, \alpha_l$, $\beta_0, \ldots, \beta_l \in S_1$, $\gamma_0, \ldots, \gamma_l \in S_r$ and $\xi_0, \ldots, \xi_l \in \Bbbk$ such that $u = \sum_{i=0}^{l} \alpha_i E^i$, $v = \sum_{i=0}^{l} \beta_i E^i$ and $w = \sum_{i=0}^{l} (\gamma_i + \xi_i D) E^i$. The coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{E}$ in $\delta^2(\eta)$ must be equal to zero, so that

$$\sum_{i=0}^{l} (-\alpha_i y + \beta_i x) \tau_1(E^i) = 0,$$

and this implies that there are scalars $\rho_1, \ldots, \rho_l \in \Bbbk$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for all $i \in \{1, \ldots, l\}$.

Looking now at the coefficient of $\hat{x} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\eta)$ and comparing with (5) we find that

$$\sum_{i=0}^{l} \left(-F\alpha_{iy}E^i - \alpha_i D\tau_r(E^i) + \gamma_i x\tau_1(E^i) + \xi_i xD\tau_1(E^i)\right) \\ = \mu_0 yD - \mu_0 y^{r+1}E. \quad (6)$$

This is an equality of two elements of $ST \oplus SDT$. Considering the components in DT, we find that $xD\sum_{i=1}^{l}(-\rho_i\tau_r(E^i) + \xi_i\tau_1(E^i)) = \mu_0 yD$, and this tells us that $\mu_0 = 0$ and that

$$\sum_{i=1}^{l} \left(-\rho_i\tau_r(E^i) + \xi_i\tau_1(E^i)\right) = 0. \quad (7)$$

On the other hand, as the components in ST of the two sides of (6) are equal, we have

$$-F\alpha_{0y} + \sum_{i=0}^{l} \gamma_i x\tau_1(E^i) = 0,$$

so that $\gamma_i = 0$ for all $i \in \{2, \ldots, l\}$ and $F\alpha_{0y} + \gamma_1 x = 0$. As x does not divide F, we must have $\alpha_{0y} = 0$ and $\gamma_1 = 0$; in particular, there is a $\rho_0 \in \Bbbk$ such that $\alpha_0 = \rho_0 x$.

Finally, considering the coefficient of $\hat{y} \wedge \hat{D} \wedge \hat{E}$ of $\delta^2(\eta)$ and of (5) we see that

$$\sum_{i=0}^{l}\Big(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i) \\ + \gamma_i y\tau_1(E^i) + \xi_i yD\tau_1(E^i) - \xi_i F(E+1)^i\Big) = \phi_0 + \psi_0 D,$$

which at this point we can rewrite (using in the process the equality (7) above and the fact that $\nabla_x^{xE^i}(F) + \nabla_x^{yE^i}(F) = F\sum_{t=0}^{r}(E+t)^i$) as

$$\rho_0 xF_x + \beta_0 F_y - F\left(\beta_{0y} + \xi_0 - \sum_{i=1}^{l}\left(\rho_i \sum_{t=1}^{r}(E+t)^i - \xi_i(E+1)^i\right)\right) \\ = \phi_0 + \psi_0 D.$$

It follows at once that $\psi_0 = 0$ and that, in fact,

$$\rho_0 xF_x + \beta_0 F_y - F\left(\beta_{0y} + \xi_0 - \sum_{i=1}^{l}\left(\rho_i \sum_{t=1}^{r} t^i - \xi_i\right)\right) = \phi_0.$$

The polynomial ϕ_0 is then in the linear span of xF_x, xF_y, yF_y and F inside S_{r+1}. Euler's relation implies that already the first three polynomials span this subspace, and we have

$$\begin{aligned}
&\delta(x \otimes \hat{x} \wedge \hat{E}) = xF_x \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta(x \otimes \hat{y} \wedge \hat{E}) = xF_y \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
&\delta(y \otimes \hat{y} \wedge \hat{E} - D \otimes \hat{D}\hat{E}) = yF_y \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned} \tag{8}$$

We conclude in this way that the only 3-coboundaries among the cocycles of the form (5) are the linear combinations of the right hand sides of the equalities (8); these three cocycles are, moreover, linearly independent. This means that there is an isomorphism

$$H^3(\mathfrak{X}) \cong \Bbbk\omega_3 \oplus S_1 D \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \oplus \frac{S_{r+1}}{\langle xF_x, xF_y, yF_y\rangle} \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

with

$$\omega_3 = (yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + y\bar{F}E \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},$$

and that, in particular, $\dim H^3(\mathfrak{X}) = r+2$, since the denominator appearing in the right hand side of this isomorphism is a 3-dimensional vector space —this follows at once from Lemma 2.3.

4.6. We consider now a 2-cocycle $\omega \in \mathfrak{X}^2$ and polynomials $a \in S_2T$, b, $c \in S_1T$, d, $e \in S_{r+1}T \oplus S_1DT$ and $f \in S_rT \oplus DT$ such that

$$\omega = a \otimes \hat{x} \wedge \hat{y} + b \otimes \hat{x} \wedge \hat{E} + c \otimes \hat{y} \wedge \hat{E} + d \otimes \hat{x} \wedge \hat{D} + e \otimes \hat{y} \wedge \hat{D} \\ + f \otimes \hat{E} \wedge \hat{D}.$$

Adding to ω an element of $\delta^1(T \otimes \hat{E})$, we can assume that $f \in S_rT$; adding an element of $\delta^1(S_1T \otimes \hat{x} \oplus S_1T \otimes \hat{y})$, we can suppose that $a = 0$; finally, adding an element of $\delta^1((S_rT \oplus DT) \otimes \hat{D})$ we can suppose that $d \in y^{r+1}T \oplus yDT$. In this situation, there are an integer $l \geq 0$, $\alpha_0, \ldots, \alpha_l$, $\beta_0, \ldots, \beta_l \in S_1$, $\lambda_0, \ldots, \lambda_l$, $\mu_0, \ldots, \mu_l \in \Bbbk$, $\phi_0, \ldots, \phi_l \in S_{r+1}$, $\psi_0, \ldots, \psi_l \in S_1$ and $\xi_0, \ldots, \xi_l \in S_r$ such that $b = \sum_{i=0}^{l} \alpha_i E^i$, $c = \sum_{i=0}^{l} \beta_i E^i$, $d = \sum_{i=0}^{l} (\lambda_i y^{r+1} + \mu_i yD)E^i$, $e = \sum_{i=0}^{l} (\phi_i + \psi_i D)E^i$ and $f = \sum_{i=0}^{l} \xi_i E^i$. As

$$\delta^1(-y \otimes \hat{x} + \bar{F}E \otimes \hat{D}) = y^{r+1} \otimes \hat{x} \wedge \hat{D} + \boxed{S_{r+1}} \otimes \hat{y} \wedge \hat{D},$$

we can assume that $\lambda_0 = 0$.

The coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{E}$ in $\delta^2(\omega)$ is $\sum_{i=0}^{l}(-\alpha_i y + \beta_i x)\tau_1(E^i) = 0$, and this implies that there are scalars $\rho_1, \ldots, \rho_l \in \Bbbk$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for each $i \in \{1, \ldots, l\}$. The coefficient of $\hat{x} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\omega)$ is

$$\sum_{i=0}^{l} \bigl(-F\alpha_{iy}E^i - \alpha_i D\tau_r(E^i) + \xi_i x\tau_1(E^i)\bigr) = 0. \tag{9}$$

It follows that $\sum_{i=0}^{l} \alpha_i D\tau_r(E^i) = 0$, so that $\alpha_1 = \cdots = \alpha_l = 0$; as a consequence of this, we have that $\rho_1 = \cdots = \rho_l = 0$ and $\beta_1 = \cdots = \beta_l = 0$. The equality (9) also tells us that $-F\alpha_{0y} + \sum_{i=0}^{l} \xi_i x \tau_1(E^i) = 0$, and from this we see that $\xi_2 = \cdots = \xi_l = 0$ and $-F\alpha_{0y} - \xi_1 x = 0$, so that $\alpha_{0y} = 0$ and $\xi_1 = 0$, since x does not divide F. In particular, there is a $\rho_0 \in \Bbbk$ such that $\alpha_0 = \rho_0 x$. The coefficient of $\hat{y} \wedge \hat{D} \wedge \hat{E}$ in $\delta^2(\omega)$ is

$$\begin{aligned}\sum_{i=0}^{l} \Big(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i) + \xi_i y \tau_1(E^i) \Big) \\ = \rho_0 x F_x + \beta_0 F_y - \beta_{0y} F \\ = \big(\rho_0 - (r+1)^{-1}\beta_{0y}\big) x F_x + \big(\beta_{0x} x + (1 - (1+r)^{-1})\beta_{0y} y\big) F_y = 0,\end{aligned}$$

and our Lemma 2.3 implies then that $\beta_0 = 0$ and $\rho_0 = 0$. Finally, we consider the coefficient of $\hat{x} \wedge \hat{y} \wedge \hat{D}$:

$$\sum_{i=0}^{l} \Big(-\lambda_i y^{r+2} \tau_1(E^i) + \mu_i y F (E+1)^i - \mu_i y^2 D \tau_1(E^i) \\ + \phi_i x \tau_1(E^i) + \psi_i x D \tau_1(E^i) \Big) = 0.$$

Looking at the terms involving D in this equation, we see that

$$\sum_{i=0}^{l} (-\mu_i y^2 + \psi_i x) D \tau_1(E^i)) = 0,$$

so $\mu_1 = \cdots = \mu_l = 0$ and $\psi_1 = \cdots = \psi_l = 0$. The terms not involving D add up to

$$\mu_0 y F + \sum_{i=0}^{l} (-\lambda_i y^{r+2} + \phi_i x) \tau_1(E^i) = 0,$$

so that $\lambda_2 = \cdots = \lambda_l = 0$, $\phi_2 = \cdots = \phi_l = 0$ and $\mu_0 y F + \lambda_1 y^{r+1} - \phi_1 x = 0$, which implies that $\lambda_1 = -\mu_0$ and $\phi_1 = \mu_0 y \bar{F}$.

After all this, we see that every 2-cocycle in our complex is cohomologous to one of the form

$$(\mu_0 y D - \mu_0 y^{r+1} E)\hat{x} \wedge \hat{D} + (\phi_0 + \psi_0 D + \mu_0 y \bar{F} E)\hat{y} \wedge \hat{D} + \xi_0 \hat{D} \wedge \hat{E} \tag{10}$$

with $\mu_0 \in \Bbbk$, $\phi_0 \in S_{r+1}$, $\psi_0 \in S_1$ and $\xi_0 \in S_r$. Computing we find that all elements of this form are in fact 2-cocycles.

Let us now suppose that the cocycle (10), which we call again ω, is a coboundary, so that there exist $k \geq 0$, $\alpha_0, \ldots, \alpha_k, \beta_0, \ldots, \beta_k \in S_1$, $\sigma_1, \ldots, \sigma_k \in S_r$, $\zeta_0, \ldots, \zeta_k \in \Bbbk$ and $u \in T$ such that if

$$\eta = \sum_{i=0}^{k} \alpha_i E^i \hat{x} + \sum_{i=0}^{k} \beta_i E^i \hat{y} + \sum_{i=0}^{k} (\sigma_i + \zeta_i D) E^i \hat{D} + u\hat{E},$$

we have $\delta^1(\eta) = \omega$. The coefficient of $\hat{D} \wedge \hat{E}$ in $\delta^1(\eta)$ is $D\tau_r(u)$ so, comparing with (10), we see that we must have $\xi_0 = 0$ and $u \in \Bbbk$; it follows from this that the coefficients of $\hat{E} \wedge \hat{E}$ and of $\hat{y} \wedge \hat{E}$ in $\delta^1(\eta)$ vanish. On the other hand, the coefficient of $\hat{x} \wedge \hat{y}$ in $\delta^1(\eta)$ is $\sum_{i=0}^{k}(-\alpha_i y + \beta_i x)\tau_1(E^i)$: as this has to be zero, we see that there exist $\rho_1, \ldots, \rho_k \in \Bbbk$ such that $\alpha_i = \rho_i x$ and $\beta_i = \rho_i y$ for each $i \in \{1, \ldots, k\}$.
The coefficient of $\hat{x} \wedge \hat{D}$ in $\delta^1(\eta)$ is

$$\sum_{i=0}^{k}\bigl(-F\alpha_{iy}E^i - \alpha_i D\tau_r(E^i) + \sigma_i x\tau_1(E^i) + \zeta_i xD\tau_1(E^i)\bigr) \\ = \mu_0 yD - \mu_0 y^{r+1}E. \quad (11)$$

This means, first, that $\sum_{i=1}^{k}\bigl(-\rho_i xD\tau_r(E^i) + \zeta_i xD\tau_1(E^i)\bigr) = \mu_0 yD$ and this is only possible if $\mu_0 = 0$ and

$$\sum_{i=1}^{k}\bigl(-\rho_i\tau_r(E^i) + \zeta_i\tau_1(E^i)\bigr) = 0. \quad (12)$$

