

(CO)HOMOLOGY OF CROSSED PRODUCTS BY WEAK HOPF ALGEBRAS

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ABSTRACT. We obtain a mixed complex simpler than the canonical one that computes the cyclic type homologies of a crossed product with invertible cocycle $A \times_f^H$, of a weak module algebra A by a weak Hopf algebra H . This complex is endowed with a filtration. The spectral sequence of this filtration generalizes the spectral sequence obtained in [12]. When f takes its values in a separable subalgebra of A that satisfies suitable conditions, the above mentioned mixed complex is provided with another filtration, whose spectral sequence generalize the Feigin-Tsygan spectral sequence.

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Introduction

Given a differential or algebraic manifold M , each group G acting on M acts in a natural way on the ring A of regular functions of M , and the algebra ${}^G A$ of invariants of this action consists of the functions that are constants on each of the orbits of M . This suggest to consider ${}^G A$ as a replacement for M/G in noncommutative geometry. Under suitable conditions the invariant algebra ${}^G A$ and the smash product $A \# k[G]$, associated with the action of G on A , are Morita equivalent. Since K -theory, Hochschild homology and cyclic homology are Morita invariant, there is no loss of information if ${}^G A$ is replaced by $A \# k[G]$. In the general case the experience has shown that smash products are better choices than invariant rings for algebras playing the role of noncommutative quotients. In fact, except when the invariant algebra and the smash product are Morita equivalent, the first one never is considered in noncommutative geometry. The problem of developing tools to compute the cyclic homology of smash products algebras $A \# k[G]$, where A is an algebra and G is a group, was considered in [17, 20, 32]. For instance, in the first paper the authors obtained a spectral sequence converging to the cyclic homology of $A \# k[G]$, and in [20] this result was derived from the theory of paracyclic modules and cylindrical modules developed by the authors. The main tool for this computation is a version for cylindrical modules of the Eilenberg-Zilber theorem. More recently, and also due to its connections with noncommutative geometry, the cyclic homology of algebras obtained through more general constructions involving Ore extensions, twisted tensor products or Hopf algebras (Hopf crossed products, Hopf Galois extensions, Braided Hopf crossed products, etcetera) has been extensively studied. See for instance [1, 12–16, 22, 23, 27, 29, 30, 34–36, 38]. Weak Hopf algebras (also called quantum groupoids) are an important generalization of Hopf algebras in which the counit is not required to be an algebra homomorphism and the unit is not required to be a coalgebra homomorphism,

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these properties being replaced by weaker axioms. Examples of weak Hopf algebras are groupoid algebras and their duals, face algebras [26], quantum groupoids constructed from subfactors [31] and generalized Kac algebras of Yamanouchi [37]. It is natural to try to extend the results of [17] to noncommutative quotients $A\#H$ of algebras A by actions of weak Hopf algebras H . More generally, in this paper we use the results obtained in [25] to study the Hochschild (co)homology and the cyclic type homologies of weak crossed products with invertible cocycle. Specifically, for a unitary crossed product $E := A \times_{\rho}^f H$, of an algebra A by a weak Hopf algebra H and a subalgebra K of A satisfying suitable conditions, we construct a chain complex, a cochain complex and a mixed complex, simpler than the canonical ones (and also simpler than the ones constructed in [25]), that compute the Hochschild (co)homology of E with coefficients in an E -bimodule M , relative to K , and the cyclic, negative and periodic homologies of E , relative to K (see [19]). It is well known that when K is separable, then these relative groups coincide with the absolute groups. These complexes are endowed with canonical filtrations whose spectral sequences generalize the Hochschild-Serre spectral sequence and the Feigin and Tsygan spectral sequence. By example, applying these results, we obtain that when $A = K$, the Hochschild homology of E with coefficients in M , relative to K , coincide with the homology of H with coefficients in the group $M/[M, K]$, considered as a left H -module via conjugation. A similar result is obtained for the Hochschild cohomology (see Subsection 1.2 and Examples 2.15 and 3.9).

The paper is organized as follows:

In Section 1 we review the notions of weak Hopf algebra and of crossed products of algebras by weak Hopf algebras, and we recall the concept of mixed complex and the perturbation lemma. In sections 2 and 3 we obtain complexes that compute the Hochschild homology and the Hochschild cohomology of a weak crossed product with invertible cocycle $E := A \times_{\rho}^f H$ with coefficients in an E -bimodule M . Then, in Section 4 we study the cup and the cap products of E , and in Section 5 we obtain a mixed complex that computes the cyclic type homologies of E .

Remark. In this paper we consider the notion of weak crossed products introduced in [6], but this is not the unique concept of weak crossed product of algebras by weak Hopf algebras in the literature. There is a notion of crossed product of an algebra A with a Hopf algebroid introduced by Böhm and Brzeziński in [7]. It is well known that weak Hopf algebras H provide examples of Hopf algebroids. The crossed products considered by us in this paper are canonically isomorphic to the Böhm-Brzeziński crossed products $A\#_f H$ with invertible cocycle whose action satisfies $h \cdot (l \cdot 1_A) = hl \cdot 1_A$ for all $h \in H$ and $l \in H^L$ (see [21]). So, our results also apply to these algebras.

We thank the referee for a careful reading of the paper and for his indications that helped improve the writing. We also thank the referee for pointing out the paper [29], in which the Hochschild homology of the twisted tensor product of algebras is studied, obtaining complete calculations for several examples. There is some intersection between the results of [29] and some results obtained in [22] and [23]. More precisely, [29, Proposition 2.1] is [22, Theorem 1.7] and the results in [29, Subsections 3.3.1 to 3.3.3] are the cases $r = n = 2$, $r = 1, n = 2$ and $r = 0, n = 2$ of [23, Proposition 1.9].

1 Preliminaries

In this article we work in the category of vector spaces over a field k . Hence we assume implicitly that all the maps are k -linear maps. The tensor product over k is denoted by \otimes_k . Given an arbitrary algebra K , a K -bimodule V and an $n \geq 0$, we let $V^{\otimes_n K}$ denote the n -fold tensor product $V \otimes_K \cdots \otimes_K V$, which is considered as a K -bimodule via

$$\lambda \cdot (v_1 \otimes_K \cdots \otimes_K v_n) \cdot \lambda' := \lambda \cdot v_1 \otimes_K \cdots \otimes_K v_n \cdot \lambda'.$$

Given k -vector spaces U, V, W and a map $g: V \rightarrow W$ we write $U \otimes_k g$ for $\text{id}_U \otimes_k g$ and $g \otimes_k U$ for $g \otimes_k \text{id}_U$. We assume that the reader is familiar with the notions of weak Hopf algebra introduced in [8, 9] and of weak crossed products introduced in [6] and studied in a series of papers (see for instance [2–4, 18, 24, 33]). We are specifically interested in the case in which A is a weak module algebra and the cocycle of the crossed product is convolution invertible (see [24, Sections 4–6]).

1.1 Weak Hopf algebras

Weak bialgebras and weak Hopf algebras are generalizations of bialgebras and Hopf algebras, introduced in [8, 9], in which the axioms about the unit, the counit and the antipode are replaced by weaker properties. Next we give a brief review of the basic properties of these structures.

Definition 1.1. Let k be a field. A weak *bialgebra* is a k -vector space H , endowed with an algebra structure and a coalgebra structure, such that $\Delta(hl) = \Delta(h)\Delta(l)$ for all $h, l \in H$, and the equalities

$$\Delta^2(1) = 1^{(1)} \otimes_k 1^{(2)} 1^{(1')} \otimes_k 1^{(2')} = 1^{(1)} \otimes_k 1^{(1')} 1^{(2)} \otimes_k 1^{(2')} \quad (1.1)$$

and

$$\epsilon(hlm) = \epsilon(hl^{(1)})\epsilon(l^{(2)}m) = \epsilon(hl^{(2)})\epsilon(l^{(1)}m) \quad \text{for all } h, l, m \in H, \quad (1.2)$$

are fulfilled, where we are using the Sweedler notation for the coproduct, with the summation symbol omitted. A *weak bialgebra morphism* is a function $g: H \rightarrow L$ that is an algebra and a coalgebra map.

In the rest of this subsection we assume that H is a weak bialgebra.

It is well known that the maps $\Pi^L, \Pi^R, \bar{\Pi}^L, \bar{\Pi}^R \in \text{End}_k(H)$, defined by

$$\Pi^L(h) := \epsilon(1^{(1)}h)1^{(2)}, \quad \Pi^R(h) := 1^{(1)}\epsilon(h1^{(2)}), \quad \bar{\Pi}^L(h) := 1^{(1)}\epsilon(1^{(2)}h), \quad \bar{\Pi}^R(h) := \epsilon(h1^{(1)})1^{(2)},$$

respectively, are idempotent (for a proof see [8, 11]). We set $H^L := \text{Im}(\Pi^L)$ and $H^R := \text{Im}(\Pi^R)$. In [11] it was also proven that $\text{Im}(\bar{\Pi}^R) = H^L$ and $\text{Im}(\bar{\Pi}^L) = H^R$.

Proposition 1.2. *For all $h, l \in H$ and $m \in H^L$, we have*

$$\bar{\Pi}^R(hl) = \bar{\Pi}^R(\bar{\Pi}^R(h)l), \quad \Pi^R(hl) = \Pi^R(\bar{\Pi}^R(h)l) \quad \text{and} \quad \bar{\Pi}^R(hm) = \bar{\Pi}^R(h)m.$$

Proof. Left to the reader. \square

An *antipode* of H is a map $S: H \rightarrow H$ (or S_H if necessary), such that

$$h^{(1)}S(h^{(2)}) = \Pi^L(h), \quad S(h^{(1)})h^{(2)} = \Pi^R(h) \quad \text{and} \quad S(h^{(1)})h^{(2)}S(h^{(3)}) = S(h),$$

for all $h \in H$. As it was shown in [8], if an antipode S exists, then it is unique. It was also shown in [8] that S is antimultiplicative, anticomultiplicative and leaves the unit and counit invariant. A *weak Hopf algebra* is a weak bialgebra that has an antipode. By [8, equalities (2.24a) and (2.24b)],

$$\Pi^L = S \circ \bar{\Pi}^L \quad \text{and} \quad \Pi^R = S \circ \bar{\Pi}^R. \quad (1.3)$$

A *morphism of weak Hopf algebras* $g: H \rightarrow L$ is simply a bialgebra morphism from H to L . In [5, Proposition 1.4] it was proven that if $g: H \rightarrow L$ is a weak Hopf algebra morphism, then $g \circ S_H = S_L \circ g$.

1.2 (Co)homology of weak Hopf algebras

Consider H^R as a right H -module via $l \cdot h := \Pi^R(lh)$ (by [8, equality (2.5b)] this is an action and the map $\Pi^R: H \rightarrow H^R$ is a morphism of right H -modules). By definition, the *homology of H with coefficients in a left H -module N* is $H_*(H, N) := \text{Tor}_*^H(H^R, N)$, while the *cohomology of H with coefficients in a right H -module N* is $H^*(H, N) := \text{Ext}_H^*(H^R, N)$.

Notations 1.3. We will use the following notations:

- (1) We set $\bar{H} := H/H^L$. Moreover, given $h \in H$ we let \bar{h} denote its class in \bar{H} .
- (2) Given $h_1, \dots, h_s \in H$ we set $\bar{\mathbf{h}}_{1s} := \bar{h}_1 \otimes_{H^L} \dots \otimes_{H^L} \bar{h}_s$.

Clearly $\bar{H}^{\otimes_{H^L} s}$ is an H^L -bimodule via $l \cdot \bar{\mathbf{h}}_{1s} \cdot l' := \bar{h}_1 \otimes_{H^L} \bar{\mathbf{h}}_{2,s-1} \otimes_{H^L} \bar{h}_s \bar{l}$.