Second, the equality (11) implies that

$$\sum_{i=0}^{k}\bigl(-F\alpha_{iy}E^i + \sigma_i x\tau_1(E^i)\bigr) = -F\alpha_{0y} + \sum_{i=1}^{k}\sigma_i x\tau_1(E^i) = 0,$$

so that $\sigma_2 = \cdots = \sigma_k = 0$ and $F\alpha_{0y} + \sigma_1 x = 0$, which tells us that $\sigma_1 = 0$ and $\alpha_{0y} = 0$; there is then a $\rho_0 \in \Bbbk$ such that $\alpha_0 = \rho_0 x$.
Finally, the coefficient of $\hat{y} \wedge \hat{D}$ in $\delta^1(\eta)$ is

$$\sum_{i=0}^{k}\bigl(\nabla_x^{\alpha_i E^i}(F) + \nabla_y^{\beta_i E^i}(F) - F\beta_{iy}E^i - \beta_i D\tau_r(E^i) + \sigma_i y\tau_1(E^i) \\ - \zeta_i F(E+1)^i + \zeta_i yD\tau_1(E^i)\bigr) = \phi_0 + \psi_0 D.$$

Looking only at the terms which are in S_1DT, we see that

$$yD\sum_{i=1}^{k}(-\rho_i\tau_r(E^i) + \zeta_i\tau(E^i)) = \psi_0 D$$

and, in view of (12), it follows from this that $\psi_0 = 0$. The terms in $S_{r+1}T$, on the other hand, are

$$\rho_0 xF_x + \beta_0 F_y + F\left(-\beta_{0y} - \zeta_0 + \sum_{i=0}^{k}\left(\rho_i\sum_{t=1}^{r}(E+t)^i - \zeta_i(E+1)^i\right)\right) = \phi_0,$$

and proceeding as before we see that ϕ_0 is in the linear span of xF_x, xF_y and yF_y. Computing, we find that

$$\begin{aligned}\delta^1(x\otimes\hat{x}) &= xF_x\otimes\hat{y}\wedge\hat{D},\\ \delta^1(x\otimes\hat{y}) &= xF_y\otimes\hat{y}\wedge\hat{D},\\ \delta^1(y\otimes\hat{y}-D\hat{D}) &= yF_y\otimes\hat{y}\wedge\hat{D}.\end{aligned}$$

We thus conclude that there is an isomorphism

$$H^2(\mathfrak{X})\cong \Bbbk\omega_2\oplus\frac{S_{r+1}}{\langle xF_x,xF_y,yF_y\rangle}\otimes\hat{y}\wedge\hat{D}\oplus S_1D\otimes\hat{y}\wedge\hat{D}\oplus S_r\otimes\hat{D}\wedge\hat{E},$$

with $\omega_2=(yD-y^{r+1}E)\otimes\hat{x}\wedge\hat{D}+y\bar{F}E\otimes\hat{y}\wedge\hat{D}$, and that, in particular, the dimension of $H^2(\mathfrak{X})$ is $2r+3$.

4.7. We can summarize our findings as follows:

Proposition. *Suppose that $r\geq 3$. For all $p\geq 4$ we have $\mathrm{HH}^p(A)=0$. There are isomorphisms*

$$\begin{aligned}&\mathrm{HH}^0(A)\cong\Bbbk,\\ &\mathrm{HH}^1(A)\cong S_r\otimes\hat{D}\oplus\Bbbk\otimes\hat{E},\\ &\mathrm{HH}^2(A)\cong\Bbbk\omega_2\oplus\frac{S_{r+1}}{\langle xF_x,xF_y,yF_y\rangle}\otimes\hat{y}\wedge\hat{D}\oplus S_1D\otimes\hat{y}\wedge\hat{D}\oplus S_r\otimes\hat{D}\wedge\hat{E},\\ &\mathrm{HH}^3(A)\cong\Bbbk\omega_3\oplus\frac{S_{r+1}}{\langle xF_x,xF_y,yF_y\rangle}\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}\oplus S_1D\otimes\hat{y}\wedge\hat{D}\wedge\hat{E},\end{aligned}$$

with

$$\begin{aligned}\omega_2&=(yD-y^{r+1}E)\otimes\hat{x}\wedge\hat{D}+y\bar{F}E\otimes\hat{y}\wedge\hat{D},\\ \omega_3&=(yD-y^{r+1}E)\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}+y\bar{F}E\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}.\end{aligned}$$

The Hilbert series of the Hochschild cohomology of A is

$$\begin{aligned}h_{\mathrm{HH}^\bullet(A)}(t)&=1+(r+2)t+(2r+3)t^2+(r+2)t^3\\ &=(1+t)(1+(r+1)t+(r+2)t^2).\end{aligned}$$ □

In fact, in each of the isomorphisms appearing in the statement of the proposition we have given a set of representing cocycles. This will be important in what follows, when we compute the Gerstenhaber algebra structure on the cohomology of A.

We have chosen a system of coordinates in the vector space containing the arrangement A in such a way that one of the lines is given by the equation $x=0$. This was useful in picking a basis for the S-module of derivations $\mathrm{Der}(\mathcal{A})$ and, as a consequence, obtaining a presentation of the algebra A amenable to the computations we wanted to carry out, but the unnaturality of our choice is reflected in the rather unpleasant form of the representatives that we have found for cohomology classes —a consequence of the combination of the truth of Hermann Weyl's dictum that the introduction of coordinates is an act of violence together with that of the everyday observation that violence does not lead to anything good. In the next section we will be able to obtain a more natural description.

4.8. In Proposition 4.7 we considered only line arrangements with $r \geq 3$, that is, with at least 5 lines. As we explained in 4.2, without the restriction the method of calculation that we followed has to be modified, and it turns out that this is not only a technical difference: the actual results are different. Let us describe what happens, starting with the factorizable cases:

- If there are no lines, so that $r = -2$, the arrangement is empty and $\mathscr{D}(\mathcal{A})$ is the second Weyl algebra $\Bbbk[x, y, \partial_x, \partial_y]$.
- If there is one line, then $\mathscr{D}(\mathcal{A})$ is $\Bbbk[x, y, x\partial_x, \partial_y]$ and this is isomorphic to $\mathscr{U}(\mathfrak{s}) \otimes A_1$, with $\mathscr{U}(\mathfrak{s})$ the enveloping algebra of the non-abelian 2-dimensional Lie algebra $\mathfrak{s}$ and A_1 the first Weyl algebra.
- If there are two lines, so that $r = 0$, then $\mathscr{D}(\mathcal{A})$ is $\Bbbk[x, y, x\partial_x, y\partial_y]$, which is isomorphic to $\mathscr{U}(\mathfrak{s}) \otimes \mathscr{U}(\mathfrak{s})$.

The Hochschild cohomology of the Weyl algebras is well-known —for example, from [16]— as is that of $\mathscr{U}(s)$. Using this and Künneth's formula (this is the claim in Theorem XI.3.1 of [4] about the "product $\vee$") we find that when $-2 \leq r \leq 0$ we have for all $i \in \mathbb{N}_0$.

$$\dim \mathrm{HH}^i(\mathscr{D}(\mathcal{A})) = \binom{r+2}{i}.$$

Finally, we have the cases of three and four lines. Up to isomorphism of arrangements, one can assume that the defining polynomials are $Q = xy(x - y)$ and $Q = xy(x - y)(x - \lambda y)$ for some $\lambda \in \Bbbk \setminus \{0, 1\}$, respectively. One can compute the Hochschild cohomology of $\mathscr{D}(\mathcal{A})$ in these cases along the lines of what we did above, but the computation is surprisingly much more involved. We have done the computation using an alternative, much more efficient approach —using a spectral sequence that computes in general the Hochschild cohomology of the enveloping algebra of a Lie–Rinehart pair— on which the first author, together with Thierry Lambre, reports in [9]. Let us here content us with describing the result: when r is 2 or 3, the Hilbert series of $\mathrm{HH}^\bullet(A)$ is

$$h_{\mathrm{HH}^\bullet(A)}(t) = 1 + (r+2)t + (2r+4)t^2 + (r+3)t^3.$$

This differs from the general case of Proposition 4.7 in the coefficients of t^2 and t^3.

For our immediate purposes, we remark that in all cases $\mathrm{HH}^1(\mathscr{D}(\mathcal{A}))$ has dimension equal to the number of lines in the arrangement $\mathcal{A}$, and that its concrete description is the same in all cases.

5 The Gerstenhaber algebra structure on $\mathrm{HH}^\bullet(\mathscr{D}(\mathcal{A}))$

5.1. Let $\mathbf{B}A$ be the usual bar resolution for A as an A-bimodule. There is a morphism of complexes $\phi : \mathbf{P} \to \mathbf{B}A$ over the identity map of A such that $\phi = \phi_K + \phi_N$ with ϕ_K, $\phi_N : \mathbf{P} \to \mathbf{B}A$ maps of A-bimodules such that

$$\phi_K(1|v_1 \wedge \cdots \wedge v_p|1) = \sum_{\pi \in S_p} (-1)^{\varepsilon(\pi)} 1|v_{\pi(1)}|\cdots|v_{\pi(p)}|1,$$

whenever $p \geq 0$ and $v_1, \ldots, v_p \in V$, with the sum running over permutations of degree p, and

$$\begin{aligned}
&\phi_N(1|1) = 0;\\
&\phi_N(1|v|1) = 0, \quad \forall v \in V;\\
&\phi_N(1|x \wedge y|1) = \phi_N(1|x \wedge E|1) = \phi_N(1|y \wedge E|1)\\
&\qquad\qquad = \phi_N(1|x \wedge D|1) = \phi_N(1|D \wedge E|1) = 0;\\
&\phi_N(1|y \wedge D|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|1 - F|1|1|1;\\
&\phi_N(1|x \wedge y \wedge E|1) = \phi_N(1|x \wedge D \wedge E|1) = 0;\\
&\phi_N(1|x \wedge y \wedge D|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|x|1 - q_{(1)}|\bar{q}_{(2)}|x|q_{(3)}|1\\
&\qquad\qquad + q_{(1)}|x|\bar{q}_{(2)}|q_{(3)}|1 - F|x|1|1|1 - F|1|1|x|1;\\
&\phi_N(1|y \wedge D \wedge E|1) = q_{(1)}|\bar{q}_{(2)}|q_{(3)}|E|1 - q_{(1)}|\bar{q}_{(2)}|E|q_{(3)}|1\\
&\qquad\qquad + q_{(1)}|E|\bar{q}_{(2)}|q_{(3)}|1 - F|E|1|1|1 - F|1|1|E|1.
\end{aligned}$$

Here $q_{(1)}|\bar{q}_{(2)}|q_{(3)}$ denotes the element $\nabla(F)$ of $S \otimes S_1 \otimes S$, with an omitted sum, à la Sweedler.
On the other hand, there is a morphism of complexes of A-bimodules $\psi : \mathbf{B}A \to \mathbf{P}$ over the identity map of A such that

$$\begin{aligned}
&\psi_0(1|1) = 1|1,\\
&\psi_1(1|w|1) = w_{(1)}|w_{(2)}|w_{(3)}, && \text{for all standard monomials } w;\\
&\psi_2(1|yD|y|1) = -y|y \wedge D|1 - q_{(1)}|q_{(2)} \wedge y|q_{(3)};\\
&\psi_2(1|y^{r+1}E|y|1) = -y^{r+1}|y \wedge E|1;\\
&\psi_2(1|E|w|1) = -w_{(1)}|w_{(2)} \wedge E|w_{(3)} && \text{for all standard monomials } w;\\
&\psi_2(1|v|w|1) = -1|w \wedge v|1, && \text{if } v, w \in \{x, y, D, E\} \text{ and } vw \text{ is not standard;}\\
&\psi_2(1|w|x|1) = -w_{(1)}|x \wedge w_{(2)}|w_{(3)} && \text{for all standard monomials } w;
\end{aligned}$$

and

$$\psi_2(1|u|v|1) = 0$$

whenever u and v are standard monomials of A such that the concatenation uv is also a standard monomial. This morphism ψ can be taken — and we will take it — to be normalized, so that it vanishes on elementary tensors of $\mathbf{B}A$ with a scalar factor.

5.2. We need the comparison morphisms that we have just described in order to compute the Gerstenhaber bracket on $\mathrm{HH}^\bullet(A)$, but we start with a more immediate application: obtaining a natural basis of the first cohomology space $\mathrm{HH}^1(A)$.

Proposition.

(*i*) *If α is a non-zero element of S_1 that divides Q, then there exists a unique derivation $\partial_\alpha : A \to A$ such that $\partial_\alpha(f) = 0$ for all $f \in S$ and*

$$\partial_\alpha(\delta) = \frac{\delta(\alpha)}{\alpha}$$

for all $\delta \in \mathrm{Der}(\mathcal{A})$.

(*ii*) *If $Q = \alpha_0 \dots \alpha_{r+1}$ is a factorization of Q as a product of elements of S_1, then the cohomology classes of the $r+2$ derivations $\partial_{\alpha_0}, \dots, \partial_{\alpha_{r+1}}$ of A freely span the vector space $\mathrm{HH}^1(A)$.*

Here we are viewing $\mathrm{HH}^1(A)$ as the vector space of outer derivations of A, as usual. It should be noticed that the derivation ∂_α associated to a linear factor of Q does not change if we replace α by one of its non-zero scalar multiples: this means that the basis of $\mathrm{HH}^1(A)$ is really indexed by the lines of the arrangement $\mathcal{A}$.

Proof. (*i*) Let us fix a non-zero element α in S_1 dividing Q. There is at most one derivation $\partial_\alpha : A \to A$ as in the statement of the proposition simply because the algebra A is generated by the set $S \cup \mathrm{Der}(\mathcal{A})$. In order to prove that there is such a derivation, we need only recall from [12, Proposition 4.8] that $\delta(\alpha) \in \alpha S$ for all $\delta \in \mathrm{Der}(\mathcal{A})$ and check that the candidate derivation respects the relations (1) of 2.2 that present the algebra A.

(*ii*) We need to pass from the description of $\mathrm{HH}^1(A)$ as the space of outer derivations to its description in terms of the complex $\mathfrak{X}$ that was used to compute it: we do this with the comparison morphism $\phi : \mathbf{P} \to \mathbf{B}A$ over the identity map that we described in 5.1. If $\delta : A \to A$ is a derivation of A and $\tilde{\delta} : A \otimes A \otimes A \to A$ is the map such that $\tilde{\delta}(a \otimes b \otimes c) = a\delta(b)c$ for all a, b, $c \in A$, which is a 1-cocycle on $\mathbf{B}A$, the composition $\bar{\delta} \circ \phi_1 : A \otimes V \otimes A \to A$ is a 1-cocycle in the complex $\mathrm{Hom}_{A^e}(\mathbf{P}, A)$ whose cohomology class corresponds to δ in the usual description of $\mathrm{HH}^1(A)$ as the space of outer derivations of A. In the notation that we used in 4.1, this cohomology class is that of

$$\delta(x) \otimes \hat{x} + \delta(y) \otimes \hat{y} + \delta(D) \otimes \hat{D} + \delta(E) \otimes \hat{E} \in A \otimes \hat{V}.$$

Using this, we can now prove the second part of the proposition. We can suppose without loss of generality that $\alpha_0 = x$, and then the class of δ_{α_0} in $\mathrm{HH}^1(A)$ is that of

$$1 \otimes \hat{E}.$$

On the other hand, for each $i \in \{1, \dots, r+1\}$, computing we find that the class of ∂_{α_i} is

$$\alpha_{iy} \frac{F}{\alpha_i} \otimes \hat{D} + 1 \otimes \hat{E}.$$

It follows easily from the second part of Lemma 2.3 that these $r+2$ classes span $\mathrm{HH}^1(A)$ and, since the dimension of this space is exactly $r+2$, do so freely. $\square$

The cup product

5.3. We describe the associative algebra structure on $\mathrm{HH}^\bullet(A)$ given by the cup product.

Proposition. *The cup product on* $\mathrm{HH}^\bullet(A)$ *is graded-commutative and such that*

$$\begin{aligned}
&S_r \otimes \hat{D} \smile S_r \otimes \hat{D} = 0; \\
&\phi\hat{D} \smile \hat{E} = \phi\hat{D} \wedge \hat{E}, && \forall \phi \in S_r; \\
&S_r \otimes \hat{D} \smile \mathrm{HH}^2(A) = 0; \\
&1 \otimes \hat{E} \smile \omega_2 = \omega_3; \\
&1 \otimes \hat{E} \smile \kappa \otimes \hat{y} \wedge \hat{D} = \kappa \otimes \hat{y} \wedge \hat{E} \wedge \hat{D}, && \forall \kappa \in S_{r+1}/\langle xF_x, xF_y, yF_y\rangle; \\
&1 \otimes \hat{E} \smile \psi D \otimes \hat{y} \wedge \hat{D} = \psi D \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, && \forall \psi \in S_1; \\
&1 \otimes \hat{E} \smile S_r \otimes \hat{D} \wedge \hat{E} = 0.
\end{aligned}$$

These equalities completely describe the multiplication of the Hochschild cohomology algebra $\mathrm{HH}^\bullet(A)$.

Proof. That the cup product is graded-commutative is a celebrated theorem of Murray Gerstenhaber [7]. There is a morphism of complexes of A-bimodules $\Delta : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ that lifts the canonical isomorphism $A \to A \otimes_A A$ such that $\Delta = \Delta_K + \Delta_N$, with

- $\Delta_K : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ the map of A-bimodules such that for whenever $p \geq 0$ and $v_1, \ldots, v_p \in V$ we have

$$\Delta_K(1|v_1 \wedge \cdots \wedge v_p|1) = \sum (-1)^\varepsilon 1|v_{i_1} \wedge \cdots \wedge v_{i_r}|1 \otimes 1|v_{j_1} \wedge \cdots \wedge v_{j_s}|1,$$

  with the sum taken over all decompositions $r+s=p$ with r, $s \geq 0$, and all permutations $(i_1, \ldots, i_r, j_1, \ldots, j_s)$ of $(1, \ldots, p)$ such that $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_s$, and where ε is the signature of the permutations,
- and $\Delta_N : \mathbf{P} \to \mathbf{P} \otimes_A \mathbf{P}$ the map of A-bimodules such that

$$\begin{aligned}
&\Delta_N(1|1) = 0; \\
&\Delta_N(1|v|1) = 0, \quad \forall v \in V; \\
&\Delta_N(1|v \wedge w|1) = 0, \quad \text{if } v,\ w \in \{x, y, D, E\},\ v \neq w \text{ and } \{v, w\} \neq \{y, D\}; \\
&\Delta_N(1|y \wedge D|1) = f_{(1)}|f_{(2)}|f_{(3)} \otimes 1|f_{(4)}|f_{(5)}; \\
&\Delta_N(1|x \wedge y \wedge D|1) = \Delta_N(1|x \wedge y \wedge E|1) = \Delta_N(1|x \wedge D \wedge E|1) = 0;
\end{aligned}$$

$$\Delta_N(1|y \wedge D \wedge E|1) = -f_{(1)}|f_{(2)} \wedge E|f_{(3)} \otimes 1|f_{(4)}|f_{(5)} \\ + f_{(1)}|f_{(2)}|f_{(3)} \otimes 1|f_{(4)} \wedge E|f_{(5)}.$$

Here we have written $f_{(1)}|f_{(2)}|f_{(3)}|f_{(4)}|f_{(5)}$ for the image of F under the composition

$$S \xrightarrow{\nabla} S \otimes S_1 \otimes S \xrightarrow{\mathrm{id}_S \otimes \mathrm{id}_{S_1} \otimes \nabla} S \otimes S_1 \otimes S \otimes S_1 \otimes S,$$

with an omitted sum, *à la* Sweedler as usual.

We leave the verification that this does define a morphism of complexes to the reader.

The cup product of the algebra $\mathrm{HH}^\bullet(A)$ can be computed using this diagonal morphism Δ. Indeed, we view $\mathrm{HH}^\bullet(A)$ as the cohomology of the complex $\mathrm{Hom}_{A^e}(\mathbf{P}, A)$, and if ϕ and ψ are a p- and a q-cocycle in that complex, the cup product of their cohomology classes is represented by the composition

$$P_{p+q} \xrightarrow{\Delta_{p,q}} P_p \otimes_A P_q \xrightarrow{\phi \otimes \psi} A \otimes_A A = A,$$

with $\Delta_{p,q}$ the component $P_{p+q} \to P_p \otimes P_q$ of the morphism Δ. The multiplication table given in the statement of the composition can be computed in this way, item by item. $\square$

5.4. PROPOSITION.

(*i*) *For all i, j, $k \in \{0, \dots, r+1\}$ we have*

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} + \partial_{\alpha_j} \smile \partial_{\alpha_k} + \partial_{\alpha_k} \smile \partial_{\alpha_i} = 0 \tag{13}$$

and $\mathrm{HH}^1(A) \smile \mathrm{HH}^1(A) = S_r \otimes \hat{D} \wedge \hat{E}$.

(*ii*) *The subalgebra $\mathscr{H}$ of $\mathrm{HH}^\bullet(A)$ generated by $\mathrm{HH}^1(A)$ is the graded-commutative algebra freely generated by its elements $\partial_{\alpha_0}, \dots, \partial_{\alpha_{r+1}}$ of degree 1 subject to the $\binom{r+2}{3}$ relations (13).*

This subalgebra $\mathscr{H}$ is isomorphic to the de Rham cohomology of the complement of the arrangement of lines $\mathcal{A}$. This follows from a direct computation of this cohomology or, in fact, from the solution of Arnold's conjecture by Brieskorn; this is discussed in detail in [12, Section 5.4].

Proof. Using Proposition 5.3 and the description given in the proof of Proposition 5.2 for the derivations ∂_{α_i} we compute immediately that

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} = -\begin{vmatrix} \alpha_{ix} & \alpha_{jx} \\ \alpha_{iy} & \alpha_{jy} \end{vmatrix} \frac{Q}{\alpha_i \alpha_j}$$

for all i, $j \in \{0, \dots, r+1\}$. Using this, we see that for all i, j, $k \in \{0, \dots, r+1\}$ we have

$$\partial_{\alpha_i} \smile \partial_{\alpha_j} + \partial_{\alpha_j} \smile \partial_{\alpha_k} + \partial_{\alpha_k} \smile \partial_{\alpha_i} = -\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \alpha_{ix} & \alpha_{jx} & \alpha_{kx} \\ \alpha_{iy} & \alpha_{jy} & \alpha_{ky} \end{vmatrix} \frac{Q}{\alpha_i \alpha_j \alpha_k} = 0,$$

as the determinant vanishes. This proves the first claim of (i). The second one follows immediately from the description of the cup product of Proposition 5.3.

(ii) Let $\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{F}_n$ be the free graded-commutative algebra generated by $r+2$ generators $w_0, \ldots, w_{r+1}$ of degree 1 subject to the relations $w_i w_j + w_j w_k + w_k w_i = 0$, one for each choice of $i, j, k \in \{0, \ldots, r+1\}$. We have $\mathcal{F}_n = 0$ if $n \geq 3$: whenever i, j and k are elements of $\{1, \ldots, r+1\}$ we have that $w_i w_j w_k = (w_i w_j + w_j w_k + w_k w_i) w_k = 0$, because of graded-commutativity. On the other hand, we have $\dim \mathcal{F}_2 \leq r+1$. To see this, we notice that $\mathcal{F}_2$ is spanned by products $w_i w_j$ with $1 \leq i < j \leq r+1$. If $i+1 < j$ then $w_i w_j = -w_{i+1} w_j - w_{i+1} w_i$: it follows from this that the set of monomials $\{w_i w_{i+1} : 0 \leq i \leq r\}$ already spans $\mathcal{F}_2$.

The first part of the proposition implies that there is a surjective morphism of graded algebras $f : \mathcal{F} \to \mathscr{H}$ such that $f(w_i) = \partial_{\alpha_i}$ for all $i \in \{0, \ldots, r+1\}$, and this map is also injective because the dimension of the component of degree 2 of $\mathscr{H}$, which is $S_r \otimes \hat{D} \wedge \hat{E}$, is $r+1$. □

5.5. Proposition 5.4 describes a part of the associative algebra $\mathrm{HH}^\bullet(A)$, the subalgebra $\mathscr{H}$ generated by $\mathrm{HH}^1(A)$, in terms of the geometry of the arrangement $\mathcal{A}$. It is not clear how to make sense of the complete algebra. We can make the following observation, though. Let us write

$$\mathrm{HH}^2(A)' = \Bbbk\omega_2 \oplus (S_{r+1}/\langle xF_x, xF_y, yF_y \rangle \oplus S_1 D) \otimes \hat{y} \wedge \hat{D},$$

which is a complement of $\mathscr{H}^2$ in $\mathrm{HH}^2(A)$, and let $Q = \alpha_0 \ldots \alpha_{r+1}$ be a factorization of Q as a product of linear factors. If $\delta : A \to A$ is derivation of A, then our description of $\mathrm{HH}^1(A)$ implies that there exist scalars $\delta_0, \ldots, \delta_{r+1} \in \Bbbk$ and an element $u \in A$ such that $\delta = \sum_{i=0}^{r+1} \delta_i \partial_{\alpha_u} + \mathrm{ad}(u)$, and it follows easily from Proposition 5.3 that the map

$$\zeta \in \mathrm{HH}^2(A)' \mapsto \delta \smile \zeta \in \mathrm{HH}^3(A)$$

is either zero or an isomorphism, provided $\sum_{i=0}^{r+1} \delta_i$ is zero or not.

The Gerstenhaber bracket

5.6. Using the comparison morphisms of 5.1, we can now compute the Gerstenhaber bracket. As usual, this is very laborious.