Proposition 1.4. *The following facts hold:*

- (1) *The homology of H with coefficients in a left H -module N is the homology of the chain complex*

$$N \xleftarrow{d_1} \bar{H} \otimes_{H^L} N \xleftarrow{d_2} \bar{H}^{\otimes_{H^L} 2} \otimes_{H^L} N \xleftarrow{d_3} \bar{H}^{\otimes_{H^L} 3} \otimes_{H^L} N \xleftarrow{d_4} \bar{H}^{\otimes_{H^L} 4} \otimes_{H^L} N \xleftarrow{d_5} \dots,$$

where $d_1(h_1 \otimes_{H^L} n) := \bar{\Pi}^R(h_1) \cdot n - h_1 \cdot n$ and, for $s > 1$,

$$\begin{aligned} d_s(\bar{\mathbf{h}}_{1s} \otimes_{H^L} n) &:= \bar{\Pi}^R(h_1) \cdot \bar{\mathbf{h}}_{2s} \otimes_{H^L} n \\ &+ \sum_{i=1}^{s-1} (-1)^i \bar{\mathbf{h}}_{1,i-1} \otimes_{H^L} \overline{h_i h_{i+1}} \otimes_{H^L} \bar{\mathbf{h}}_{i+2,s} \otimes_{H^L} n + (-1)^s \bar{\mathbf{h}}_{1,s-1} \otimes_{H^L} h_s \cdot n. \end{aligned}$$

- (2) *The cohomology of H with coefficients in a right H -module N is the cohomology of the cochain complex*

$$N \xrightarrow{d^1} \text{Hom}_{H^L}(\bar{H}, N) \xrightarrow{d^2} \text{Hom}_{H^L}(\bar{H}^{\otimes_{H^L} 2}, N) \xrightarrow{d^3} \text{Hom}_{H^L}(\bar{H}^{\otimes_{H^L} 3}, N) \xrightarrow{d^4} \dots,$$

where $d^1(n)(h_1) := n \cdot \bar{\Pi}^R(h_1) - n \cdot h_1$ and, for $s > 1$,

$$d_s(\beta)(\bar{\mathbf{h}}_{1s}) := \beta(\bar{\Pi}^R(h_1) \cdot \bar{\mathbf{h}}_{2s}) + \sum_{i=1}^{s-1} (-1)^i \beta(\bar{\mathbf{h}}_{1,i-1} \otimes_{H^L} \overline{h_i h_{i+1}} \otimes_{H^L} \bar{\mathbf{h}}_{i+2,s}) + (-1)^s \beta(\bar{\mathbf{h}}_{1,s-1}) \cdot h_s.$$

Proof. For $s \in \mathbb{N}$ and $0 \leq i \leq s$, let $d''_{si}: H^{\otimes_k^{s+1}} \rightarrow H^{\otimes_k^{s+1}}$ be the map defined by

$$d''_{si}(\mathbf{x}) := \begin{cases} \overline{\Pi}^R(h_1)h_2 \otimes_k h_3 \otimes_k \cdots \otimes_k h_{s+1} & \text{if } i = 0, \\ h_1 \otimes_k \cdots \otimes_k h_{i-1} \otimes_k h_i h_{i+1} \otimes_k h_{i+2} \otimes_k \cdots \otimes_k h_{s+1} & \text{if } 1 \leq i \leq s, \end{cases}$$

where $\mathbf{x} := h_1 \otimes_k \cdots \otimes_k h_{s+1}$. For each $s > 1$ and $0 \leq i \leq s$, we let $d'_{si}: H^{\otimes_k^{s+1}} \rightarrow \overline{H}^{\otimes_{H^L}^{s-1}} \otimes_{H^L} H$ denote the morphism induced by d''_{si} . Moreover we set $d'_{10} = d''_{10}$ and $d'_{11} = d''_{11}$. We claim that

- (1) If $s = 1$ and $h_1 \in H^L$, then $d'_{10}(\mathbf{x}) = d'_{11}(\mathbf{x})$,
- (2) If $s > 1$ and $h_1 \in H^L$, then $d'_{s0}(\mathbf{x}) = d'_{s1}(\mathbf{x})$ and $d'_{si}(\mathbf{x}) = 0$ for $1 < i$,
- (3) If $s > 1$ and $h_j \in H^L$ for some $j > 1$, then $d'_{s,j-1}(\mathbf{x}) = d'_{sj}(\mathbf{x})$ and $d'_{si}(\mathbf{x}) = 0$ for $i \notin \{j-1, j\}$,
- (4) $d'_{si}(h_1 l \otimes_k h_2 \otimes_k \cdots \otimes_k h_{s+1}) = d'_{si}(h_1 \otimes_k l h_2 \otimes_k \cdots \otimes_k h_{s+1})$ for $0 \leq i \leq s$ and $l \in H^L$,
- (5) For $0 \leq i \leq s$, $1 \leq j \leq s$ and $l \in H^L$, we have

$$d'_{si}(h_1 \otimes_k \cdots \otimes_k h_j l \otimes_k h_{j+1} \otimes_k \cdots \otimes_k h_{s+1}) = d'_{si}(h_1 \otimes_k \cdots \otimes_k h_j \otimes_k l h_{j+1} \otimes_k \cdots \otimes_k h_{s+1}). \quad (1.4)$$

Items (1) and the first assertion in item (2) hold since $\overline{\Pi}^R(h) = h$, for all $h \in H^L$; while the second assertion in item (2) is clear. We next prove item (3). The fact that $d'_{sj}(\mathbf{x}) = 0$, for $i > 0$ and $i \notin \{j-1, j\}$, is trivial; while the equality $d'_{s,j-1}(\mathbf{x}) = d'_{sj}(\mathbf{x})$ is straightforward. Finally if $j = 2$, then $d'_{s0}(\mathbf{x}) = 0$, since $\overline{\Pi}^R(h_1)h_2 \in H^L$. The case $i = 0$ in item (4) holds, since $\overline{\Pi}^R(h_1 l)h_2 = \overline{\Pi}^R(h_1)l h_2$, by the third identity in Proposition 1.2; while the cases $i > 0$ are straightforward. It remains to check item (4). Equality (1.4) for $i = 0$ and $j = 1$ holds, since $\overline{\Pi}^R(h_1 l)h_2 = \overline{\Pi}^R(h_1)l h_2$, by the third identity in Proposition 1.2; while the other cases are trivial. This ends the proof of the claim. Consequently, $\sum_{i=0}^s (-1)^i d'_{si}$ induces a morphism $d'_s: \overline{H}^{\otimes_{H^L}^s} \otimes_{H^L} H \rightarrow \overline{H}^{\otimes_{H^L}^{s-1}} \otimes_{H^L} H$, for each $s \geq 1$. Consider the diagram

$$H^R \xleftarrow{\Pi^R} H \xleftarrow{d'_1} \overline{H} \otimes_{H^L} H \xleftarrow{d'_2} \overline{H}^{\otimes_{H^L}^2} \otimes_{H^L} H \xleftarrow{d'_3} \overline{H}^{\otimes_{H^L}^3} \otimes_{H^L} H \xleftarrow{d'_4} \cdots, \quad (1.5)$$

where $\overline{H}^{\otimes_{H^L}^s} \otimes_{H^L} H$ is a right H -module via the canonical action. Clearly the d'_s 's are right H -linear maps. Next we will prove that (1.5) is a chain complex. Assume that $s > 2$. Since

$$\begin{aligned} d''_{s-1,i} \circ d''_{sj} &= d''_{s-1,j-1} \circ d''_{si} & \text{for } 1 \leq i < j, \\ d''_{s-1,0} \circ d''_{sj} &= d''_{s-1,j-1} \circ d''_{s0} & \text{for } 1 < j, \end{aligned}$$

the composition $d'_{s-1} \circ d'_s$ is the map induced by $d''_{s-1,0} \circ d''_{s0} - d''_{s-1,0} \circ d''_{s1}$. Thus,

$$d'_{s-1} \circ d'_s(\overline{\mathbf{h}}_{1s} \otimes_{H^L} h_{s+1}) = \overline{\Pi}^R(\overline{\Pi}^R(h_1)h_2)h_3 \otimes_{H^L} \overline{\mathbf{h}}_{4s} \otimes_{H^L} h_{s+1} - \overline{\Pi}^R(h_1 h_2)h_3 \otimes_{H^L} \overline{\mathbf{h}}_{4s} \otimes_{H^L} h_{s+1} = 0,$$

where the last equality follows from the first identity in Proposition 1.2. In order to finish the proof that (1.5) is a chain complex, we must check that $\Pi^R \circ d'_1 = 0$ and $d'_1 \circ d'_2 = 0$. But, this follows from the first two identities in Proposition 1.2. We claim that the family of morphisms

$$\hbar_0: H^R \longrightarrow H, \quad \hbar_{s+1}: \overline{H}^{\otimes_{H^L}^s} \otimes_{H^L} H \longrightarrow \overline{H}^{\otimes_{H^L}^{s+1}} \otimes_{H^L} H \quad (s \geq 0),$$

given by $\hbar_0(h) := h$ and $\hbar_{s+1}(\overline{\mathbf{h}}_{1s} \otimes_{H^L} h_{s+1}) := (-1)^{s+1} \overline{\mathbf{h}}_{1,s+1} \otimes_{H^L} 1$ for $s \geq 0$, is a contracting homotopy of (1.5) as a complex of right H^L -modules. In fact, this follows immediately from the equalities

$$\begin{aligned} \Pi^R \circ \hbar_0(h) &= h, & \hbar_0 \circ \Pi^R(h) &= \Pi^R(h), & d'_1 \circ \hbar_1(h) &= -\overline{\Pi}^R(h) + h, \\ \hbar_s \circ d'_s(\overline{\mathbf{h}}_{1s} \otimes_{H^L} h_{s+1}) &= (-1)^s \overline{\Pi}^R(h_1)h_2 \otimes_{H^L} \overline{\mathbf{h}}_{3,s+1} \otimes_{H^L} 1 \\ &\quad + \sum_{i=1}^s (-1)^{s+i} \overline{\mathbf{h}}_{1,i-1} \otimes_{H^L} \overline{h_i h_{i+1}} \otimes_{H^L} \overline{\mathbf{h}}_{i+2,s+1} \otimes_{H^L} 1 \end{aligned}$$

and

$$\begin{aligned} d'_{s+1} \circ \hbar_{s+1}(\overline{\mathbf{h}}_{1s} \otimes_{H^L} h_{s+1}) &= (-1)^{s+1} \overline{\Pi}^R(h_1)h_2 \otimes_{H^L} \overline{\mathbf{h}}_{3,s+1} \otimes_{H^L} 1 \\ &\quad + \sum_{i=1}^s (-1)^{s+i+1} \overline{\mathbf{h}}_{1,i-1} \otimes_{H^L} \overline{h_i h_{i+1}} \otimes_{H^L} \overline{\mathbf{h}}_{i+2,s+1} \otimes_{H^L} 1 \\ &\quad + \overline{\mathbf{h}}_{1s} \otimes_{H^L} h_{s+1}. \end{aligned}$$

Note that, since H^L is separable, the right H^L -modules $\overline{H}^{\otimes_{H^L}^s}$ are projective. Hence, $\overline{H}^{\otimes_{H^L}^s} \otimes_{H^L} H$ is a right projective H -module for each $s \geq 0$ and, consequently, (1.5) is a projective resolution of H^R as a right H -module.

Now items (1) and (2) follows immediately from this and the fact that $(\overline{H}^{\otimes_{H^L} s} \otimes_{H^L} H) \otimes_H N \simeq \overline{H}^{\otimes_{H^L} s} \otimes_{H^L} N$ and $\text{Hom}_H(\overline{H}^{\otimes_{H^L} s} \otimes_{H^L} H, N) \simeq \text{Hom}_{H^L}(\overline{H}^{\otimes_{H^L} s}, N)$. \square

1.3 Crossed products by weak Hopf algebras

Let H be a weak-Hopf algebra, A an algebra and $\rho: H \otimes_k A \rightarrow A$ a linear map. For $h \in H$ and $a \in A$, we set $h \cdot a := \rho(h \otimes_k a)$. We say that ρ is a *weak measure of H on A* if

$$h \cdot (aa') = (h^{(1)} \cdot a)(h^{(2)} \cdot a') \quad \text{for all } h \in H \text{ and } a, a' \in A. \quad (1.6)$$

From now on ρ denotes a weak measure of H on A . Let $\chi_\rho: H \otimes_k A \rightarrow A \otimes_k H$ be the map defined by $\chi_\rho(h \otimes_k a) := h^{(1)} \cdot a \otimes_k h^{(2)}$. By equality (1.6) the triple (A, H, χ_ρ) is a twisted space (see [24, Definition 1.6]). By [24, Subsection 1.2] we know that $A \otimes_k H$ is a non unitary A -bimodule via

$$a' \cdot (a \otimes_k h) := a'a \otimes_k h \quad \text{and} \quad (a \otimes_k h) \cdot a' := a(h^{(1)} \cdot a') \otimes_k h^{(2)},$$

and that the map $\nabla_\rho: A \otimes_k H \rightarrow A \otimes_k H$, defined by $\nabla_\rho(a \otimes_k h) := a \cdot \chi_\rho(h \otimes_k 1_A)$, is a left and right A -linear idempotent. In the sequel we will write $a \times h := \nabla_\rho(a \otimes_k h)$. It is easy to check that the A -subbimodule $A \times H := \nabla_\rho(A \otimes_k H)$ of $A \otimes_k H$ is unitary. Let $\gamma: H \rightarrow A \times H$, $\nu: k \rightarrow A \otimes_k H$ and $j_\nu: A \rightarrow A \times H$, be the maps defined by $\gamma(h) := 1_A \times h$, $\nu(\lambda) := \lambda 1_A \times 1$ and $j_\nu(a) := a \times 1$, respectively. Given a morphism $f: H \otimes_k H \rightarrow A$, we define $\mathcal{F}_f: H \otimes_k H \rightarrow A \otimes_k H$ by $\mathcal{F}_f(h \otimes_k l) := f(h^{(1)} \otimes_k l^{(1)}) \otimes_k h^{(2)} l^{(2)}$. Assume that

$$f(h \otimes_k l) = f(h^{(1)} \otimes_k l^{(1)})(h^{(2)} l^{(2)} \cdot 1_A) \quad \text{for all } h, l \in H.$$