Proposition. *In* $\mathrm{HH}^\bullet(A)$ *we have*

$$
\begin{array}{ll}
[0,\bullet] & \left\{ \; [\mathrm{HH}^0(A), \mathrm{HH}^\bullet(A)] = 0, \right. \\
[1,1] & \left\{ \; [\mathrm{HH}^1(A), \mathrm{HH}^1(A)] = 0, \right. \\
[1,2] & \left\{ \begin{array}{l}
[\mathrm{HH}^1(A), S_r \otimes \hat{D} \wedge \hat{E}] = 0, \\
[u \otimes \hat{D} + \lambda \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] = uw \otimes \hat{y} \wedge \hat{D}, \\
[u \otimes \hat{D} + \lambda \otimes \hat{E}, \omega_2] = ((\mu - \lambda) y F_x + \mu y \bar{F} - y^2 \bar{u}) \otimes \hat{y} \wedge \hat{D},
\end{array} \right. \\
[1,3] & \left\{ \begin{array}{l}
[u \otimes \hat{D} + \lambda \otimes \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}] = uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
[u \otimes \hat{D} + \lambda \otimes \hat{E}, \omega_3] = ((\mu - \lambda) y F_x + \mu y \bar{F} - y^2 \bar{u}) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E},
\end{array} \right. \\
[2,2] & \left\{ \begin{array}{l}
[S_r \otimes \hat{D} \wedge \hat{E}, S_r \otimes \hat{D} \wedge \hat{E}] = 0, \\
[u \otimes \hat{D} \wedge \hat{E}, (v + wD) \otimes \hat{y} \wedge \hat{D}] = uw \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
[u \otimes \hat{D} \wedge \hat{E}, \omega_2] = (\mu y F x + \mu y \bar{F} - y^2 \bar{u}) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
[(S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}, (S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}] = 0, \\
[(S_{r+1} + S_1 D) \otimes \hat{y} \wedge \hat{D}, \omega_2] = 0, \\
[\omega_2, \omega_2] = 0.
\end{array} \right.
\end{array}
$$

Here $u \in S_r$, $\lambda \in \Bbbk$, $v \in S_{r+1}$, $w \in S_1$ *and* $\mu \in \Bbbk$ *and* $\bar{u} \in S_{r-1}$ *are such that* $u = \lambda y^r + x\bar{u}$.

Proof. Let us first recall from [7] how one can compute the Gerstenhaber bracket in the standard complex $\mathrm{Hom}_{A^e}(\mathbf{B}A, A)$. If $f : A^{\otimes q} \to A$ is a q-cochain in the standard complex $\mathrm{Hom}_{A^e}(\mathbf{B}A, A)$, which we identify as usual with $\mathrm{Hom}(A^{\otimes\bullet}, A)$, and $p \geq q$, we denote $\mathsf{w}_p(f) : A^{\otimes p} \to A^{p-q+1}$ the p-cochain in the same complex such that

$$
\mathsf{w}_p(f)(a_1 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p-q+1} (-1)^{(q-1)(i-1)} a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+q-1}) \otimes a_{i+q} \otimes \cdots \otimes a_p.
$$

If now α and β are a p- and a q-cocycle in the standard complex, the Gerstenhaber composition $\diamond$ (which is usually written simply $\circ$) of α and β is the $(p+q-1)$-cochain

$$
\alpha \diamond \beta = \alpha \circ \mathsf{w}_{p+q-1}(\beta)
$$

and the Gerstenhaber bracket is the graded commutator for this composition, so that

$$
[\alpha, \beta] = \alpha \diamond \beta - (-1)^{(p-1)(q-1)} \beta \diamond \alpha.
$$

Next, if α and β are now a p- and a q-cochain in the complex $\mathrm{Hom}_{A^e}(\mathbf{P}, A)$, we can lift them to a p-cochain $\tilde{\alpha} = \alpha \circ \psi_p$ and a q-cochain $\tilde{\beta} = \beta \circ \psi_q$ in the standard complex $\mathrm{Hom}_{A^e}(\mathbf{B}A, A)$, and the Gerstenhaber bracket of the classes of α and β is then represented by the $(p+q-1)$-cochain $[\tilde{\alpha},\tilde{\beta}]\circ\phi_{p+q-1}$. This is the computation we have to do in order to compute brackets in $\mathrm{HH}^\bullet(A)$, except that in some favorable circumstances we can take advantage of the compatibility of the bracket with the product to cut down the work. We do this in several steps.

- Since the morphism ψ is normalized and $\mathrm{HH}^0(A)$ is spanned by $1\in\Bbbk$, it follows immediately that
$$[\mathrm{HH}^0(A),\mathrm{HH}^\bullet(A)]=0.$$
- The Gerstenhaber bracket on $\mathrm{HH}^1(A)$ is induced by the commutator of derivations. From Proposition 5.2 we have a basis of $\mathrm{HH}^1(A)$ whose elements are classes of certain derivations, and it is immediate to check that those derivations commute, so that
$$[\mathrm{HH}^1(A),\mathrm{HH}^1(A)]=0. \tag{14}$$
- We know that the subspace $S_r\otimes\hat{D}\wedge\hat{E}$ of $\mathrm{HH}^2(A)$ is $\mathrm{HH}^1(A)\smile\mathrm{HH}^1(A)$. Since $\mathrm{HH}^\bullet(A)$ is a Gerstenhaber algebra and we know that (14) holds, it follows that
$$[\mathrm{HH}^1(A),S_r\otimes\hat{D}\wedge\hat{E}]=0.$$
For exactly the same reasons we also have that
$$[S_r\otimes\hat{D}\wedge\hat{E},S_r\otimes\hat{D}\wedge\hat{E}]=0.$$
- Let $\alpha=u\otimes\hat{D}+\lambda\otimes\hat{E}$, with $u\in S_r$ and $\lambda\in\Bbbk$. If $\beta=(v+wD)\otimes\hat{y}\wedge\hat{D}$, with $v\in S_{r+1}$ and $w\in S_1$, one can compute that $(\tilde{\alpha}\diamond\tilde{\beta})\circ\phi=uw\otimes\hat{y}\wedge\hat{D}$ and that $(\tilde{\beta}\diamond\tilde{\alpha})\circ\phi=0$: it follows from this that
$$[\alpha,(v+wD)\otimes\hat{y}\wedge\hat{D}]=uw\otimes\hat{y}\wedge\hat{D}.$$
On the other hand, we have $(\tilde{\omega}_2\diamond\tilde{\alpha})\circ\phi=0$ and
$$\begin{aligned}[\tilde{\alpha},\tilde{\omega}_2]\circ\phi=(\tilde{\alpha}\diamond\tilde{\omega}_2)\circ\phi&=(yu-\lambda y^{r+1})\otimes\hat{x}\wedge\hat{D}+\lambda y\bar{F}\otimes\hat{y}\wedge\hat{D}\\&=\big((\mu-\lambda)yF_x+\mu y\bar{F}-y^2\bar{u}\big)\otimes\hat{y}\wedge\hat{D}\\&\qquad-\delta^1\big(((\mu-\lambda)\bar{F}-y\bar{u})E\otimes\hat{D}+(\lambda-\mu)y\otimes\hat{x}\big)\end{aligned}$$
with $\bar{u}\in S_{r-1}$ and $\mu\in\Bbbk$ chosen so that $u=\mu y^r+x\bar{u}$.
Finally, if $v\in S_{r+1}$ and $w\in S_1$, using the compatibility of the bracket and the product and what we know so far we see that
$$\begin{aligned}[\alpha,(v+wD)\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}]&=[\alpha,1\otimes E\smile(v+wD)\otimes\hat{y}\wedge\hat{D}]\\&=1\otimes E\smile[\alpha,(v+wD)\otimes\hat{y}\wedge\hat{D}]\\&=1\otimes E\smile uw\otimes\hat{y}\wedge\hat{D}\\&=uw\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}\end{aligned}$$

and, similarly, that

$$\begin{aligned}[\alpha,\omega_3] &= [\alpha,\omega_2\smile 1\otimes\hat{E}] = [\alpha,\omega_2]\smile 1\otimes\hat{E}+\omega_2\smile[\alpha,1\otimes\hat{E}]\\ &= \big((\mu-\lambda)yF_x+\mu y\bar{F}-y^2\bar{u}\big)\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}.\end{aligned}$$

- Let $u\in S_r$. If $v\in S_{r+1}$ and $w\in S_1$, we have

$$\begin{aligned}&[u\otimes\hat{D}\wedge\hat{E},(v+wD)\otimes\hat{y}\wedge\hat{D}] = [u\otimes\hat{D}\smile 1\otimes\hat{E},(v+wD)\otimes\hat{y}\wedge\hat{D}]\\ &\qquad= [u\otimes\hat{D},(v+wD)\otimes\hat{y}\wedge\hat{D}]\smile 1\otimes\hat{E}\\ &\qquad\qquad\qquad\qquad+u\otimes\hat{D}\smile[1\otimes\hat{E},(v+wD)\otimes\hat{y}\wedge\hat{D}]\\ &\qquad= uw\otimes\hat{y}\wedge\hat{D}\smile 1\otimes\hat{E} = uw\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}.\end{aligned}$$

  Similarly,

$$\begin{aligned}[u\otimes\hat{D}\wedge\hat{E},\omega_2] &= [u\otimes\hat{D}\smile 1\otimes\hat{E},\omega_2]\\ &= [u\otimes\hat{D},\omega_2]\smile 1\otimes\hat{E}+u\otimes\hat{D}\smile[1\otimes\hat{E},\omega_2]\\ &= \big(\mu yF_x+\mu y\bar{F}-y^2\bar{u}\big)\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}.\end{aligned}$$

  if $u=\mu y^r+x\bar{u}$ with $\mu\in\Bbbk$ and $\bar{u}\in S_{r-1}$.
- Let now $\alpha=(v+wD)\otimes\hat{y}\wedge\hat{D}$ and $\beta=(s+tD)\otimes\hat{y}\wedge\hat{D}$, with v, $s\in S_{r+1}$ and w, $t\in S_1$. We claim that $(\tilde{\alpha}\diamond\tilde{\beta})\circ\phi=0$, so that, by symmetry, we have $[\tilde{\alpha},\tilde{\beta}]\circ\phi=0$. To verify our claim, we compute:

$$1|x\wedge y\wedge E|1 \overset{\phi}{\longmapsto} \boxed{\Bbbk[x,y,E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\beta})} 0;$$

$$1|x\wedge D\wedge E|1 \overset{\phi}{\longmapsto} \boxed{\Bbbk[x,D,E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\beta})} 0;$$

$$\begin{aligned}1|x\wedge y\wedge D|1 &\overset{\phi}{\longmapsto} 1|x|y|D|1-1|x|D|y|1+1|D|x|y|1\\ &\qquad\qquad -1|D|y|x|1+1|y|D|x|1-1|y|x|D|1+\boxed{S^{\otimes 5}}\\ &\xmapsto{\mathsf{w}_3(\tilde{\beta})} 1|(s+tD)|x|1-1|x|(s+tD)|1\\ &\overset{\psi}{\longmapsto} -s_{(1)}|x\wedge s_{(2)}|s_{(3)}-t_{(1)}|x\wedge t_{(2)}|t_{(3)}D-t|x\wedge D|1\\ &\overset{\alpha}{\longmapsto} 0;\end{aligned}$$

$$\begin{aligned}1|y\wedge D\wedge E|1 &\overset{\phi}{\longmapsto} 1|y|D|E|1-1|y|E|D|1+1|E|y|D|1-1|E|D|y|1\\ &\qquad\qquad +1|D|E|y|1-1|D|y|E|1+\boxed{\Bbbk[x,y,E]^{\otimes 5}}\\ &\xmapsto{\mathsf{w}_3(\tilde{\beta})} 1|(s+tD)|E|1-1|E|(s+tD)|1\\ &\overset{\psi}{\longmapsto} s_{(1)}|s_{(2)}\wedge E|s_{(3)}+t_{(1)}|t_{(2)}\wedge E|t_{(3)}D+t|D\wedge E|1\\ &\overset{\alpha}{\longmapsto} 0.\end{aligned}$$

- Let again $\alpha = (v + wD) \otimes \hat{y} \wedge \hat{D}$, with $v \in S_{r+1}$ and $w \in S_1$, and let us compute that $(\tilde{\omega}_2 \diamond \tilde{\alpha}) \circ \phi_3 = -w(yD - y^{r+1}E) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$.

$$1|x \wedge y \wedge z|1 \xmapsto{\phi_3} \boxed{\Bbbk[x,y,E]^{\otimes 5}} \xmapsto{\mathsf{w}_2(\tilde{\alpha})} 0$$

$$1|x \wedge D \wedge E|1 \xmapsto{\phi_3} \boxed{\Bbbk[x,D,E]^{\otimes 5}} \xmapsto{\mathsf{w}_3(\tilde{\alpha})} 0$$

$$\begin{aligned}
1|x \wedge y \wedge D|1 &\xmapsto{\phi_3} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde{\alpha})} 1|(v+wD)|x|1 + 1|x|(v+wD)|1 \\
&\xmapsto{\psi_2} -v_{(1)}|x \wedge v_{(2)}|v_{(3)} - w_{(1)}|x \wedge w_{(2)}|w_{(3)}D - w|x \wedge D|1 \\
&\xmapsto{\omega_2} -w(yD - y^{r+1}E)
\end{aligned}$$

$$\begin{aligned}
1|y \wedge D \wedge E|1 &\xmapsto{\phi_3} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 - 1|E|D|y|1 \\
&\qquad + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde{\alpha})} 1|(v+wD)|E|1 - 1|E|(v+wD)|1 \\
&\xmapsto{\psi_2} v_{(1)}|v_{(2)} \wedge E|v_{(3)} + w_{(1)}|w_{(2)} \wedge E|w_{(3)}D + w|D \wedge E|1 \\
&\xmapsto{\omega_2} 0.
\end{aligned}$$

Similarly, we have that $(\tilde{\alpha} \diamond \tilde{\omega}_2) \circ \phi_3 = y(v + wD) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}$:

$$\begin{aligned}
1|x \wedge y \wedge z|1 &\xmapsto{\phi_3} \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_2(\tilde{\omega}_2)} 0
\end{aligned}$$

$$\begin{aligned}
1|x \wedge D \wedge E|1 &\xmapsto{\phi_3} 1|x|D|E|1 - 1|x|E|D|1 + 1|E|x|D|1 \\
&\qquad - 1|E|D|x|1 + 1|D|E|x|1 - 1|D|x|E|1 \\
&\xmapsto{\mathsf{w}_3(\tilde{\omega}_2)} -1|E|(yD - y^{r+1}E)|1 + 1|(yD - y^{r+1})|E|1 \\
&\xmapsto{\psi_2} -1|y \wedge E|D - y|D \wedge E|1 + \sum_{i=0}^{r} y^i|y \wedge E|y^{r-i} \\
&\xmapsto{\alpha} 0
\end{aligned}$$

$$\begin{aligned}
1|x \wedge y \wedge D|1 &\xmapsto{\phi_3} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\tilde{\omega}_2)} -1|x|y\bar{F}E|x|1 - 1|(yD - y^{r+1}E)|y|1 \\
&\qquad + 1|y\bar{F}E|x|1 + 1|y|(yD - y^{r+1}E)|1 \\
&\xmapsto{\psi_2} y|y \wedge D|1 - y^{r+1}|y \wedge E|1 \\
&\qquad - (y\bar{F}E)_{(1)}|x \wedge (y\bar{F}E)_{(2)}|(y\bar{F}E)_{(3)}
\end{aligned}$$

$$
\begin{aligned}
&\overset{\alpha}{\longmapsto} -y(v+wD) \\
1|y\wedge D\wedge E|1 &\overset{\phi_3}{\longmapsto} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\
&\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xrightarrow{\mathsf{w}_3(\tilde\omega_2)} -1|E|y\bar{F}E|1 + 1|y\bar{F}E|E|1 \\
&\overset{\psi_2}{\longmapsto} (y\bar{F}E)_{(1)}|(y\bar{F}E)_{(2)}\wedge E|(y\bar{F}E)_{(3)} \\
&\overset{\alpha}{\longmapsto} 0.
\end{aligned}
$$

It follows from this that

$$
\begin{aligned}
[\tilde\omega_2,\tilde\alpha]\circ\phi_3 &= -w(yD-y^{r+1}E)\otimes\hat{x}\wedge\hat{y}\wedge\hat{D} + y(v+wD)\otimes\hat{x}\wedge\hat{y}\wedge\hat{D} \\
&= (yv+y^{r+1}E)\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}
\end{aligned}
$$

and, as we say in 4.5, this is a coboundary.

- The one computation that remains is that of the bracket of ω_2 with itself, which is represented by the 3-cocycle

$$
[\tilde\omega_2,\tilde\omega_2]\circ\phi_3 = 2(\tilde\omega_2\diamond\tilde\omega_2)\circ\phi_3 = 2y^2\bar{F}E\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}, \tag{15}
$$

as can be seen from the following calculation:

$$
\begin{aligned}
1|x\wedge y\wedge z|1 &\overset{\phi_3}{\longmapsto} \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xrightarrow{\mathsf{w}_2(\tilde\omega_2)} 0 \\
1|x\wedge D\wedge E|1 &\overset{\phi_3}{\longmapsto} 1|x|D|E|1 - 1|x|E|D|1 + 1|E|x|D|1 \\
&\qquad\qquad - 1|E|D|x|1 + 1|D|E|x|1 - 1|D|x|E|1 \\
&\xrightarrow{\mathsf{w}_3(\tilde\omega_2)} -1|E|(yD-y^{r+1}E)|1 + 1|(yD-y^{r+1})|E|1 \\
&\overset{\psi_2}{\longmapsto} -1|y\wedge E|D - y|D\wedge E|1 + \sum_{i=0}^{r} y^i|y\wedge E|y^{r-i} \\
&\overset{\omega_2}{\longmapsto} 0 \\
1|x\wedge y\wedge D|1 &\overset{\phi_3}{\longmapsto} 1|x|y|D|1 - 1|x|D|y|1 + 1|D|x|y|1 \\
&\qquad - 1|D|y|x|1 + 1|y|D|x|1 - 1|y|x|D|1 + \boxed{S^{\otimes 5}} \\
&\xrightarrow{\mathsf{w}_3(\tilde\omega_2)} -1|x|y\bar{F}E|x|1 - 1|(yD-y^{r+1}E)|y|1 \\
&\qquad\qquad + 1|y\bar{F}E|x|1 + 1|y|(yD-y^{r+1}E)|1 \\
&\overset{\psi_2}{\longmapsto} y|y\wedge D|1 - y^{r+1}|y\wedge E|1 \\
&\qquad\qquad - (y\bar{F}E)_{(1)}|x\wedge(y\bar{F}E)_{(2)}|(y\bar{F}E)_{(3)} \\
&\overset{\omega_2}{\longmapsto} -y^2\bar{F}E
\end{aligned}
$$

$$
\begin{aligned}
1|y \wedge D \wedge E|1 &\xmapsto{\phi_3} 1|y|D|E|1 - 1|y|E|D|1 + 1|E|y|D|1 \\
&\qquad - 1|E|D|y|1 + 1|D|E|y|1 - 1|D|y|E|1 + \boxed{\Bbbk[x,y,E]^{\otimes 5}} \\
&\xmapsto{\mathsf{w}_3(\bar\omega_2)} -1|E|y\bar{F}E|1 + 1|y\bar{F}E|E|1 \\
&\xmapsto{\psi_2} (y\bar{F}E)_{(1)}|(y\bar{F}E)_{(2)} \wedge E|(y\bar{F}E)_{(3)} \\
&\xmapsto{\omega_2} 0.
\end{aligned}
$$

Now the 3-cocycle (15) is a coboundary, again by what we saw in 4.5, so that we have $[\omega_2, \omega_2] = 0$.

This completes the proof of the proposition. □

6 Hochschild homology, cyclic homology and K-theory

6.1. For completeness, we determine the rest of the 'usual' homological invariants of our algebra A. Recall that our ground field $\Bbbk$ is of characteristic zero.

Proposition. *The inclusion $T = \Bbbk[E] \to A$ induces an isomorphism in Hochschild homology and in cyclic homology. In particular, there are isomorphisms of vector spaces*

$$
\mathrm{HH}_i(A) \cong \begin{cases} T, & \text{if } i = 0 \text{ or } i = 1; \\ 0, & \text{if } i \geq 2; \end{cases} \qquad \mathrm{HC}_i(A) \cong \begin{cases} T, & \text{if } i = 0; \\ \mathrm{HC}_i(\Bbbk), & \text{if } i > 0. \end{cases}
$$

On the other hand, the inclusion $\Bbbk \to A$ induces an isomorphism in periodic cyclic homology and in K-theory.

Proof. As we know, the algebra A is $\mathbb{N}_0$-graded and for each $n \in \mathbb{N}_0$ its homogeneous component A_n of degree n is the eigenspace corresponding to the eigenvalue n of the derivation $\mathrm{ad}(E) : A \to A$. On one hand, this grading of A induces as usual an $\mathbb{N}_0$-grading on the Hochschild homology $\mathrm{HH}_\bullet(A)$ of A; on the other, the derivation $\mathrm{ad}(E)$ induces a linear map $L_{\mathrm{ad}(E)} : \mathrm{HH}_\bullet(A) \to \mathrm{HH}_\bullet(A)$ as in [10, §4.1.4] and, in fact, for all $n \in \mathbb{N}_0$ the homogeneous component $\mathrm{HH}_\bullet(A)_n$ of degree n for that grading coincides with the eigenspace corresponding to the eigenvalue n of $L_{\mathrm{ad}(E)}$. As the derivation $\mathrm{ad}(E)$ is inner, it follows from [10, Proposition 4.1.5] that the map $L_{\mathrm{ad}(E)}$ is actually the zero map and this tells us in our situation that $\mathrm{HH}_\bullet(A)_n = 0$ for all $n \neq 0$. Of course, this means that $\mathrm{HH}_\bullet(A) = \mathrm{HH}_\bullet(A)_0$ and, since A is non-negatively graded, it is immediate that the 0th homogeneous component $\mathrm{HH}_\bullet(A)_0$ coincides with the Hochschild homology $\mathrm{HH}_\bullet(A_0)$ of A_0 and that the map $\mathrm{HH}_\bullet(A_0) \to \mathrm{HH}_\bullet(A)$ induced by the inclusion $A_0 \hookrightarrow A$ is an isomorphism. Now, in the notation of [10, Theorem 4.1.13], this tells us that $\widetilde{\widetilde{\mathrm{HH}}}_\bullet(A) = 0$ so that by that theorem we also have $\widetilde{\widetilde{\mathrm{HC}}}_\bullet(A) = 0$: this means precisely that the inclusion $A_0 \hookrightarrow A$ induces an isomorphism $\mathrm{HC}_\bullet(A_0) \to \mathrm{HC}_\bullet(A)$ in cyclic homology. Together with

the well-known computation of the Hochschild homology of a polynomial ring and that of the cyclic homology of symmetric algebras [10, Theorem 3.2.5], this proves the first claim of the statement.
In the proof of the lemma of 3.4 we constructed an increasing filtration F on the algebra A with $F_{-1}A = 0$ and such that the corresponding graded algebra is the commutative polynomial ring $\operatorname{gr} A = \Bbbk[x, y, D, E]$ with generators x and y in degree 0 and D and E in degree 1. In particular, both $\operatorname{gr} A$ and its subalgebra $\operatorname{gr}_0 A$ of degree 0 have finite global dimension. It follows from a theorem of D. Quillen [13, p. 117, Theorem 7] that the inclusion $\Bbbk[x, y] = F_0A \to A$ induces an isomorphism $K_i(\Bbbk[x, y]) \to K_i(A)$ in K-theory for all $i \geq 0$. Similarly, the theorem of J. Block [2, Theorem 3.4] tells us that that inclusion induces an isomorphism $\mathrm{HP}_\bullet(\Bbbk[x, y]) \to \mathrm{HP}_\bullet(A)$ in periodic cyclic homology. As the inclusion $\Bbbk \to \Bbbk[x, y]$ induces an isomorphism in K-theory and in periodic cyclic homology, we see that the second claim of the proposition holds. □

7 The Calabi–Yau property

7.1. The enveloping algebra A^e of A is a bimodule over itself, with left and right actions $\triangleright$ and $\triangleleft$ given by 'outer' and 'inner' multiplication, respectively, so that if $a \otimes b$, $c \otimes d$ and $e \otimes f$ are elementary tensors in A^e, we have

$$a \otimes b \triangleright c \otimes d \triangleleft e \otimes f = ace \otimes fdb.$$

From this bimodule structure we obtain a duality functor

$$\operatorname{Hom}_{A^e}(-, A^e) : {}_{A^e}\mathsf{Mod} \to \mathsf{Mod}_{A^e}.$$

On the other hand, using the anti-automorphism $\tau : A^e \to A^e$ such that $\tau(a \otimes b) = b \otimes a$ for all $a, b \in A$, we can turn a right A^e-module M into a left A^e-module, with action $u \triangleright m = m \triangleleft \tau(u)$ for all $u \in A^e$ and all $m \in M$. In this way, we obtain an isomorphism of categories $\tau^* : \mathsf{Mod}_{A^e} \to {}_{A^e}\mathsf{Mod}$. We denote $(-)^\vee : {}_{A^e}\mathsf{Mod} \to {}_{A^e}\mathsf{Mod}$ the composition $\tau^* \circ \operatorname{Hom}_{A^e}(-, A^e)$.
Let now W be a finite dimensional vector space, let W^* be the vector space dual to W, and view $A \otimes W \otimes A$ and $A \otimes W^* \otimes A$ as left A^e-modules using the usual 'exterior' action. There is a unique $\Bbbk$-linear map

$$\Phi : A \otimes W^* \otimes A \to (A \otimes W \otimes A)^\vee$$

such that $\Phi(a \otimes \phi \otimes b)(1 \otimes w \otimes 1) = \phi(w)b \otimes a$ and it is an isomorphism of left A^e-modules: we will view it in all that follows as an identification.

7.2. Proposition. *The algebra A is twisted Calabi-Yau of dimension 4 with modular automorphism $\sigma : A \to A$ such that*

$$\sigma(x) = x, \qquad \sigma(y) = y, \qquad \sigma(D) = D + F_y, \qquad \sigma(E) = E + r + 2.$$

Let us recall from [8] that this means that A has a resolution of finite length by finitely generated projective A-bimodules, that $\operatorname{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq 4$ and that $\operatorname{Ext}^4_{A^e}(A, A^e) \cong A_\sigma$, the A-bimodule obtained from A by twisting its right action using the automorphism σ, so that $a \triangleright x \triangleleft b = ax\sigma(b)$ for all a, $b \in A$ and all $x \in A_\sigma$.