By [24, Proposition 2.4] we know that $(A, H, \chi_\rho, \mathcal{F}_f)$ is a crossed product system (see [24, Definition 1.7]). We say that f satisfies the *twisted module condition* if

$$f(h^{(1)} \otimes_k l^{(1)})(h^{(2)} l^{(2)} \cdot a) = (h^{(1)} \cdot (l^{(1)} \cdot a))f(h^{(2)} \otimes_k l^{(2)}) \quad \text{for all } h, l \in H \text{ and } a \in A,$$

and that f is a *cocycle* if

$$f(h^{(1)} \otimes_k l^{(1)})f(h^{(2)} l^{(2)} \otimes_k m) = (h^{(1)} \cdot f(l^{(1)} \otimes_k m^{(1)}))f(h^{(2)} \otimes_k l^{(2)} m^{(2)}) \quad \text{for all } h, l, m \in H.$$

Let E be $A \times H$ endowed with the multiplication map μ_E introduced in [24, Notation 1.9]. A direct computation using the twisted module condition shows that $(a \times h)(b \times l) = a(h^{(1)} \cdot b)f(h^{(2)} \otimes_k l^{(1)}) \times h^{(3)} l^{(2)}$.

Theorem 1.5. *Assume that*

- (1) $f(h \otimes_k l) = f(h^{(1)} \otimes_k l^{(1)})(h^{(2)} l^{(2)} \cdot 1_A)$ for all $h, l \in H$,
- (2) $h \cdot 1_A = (h^{(1)} \cdot (1^{(1)} \cdot 1_A))f(h^{(2)} \otimes_k 1^{(2)})$ for all $h \in H$,
- (3) $h \cdot 1_A = (1^{(1)} \cdot 1_A)f(1^{(2)} \otimes_k h)$ for all $h \in H$,
- (4) $a \times 1 = 1^{(1)} \cdot a \otimes_k 1^{(2)}$ for all $a \in A$,
- (5) f is a cocycle that satisfies the twisted module condition.

Then,

- (a) $(A, H, \chi_\rho, \mathcal{F}_f, \nu)$ is a crossed product system with preunit (see [24, Definition 1.11]),
- (b) \mathcal{F}_f is a cocycle that satisfies the twisted module condition (see [24, Definitions 1.8]),
- (c) μ_E is left and right A -linear, associative and has unit $1_A \times 1$,
- (d) The morphism $j_\nu: A \rightarrow E$ is left and right A -linear, multiplicative and unitary,
- (e) $j_\nu(a)\mathbf{x} = a \cdot \mathbf{x}$ and $\mathbf{x}j_\nu(a) = \mathbf{x} \cdot a$, for all $a \in A$ and $\mathbf{x} \in E$,
- (f) $\chi_\rho(h \otimes_k a) = \gamma(h)j_\nu(a)$ and $\mathcal{F}_f(h \otimes_k l) = \gamma(h)\gamma(l)$, for all $h, l \in H$ and $a \in A$.

Proof. By [24, Propositions 2.4, 2.6 and 2.10, and Theorems 1.12(7) and 2.11]. \square

In the rest of this subsection we assume that the hypotheses of Theorem 1.5 are fulfilled, and we say that E is the *unitary crossed product of A with H associated with ρ and f* . Note that by item (10) of that theorem we have

$$j_\nu(a)\gamma(h) = a \cdot \nabla_\rho(1_A \otimes_k h) = \nabla_\rho(a \otimes_k h) = a \times h. \quad (1.7)$$

Moreover, by [25, Equalities (1.6) and (1.7)] and the definitions of χ_ρ and \mathcal{F}_f , we have

$$\gamma(h)j_\nu(a) = j_\nu(h^{(1)} \cdot a)\gamma(h^{(2)}) \quad \text{and} \quad \gamma(h)\gamma(l) = j_\nu(f(h^{(1)} \otimes_k l^{(1)}))\gamma(h^{(2)} l^{(2)}). \quad (1.8)$$

1.3.1 The weak comodule structure of E

Let B be a right H -comodule. The tensor product $B \otimes_k B$ is a (not necessarily counitary) right H -comodule, via $\delta_{B \otimes_k B}(b \otimes_k c) := b^{(0)} \otimes_k c^{(0)} \otimes_k b^{(1)} c^{(1)}$.

Definition 1.6. A unitary associative algebra B , which is also a counitary right H -comodule, is a *right H -comodule algebra* if μ_B is right H -colinear and one of the following equivalent conditions is satisfied:

- (1) $1_B^{(0)} \otimes_k 1_B^{(1)} \otimes_k 1_B^{(2)} = 1_B^{(0)} \otimes_k 1_B^{(1)} 1_B^{(2)}$.
- (2) $1_B^{(0)} \otimes_k 1_B^{(1)} \otimes_k 1_B^{(2)} = 1_B^{(0)} \otimes_k 1_B^{(1)} 1_B^{(2)}$.
- (3) $b^{(0)} \otimes_k \bar{\Pi}^R(b^{(1)}) = b 1_B^{(0)} \otimes_k 1_B^{(1)}$ for all $b \in B$.
- (4) $b^{(0)} \otimes_k \Pi^L(b^{(1)}) = 1_B^{(0)} b \otimes_k 1_B^{(1)}$ for all $b \in B$.
- (5) $1_B^{(0)} \otimes_k \bar{\Pi}^R(1_B^{(1)}) = 1_B^{(0)} \otimes_k 1_B^{(1)}$.
- (6) $1_B^{(0)} \otimes_k \Pi^L(1_B^{(1)}) = 1_B^{(0)} \otimes_k 1_B^{(1)}$.

Proposition 1.7 (Comodule algebra structure on E). *Assume that the hypotheses of Theorem 1.5 are satisfied. Then E is a weak H -comodule algebra via the map $\delta_E: E \rightarrow E \otimes_k H$ defined by*

$$\delta_E\left(\sum a_i \otimes_k h_i\right) := \sum \nabla_\rho(a_i \otimes_k h_i^{(1)}) \otimes_k h_i^{(2)} = \sum (a_i \times h_i^{(1)}) \otimes_k h_i^{(2)}.$$

Proof. See [24, Proposition 2.27]. □

1.4 Weak crossed products of weak module algebras

Let H be a weak bialgebra, A a k -algebra and $\rho: H \otimes_k A \rightarrow A$ a map. In this subsection we study the weak crossed products of A with H in which A is a left weak H -module algebra.

Definition 1.8. We say that A is a *left weak H -module algebra* via ρ , if

- (1) $1 \cdot a = a$ for all $a \in A$,
- (2) $h \cdot (aa') = (h^{(1)} \cdot a)(h^{(2)} \cdot a')$ for all $h \in H$ and $a, a' \in A$,
- (3) $h \cdot (l \cdot 1_A) = (hl) \cdot 1_A$ for all $h, l \in H$ and $a \in A$,

In this case we say that ρ is a weak left action of H on A . If we also have

$$h \cdot (l \cdot a) = (hl) \cdot a \quad \text{for all } h, l \in H \text{ and all } a \in A, \tag{1.9}$$

then we say that A is a *left H -module algebra*.

Proposition 1.9. *For each weak H -module algebra A the following assertions hold:*

- (1) $\Pi^L(h) \cdot a = (h \cdot 1_A)a$ for all $h \in H$ and $a \in A$.
- (2) $\bar{\Pi}^L(h) \cdot a = a(h \cdot 1_A)$ for all $h \in H$ and $a \in A$.
- (3) $\Pi^L(h) \cdot 1_A = h \cdot 1_A$ for all $h \in H$.
- (4) $\bar{\Pi}^L(h) \cdot 1_A = h \cdot 1_A$ for all $h \in H$.
- (5) $h \cdot (l \cdot 1_A) = (h^{(1)} \cdot 1_A)\epsilon(h^{(2)}l)$ for all $h, l \in H$.
- (6) $h \cdot (l \cdot 1_A) = (h^{(2)} \cdot 1_A)\epsilon(h^{(1)}l)$ for all $h, l \in H$.

Proof. In [11] it was proven that these items are equivalent and the proof of item (3) it was kindly communicated to us by José Nicanor Alonso Álvarez and Ramón González Rodríguez (see [24, Proposition 4.2]). □

Example 1.10. The algebra H^L is a left H -module algebra via $h \cdot l := \Pi^L(hl)$. In fact item (1) of Definition 1.8 is trivial. By [8, equality (2.7a)] we have

$$h^{(1)}l^{(1)} \otimes_k h^{(2)}l^{(2)} = h^{(1)}l \otimes_k h^{(2)} \quad \text{for all } h \in H \text{ and } l \in H^L. \tag{1.10}$$

Using this, equality [8, equality (2.5a)] and the fact that $\Pi^L * \text{id} = \text{id}$, we obtain

$$\Pi^L(h^{(1)}l)\Pi^L(h^{(2)}m) = \Pi^L(\Pi^L(h^{(1)}l^{(1)})\Pi^L(h^{(2)}l^{(1)}m)) = \Pi^L(\Pi^L(h^{(1)}l^{(1)})h^{(2)}l^{(1)}m) = \Pi^L(hlm),$$

$h \in H$ and $l, m \in H^L$. This proves item (2). Finally equality (1.9) follows from [8, equality (2.5a)]. This example is known as the trivial representation.

Example 1.11. Let A be a left H -module algebra. The map $f(h \otimes_k l) := hl \cdot 1_A$, named the *trivial cocycle* of A , satisfies the hypotheses of Theorem 1.5.

Definition 1.12. A subalgebra K of A is *stable under ρ* if $h \cdot \lambda \in K$ for all $h \in H$ and $\lambda \in K$.

Remark 1.13. A subalgebra K of A is stable under ρ if and only if $\chi_\rho(H \otimes_k K) \subseteq K \otimes_k H$.

Example 1.14. Let $K := \{h \cdot 1_A : h \in H\}$. By Definition 1.8 and Proposition 1.9, we know that K is a subalgebra of A , which is stable under ρ , and $K = \{h \cdot 1_A : h \in H^L\} = \{h \cdot 1_A : h \in H^R\}$. Moreover, the map $\pi^L : H^L \rightarrow K$, defined by $\pi^L(h) := h \cdot 1_A$, is a surjective morphism of algebras, and the map $\pi^R : H^R \rightarrow K$, defined by the same formula, is a surjective anti-morphism of algebras. By [8, Proposition 2.11], this implies that K is separable.

Remark 1.15. Each subalgebra of A , which is stable under ρ , includes $\{h \cdot 1_A : h \in H\}$.

From here until the end of this subsection A is a left weak H -module algebra and E is the unitary crossed product of A by H associated with ρ and a map $f : H \otimes_k H \rightarrow A$. Thus, we assume that the hypotheses of Theorem 1.5 are fulfilled. In particular ν, j_ν and γ are as at the beginning of Subsection 1.3. By [8, Proposition 2.4] we know that $H^L H^R$ is a subalgebra H .

Proposition 1.16. *The map f factorizes throughout $H \otimes_{H^L H^R} H$.*

Proof. It suffices to prove that $f(hl \otimes_k m) = f(hl \otimes_k m)$ for all $h, m \in H$ and $l \in H^L \cup H^R$. For $l \in H^R$ this follows from [24, Proposition 2.7]. Suppose now that $l \in H^L$. By Definition 1.8(3), the maps $u_2, v_2 : H \otimes_k H \rightarrow A$, defined by $u_2(h \otimes_k l) := hl \cdot 1_A$ and $v_2(h \otimes_k l) := h \cdot (l \cdot 1_A)$ coincide. Consequently $f(hl \otimes_k m) = f(hl \otimes_k m)$ by [24, Proposition 2.8 and Remarks 2.16 and 2.17]. \square

1.4.1 Invertible cocycles and cleft extensions

Let $u_2 : H \otimes_k H \rightarrow A$ be the map defined by $u_2(h \otimes_k l) := hl \cdot 1_A$. We say that the cocycle f is invertible if there exists a (unique) map $f^{-1} : H \otimes_k H \rightarrow A$ such that $u_2 * f^{-1} = f^{-1} * u_2 = f^{-1}$ and $f^{-1} * f = f * f^{-1} = u_2$.