Proof. A direct computation shows that there is indeed an automorphism σ of A as in the statement of the proposition. We already know that A has a resolution $\mathbf{P}$ of length 4 by finitely generated free A-bimodules, so we need only compute $\operatorname{Ext}^\bullet_{A^e}(A, A^e)$, and this is the cohomology of the complex $\mathbf{P}^\vee$ obtained by applying the functor described in 7.1 to $\mathbf{P}$. Using the identifications introduced there, this complex $\mathbf{P}^\vee$ is

$$A \mid A \xrightarrow{d_1^\vee} A \mid V^* \mid A \xrightarrow{d_2^\vee} A \mid \Lambda^2 V^* \mid A \xrightarrow{d_3^\vee} A \mid \Lambda^3 V^* \mid A \xrightarrow{d_4^\vee} A \mid \Lambda^4 V^* \mid A$$

with left A^e-linear differentials such that

$$\begin{aligned}
d_1^\vee(1 \otimes 1) &= -[x, 1 \otimes \hat{x} \otimes 1] - [y, 1 \otimes \hat{y} \otimes 1] - [D, 1 \otimes \hat{D} \otimes 1] \\
&\qquad - [E, 1 \otimes \hat{E} \otimes 1]; \\
d_2^\vee(1 \otimes \hat{x} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \otimes 1] + [D, 1 \otimes \hat{x} \wedge \hat{D} \otimes 1] \\
&\qquad + [E, 1 \otimes \hat{x} \wedge \hat{E} \otimes 1] + 1 \otimes \hat{x} \wedge \hat{E} \otimes 1 + \tilde{\nabla}_x^{\hat{y} \wedge \hat{D}}(F); \\
d_2^\vee(1 \otimes \hat{y} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{y} \otimes 1] + [D, 1 \otimes \hat{y} \wedge \hat{D} \otimes 1] \\
&\qquad + [E, 1 \otimes \hat{y} \wedge \hat{E} \otimes 1] + 1 \otimes \hat{y} \wedge \hat{E} \otimes 1 + \tilde{\nabla}_y^{\hat{y} \wedge \hat{D}}(F); \\
d_2^\vee(1 \otimes \hat{D} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{D} \otimes 1] - [y, 1 \otimes \hat{y} \wedge \hat{D} \otimes 1] \\
&\qquad + [E, 1 \otimes \hat{D} \wedge \hat{E} \otimes 1] + r \otimes \hat{D} \wedge \hat{E} \otimes 1; \\
d_2^\vee(1 \otimes \hat{E} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{E} \otimes 1] - [y, 1 \otimes \hat{y} \wedge \hat{E} \otimes 1] \\
&\qquad - [D, 1 \otimes \hat{D} \wedge \hat{E} \otimes 1]; \\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{y} \otimes 1) &= -[D, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \otimes 1] - \tilde{\nabla}_y^{\hat{x} \wedge \hat{y} \wedge \hat{D}}(F) \\
&\qquad - [E, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1] - 2 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1; \\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{D} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \otimes 1] - [E, 1 \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1] \\
&\qquad - (r+1) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1; \\
d_3^\vee(1 \otimes \hat{x} \wedge \hat{E} \otimes 1) &= [y, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1] + [D, 1 \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1] \\
&\qquad + \tilde{\nabla}_x^{\hat{y} \wedge \hat{D} \wedge \hat{E}}(F); \\
d_3^\vee(1 \otimes \hat{y} \wedge \hat{D} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \otimes 1] - [E, 1 \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \otimes 1] \\
&\qquad - (r+1) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \otimes 1; \\
d_3^\vee(1 \otimes \hat{y} \wedge \hat{E} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} \otimes 1] + [D, 1 \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \otimes 1] \\
&\qquad + \tilde{\nabla}_y^{\hat{y} \wedge \hat{D} \wedge \hat{E}}(F); \\
d_3^\vee(1 \otimes \hat{D} \wedge \hat{E} \otimes 1) &= -[x, 1 \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \otimes 1] - [y, 1 \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \otimes 1];
\end{aligned}$$

$$\begin{aligned}
d_4^\vee(1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\otimes 1) &= [E, 1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1] \\
&\qquad + (r+2)\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1; \\
d_4^\vee(1\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}\otimes 1) &= -[D, 1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1] - \tilde{\nabla}_y^{\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}}(F); \\
d_4^\vee(1\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}\otimes 1) &= [y, 1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1]; \\
d_4^\vee(1\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1) &= -[x, 1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1],
\end{aligned}$$

where each $\tilde{\nabla}_x^u$ is the image of ∇_x^u under the map $a\otimes u\otimes b\mapsto b\otimes u\otimes a$, and the same with each $\tilde{\nabla}_y^u$.
Let us now identify $\mathbf{P}\otimes_A A_\sigma$ with $\mathbf{P}$ as vector spaces, remembering that the bimodule structure on $\mathbf{P}$ with this identification is given by $a\triangleright x\triangleleft b = ax\sigma(b)$ for all a, $b\in A$ and all $x\in\mathbf{P}$. There is a morphism of complexes of A-bimodules $\psi:\mathbf{P}^\vee\to\mathbf{P}\otimes_A A_\sigma$ such that

$$\begin{aligned}
&\psi(1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1) = 1\otimes 1; \\
&\psi(1\otimes\hat{y}\wedge\hat{D}\wedge\hat{E}\otimes 1) = -1\otimes x\otimes 1; \\
&\psi(1\otimes\hat{x}\wedge\hat{D}\wedge\hat{E}\otimes 1) = 1\otimes y\otimes 1; \\
&\psi(1\otimes\hat{x}\wedge\hat{y}\wedge\hat{E}\otimes 1) = -1\otimes D\otimes 1 - \xi; \\
&\psi(1\otimes\hat{x}\wedge\hat{y}\wedge\hat{D}\otimes 1) = 1\otimes E\otimes 1; \\
&\psi(1\otimes\hat{D}\wedge\hat{E}\otimes 1) = -1\otimes x\wedge y\otimes 1; \\
&\psi(1\otimes\hat{x}\wedge\hat{D}\otimes 1) = 1\otimes y\wedge E\otimes 1; \\
&\psi(1\otimes\hat{y}\wedge\hat{D}\otimes 1) = -1\otimes x\wedge E\otimes 1; \\
&\psi(1\otimes\hat{y}\wedge\hat{E}\otimes 1) = 1\otimes x\wedge D\otimes 1 + x\wedge\xi; \\
&\psi(1\otimes\hat{x}\wedge\hat{E}\otimes 1) = -1\otimes y\wedge D\otimes 1 + \zeta; \\
&\psi(1\otimes\hat{x}\wedge\hat{y}\otimes 1) = -1\otimes D\wedge E\otimes 1 - \xi\wedge E; \\
&\psi(1\otimes\hat{E}\otimes 1) = 1\otimes x\wedge y\wedge D\otimes 1; \\
&\psi(1\otimes\hat{D}\otimes 1) = -1\otimes x\wedge y\wedge E\otimes 1; \\
&\psi(1\otimes\hat{y}\otimes 1) = 1\otimes x\wedge D\wedge E\otimes 1 + x\wedge\xi\wedge E; \\
&\psi(1\otimes\hat{x}\otimes 1) = -1\otimes y\wedge D\wedge E\otimes 1 + \zeta\wedge E; \\
&\psi(1\otimes 1) = 1\otimes x\wedge y\wedge D\wedge E\otimes 1,
\end{aligned}$$

where $\xi\in A\otimes V\otimes A$ and $\zeta\in A\otimes\Lambda^2V\otimes A$ are chosen so that

$$d_1(\xi) = \tilde{\nabla}_y(F) - 1|F_y, \qquad d_2(\zeta) = \xi y - y\xi - 1|y|F_y - \tilde{\nabla}_x^x(F) + \nabla(F).$$

That there are elements which satisfy these two conditions follows immediately from the exactness of the Koszul resolution of S as an S-bimodule — indeed, the right hand sides of the two conditions are cycles in that complex —

but we can exhibit a specific choice: if we write $F = \sum_{a+b=r+1} c_a x^a y^b$, with $c_0, \ldots, c_{r-1} \in kk$, then we can pick

$$\xi = \sum_{\substack{a+b=r+1\\ s+t+1=b-1}} (t+1)c_a y^s|y|x^a y^t, \qquad \zeta = \sum_{\substack{a+b=r+1\\ s+t+1=b\\ s'+t'+1=a}} c_a x^{s'} y^s|x\wedge y|x^{t'} y^t.$$

That these formulas for ψ do indeed define a morphism of complexes follows from a direct computation and it is easy to see that it is in fact an isomorphism, as for an appropriate ordering of the bases of the bimodules involved the matrices for the components of ψ are upper triangular. Of course, it therefore induces an isomorphism in cohomology and, since A_σ is A-projective on the left, we conclude that there are isomorphisms of A-bimodules

$$H^i(\mathbf{P}^\vee) \cong H^i(\mathbf{P}\otimes_A A_\sigma) \cong \begin{cases} A_\sigma & \text{if } i=4;\\ 0 & \text{if } i\neq 4.\end{cases}$$

This completes the proof. □

8 Automorphisms, isomorphisms and normal elements

8.1. Our next objective is to compute the group of automorphisms of the algebra A. We start by describing some graded automorphisms of A. Later we will see that these are, in fact, *all* the graded automorphisms of our algebra, and that together with the exponentials of locally ad-nilpotent elements they generate the whole group $\operatorname{Aut}(A)$.

Lemma. *If* $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2(\Bbbk)$ *and* $e\in\Bbbk^\times$ *are such that*

$$\frac{1}{(ad-bc)e} Q(ax+by, cx+dy) = Q(x,y),$$

and $v\in\Bbbk$ *and* $\phi_0 \in S_r$*, then there is a homogeneous algebra automorphism* $\theta : A\to A$ *such that*

$$\theta(x) = ax+by, \qquad \theta(y) = cx+dy, \qquad \theta(E) = E+v$$

and

$$\theta(D) = \phi_0 - \frac{ebF}{ax+by}E + eD. \tag{16}$$

Proof. This is proved by a straightforward calculation. It should be noted that the quotient appearing in the formula (16) is always a polynomial. □

8.2. Recall that a *higher derivation* of A is a sequence $d = (d_i)_{i\geq 0}$ of linear maps $A\to A$ such that $d_0 = \mathrm{id}_A$ and for all a, $b\in A$ and all $i\geq 0$ we have the *higher Leibniz identity*

$$d_i(ab) = \sum_{s+t=i} d_s(a)d_t(b).$$

It is clear that if $d = (d_i)_{i\geq 0}$ is a higher derivation and m is a positive integer, then the sequence $d^{[m]} = (d_i^{[m]})_{i\geq 0}$ with

$$d_i^{[m]} = \begin{cases} d_{i/m}, & \text{if } i \text{ is divisible by } m; \\ 0, & \text{if not} \end{cases}$$

is also a higher derivation. On the other hand, if $d = (d_i)_{i\geq 0}$ and $d' = (d'_i)_{i\geq 0}$ are higher derivations of A, we can construct a new higher derivation $(d''_i)_{i\geq 0}$, which we denote $d \circ d'$, putting $d''_i = \sum_{s+t=i} d_s \circ d'_t$ for all $i \geq 0$. Finally, if $\delta : A \to A$ is a derivation of A, then the sequence $(\frac{1}{i!}\delta^i)_{i\geq 0}$ is a higher derivation, which we denote by $\exp(\delta)$; notice that this makes sense because our ground field $\Bbbk$ has characteristic zero.

We let $D(A)$ be the subalgebra of $\operatorname{End}_{\Bbbk}(A)$ generated by $\operatorname{Der}(A)$, and say that two higher derivations $d = (d_i)_{i\geq 0}$ and $d' = (d'_i)_{i\geq 0}$ of A are equivalent, and write $d \sim d'$, if for all $i \geq 0$ the map $d_i - d'_i$ is in the subalgebra of $\operatorname{End}_{\Bbbk}(A)$ generated by $D(A)$ and $d_0, \ldots, d_{i-1}$; one can check that this is indeed an equivalence relation on the set of higher derivations.

8.3. We recall the following very useful lemma from [1] (where it appears as the *sous-lemme* of Section 1.4):

LEMMA. *If $d = (d_i)_{i\geq 0}$ is a higher derivation of A, then $d_i \in D(A)$ for all non-negative integers i.*

Proof. The result is an easy consequence of the fact that

> *if d is a higher derivation of A and $j \geq 1$, then there exists a higher derivation $d' = (d'_i)_{i\geq 0}$ such that $d' \sim d$, $d'_i = 0$ if $1 < i < j$, and d'_j is an element of* $\operatorname{Der}(A)$. (17)

To prove that this holds, let $d = (d_i)_{i\geq 0}$ and suppose there is an $j \geq 1$ such that that $d_i = 0$ if $1 < i < j$. The higher Leibniz identity implies that d_j is an element of $\operatorname{Der}(A)$, and then we can consider the higher derivation $\exp(-d_j)^{[j]}$. We let $d' = (d'_i)_{i\geq 0}$ be the composition $\exp(-d_j)^{[j]} \circ d$. It is immediate that $d \sim d'$ and a simple computation shows that $d'_i = 0$ if $1 < i < j + 1$. The claim (17) follows inductively from this. ☐

8.4. LEMMA. *An element of A commutes with x and with y if and only if it belongs to S.*

Proof. The sufficiency of the condition is clear. To prove the necessity, let $e \in A$ be such that $[x, e] = [y, e] = 0$. There are an integer $m \geq 0$ and elements $\phi_0, \ldots, \phi_m$ in the subalgebra generated by x, y and D in A such that $e = \sum_{i=0}^{m} \phi_i E^i$, and we have $0 = [x, e] = \sum_{i=0}^{m} \phi_i \tau_1(E^i)$: this tells us that $\phi_i = 0$ if $i > 0$, and that $e = \phi_0$. In particular, there are an integer $n \geq 0$ and elements $\psi_0, \ldots, \psi_n$ in S such that $e = \sum_{i=0}^{n} \psi_i D^i$. If $i \geq 0$ we have

$[D^i, y] \equiv iFD^{i-1} \mod \bigoplus_{j=0}^{i-2} SD^j$, so that

$$0 = [e, y] = \sum_{i=0}^{n} \psi_i [D^i, y] \equiv n\psi_n FD^{n-1} \mod \bigoplus_{j=0}^{n-2} SD^i.$$

Proceeding by descending induction we see from this that $\psi_i = 0$ if $i > 0$, so that $e = \psi_0 \in S$. □

8.5. Proposition. *If $\theta : A \to A$ is an automorphism of A such that for all $i \geq 0$ and all $a \in A_i$ we have $\theta(a) \in a + \bigoplus_{j>i} A_j$, then there exists an $f \in S$, uniquely determined up to the addition of a constant, such that*

$$\theta(x) = x, \qquad \theta(y) = y, \qquad \theta(D) = D - Ff_y, \qquad \theta(E) = E - [E, f].$$

Conversely, every $f \in S$ determines in this way an automorphism of A satisfying that condition.

Proof. Let $\theta : A \to A$ be an automorphism of A as in the statement. For each $j \geq 0$ there is a unique linear map $\theta_j : A \to A$ of degree j such that for each $i \geq 0$ and each $a \in A_i$ the element $\theta_j(a)$ is the $(i + j)$th homogeneous component of $\theta(a)$. We have that for all $a \in A$ we have $\theta_j(a) = 0$ for $j \geq 0$ and $\theta(a) = \sum_{j\geq 0} \theta_i(a)$ and, moreover, the sequence $(\theta_j)_{j\geq 0}$ is a higher derivation of A. In particular, it follows from Lemma 8.3 that

$$\theta_i \in D(A) \text{ for all } i \geq 0. \tag{18}$$

We know, from Proposition 4.7, that $\mathrm{Der}(A) = S_r\hat{D} \oplus \Bbbk\hat{E} \oplus \mathrm{InnDer}(A)$. If u is an irreducible factor of xF, then $(\phi\hat{D})(uA)$, $\hat{E}(uA)$ and $[a, uA]$ are all contained in uA for all $\phi \in S_r$ and all $a \in A$, and therefore (18) implies that that $\theta(uA) \subseteq uA$. As our argument also applies to the inverse automorphism θ^{-1}, we have $\theta^{-1}(uA) \subseteq uA$ and, therefore, $\theta(uA) = uA$. Since all units of A are in $\Bbbk$, we see that $\theta(u) = u$. Since of xF has two linearly independent linear factors, we can conclude that $\theta(x) = x$ and $\theta(y) = y$.
Let $\theta(E) = E + e_1 + \cdots + e_l$ with $e_i \in A_i$ for each $i \in \{1, \ldots, l\}$. We have

$$x = \theta(x) = [\theta(E), \theta(x)] = [E, x] + [e_1, x] + \cdots + [e_l, x]$$

and, by looking at homogeneous components, we see that $[e_i, x] = 0$ for all $i \in \{1, \ldots, l\}$ Similarly, $[e_i, y] = 0$ for such i, and therefore Lemma 8.4 tells us that $e_1, \ldots, e_l \in S$.
Suppose now that $\theta(D) = D + d_{r+1} + \cdots + d_l$ with $d_j \in A_j$ for each $j \in \{r + 1, \ldots, l\}$. Considering the equality $[\theta(E), \theta(D)] = r\theta(D)$ we see that $d_{r+i} = \frac{1}{i} Fe_{iy}$ for each $i \in \{1, \ldots, l\}$. Putting $f = -\sum_{i=1}^{l} \frac{1}{i} e_i$, we obtain the first part of the lemma. The second part follows from a direct verification. □

8.6. The automorphisms described in Proposition 8.5 are precisely the exponentials of the inner derivations corresponding to locally ad-nilpotent elements of A. This is a consequence of the following result:

Proposition. *An element of A is locally* ad*-nilpotent if and only if it belongs to S. If $f \in S$, then the automorphism* $\exp \operatorname{ad}(f)$ *maps x, y, D and E to x, y, $D - Ff_y$ and $E - [E, f]$, respectively.*

Proof. Suppose that $e \in A$ is a locally ad-nilpotent element. The kernel $\ker \operatorname{ad}(e)$ is a factorially closed subalgebra of A, so that whenever a, $b \in A$ and $\operatorname{ad}(e)(ab) = 0$ we have $\operatorname{ad}(e)(a) = 0$ or $\operatorname{ad}(e)(b) = 0$; see [6] for the proof of this in the commutative case, which adapts to ours.

Since $[x^i y^j D^k E^l, x] = -x^{i+1} y^j D^k \tau_1(E^l)$ for all i, j, k, $l \geq 0$, we have $[A, x] \subseteq xA$ and from this we see immediately that $[A, xA] \subseteq xA$. This implies that there is a sequence $(u_k)_{k\geq 0}$ in A such that $\operatorname{ad}(e)^k(x) = xu_k$ for all $k \geq 0$. Since e is locally ad-nilpotent, we can consider the integer $k_0 = \max\{k \in \mathbb{N}_0 : \operatorname{ad}(e)^k(x) \neq 0\}$, and then we have $0 \neq xu_{k_0} \in \ker \operatorname{ad}(e)$. As $\ker \operatorname{ad}(e)$ is factorially closed, we see that $\operatorname{ad}(e)(x) = 0$. In other words, the element e commutes with x.

There are an integer $m \geq 0$ and elements $\phi_0, \ldots, \phi_m$ in the subalgebra generated by x, y and D in A such that $e = \sum_{i=0}^m \phi_i E^i$, and we have $0 = [x, e] = \sum_{i=0}^m \phi_i \tau_1(E^i)$: this tells us that $\phi_i = 0$ if $i > 0$, and that $e = \phi_0$. In particular, there are an integer $n \geq 0$ and elements $\psi_0, \ldots, \psi_n$ in S such that $e = \sum_{i=0}^n \psi_i D^i$.

An induction shows that $[D^i, F] \in FA$ for all $i \geq 0$, and using this we see that $[e, F] = \sum_{i=0}^n \psi_i [D^i, F] \in FA$, from which it follows that in fact $[e, FA] \subseteq FA$. There is therefore a sequence $(v_i)_{i\geq 0}$ of elements of A such that $\operatorname{ad}(e)^i(F) = Fv_i$ for all $i \geq 0$. The local nilpotence of the map $\operatorname{ad}(e)$ allows us to consider the integer

$$i_0 = \max\{i \in \mathbb{N}_0 : \operatorname{ad}(e)^i(F) \neq 0\},$$

and then $0 \neq Fv_{i_0} \in \ker \operatorname{ad}(e)$. If $ax + by$ is any of the factors of F, we have $b \neq 0$ and $ax + by \in \ker \operatorname{ad}(e)$: clearly, this implies that y commutes with e.

In view of Lemma 8.4, we see that $e \in S$: this proves the necessity of the condition for local ad-nilpotency given in the lemma. Its sufficiency is a direct consequence of the fact that the graded algebra associated to the filtration on A described in 2.2 is commutative. Finally, the truth of the last sentence of the proposition can be verified by an easy computation. □

8.7. We write $\operatorname{Aut}_0(A)$ the set all automorphisms of A described in Lemma 8.1, and $\operatorname{Exp}(A)$ the set of all automorphisms of A described in Proposition 8.5; they are subgroups of the full group of automorphisms $\operatorname{Aut}(A)$.

Proposition. *The group* $\operatorname{Aut}(A)$ *is the semidirect product* $\operatorname{Aut}_0(A) \ltimes \operatorname{Exp}(A)$, *corresponding to the action of* $\operatorname{Aut}_0(A)$ *on* $\operatorname{Exp}(A)$ *given by*

$$\theta_0 \cdot \exp \operatorname{ad}(f) = \exp \operatorname{ad}(\theta^{-1}(f))$$

for all $\theta_0 \in \mathrm{Aut}_0(A)$ *and* $f \in S$. *The subgroup* $\mathrm{Aut}_0(A)$ *is precisely the set of automorphisms of* A *preserving the grading and* $\mathrm{Exp}(A)$ *is the set of exponentials of locally nilpotent inner derivations of* A.

Notice that the action described in this statement makes sense, as $\theta_0(S) = S$ whenever θ_0 belongs to $\mathrm{Aut}_0(A)$.

Proof. Let $\theta : A \to A$ be an automorphism and let us write $\theta(E) = e_0 + \cdots + e_l$, $\theta(x) = x_0 + \cdots + x_l$, $\theta(y) = e_0 + \cdots + y_l$, $\theta(D) = d_0 + \cdots + d_l$ with $e_i, x_i, y_i, d_i \in A_i$ for each $i \in \{0, \ldots, l\}$. Since θ is an automorphism, we have

$$[\theta(E), \theta(x)] = \theta(x), \qquad [\theta(E), \theta(y)] = \theta(y), \qquad [\theta(E), \theta(D)] = r\theta(D). \tag{19}$$

Looking at the degree zero parts of these equalities, and remembering that A_0 is a commutative ring, we see $x_0 = y_0 = d_0 = 0$. As $\theta(x) \neq 0$, we can consider the number $s = \min\{i \in \mathbb{N}_0 : x_i \neq 0\}$ and we have $s > 0$. Looking that the component of degree s of the first equality in (19), we see that $[e_0, x_s] = x_s$. This means that the restriction $\mathrm{ad}(e_0) : A_s \to A_s$ has a nonzero fixed vector. Now A_s as a right $\Bbbk[E]$-module is free with basis $\{x^i y^j D^k : i + j + rk = s\}$, the map $\mathrm{ad}(e_0)$ is right $\Bbbk[E]$-linear, and coincides with right multiplication by $-\tau_s(e_0)$ on A_s. Clearly, the existence of nonzero fixed vector implies that $-\tau_s(e_0) = 1$, so that $e_0 = uE + v$ for some $u \in \Bbbk^\times$ and $v \in \Bbbk$ with $su = 1$. Putting now $s' = \min\{i \in \mathbb{N}_0 : y_i \neq 0\}$ and $s'' = \min\{u \in \mathbb{N}_0 : d_i \neq 0\}$ and looking at the components in the least possible degree in the second and third equations of (19), we find that $s'u = 1$ and $s''u = r$. In particular, $s = s'$ and $s'' = rs$.
Suppose for a moment that $s > 1$. As $\theta(x)$, $\theta(y)$ and $\theta(D)$ are in the ideal (A_s) generated by A_s, the composition $q : A \to A$ of θ with the quotient map $A \to A/(A_s)$ is a surjection such that $q(A_0) = A/(A_s)$. This is impossible, as A_0 is a commutative ring and $A/(A_s)$ is not: we therefore have $s = 1$ and, as a consequence, $u = 1$.
There exist a, b, c, $d \in \Bbbk[E]$ such that $x_1 = xa + yb$ and $y_1 = xc + yd$. The four elements $\theta(E)$, $\theta(x)$, $\theta(y)$ and $\theta(D)$ generate A and, as $\theta(D)$ is in $\bigoplus_{i \geq r} A_i$, the elements x and y are in the subalgebra generated by the first three. It follows at once that x, $y \in x_1\Bbbk[E] + y_1\Bbbk[E]$ and, therefore, that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\Bbbk[E])$.
Let us write $f \in \Bbbk[E] \mapsto \vec{f} \in \Bbbk[E]$ the unique algebra morphism such that $\vec{E} = E + 1$. We have $[\theta(x), \theta(y)] = 0$ and in degree 2 this tells us that

$$x^2(a\vec{c} - \vec{a}c) + xy\boxed{T} + y^2(b\vec{d} - \vec{b}d) = 0,$$

so that

$$a\vec{c} = \vec{a}c, \qquad\qquad b\vec{d} = \vec{c}d. \tag{20}$$

Suppose that a is not constant. As the characteristic of $\Bbbk$ is zero (and possibly after replacing $\Bbbk$ by an algebraic extension, which does not change anything) there is then a $\xi \in \Bbbk$ such that $a(\xi) = 0$ and $\vec{a}(\xi) = a(\xi + 1) \neq 0$, and the

first equality in (20) implies that $c(\xi) = 0$. The determinant of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is thus divisible by $E - \xi$, and this is impossible. Similarly, we find that all of b, c, d must be constant.
Since $d_r \in A_r$, there exist $k \geq 0$, $\phi_0, \ldots, \phi_k \in S_r$ and $h \in \Bbbk[E]$ such that $d_r = \sum_{i=0}^{k} \phi_i E^i + Dh$. The component of degree $r+1$ of $[\theta(D), \theta(x)]$ is

$$0 = [d_r, x_1] = -\sum_{i=0}^{k} (ax+by)\phi_i \tau_1(E^i) - (ax+by)D\tau_1(h) + bF\vec{h}.$$