Remark 1.17. Condition (1) in Theorem 1.5 says that $f * u_2 = f$. By [24, Remark 2.17] and the comment above [24, Proposition 4.4], this implies that $u_2 * f = f$.

Example 1.18. Assume that A is a left H -module algebra and that f is the trivial cocycle. By the previous remark and Definition 1.8(2), the cocycle f is invertible and $f^{-1} = f$.

Definition 1.19. A map $g : H \otimes_k H \rightarrow A$ is *normal* if $g(1 \otimes_k h) = g(h \otimes_k 1) = h \cdot 1_A$ for all h .

By [24, Proposition 2.17 and Remark 2.19] we know that the cocycle f is normal if and only if the equalities in items (2) and (3) of Theorem 1.5 hold. When f is invertible, we define $\gamma^{-1} : H \rightarrow E$ by

$$\gamma^{-1}(h) := j_\nu(f^{-1}(S(h^{(2)}) \otimes_k h^{(3)}))\gamma(S(h^{(1)})). \quad (1.11)$$

Assume for instance that A is a left H -module algebra and that f is the trivial cocycle. By Example 1.18 and the fact that $S * \text{id} = \Pi^R$, we have

$$\gamma^{-1}(h) = j_\nu(S(h^{(2)})h^{(3)} \cdot 1_A)\gamma(S(h^{(1)})) = j_\nu(\Pi^R(h^{(2)}) \cdot 1_A)\gamma(S(h^{(1)})).$$

Using this, Proposition 1.9(3) and the fact that $\Pi^L \circ \Pi^R = \Pi^L \circ \overline{\Pi}^L \circ S = \Pi^L \circ S$, we obtain

$$\gamma^{-1}(h) = j_\nu(\Pi^L(S(h^{(2)})) \cdot 1_A)\gamma(S(h^{(1)})) = \gamma(\Pi^L(S(h^{(2)}))S(h^{(1)})) = \gamma(S(h)),$$

where the second equality holds by [24, Propositions 2.22 and 4.5]; and the last one, since S is anticommutative and $\Pi^L * \text{id} = \text{id}$.

1.5 Mixed complexes

In this subsection we recall briefly the notion of mixed complex. For more details about this concept we refer to [10] and [28].

A *mixed complex* $\mathcal{X} := (X, b, B)$ is a graded k -module $(X_n)_{n \geq 0}$, endowed with morphisms

$$b : X_n \longrightarrow X_{n-1} \quad \text{and} \quad B : X_n \longrightarrow X_{n+1},$$

such that $b \circ b = 0$, $B \circ B = 0$ and $B \circ b + b \circ B = 0$. A *morphism of mixed complexes* $g : (X, b, B) \longrightarrow (Y, d, D)$ is a family of maps $g : X_n \rightarrow Y_n$, such that $d \circ g = g \circ b$ and $D \circ g = g \circ B$. Let u be a degree 2 variable. A

mixed complex $\mathcal{X} := (X, b, B)$ determines a double complex

$$\text{BP}(\mathcal{X}) = \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots & & \vdots \\ & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\ \dots & \xleftarrow{B} & X_3 u^{-1} & \xleftarrow{B} & X_2 u^0 & \xleftarrow{B} & X_1 u & \xleftarrow{B} & X_0 u^2 \\ & & \downarrow b & & \downarrow b & & \downarrow b & & \\ \dots & \xleftarrow{B} & X_2 u^{-1} & \xleftarrow{B} & X_1 u^0 & \xleftarrow{B} & X_0 u & & \\ & & \downarrow b & & \downarrow b & & & & \\ \dots & \xleftarrow{B} & X_1 u^{-1} & \xleftarrow{B} & X_0 u^0 & & & & \\ & & \downarrow b & & & & & & \\ \dots & \xleftarrow{B} & X_0 u^{-1} & & & & & & \end{array},$$

where $b(\mathbf{x}u^i) := b(\mathbf{x})u^i$ and $B(\mathbf{x}u^i) := B(\mathbf{x})u^{i-1}$. By deleting the positively numbered columns we obtain a subcomplex $\text{BN}(\mathcal{X})$ of $\text{BP}(\mathcal{X})$. Let $\text{BN}'(\mathcal{X})$ be the kernel of the canonical surjection from $\text{BN}(\mathcal{X})$ to (X, b) . The quotient double complex $\text{BP}(\mathcal{X})/\text{BN}'(\mathcal{X})$ is denoted by $\text{BC}(\mathcal{X})$. The homology groups $\text{HC}_*(\mathcal{X})$, $\text{HN}_*(\mathcal{X})$ and $\text{HP}_*(\mathcal{X})$, of the total complexes of $\text{BC}(\mathcal{X})$, $\text{BN}(\mathcal{X})$ and $\text{BP}(\mathcal{X})$, respectively, are called the *cyclic*, *negative* and *periodic homology groups* of \mathcal{X} . The homology $\text{HH}_*(\mathcal{X})$, of (X, b) , is called the *Hochschild homology* of \mathcal{X} . Finally, it is clear that a morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ of mixed complexes induces a morphism from the double complex $\text{BP}(\mathcal{X})$ to the double complex $\text{BP}(\mathcal{Y})$.

Notation 1.20. Let C be an algebra. If K is a subalgebra of C we will say that C is a K -algebra. Given a K -bimodule M , we let $M \otimes_K$ denote the quotient $M/[M, K]$, where $[M, K]$ is the k -submodule of M generated by all the commutators $m\lambda - \lambda m$, with $m \in M$ and $\lambda \in K$. Moreover, for $m \in M$, we let $[m]$ denote the class of m in $M \otimes_K$.

By definition, the *normalized mixed complex of the K -algebra C* is $(C \otimes_K \overline{C}^{\otimes_K} \otimes_K, b_*, B_*)$, where $\overline{C} := C/K$, b_* is the canonical Hochschild boundary map and the Connes operator B_* is given by

$$B([c_0 \otimes_K \cdots \otimes_K c_r]) := \sum_{i=0}^r (-1)^{ir} [1 \otimes_K c_i \otimes_K \cdots \otimes_K c_r \otimes_K c_0 \otimes_K c_1 \otimes_K \cdots \otimes_K c_{i-1}].$$

The *cyclic*, *negative*, *periodic* and *Hochschild homology groups* $\text{HC}_*^K(C)$, $\text{HN}_*^K(C)$, $\text{HP}_*^K(C)$ and $\text{HH}_*^K(C)$ of C are the respective homology groups of $(C \otimes_K \overline{C}^{\otimes_K} \otimes_K, b_*, B_*)$.

2 Hochschild homology of cleft extensions

Let H be a weak Hopf algebra, A an algebra, $\rho: H \rightarrow A$ a weak left action and $f: H \otimes_k H \rightarrow A$ a linear map. Let $\chi_\rho, \gamma, \nu, j_\nu$ and \mathcal{F}_f be as at the beginning of Subsection 1.3. Assume that the hypotheses of Theorem 1.5 are fulfilled. Let E be the crossed product associated with ρ and f and let K be a subalgebra of A , which is stable under ρ . For instance we can take K as the minimal subalgebra of A that is stable under ρ (see Example 1.14). By Theorem 1.5, the tuple $(A, \chi_\rho, \mathcal{F}_f, \nu)$ is a crossed product system with preunit, \mathcal{F}_f is a cocycle that satisfies the twisted module condition and E is an associative algebra with unit $1_E := 1_A \times 1$. Moreover, by Remark 1.15 we know that $1_E \in K \otimes_k H$. So, the hypotheses of [25, Section 3] are satisfied. For the sake of simplicity in the sequel we will write \otimes instead of \otimes_K . Let M be an E -bimodule. By definition the Hochschild homology $\text{H}_*^K(E, M)$, of the K -algebra E with coefficients in M , is the homology of the normalized Hochschild chain complex $(M \otimes \overline{E}^{\otimes} \otimes, b_*)$, where b_* is the canonical Hochschild boundary map. In [25, Section 3] a chain complex $(\widehat{X}_*(M), \widehat{d}_*)$ was obtained, simpler than the canonical one, that gives the Hochschild homology $\text{H}_*^K(E, M)$, of the K -algebra E with coefficients in M . From now on we assume that the cocycle f is convolution invertible. In this Section we prove that $(\widehat{X}_*(M), \widehat{d}_*)$ is isomorphic to a simpler complex $(\overline{X}_*(M), \overline{d}_*)$. If K is separable (for instance when $K := H \cdot 1_A$), then $(\overline{X}_*(M), \overline{d}_*)$ gives the absolute Hochschild homology of E with coefficients in M . Recall that M is an A -bimodule via the map $j_\nu: A \rightarrow E$. Let $\overline{A} := A/K$, $\overline{E} := E/j_\nu(K)$ and $\widetilde{E} := E/j_\nu(A)$. We recall from [25, Section 3] that

$$\widehat{X}_n(M) = \bigoplus_{\substack{r, s \geq 0 \\ r+s=n}} \widehat{X}_{rs}(M), \quad \text{where } \widehat{X}_{rs}(M) := M \otimes_A \widetilde{E}^{\otimes_A^s} \otimes \overline{A}^{\otimes^r} \otimes,$$

and that there exist maps $\widehat{d}_{rs} : \widehat{X}_{rs}(M) \rightarrow \widehat{X}_{r-l, s+l-1}(M)$ such that

$$\widehat{d}_n(\mathbf{x}) := \begin{cases} \sum_{l=1}^n \widehat{d}_{0n}(\mathbf{x}) & \text{if } \mathbf{x} \in \widehat{X}_{0n}, \\ \sum_{l=0}^{n-r} \widehat{d}_{r, n-r}(\mathbf{x}) & \text{if } \mathbf{x} \in \widehat{X}_{r, n-r} \text{ with } r > 0. \end{cases}$$

Notations 2.1. We will use the following notations:

- (1) For each $a \in A$ we let \bar{a} denote its class in \bar{A} . For each $x \in E$, we let \bar{x} and \tilde{x} denote its class in \bar{E} and \tilde{E} , respectively.
- (2) Given $a_1, \dots, a_r \in A$ and $1 \leq i \leq j \leq r$, we set $\mathbf{a}_{ij} := a_i \otimes \dots \otimes a_j$, and we let $\bar{\mathbf{a}}_{ij}$ denote the class of \mathbf{a}_{ij} in $\bar{A}^{\otimes j-i+1}$.
- (3) Given $h_1, \dots, h_s \in H$ we set $\mathfrak{h}_{1s} := h_1 \cdots h_s$ and $\mathfrak{h}_{1s} := h_1 \otimes_k \dots \otimes_k h_s$.
- (4) Given $h_1, \dots, h_s \in H$ we set

$$\mathfrak{h}_{1s}^{(1)} \otimes_k \mathfrak{h}_{1s}^{(2)} := (\mathfrak{h}_1^{(1)} \otimes_k \dots \otimes_k \mathfrak{h}_s^{(1)}) \otimes_k (\mathfrak{h}_1^{(2)} \otimes_k \dots \otimes_k \mathfrak{h}_s^{(2)})$$

and

$$\bar{\mathfrak{h}}_{1s}^{(1)} \otimes_k \bar{\mathfrak{h}}_{1s}^{(2)} := (\bar{\mathfrak{h}}_1^{(1)} \otimes_{H^L} \dots \otimes_{H^L} \bar{\mathfrak{h}}_s^{(1)}) \otimes_k (\bar{\mathfrak{h}}_1^{(2)} \otimes_{H^L} \dots \otimes_{H^L} \bar{\mathfrak{h}}_s^{(2)}).$$

- (5) Given $h \in H$ and $a_1, \dots, a_r \in A$ we set $h \cdot \bar{\mathbf{a}}_{1r} := \overline{h^{(1)} \cdot a_1} \otimes \dots \otimes \overline{h^{(r)} \cdot a_r}$.
- (6) We let $\bar{\gamma} : H \rightarrow \bar{E}$ and $\tilde{\gamma} : H \rightarrow \tilde{E}$ denote the maps induced by γ .
- (7) Given $h_1, \dots, h_s \in H$ we set $\tilde{\gamma}_A(\mathfrak{h}_{1s}) := \tilde{\gamma}(h_1) \otimes_A \dots \otimes_A \tilde{\gamma}(h_s)$.
- (8) Given $h_1, \dots, h_s \in H$ we set $\gamma_\times(\mathfrak{h}_{1s}) := \gamma(h_1) \cdots \gamma(h_s)$.
- (9) Given $h_1, \dots, h_s \in H$ we set $\gamma_\times^{-1}(\mathfrak{h}_{1s}) := \gamma^{-1}(h_s) \cdots \gamma^{-1}(h_1)$, where γ^{-1} is as in (1.11).