We thus see that h is constant, that $\phi_i = 0$ if $i \geq 2$, and that

$$(ax+by)\phi_1 + bhF = 0.$$

If $b = 0$, then $\phi_1 = 0$, and if instead $b \neq 0$, then either $h \neq 0$ and we see that $ax + by$ divides F and that $\phi_1 = -bhF/(ax+by)$, or $h = 0$ and $\phi_1 = 0$. In any case, we see that

$$d_r = \begin{cases} \phi_0 - \frac{hbF}{ax+by}E + hD, & \text{if } b \neq 0; \\ \phi_0 + hD, & \text{if not.} \end{cases}$$

Finally, the component of degree $r+1$ of the equality $[\theta(D), \theta(y)] = \theta(F)$ tells us that

$$F(ax+by, cx+dy) = (ad-bc)h\frac{xF}{ax+by}.$$

It follows now from Lemma 8.1 that there is a graded automorphism $\theta_0 : A \to A$ such that $\theta_0(x) = ax+by$, $\theta_0(y) = cx+dy$, $\theta_0(E) = E+v$ and $\theta_0(D) = d_r$. The composition $\theta_0^{-1} \circ \theta$ satisfies the hypothesis of Proposition 8.5, and then there exists an $f \in S$ such that $\theta = \theta_0 \circ \exp\operatorname{ad}(f)$. This shows that $\operatorname{Aut}(A) = \operatorname{Aut}_0(A) \cdot \operatorname{Exp}(A)$. Moreover, if θ is a graded automorphism, then so is $\exp\operatorname{ad}(f) = \theta_0^{-1} \circ \theta$ and, since it maps E to $E - [E, f]$, this is possible if and only if $f \in \Bbbk$, that is, if and only if $\exp\operatorname{ad}(f) = \mathrm{id}_A$; this proves the last claim of the theorem.
Finally, computing the action of both sides of the equation on the generators of A, we see that

$$\exp\operatorname{ad}(f) \circ \theta_0 = \theta_0 \circ \exp\operatorname{ad}(\theta^{-1}(f))$$

for all $f \in S$ and all $\theta_0 \in \operatorname{Aut}_0(A)$, and this tells us that $\operatorname{Aut}(A)$ is indeed a semidirect product $\operatorname{Aut}_0(A) \ltimes \operatorname{Exp}(A)$. □

8.8. As usual, we say that an element u of A is *normal* if $uA = Au$. Such an element, since it is not a zero-divisor, determines an automorphism $\theta_u : A \to A$ uniquely by the condition that $ua = \theta_u(a)u$ for all $u \in A$.

8.9. Proposition. *Let $Q = \alpha_0 \cdots \alpha_{r+1}$ be a factorization of Q as a product of linear factors. The set of non-zero normal elements of A is*

$$\mathscr{N}(A) = \{\lambda\alpha_0^{i_0} \cdots \alpha_{r+1}^{i_{r+1}} : \lambda \in \Bbbk^\times, i_0, \ldots, i_{r+1} \in \mathbb{N}_0\}.$$

This set is the saturated multiplicatively closed subset of A generated by Q.

Proof. A direct computation shows that each of the factors $\alpha_0, \ldots, \alpha_{r+1}$ of Q is normal in A, so the set $\mathscr{N}(A)$ is contained in the set of non-zero normal elements of A, for the latter is multiplicatively closed. The set $\mathscr{N}(A)$ is multiplicatively closed and it is saturated because S is closed under divisors in A, and it is clear that as a saturated multiplicatively closed it is generated by Q. To conclude the proof, we have to show that every non-zero normal element of A belongs to $\mathscr{N}(A)$.

Let u be a non-zero normal element in A and let $\theta_u : A \to A$ be the associated automorphism, so that $ua = \theta_u(a)u$ for all $a \in A$. There are $k, l \in \mathbb{N}_0$ with $k \leq l$ and elements $u_k, \ldots u_l \in A$ such that $u = u_k + \cdots + u_l$, $u_i \in A_i$ if $k \leq i \leq l$, and $u_k \neq 0 \neq u_l$. Similarly, there are $s, t \in \mathbb{N}_0$ with $s \leq t$ and elements $e_s, \ldots, e_t \in A$ such that $\theta_u(E) = e_s + \cdots + e_t$, $e_i \in A_i$ if $s \leq i \leq t$, and $e_s \neq 0 \neq e_t$. As we have

$$u_k E + \cdots + u_l E = uE = \theta_u(E)u = e_s u_k + \cdots + e_t u_l$$

with $u_k E$, $u_l E$, $e_s u_k$ and $e_t u_l$ all non-zero, looking at the homogeneous components of both sides we see that $s = t = 0$. This means that $\theta_u(E) = f(E) \in \Bbbk[E]$, and therefore the above equality is really of the form

$$u_k E + \cdots + u_l E = f(E)u_k + \cdots + f(E)u_l.$$

It follows from this that $u_i E = f(E)u_i = u_i f(E+i)$ for all $i \in \{k, \ldots, l\}$ and therefore that $E = f(E+k)$ and that $E = f(E+l)$. Since our ground field has characteristic zero, this is only possible if $k = l$: the element u is homogeneous of degree l.

Now, since $ua = \theta_u(a)u$ for all $a \in A$, the homogeneity of u implies immediately that θ_u is a homogeneous map. There are $n \in \mathbb{N}_0$ and $\phi_0, \ldots, \phi_n$ in the subalgebra of A generated by x, y and D, such that $\phi_n \neq 0$ and $u = \sum_{i=0}^n \phi_i E^i$. As $\theta_u(x)$ has degree 1, it belongs to S_1 and we have

$$\theta_u(x) \sum_{i=0}^n \phi_i E^i = \theta_u(x)u = ux = \sum_{i=0}^n \phi_i E^i x = x \sum_{i=0} \phi_i (E+1)^i.$$

Considering only the terms that have E^n as a factor we see that $\theta_u(x) = x$, and then the equality tells us that in fact $\sum_{i=0}^n \phi_i E^i = \sum_{i=0} \phi_i (E+1)^i$. Looking now at the terms which have E^{n-1} as a factor here we see that moreover $n = 0$, so that $u \in \Bbbk[x, y, D]$. There exist then $m \in \mathbb{N}_0$ and $\psi_0, \ldots, \psi_m \in S$ such that $\psi_m \neq 0$ and $u = \sum_{i=0}^m \psi_i D^i$. As $\theta_u(y)$ has degree 1, it belongs to S_1 and we have

$$\theta_u(y) \sum_{i=0}^m \psi_i D^i = \theta_u(y)u = uy = \sum_{i=0}^m \psi_i D^i y = \sum_{i=0}^m y \psi_i D^i + \sum_{i=0}^m \psi_i [D^i, y].$$

Comparing the terms that have D^m as a factor we conclude that also $\theta_u(y) = y$. As θ_u fixes x and y, the element u commutes with x and y, and Lemma 8.4 allows us to conclude that u is in S_l. Moreover, we know that all homogeneous automorphisms of A are those described in Lemma 8.1, so there exist $\phi \in S_r$ and $e \in \Bbbk^\times$ such that $\theta_u(D) = \phi + eD$. We then have that

$$uD = \theta_u(D)u = (\phi + eD)u = \phi u + euD + eu_y F$$

and this implies that $e = 1$ and $\phi u + u_y F = 0$. Suppose now that α is a linear factor of u and let $k \in \mathbb{N}$ and $v \in S$ be such that $u = \alpha^k v$ and v is not divisible by α. The last equality becomes $\phi\alpha^k v + k\alpha^{k-1}\alpha_y vF + \alpha^k v_y F = 0$ and implies that α divides $\alpha_y F$: this means that α is a non-zero multiple of x or a linear factor of F. As u can be factored as a product of linear factors, we can therefore conclude that u belongs to the set described in the statement of the proposition. ☐

8.10. There is a close connection between normal elements, the first Hochschild cohomology space that we computed in Section 4 and the modular automorphisms of A.

PROPOSITION. *Let $Q = \alpha_0 \cdots \alpha_{r+1}$ be a factorization of Q as a product of linear factors.*

(*i*) *Every linear combination of the derivations $\partial_{\alpha_0}, \ldots, \partial_{\alpha_{r+1}} : A \to A$ described in Proposition 5.2 is locally nilpotent.*

(*ii*) *If $u = \lambda\alpha_0^{i_0} \cdots \alpha_{r+1}^{i_{r+1}}$, with $\lambda \in \Bbbk^\times$ and $i_0, \ldots, i_{r+1} \in \mathbb{N}_0$, is a normal element of A, then the automorphism $\theta_u : A \to A$ associated to u is*

$$\theta_u = \exp\left(-\sum_{j=0}^{r+1} i_j \partial_{\alpha_j}\right).$$

This automorphism is such that $\theta_u(f) = f$ for all $f \in S$ and

$$\theta_u(\delta) = \delta + \frac{\delta(u)}{u}$$

for all $\delta \in \mathrm{Der}(\mathcal{A})$.

(*iii*) *The modular automorphism $\sigma : A \to A$ described in Proposition 7.2 coincides with the automorphism θ_Q associated to the normal element Q.*

8.11. Another immediate application of the determination of the set of normal elements is the classification under isomorphisms of our algebras.

PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two central arrangements of lines in $\mathbb{A}^2$. The algebras $\mathscr{D}(\mathcal{A})$ and $\mathscr{D}(\mathcal{A}')$ are isomorphic if and only if the arrangements $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic.*

Proof. The sufficiency of the condition being obvious, we prove only its necessity. We will denote with primes the objects associated to the arrangement $\mathcal{A}'$, so that for example $A' = \mathscr{D}(\mathcal{A}')$ and so on. Moreover, in view of the sufficiency of the condition we can suppose without loss of generality that both arrangements $\mathcal{A}$ and $\mathcal{A}'$ contain the line with equation $x = 0$.

Let us suppose that there is an isomorphism of algebras $\phi : A \to A'$. Since ϕ maps locally ad-nilpotent elements to locally ad-nilpotent elements, it follows from Proposition 8.6 that $\phi(S) = S'$ and therefore that ϕ restricts to an isomorphism of algebras $\phi : S \to S'$. On the other hand, ϕ also maps normal elements to normal elements, so that ϕ restricts to a monoid homomorphism $\phi : \mathscr{N}(A) \to \mathscr{N}(A')$. Let $Q = \alpha_0 \cdots \alpha_{r+1}$ and $Q' = \alpha'_0 \cdots \alpha'_{r'+1}$ be the factorizations of Q and of Q' as products of linear factors. The invertible elements of the monoid $\mathscr{N}(A)$ are the units of $\Bbbk$ and the quotient $\mathscr{N}(A)/\Bbbk^\times$ is the free abelian monoid generated by (the classes of) $\alpha_0, \ldots, \alpha_{r+1}$ and, of course, a similar statement holds for the other arrangement. Since ϕ induces an isomorphism $\mathscr{N}(A)/\Bbbk^\times \to \mathscr{N}(A')/\Bbbk^\times$ we see, first, that $r = r'$ and, second, that there are a permutation π of the set $\{0, \ldots, r+1\}$ and a function $\lambda : \{0, \ldots, r+1\} \to \Bbbk^\times$ such that $\phi(\alpha_i) = \lambda(i)\alpha'_{\pi(i)}$ for all $i \in \{0, \ldots, r+1\}$. As there are at least two lines in each arrangement, this implies that the restriction $\phi|_S : S \to S'$ restricts to an isomorphism of vector spaces $\phi : S_1 \to S'_1$, so that $\phi|_S$ is linear, and that $\phi(Q) = Q'$. It is clear that this implies that the arrangements $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic. □

8.12. A simple and final observation that we can make at this point is that our algebra A and the full algebra $\mathscr{D}(S)$ of regular differential operators of S are birational, that is, that they have the same fields of quotients. In fact, the two algebras become isomorphic already after localization at a single element:

8.13. Proposition. *The inclusion $A \to \mathscr{D}(S)$ induces after localization at Q an isomorphism $A[\frac{1}{Q}] \to \mathscr{D}(S)[\frac{1}{Q}]$ and, in particular, A and $\mathscr{D}(S)$ have isomorphic fields of fractions.*

That both localizations actually exist follows from the usual characterization of quotient rings; see, for example, [11, Chapter 2].

Proof. Clearly the map $A[\frac{1}{Q}] \to \mathscr{D}(S)[\frac{1}{Q}]$ induced by the inclusion is injective, and it is surjective since S is contained in its image as are $\partial_y = \frac{1}{F}D$ and $\partial_x = \frac{1}{x}E - \frac{y}{Q}D$. □

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