2.1 Technical results

Lemma 2.2. *Let $a, a' \in A$ and $h \in H$. The following equalities hold:*

- (1) $\delta_E(j_\nu(a)\gamma(l)j_\nu(a')\gamma(h)) = j_\nu(a)\gamma(l)j_\nu(a')\gamma(h^{(1)}) \otimes_k h^{(2)}$, for all $l \in H^L$.
- (2) $\delta_E(j_\nu(a')\gamma(h)j_\nu(a)\gamma(l)) = j_\nu(a')\gamma(h^{(1)})j_\nu(a)\gamma(l) \otimes_k h^{(2)}$, for all $l \in H^L$.
- (3) $\delta_E(j_\nu(A)\gamma(H^L)) \subseteq E \otimes_k H^L$.
- (4) $j_\nu(a)\gamma^{-1}(h) = \gamma^{-1}(h^{(1)})j_\nu(h^{(2)} \cdot a)$.

Proof. 1) We have

$$\begin{aligned} \delta_E(j_\nu(a)\gamma(l)j_\nu(a')\gamma(h)) &= \delta_E(j_\nu(a(l^{(1)} \cdot a'))\gamma(l^{(2)})\gamma(h)) \\ &= \delta_E(j_\nu(a(l^{(1)} \cdot a'))\gamma(l^{(2)}h)) \\ &= j_\nu(a(l^{(1)} \cdot a'))\gamma(l^{(2)}h^{(1)}) \otimes_k l^{(3)}h^{(2)} \\ &= j_\nu(a(l^{(1)} \cdot a'))\gamma(l^{(2)}h^{(1)}) \otimes_k h^{(2)} \\ &= j_\nu(a(l^{(1)} \cdot a'))\gamma(l^{(2)})\gamma(h^{(1)}) \otimes_k h^{(2)} \\ &= j_\nu(a)\gamma(l)j_\nu(a')\gamma(h^{(1)}) \otimes_k h^{(2)}, \end{aligned}$$

where the first and the last equality hold by the first identity in (1.8) and Theorem 1.5(d); the second one, by [8, (2.6a)] and [24, Proposition 2.22]; the third one because, by equality (1.7), the definition of δ_E , and the fact that $j_\nu(A)\gamma(H) \subseteq E$, for all $a \in A$ and $h \in H$, we have

$$\delta_E(j_\nu(a)\gamma(h)) = \nabla_\rho(a \times h^{(1)}) \otimes_k h^{(2)} = \nabla_\rho(j_\nu(a)\gamma(h^{(1)})) \otimes_k h^{(2)} = j_\nu(a)\gamma(h^{(1)}) \otimes_k h^{(2)}; \quad (2.1)$$

the fourth one, by [8, (2.6a)] and the fact that by [8, (2.4), (2.7a) and (2.10)]

$$l^{(1)}h^{(1)} \otimes_k l^{(2)}h^{(2)} = lh^{(1)} \otimes_k h^{(2)} \quad \text{for all } h \in H \text{ and } l \in H^L; \quad (2.2)$$

and the fifth one, by [24, Proposition 2.22].

2) Mimic the proof of item (1).

3) This follows from equality (2.1) and [8, (2.6a)].

4) This is [24, Proposition 5.22]. □

Remark 2.3. Since $j_\nu(1_A) = 1_E$, from equality (2.1) it follows that γ is H -colinear.

Lemma 2.4. For all $h \in H$ and $l \in H^L$, we have $\gamma^{-1}(hl) = \gamma(S(l))\gamma^{-1}(h)$ and $\gamma^{-1}(lh) = \gamma^{-1}(h)\gamma(S(l))$.

Proof. We prove the first equality and leave the second one, which is similar, to the reader. We have

$$\begin{aligned}\gamma^{-1}(hl) &= j_\nu(f^{-1}(S(h^{(2)}l^{(2)})) \otimes_k h^{(3)}l^{(3)})\gamma(S(h^{(1)}l^{(1)})) \\ &= j_\nu(f^{-1}(S(h^{(2)}) \otimes_k h^{(3)})\gamma(S(h^{(1)}l)) \\ &= j_\nu(f^{-1}(S(h^{(2)}) \otimes_k h^{(3)})\gamma(S(l))\gamma(S(h^{(1)})) \\ &= \gamma(S(l))j_\nu(f^{-1}(S(h^{(2)}) \otimes_k h^{(3)})\gamma(S(h^{(1)})) \\ &= \gamma(S(l))\gamma^{-1}(h),\end{aligned}$$

where the first and last equality hold by the definition of γ^{-1} ; the second one, by [8, (2.6a)] and identity (1.10); the third one, by [24, Proposition 2.22] and the fact that S is antimultiplicative; and the fourth one, by [24, Proposition 4.6]. \square

Lemma 2.5. Let $h_1, \dots, h_s \in H$ and $a \in A$. The following equality holds:

$$\gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \cdot j_\nu(a) = j_\nu(a)\gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}).$$

Proof. We proceed by induction on s . By the first identity in (1.8) and Lemma 2.2(4), for $s = 1$ we have

$$\gamma^{-1}(h_1^{(1)}) \otimes_A \tilde{\gamma}(h_1^{(2)})j_\nu(a) = \gamma^{-1}(h_1^{(1)})j_\nu(h_1^{(2)} \cdot a) \otimes_A \tilde{\gamma}(h_1^{(3)}) = j_\nu(a)\gamma^{-1}(h_1^{(1)}) \otimes_A \tilde{\gamma}(h_1^{(2)}).$$

Assume $s > 1$ and that the result is valid for $s - 1$. Let $T := \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \cdot j_\nu(a)$. By the first identity in (1.8),

$$T = \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes_A j_\nu(h_s^{(2)} \cdot a)\tilde{\gamma}(h_s^{(3)}) = \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \cdot j_\nu(h_s^{(2)} \cdot a) \otimes_A \tilde{\gamma}(h_s^{(3)}).$$

Consequently, by the inductive hypothesis and Lemma 2.2(4),

$$T = \gamma^{-1}(h_s^{(1)})j_\nu(h_s^{(2)} \cdot a)\gamma_x^{-1}(h_{1,s-1}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes_A \tilde{\gamma}(h_s^{(3)}) = j_\nu(a)\gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}),$$

as desired. \square

Lemma 2.6. Let $s \geq 1$. for all $h_1, \dots, h_s \in H$, we have $\gamma_x(h_{1s}^{(1)})\gamma_x^{-1}(h_{1s}^{(2)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(3)}) = 1_E \otimes_A \tilde{\gamma}_A(h_{1s})$.

Proof. We proceed by induction on s . Using the first equality in (1.3) and arguing as in [8, equality (2.8b)], we obtain that

$$\Pi^L(h^{(1)}) \otimes_k h^{(2)} = S(\overline{\Pi}^L(h^{(1)})) \otimes_k h^{(2)} = S(1^{(1)}) \otimes_k 1^{(2)}h \quad \text{for all } h \in H. \quad (2.3)$$

Using this, [24, Lemma 2.20 and Propositions 4.5 and 5.19], we obtain

$$\gamma(h_1^{(1)})\gamma^{-1}(h_1^{(2)}) \otimes_A \tilde{\gamma}(h_1^{(3)}) = \gamma(\Pi^L(h_1^{(1)})) \otimes_A \tilde{\gamma}(h_1^{(2)}) = \gamma(S(1^{(1)})) \otimes_A \tilde{\gamma}(1^{(2)}h_1) = 1_E \otimes_A \tilde{\gamma}(h_1).$$

This proves the case $s = 1$. Assume that the result is true for s and set $T := \gamma_x(h_{1,s+1}^{(1)})\gamma_x^{-1}(h_{1,s+1}^{(2)}) \otimes_A \tilde{\gamma}_A(h_{1,s+1}^{(3)})$. By equality (2.3), [24, Propositions 4.5 and 5.19],

$$\gamma(h_{s+1}^{(1)})\gamma^{-1}(h_{s+1}^{(2)}) \otimes_k h_{s+1}^{(3)} = \gamma(\Pi^L(h_{s+1}^{(1)})) \otimes_k h_{s+1}^{(2)} = j_\nu(\Pi^L(h_{s+1}^{(1)}) \cdot 1_A) \otimes_k h_{s+1}^{(2)} = j_\nu(S(1^{(1)}) \cdot 1_A) \otimes_k 1^{(2)}h_{s+1}.$$

Hence

$$T = \gamma_x(h_{1s}^{(1)})j_\nu(S(1^{(1)}) \cdot 1_A)\gamma_x^{-1}(h_{1s}^{(2)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(3)}) \otimes_A \tilde{\gamma}(1^{(2)}h_{s+1}).$$

Using now the first identity in (1.8) and Definition 1.8(3) again and again, we obtain

$$T = j_\nu(h_{1s}^{(1)}S(1^{(1)}) \cdot 1_A)\gamma_x(h_{1s}^{(2)})\gamma_x^{-1}(h_{1s}^{(3)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(4)}) \otimes_A \tilde{\gamma}(1^{(2)}h_{s+1}).$$

Consequently, by the inductive hypothesis,

$$T = j_\nu(h_{1s}^{(1)}S(1^{(1)}) \cdot 1_A) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes_A \tilde{\gamma}(1^{(2)}h_{s+1}) = 1_E \otimes_A \tilde{\gamma}_A(h_{1s}) \otimes_A j_\nu(S(1^{(1)}) \cdot 1_A)\tilde{\gamma}(1^{(2)}h_{s+1}),$$

where the last equality follows using Definition 1.8(3) and the first identity in (1.8) again and again. Hence, by [24, Lemma 2.20 and Proposition 4.5], we have $T = 1_E \otimes_A \tilde{\gamma}_A(h_{1,s+1})$, as desired. \square

Lemma 2.7. Let $s \geq 1$. For all $z \in H^R$ and $h_1, \dots, h_s \in H$, we have

$$\gamma_x^{-1}(h_{1s}^{(1)})\gamma(z)\gamma_x(h_{1s}^{(2)}) \otimes_k \bar{\mathbf{h}}_{1s}^{(3)} = \gamma(1^{(1)}) \otimes_k \overline{\Pi}^R(z) \cdot \bar{\mathbf{h}}_{1s} \cdot 1^{(2)}.$$

Proof. Set $T := \gamma_x^{-1}(h_{1s}^{(1)})\gamma(z)\gamma_x(h_{1s}^{(2)}) \otimes_k \bar{\mathbf{h}}_{1s}^{(3)}$. We proceed by induction on s . Let $s = 1$. By [8, equality (2.7b)], [24, Propositions 2.22 and 5.19], we have

$$T = \gamma^{-1}(h_1^{(1)})\gamma(zh_1^{(2)}) \otimes_k \overline{h_1^{(3)}} = \gamma^{-1}(z^{(1)}h_1^{(1)})\gamma(z^{(2)}h_1^{(2)}) \otimes_k \overline{h_1^{(3)}} = \gamma(\Pi^R(zh_1^{(1)})) \otimes_k \overline{h_1^{(2)}}.$$

Consequently, by the definition of Π^R , [8, (2.6a)] and identities (1.2) and (1.10),

$$T = \gamma(1^{(1)}) \otimes_k \epsilon(zh_1^{(1)}1^{(2)})\overline{h_1^{(2)}} = \gamma(1^{(1)}) \otimes_k \epsilon(z1^{(1)})\epsilon(1^{(2)}h_1^{(1)}1^{(2)})\overline{1^{(3)}h_1^{(2)}1^{(3)}}.$$

So, by the definition of $\overline{\Pi}^R$,

$$T = \gamma(1^{(1)}) \otimes_k \epsilon(z1^{(1')}) \overline{1^{(2')}h_1 1^{(2)}} = \gamma(1^{(1)}) \otimes_k \overline{\overline{\Pi}^R(z)h_1 1^{(2)}}.$$

Assume now that $s > 1$ and the result holds by $s - 1$. By the inductive hypothesis, we have

$$T = \gamma^{-1}(h_s^{(1)})\gamma(1^{(1)})\gamma(h_s^{(2)}) \otimes_k (\overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1,s-1} \cdot 1^{(2)} \otimes_{H^L} \overline{h_s^{(3)}}).$$

Thus, by the case $s = 1$, [8, (2.4)] and [24, Proposition 2.22], we have

$$\begin{aligned} T &= \gamma^{-1}(h_s^{(1)})\gamma(1^{(1)}h_s^{(2)}) \otimes_k (\overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1,s-1} \otimes_{H^L} \overline{1^{(2)}h_s^{(3)}}) \\ &= \gamma^{-1}(h_s^{(1)})\gamma(h_s^{(2)}) \otimes_k (\overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1,s-1} \otimes_{H^L} \overline{h_s^{(3)}}) \\ &= \gamma(1^{(1)}) \otimes_k \overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1s} \cdot 1^{(2)}, \end{aligned}$$

as desired. \square

2.2 Main results

Let $r, s \geq 0$. By [24, Propositions 2.22 and 4.6], we know that $M \otimes \overline{A}^{\otimes r}$ is a left H^L -module via

$$l \cdot [m \otimes \overline{\mathbf{a}}_{1r}] := [m \cdot \gamma(S(l)) \otimes \overline{\mathbf{a}}_{1r}], \quad (2.4)$$

where $[m \otimes \overline{\mathbf{a}}_{1r}]$ denotes the class of $m \otimes \overline{\mathbf{a}}_{1r}$ in $M \otimes \overline{A}^{\otimes r}$ (see Notation 1.20 and remember that \otimes stands for \otimes_K). Write

$$\overline{X}_{rs}(M) := \overline{H}^{\otimes_{H^L} s} \otimes_{H^L} (M \otimes \overline{A}^{\otimes r}).$$

Since $\overline{H}^{\otimes_{H^L} 0} = H^L$ and $\overline{A}^{\otimes 0} = K$, we have

$$\overline{X}_{r0}(M) \simeq M \otimes \overline{A}^{\otimes r} \otimes \quad \text{and} \quad \overline{X}_{0s}(M) \simeq \overline{H}^{\otimes_{H^L} s} \otimes_{H^L} (M \otimes). \quad (2.5)$$

Let $\Theta'_{rs}: M \otimes_k E^{\otimes_k s} \otimes_k \overline{A}^{\otimes r} \rightarrow \overline{X}_{rs}(M)$ and $\Lambda'_{rs}: H^{\otimes_k s} \otimes_k M \otimes_k \overline{A}^{\otimes r} \rightarrow \widehat{X}_{rs}(M)$ be the maps defined by

$$\Theta'(\mathbf{x}) := (-1)^{rs} \overline{\mathbf{h}}_{1s}^{(2)} \otimes_{H^L} [m \cdot J_\nu(a_1)\gamma(h_1^{(1)}) \cdots J_\nu(a_s)\gamma(h_s^{(1)}) \otimes \overline{\mathbf{a}}_{s+1,s+r}]$$

and

$$\Lambda'(\mathbf{y}) := (-1)^{rs} [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}],$$

where $\mathbf{x} := m \otimes_k J_\nu(a_1)\gamma(h_1) \otimes_k \cdots \otimes_k J_\nu(a_s)\gamma(h_s) \otimes_k \overline{\mathbf{a}}_{s+1,s+r}$ and $\mathbf{y} := h_{1s} \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r}$.

Proposition 2.8. *For each $r, s \geq 0$ the maps Θ'_{rs} and Λ'_{rs} induce morphisms*

$$\Theta_{rs}: \widehat{X}_{rs}(M) \rightarrow \overline{X}_{rs}(M) \quad \text{and} \quad \Lambda_{rs}: \overline{X}_{rs}(M) \rightarrow \widehat{X}_{rs}(M),$$

which are inverse one of each other.

Proof. First we show that Θ_{rs} is well defined. Let \mathbf{x} be as in the definition of Θ'_{rs} . We must prove that

- (1) If there exists i such that $h_i \in H^L$, then $\Theta'(\mathbf{x}) = 0$,
- (2) Θ' is A -balanced in the first s tensors of $M \otimes_k E^{\otimes_k s} \otimes_k \overline{A}^{\otimes r}$,
- (3) Θ' is K -balanced in the $(s+1)$ -tensor of $M \otimes_k E^{\otimes_k s} \otimes_k \overline{A}^{\otimes r}$,
- (4) $\Theta'(J_\nu(\lambda) \cdot \mathbf{x}) = \Theta'(\mathbf{x} \cdot J_\nu(\lambda))$ for all $\lambda \in K$.

Condition (1) follows from Lemma 2.2(3). Next we prove that Condition (2) is satisfied at the first tensor. Let $a \in A$. By Lemma 2.2(1), we have

$$\begin{aligned} &\Theta'(m \cdot J_\nu(a) \otimes_k J_\nu(a_1)\gamma(h_1) \otimes_k \cdots \otimes_k J_\nu(a_s)\gamma(h_s) \otimes_k \overline{\mathbf{a}}_{s+1,s+r}) \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s}^{(2)} \otimes_{H^L} [m \cdot J_\nu(a)J_\nu(a_1)\gamma(h_1^{(1)}) \cdots J_\nu(a_s)\gamma(h_s^{(1)}) \otimes \overline{\mathbf{a}}_{s+1,s+r}] \\ &= \Theta'(m \otimes_k J_\nu(a)J_\nu(a_1)\gamma(h_1) \otimes_k \cdots \otimes_k J_\nu(a_s)\gamma(h_s) \otimes_k \overline{\mathbf{a}}_{s+1,s+r}), \end{aligned}$$

A similar argument using items (1) and (2) of Lemma 2.2 proves Condition (2) at the i -th tensor with $2 \leq i \leq s$.

We now prove Condition (3). Let $\lambda \in K$. By Lemma 2.2(2), we have

$$\begin{aligned} &\Theta'(m \otimes_k J_\nu(a_1)\gamma(h_1) \otimes_k \cdots \otimes_k J_\nu(a_s)\gamma(h_s)J_\nu(\lambda) \otimes_k \overline{\mathbf{a}}_{s+1,s+r}) \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s}^{(2)} \otimes_{H^L} [m \cdot J_\nu(a_1)\gamma(h_1^{(1)}) \cdots J_\nu(a_s)\gamma(h_s^{(1)})J_\nu(\lambda) \otimes \overline{\mathbf{a}}_{s+1,s+r}] \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s}^{(2)} \otimes_{H^L} [m \cdot J_\nu(a_1)\gamma(h_1^{(1)}) \cdots J_\nu(a_s)\gamma(h_s^{(1)}) \otimes \overline{\lambda a_{s+1}} \otimes \overline{\mathbf{a}}_{s+2,s+r}] \\ &= \Theta'(m \otimes_k J_\nu(a_1)\gamma(h_1) \otimes_k \cdots \otimes_k J_\nu(a_s)\gamma(h_s) \otimes_k \overline{\lambda a_{s+1}} \otimes_k \overline{\mathbf{a}}_{s+2,s+r}). \end{aligned}$$

Finally, when $r \geq 1$ the fourth assertion is trivial, while, when $r = 0$ it follows from Lemma 2.2(2).

We next show that $\Lambda_{r,s}$ is well defined. Let \mathbf{y} be as in the definition of $\Lambda'_{r,s}$. We must prove that

- (1) If some $h_i \in H^L$, then $\Lambda'(\mathbf{y}) = 0$,
- (2) Λ' is H^L -balanced in the first $(s-1)$ -th tensors of $H^{\otimes_k^s} \otimes_k M \otimes_k \overline{A}^{\otimes r}$,
- (3) If $r > 0$, then Λ' is K -balanced in the $(s+1)$ -th tensor of $H^{\otimes_k^s} \otimes_k M \otimes_k \overline{A}^{\otimes r}$,
- (4) $\Lambda'(h_{1s} \otimes_k J_\nu(\lambda) \cdot m \otimes_k \overline{\mathbf{a}}_{1r}) = \Lambda'_{r,s}(h_{1s} \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r} \cdot \lambda)$ for all $\lambda \in K$,
- (5) If $s > 0$, then $\Lambda'(h_{1s} \otimes_k m \cdot \gamma(S(l)) \otimes_k \overline{\mathbf{a}}_{1r}) = \Lambda'(h_{1s} \cdot l \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r})$ for all $l \in H^L$,

Item (1) holds since, by [8, (2.6a)] and [24, Proposition 4.5], $\gamma^{-1}(l^{(1)}) \otimes_k \gamma(l^{(2)}) \in E \otimes_k J_\nu(A)$ for all $l \in H^L$. In order to prove item (2) we must check that $\Lambda'(h_{1i} \cdot l \otimes_k h_{i+1,s} \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r}) = \Lambda'(h_{1i} \otimes_k l \cdot h_{i+1,s} \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r})$, for all $i < s$ and $l \in H^L$. But this follows from Lemma 2.4 and identities (1.10) and (2.2). Item (3) holds since, by Lemma 2.5,

$$\begin{aligned} \Lambda'(h_{1s} \otimes_k m \cdot J_\nu(\lambda) \otimes_k \overline{\mathbf{a}}_{1r}) &= [m \cdot J_\nu(\lambda) \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \cdot J_\nu(\lambda) \otimes \overline{\mathbf{a}}_{1r}] \\ &= [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \lambda \cdot \overline{\mathbf{a}}_{1r}] \\ &= \Lambda'(h_{1s} \otimes_k m \otimes_k \lambda \cdot \overline{\mathbf{a}}_{1r}). \end{aligned}$$

When $r \geq 1$, Item (4) is trivial, while, when $r = 0$, it holds since, by Lemma 2.5,

$$\begin{aligned} \Lambda'(h_{1s} \otimes_k J_\nu(\lambda) \cdot m) &= [J_\nu(\lambda) \cdot m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)})] \\ &= [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \cdot J_\nu(\lambda)] \\ &= [m \cdot J_\nu(\lambda) \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)})] \\ &= \Lambda'_{r,s}(h_{1s} \otimes_k m \cdot J_\nu(\lambda)), \end{aligned}$$

Finally, for Item (5) we have

$$\begin{aligned} \Lambda'_{r,s}(h_{1s} \otimes_k m \cdot \gamma(S(l)) \otimes_k \overline{\mathbf{a}}_{1r}) &= [m \cdot \gamma(S(l)) \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= [m \cdot \gamma^{-1}(h_s^{(1)} l) \gamma_x^{-1}(h_{1,s-1}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= [m \cdot \gamma^{-1}(h_s^{(1)} l^{(1)}) \gamma_x^{-1}(h_{1,s-1}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes_A \tilde{\gamma}(h_s^{(2)} l^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= \Lambda'_{r,s}(h_{1s} \cdot l \otimes_k m \otimes_k \overline{\mathbf{a}}_{1r}), \end{aligned}$$

where the second equality holds by Lemma 2.4; and the third one, by identity (1.10).

We next prove that $\Theta_{r,s}$ and $\Lambda_{r,s}$ are inverse one of each other. To begin with, note that under the first identifications in [25, (3.1)] and (2.5),

$$\Theta_{*0} = \text{id} \quad \text{and} \quad \Lambda_{*0} = \text{id}, \tag{2.6}$$

which proves the case $s = 0$. Assume $s \geq 1$ and let $m \in M$, $h_1, \dots, h_s \in H$ and $a_1, \dots, a_r \in A$. Set

$$\mathbf{x} := [m \otimes_A \tilde{\gamma}_A(h_{1s}) \otimes \overline{\mathbf{a}}_{1r}] \in \widehat{X}_{r,s}(M) \quad \text{and} \quad \mathbf{y} := \overline{\mathbf{h}}_{1s} \otimes_{H^L} [m \otimes \overline{\mathbf{a}}_{1r}] \in \overline{X}_{r,s}(M).$$

By Lemma 2.6

$$\begin{aligned} \Lambda(\Theta(\mathbf{x})) &= (-1)^{rs} \Lambda(\overline{\mathbf{h}}_{1s}^{(2)} \otimes_{H^L} [m \cdot \gamma_x(h_{1s}^{(1)}) \otimes \mathbf{a}_{1r}]) \\ &= [m \cdot \gamma_x(h_{1s}^{(1)}) \gamma_x^{-1}(h_{1s}^{(2)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(3)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= [m \otimes_A \tilde{\gamma}_A(h_{1s}) \otimes \overline{\mathbf{a}}_{1r}], \end{aligned}$$

as desired. For the other composition, by Lemma 2.7 and [24, Lemma 2.20], we have

$$\begin{aligned} \Theta(\Lambda(\mathbf{y})) &= (-1)^{rs} \Theta([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}]) \\ &= \overline{\mathbf{h}}_{1s}^{(3)} \otimes_{H^L} [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \gamma_x(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= \overline{\mathbf{h}}_{1s} \cdot 1^{(2)} \otimes_{H^L} [m \cdot \gamma(1^{(1)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= \overline{\mathbf{h}}_{1s} \otimes_{H^L} [m \cdot \gamma(1^{(1)}) \gamma(S(1^{(2)})) \otimes \overline{\mathbf{a}}_{1r}] \\ &= \overline{\mathbf{h}}_{1s} \otimes_{H^L} [m \otimes \overline{\mathbf{a}}_{1r}], \end{aligned}$$

which finishes the proof. \square

For each $0 \leq l \leq s$ and $r \geq 0$ such that $r+l \geq 1$, let $\overline{d}_{r,s}^l: \overline{X}_{r,s}(M) \rightarrow \overline{X}_{r+l-1,s-l}(M)$ be the map defined by $\overline{d}_{r,s}^l := \Theta_{r+l-1,s-l} \circ \widehat{d}_{r,s}^l \circ \Lambda_{r,s}$.

Theorem 2.9. *The Hochschild homology of the K -algebra E with coefficients in M is the homology of the chain complex $(\overline{X}_*(M), \overline{d}_*)$, where*

$$\overline{X}_n(M) := \bigoplus_{r+s=n} \overline{X}_{rs}(M) \quad \text{and} \quad \overline{d}_n(\mathbf{x}) := \begin{cases} \sum_{j=0}^n \overline{d}_{0n}^j(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{X}_{0n}, \\ \sum_{l=0}^{n-r} \overline{d}_{r,n-r}^l(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{X}_{r,n-r} \text{ with } r > 0. \end{cases}$$

Proof. By Proposition 2.8 and the definition of $(\overline{X}_*(M), \overline{d}_*)$, the maps

$$\Theta_* : (\widehat{X}_*(M), \widehat{d}_*) \longrightarrow (\overline{X}_*(M), \overline{d}_*) \quad \text{and} \quad \Lambda_* : (\overline{X}_*(M), \overline{d}_*) \longrightarrow (\widehat{X}_*(M), \widehat{d}_*),$$

given by $\Theta_n := \bigoplus_{r+s=n} \Theta_{rs}$ and $\Lambda_n := \bigoplus_{r+s=n} \Lambda_{rs}$, are inverse one of each other. \square

By [25, Remark 3.2] if f takes its values in K , then $(\overline{X}_*(M), \overline{d}_*)$ is the total chain complex of the double complex $(\overline{X}_{**}(M), \overline{d}_{**}^0, \overline{d}_{**}^1)$; while if $K = A$, then $(\overline{X}_*(M), \overline{d}_*) = (\overline{X}_{0*}(M), \overline{d}_{0*}^1)$.

Lemma 2.10. *Let $m \in M$, $a, a_1, \dots, a_r \in A$, $h_1, \dots, h_s \in H$ and $z \in H^R$.*

(1) *For $\mathbf{x} := [m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes_A \overline{\gamma}(h_s^{(2)}) J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}]$, we have*

$$\Theta(\mathbf{x}) = (-1)^{rs} \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \cdot J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}].$$

(2) *For $\mathbf{x} := [m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,i-1}^{(2)}) \otimes_A \overline{\gamma}(h_i^{(2)}) \gamma(h_{i+1}^{(2)}) \otimes_A \tilde{\gamma}_A(h_{i+2,s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}]$, we have*

$$\Theta(\mathbf{x}) = (-1)^{rs} (\overline{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \overline{\mathbf{h}}_{i+2,s}) \otimes_{HL} [m \otimes \overline{\mathbf{a}}_{1r}]$$

(of course, we are assuming that $s \geq 2$ and $1 \leq i < s$).

(3) *For $\mathbf{x} := [m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \gamma(z) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}]$, we have $\Theta(\mathbf{x}) = (-1)^{rs} \overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \overline{\mathbf{a}}_{1r}]$.*

Proof. 1) Under the first identifications in [25, (3.1)] and (2.5), for $s = 0$ the equality in item (1) becomes $\Theta([m \cdot J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}]) = [m \cdot J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}]$, which follows immediately from equality (2.6). Assume now that $s \geq 1$. By [24, Proposition 2.21], we know that

$$J_{\nu}(a) \gamma(h_1^{(1)}) J_{\nu}(b) \gamma(S(1^{(2)})) l = J_{\nu}(a) \gamma(h) J_{\nu}(b) \gamma(l) \quad \text{for all } a, b \in A \text{ and } l, h \in H. \quad (2.7)$$

By this, the first identity in (1.8), Lemma 2.7, [8, (2.4)] and the definition of the action in (2.4),

$$\begin{aligned} \Theta(\mathbf{x}) &= \Theta \left(\left[m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes_A \overline{\gamma}(h_s^{(2)}) \cdot a \gamma(h_s^{(3)}) \otimes \overline{\mathbf{a}}_{1r} \right] \right) \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s}^{(3)} \otimes_{HL} [m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \gamma_{\times}(h_{1s}^{(2)}) J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s} \cdot 1^{(2)} \otimes_{HL} [m \cdot \gamma(1^{(1)}) J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \cdot \gamma(1^{(1)}) J_{\nu}(a) \gamma(S(1^{(2)})) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \cdot J_{\nu}(a) \otimes \overline{\mathbf{a}}_{1r}], \end{aligned}$$

as desired.

2) By the second identity in (1.8), Lemma 2.7, [8, (2.4)], the definition of the action in (2.4) and equality (2.7),

$$\begin{aligned} \Theta(\mathbf{x}) &= \Theta \left(\left[m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,i-1}^{(2)}) \otimes_A J_{\nu}(f(h_i^{(2)} \otimes_k h_{i+1}^{(2)})) \gamma(h_i^{(3)} h_{i+1}^{(3)}) \otimes_A \tilde{\gamma}_A(h_{i+2,s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r} \right] \right) \\ &= (-1)^{rs} (\overline{\mathbf{h}}_{1,i-1}^{(3)} \otimes_{HL} \overline{h_i^{(3)} h_{i+1}^{(3)}} \otimes_{HL} \overline{\mathbf{h}}_{i+2,s}^{(3)}) \otimes_{HL} [m \cdot \gamma_{\times}^{-1}(h_{1s}^{(1)}) \gamma_{\times}(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} (\overline{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \overline{\mathbf{h}}_{i+2,s} \cdot 1^{(2)}) \otimes_{HL} [m \cdot \gamma(1^{(1)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} (\overline{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \overline{\mathbf{h}}_{i+2,s}) \otimes_{HL} [m \cdot \gamma(1^{(1)}) \gamma(S(1^{(2)})) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} (\overline{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \overline{\mathbf{h}}_{i+2,s}) \otimes_{HL} [m \otimes \overline{\mathbf{a}}_{1r}], \end{aligned}$$

as desired.

3) Under the first identifications in [25, (3.1)] and (2.5), for $s = 0$ item (3) becomes

$$\Theta([m \cdot \gamma(z) \otimes \overline{\mathbf{a}}_{1r}]) = [m \cdot \gamma(S(\overline{\Pi}^R(z))) \otimes \overline{\mathbf{a}}_{1r}],$$

which follows immediately from identities (1.3) and (2.6). Assume now that $s \geq 1$. We have

$$\begin{aligned} \Theta(\mathbf{x}) &= (-1)^{rs} \overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1s} \cdot 1^{(2)} \otimes_{HL} [m \cdot \gamma(1^{(1)}) \otimes \overline{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} \overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \cdot \gamma(1^{(1)}) \gamma(S(1^{(2)})) \otimes \overline{\mathbf{a}}_{1r}] \end{aligned}$$

$$= (-1)^{rs} \bar{\Pi}^R(z) \cdot \bar{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}],$$

where the first equality holds by the definition of Θ and Lemma 2.7; the second one, by [8, (2.4)] and the definition of the action in (2.4); and the last one, by equality (2.7). \square

Notation 2.11. Given a k -subalgebra R of A and $0 \leq u \leq r$, we let $\bar{X}_{rs}^u(R, M)$ denote the k -submodule of $\bar{X}_{rs}(M)$ generated by all the elements $\bar{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}]$ with $m \in M$, $a_1, \dots, a_r \in A$, $h_1, \dots, h_s \in H$, and at least u of the a_j 's in R .

Theorem 2.12. Let $\mathbf{y} := \bar{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] \in \bar{X}_{rs}(M)$, where $m \in M$, $a_1, \dots, a_r \in A$ and $h_1, \dots, h_s \in H$. The following assertions hold:

(1) For $r \geq 1$ and $s \geq 0$, we have

$$\begin{aligned} \bar{d}^0(\mathbf{y}) &= \bar{\mathbf{h}}_{1s} \otimes_{HL} [m \cdot J_\nu(a_1) \otimes \bar{\mathbf{a}}_{2r}] \\ &\quad + \sum_{i=1}^{r-1} (-1)^i \bar{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1,i-1} \otimes \overline{a_i a_{i+1}} \otimes \bar{\mathbf{a}}_{i+2,r}] \\ &\quad + (-1)^r \bar{\mathbf{h}}_{1s} \otimes_{HL} [J_\nu(a_r) \cdot m \otimes \bar{\mathbf{a}}_{1,r-1}]. \end{aligned}$$

(2) For $r \geq 0$ and $s = 1$, we have

$$\bar{d}^1(\mathbf{y}) = (-1)^r [m \cdot \gamma(\Pi^R(h_1)) \otimes \bar{\mathbf{a}}_{1r}] - (-1)^r [\gamma(h_1^{(3)}) \cdot m \cdot \gamma^{-1}(h_1^{(1)}) \otimes h_1^{(2)} \cdot \bar{\mathbf{a}}_{1r}], \quad (2.8)$$

while, for $r \geq 0$ and $s > 1$, we have

$$\begin{aligned} \bar{d}^1(\mathbf{y}) &= (-1)^r \bar{\Pi}^R(h_1) \cdot \bar{\mathbf{h}}_{2s} \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] \\ &\quad + \sum_{i=1}^{s-1} (-1)^{r+i} (\bar{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \bar{\mathbf{h}}_{i+2,s}) \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] \\ &\quad + (-1)^{r+s} \bar{\mathbf{h}}_{1,s-1} \otimes_{HL} [\gamma(h_s^{(3)}) \cdot m \cdot \gamma^{-1}(h_s^{(1)}) \otimes h_s^{(2)} \cdot \bar{\mathbf{a}}_{1r}]. \end{aligned}$$

(3) For $r \geq 0$ and $s \geq 2$, we have

$$\bar{d}^2(\mathbf{y}) = -\bar{\mathbf{h}}_{1,s-2} \otimes_{HL} [\gamma(h_{s-1}^{(3)} h_s^{(3)}) \cdot m \cdot \gamma^{-1}(h_s^{(1)}) \gamma^{-1}(h_{s-1}^{(1)}) \otimes \mathfrak{T}(h_{s-1}^{(2)}, h_s^{(2)}, \bar{\mathbf{a}}_{1r})],$$

where

$$\mathfrak{T}(h_{s-1}, h_s, \bar{\mathbf{a}}_{1r}) := \sum_{i=0}^r (-1)^i h_{s-1}^{(1)} \cdot (h_s^{(1)} \cdot \bar{\mathbf{a}}_{1i}) \otimes f(h_{s-1}^{(2)} \otimes_k h_s^{(2)}) \otimes h_{s-1}^{(3)} h_s^{(3)} \cdot \bar{\mathbf{a}}_{i+1,r}.$$

(4) Let R be a k -subalgebra of A . If R is stable under ρ and f takes its values in R , then

$$\bar{d}^l(\bar{X}_{rs}(M)) \subseteq \bar{X}_{r+l-1, s-l}^{l-1}(R, M)$$

for each $r \geq 0$ and $1 < l \leq s$.

Proof. 1) By the definition of Λ and [25, Theorem 3.5(1)], we have

$$\begin{aligned} \bar{d}^0(\mathbf{y}) &= (-1)^{rs} \Theta \circ \widehat{d}^0([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}]) \\ &= (-1)^{rs+s} \Theta([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \cdot J_\nu(a_1) \otimes \bar{\mathbf{a}}_{2r}]) \\ &\quad + \sum_{i=1}^{r-1} (-1)^{rs+s+i} \Theta([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1,i-1} \otimes \overline{a_i a_{i+1}} \otimes \bar{\mathbf{a}}_{i+2,r}]) \\ &\quad + (-1)^{rs+s+r} \Theta([J_\nu(a_r) \cdot m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1,r-1}]). \end{aligned}$$

The formula for \bar{d}^0 follows from this using Lemma 2.10(1).

2) By the definition of Λ , [24, Proposition 5.19] and [25, Theorem 3.5(2)], we have

$$\begin{aligned} \bar{d}^1(\mathbf{y}) &= (-1)^{rs} \Theta \circ \widehat{d}^1([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}]) \\ &= (-1)^{rs} \Theta([m \cdot \gamma_x^{-1}(h_{2s}^{(1)}) \gamma(\Pi^R(h_1)) \otimes_A \tilde{\gamma}_A(h_{2s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}]) \\ &\quad + \sum_{i=1}^{r-1} (-1)^{rs+i} \Theta([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,i-1}^{(2)}) \otimes_A \overline{\gamma(h_i^{(2)}) \gamma(h_{i+1}^{(2)})} \otimes_A \tilde{\gamma}_A(h_{i+2,s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}]) \\ &\quad + (-1)^{rs+r} \Theta([\gamma(h_s^{(3)}) \cdot m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-1}^{(2)}) \otimes h_s^{(2)} \cdot \bar{\mathbf{a}}_{1r}]). \end{aligned}$$

The formula for \bar{d}^1 follows from this using Lemma 2.10 and the fact that $\bar{\Pi}^R \circ \Pi^R = \bar{\Pi}^R$.

3) By the definition of Λ and [25, Theorem 3.6], we have

$$\begin{aligned} \vec{d}^2(\mathbf{y}) &= (-1)^{rs} \Theta \circ \vec{d}^2([m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}]) \\ &= (-1)^{rs+s+1} \Theta([\gamma(h_{s-1}^{(3)} h_s^{(3)}) \cdot m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1,s-2}^{(2)}) \otimes \mathfrak{T}(h_{s-1}^{(2)}, h_s^{(2)}, \bar{\mathbf{a}}_{1r})]). \end{aligned}$$

The formula for \vec{d}^2 follows from this using Lemma 2.10(1).

4) Let $\widehat{X}_{r+l-1, s-l}^{l-1}(R, M)$ be as in [25, Notation 3.4]. By Remark 1.13 and [25, Theorem 3.5(3)] this item follows from the fact that $\overline{X}_{r+l-1, s-l}^{l-1}(R, M) = \Theta(\widehat{X}_{r+l-1, s-l}^{l-1}(R, M))$. \square

Remark 2.13. By the second equality in (1.3), we have

$$\overline{\Pi}^R(h_1) \cdot [m \otimes \bar{\mathbf{a}}_{1r}] = [m \cdot \gamma(\Pi^R(h)) \otimes \bar{\mathbf{a}}_{1r}] \quad \text{for all } h \in H, m \in M \text{ and } a_1, \dots, a_r \in A.$$

This gives an alternative formula for $\vec{d}^1(\mathbf{y})$ in (2.8). We will use this fact in the proof of Proposition 2.16.

Proposition 2.14. *For each $h \in H$, the map $F_*^h: (M \otimes \overline{A}^{\otimes*} \otimes, b_*) \rightarrow (M \otimes \overline{A}^{\otimes*} \otimes, b_*)$, defined by*

$$F_r^h([m \otimes \bar{\mathbf{a}}_{1r}]) := [\gamma(h^{(3)}) \cdot m \cdot \gamma^{-1}(h^{(1)}) \otimes h^{(2)} \cdot \bar{\mathbf{a}}_{1r}],$$

is a morphism of complexes. Moreover the following facts hold:

- (1) *For each $h, l \in H$, the endomorphisms of $H_*^K(A, M)$ induced by $F_*^h \circ F_*^l$ and F_*^{hl} coincide.*
- (2) *$F_r^l([m \otimes \bar{\mathbf{a}}_{1r}]) = l \cdot [m \otimes \bar{\mathbf{a}}_{1r}]$, for all $l \in H^L$ (see (2.4)). In particular F_*^1 is the identity map.*

Consequently, $H_^K(A, M)$ is a left H -module.*

Proof. Using the first equality in Proposition 1.8 and Lemma 2.2(4), it is easy to see that the maps F_*^h are well defined. Moreover by Definition 1.8(2), the first identity in (1.8) and Lemma 2.2(4), they are morphisms of complexes.

For $h, l \in H$, let $(\mathfrak{h}_r: M \otimes \overline{A}^{\otimes r} \otimes \rightarrow M \otimes \overline{A}^{\otimes r+1} \otimes)_{r \geq 0}$ be the family of maps, defined by

$$\mathfrak{h}_r([m \otimes \bar{\mathbf{a}}_{1r}]) := -[\gamma(h^{(3)} l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma^{-1}(h^{(1)}) \otimes \mathfrak{T}(h^{(2)}, l^{(2)}, \bar{\mathbf{a}}_{1r})],$$

where $\mathfrak{T}(h, l, \bar{\mathbf{a}}_{1r})$ is as in Theorem 2.12(3). In order to prove item (1) it suffices to show that $(\mathfrak{h}_r)_{r \geq 0}$ is a homotopy from F_*^{hl} to $F_*^h \circ F_*^l$. For this we must check that

$$(F_r^{hl} - F_r^h \circ F_r^l)([m \otimes \bar{\mathbf{a}}_{1r}]) = \begin{cases} (b \circ \mathfrak{h}_0)([m]) & \text{if } r = 0, \\ (b \circ \mathfrak{h}_r + \mathfrak{h}_{r-1} \circ b)([m \otimes \bar{\mathbf{a}}_{1r}]) & \text{if } r > 0. \end{cases} \quad (2.9)$$

Let $\mathbf{y} := (\bar{h} \otimes_{HL} \bar{l}) \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] \in \overline{X}_{r2}(M)$. Since $(\overline{X}_*(M), \vec{d}_*)$ is a chain complex,

$$\vec{d}^1(\vec{d}^1(\mathbf{y})) = \begin{cases} \vec{d}^0(\vec{d}^2(\mathbf{y})) & \text{if } r = 0, \\ \vec{d}^0(\vec{d}^2(\mathbf{y})) + \vec{d}^2(\vec{d}^0(\mathbf{y})) & \text{if } r > 0. \end{cases}$$

By this fact and Theorem 2.12, to prove equality (2.9) it suffices to show that

$$\begin{aligned} \vec{d}^1(\vec{d}^1(\mathbf{y})) &= [\gamma(h^{(3)} l^{(3)}) \cdot m \cdot \gamma^{-1}(h^{(1)} l^{(1)}) \otimes h^{(2)} l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &\quad - [\gamma(h^{(3)}) \gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma^{-1}(h^{(1)}) \otimes h^{(2)} \cdot (l^{(2)} \cdot \bar{\mathbf{a}}_{1r})]. \end{aligned} \quad (2.10)$$

Now, a direct computation shows that

$$\begin{aligned} \vec{d}^1(\vec{d}^1(\mathbf{y})) &= (-1)^r \vec{d}^1(\overline{\Pi}^R(h) \bar{l}) \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] - (-1)^r \vec{d}^1(\bar{h} \bar{l}) \otimes_{HL} [m \otimes \bar{\mathbf{a}}_{1r}] \\ &\quad + (-1)^r \vec{d}^1(\bar{h} \otimes_{HL} [\gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}]) \\ &= [m \cdot \gamma(\Pi^R(\overline{\Pi}^R(h) \bar{l})) \otimes \bar{\mathbf{a}}_{1r}] \\ &\quad - [\gamma(\overline{\Pi}^R(h)^{(3)} l^{(3)}) \cdot m \cdot \gamma^{-1}(\overline{\Pi}^R(h)^{(1)} l^{(1)}) \otimes \overline{\Pi}^R(h)^{(2)} l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &\quad - [m \cdot \gamma(\Pi^R(hl)) \otimes \bar{\mathbf{a}}_{1r}] + [\gamma(h^{(3)} l^{(3)}) \cdot m \cdot \gamma^{-1}(h^{(1)} l^{(1)}) \otimes h^{(2)} l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &\quad + [\gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma(\Pi^R(h)) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &\quad - [\gamma(h^{(3)}) \gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma^{-1}(h^{(1)}) \otimes h^{(2)} \cdot (l^{(2)} \cdot \bar{\mathbf{a}}_{1r})]. \end{aligned}$$

Using the second equality in (1.3), and the fact that, by Lemma 2.4 and identity (2.2),

$$\begin{aligned} [\gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma(\Pi^R(h)) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] &= [\gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(l^{(1)}) \gamma(S(\overline{\Pi}^R(h))) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &= [\gamma(l^{(3)}) \cdot m \cdot \gamma^{-1}(\overline{\Pi}^R(h) l^{(1)}) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}] \\ &= [\gamma(\overline{\Pi}^R(h)^{(3)} l^{(3)}) \cdot m \cdot \gamma^{-1}(\overline{\Pi}^R(h)^{(1)} l^{(1)}) \otimes (\overline{\Pi}^R(h)^{(2)} l^{(2)}) \cdot \bar{\mathbf{a}}_{1r}], \end{aligned}$$

we obtain that equality (2.10) holds.

We next prove item (2). Let $l \in H^L$. For $r = 1$, we have

$$F_0^l([m]) = [\gamma(l^{(2)}) \cdot m \cdot \gamma^{-1}(l^{(1)})] = [m \cdot \gamma^{-1}(l^{(1)})\gamma(l^{(2)})] = [m \cdot \gamma(\Pi^R(l))] = [m \cdot \gamma(S(l))],$$

where the second equality holds by [8, (2.6a)], [24, Proposition 4.5] and Remark 1.15; the third one, by [24, Proposition 5.19]; and the fourth one, by the second identity in (1.3) and the fact that $\bar{\Pi}^R(l) = l$. Assume now $r \geq 1$. To begin note that, by Remark 1.15, [24, Proposition 4.5] and Definition 1.8(2),

$$F_r^l([m \otimes \bar{\mathbf{a}}_{1r}]) = [m \cdot \gamma^{-1}(l^{(1)}) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1,r-1} \otimes (l^{(3)} \cdot a_r)(l^{(4)} \cdot 1_A)] = [m \cdot \gamma^{-1}(l^{(1)}) \otimes l^{(2)} \cdot \bar{\mathbf{a}}_{1r}].$$

We claim that

$$l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1r} = l^{(1)} \otimes_k (l^{(2)} \cdot 1_A) a_1 \otimes \mathbf{a}_{2r}. \quad (2.11)$$

By [8, (2.6a)] and Proposition 1.9(1),

$$l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1r} = l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1,r-1} \otimes l^{(3)} \cdot a_r = l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1,r-1} \otimes (l^{(3)} \cdot 1_A) a_r.$$

If $r = 1$ this ends the proof of the claim. Assume that $r > 1$. In this case, by Remark 1.15 and Definition 1.8(2),

$$l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1r} = l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1,r-2} \otimes (l^{(3)} \cdot a_{r-1})(l^{(4)} \cdot 1_A) \otimes a_r = l^{(1)} \otimes_k l^{(2)} \cdot \mathbf{a}_{1,r-1} \otimes a_r,$$

and the claim follows from an evident inductive argument. Thus, by [24, Proposition 4.5] and Remark 1.15,

$$F_r^l([m \otimes \bar{\mathbf{a}}_{1r}]) = [m \cdot \gamma^{-1}(l^{(1)}) \otimes (l^{(2)} \cdot 1_A) a_1 \otimes \mathbf{a}_{2r}] = [m \cdot \gamma^{-1}(l^{(1)}) J_\nu(l^{(2)} \cdot 1_A) \otimes \mathbf{a}_{1r}],$$

and so, by [24, Propositions 4.5 and 5.19], the second identity in (1.3) and the fact that $\bar{\Pi}^R(l) = l$,

$$F_r^l([m \otimes \bar{\mathbf{a}}_{1r}]) = [m \cdot \gamma(\Pi^R(l)) \otimes \mathbf{a}_{1r}] = [m \cdot \gamma(S(l)) \otimes \mathbf{a}_{1r}],$$

as desired. \square

Example 2.15. If $A = K$, then $H_*^K(E, M) = H_*(H, M \otimes)$, where $M \otimes$ is considered as a left H -module via the action given by $h \cdot [m] := [\gamma(h^{(2)}) \cdot m \cdot \gamma^{-1}(h^{(1)})]$.

In the following proposition, for each $r \geq 0$, we consider $M \otimes \bar{A}^{\otimes r}$ as a left H -module via the action introduced in Proposition 2.14, and we consider $M \otimes \bar{A}^{\otimes r}$ as a left H^L -module via the canonical inclusion of H^L into H . In Proposition 2.14(2) we prove that these structures coincide with the ones introduced in (2.4) (which are the ones used in Theorem 2.12).

Proposition 2.16. *The spectral sequence of [25, (3.3)] satisfies*

$$E_{rs}^1 = \bar{H}^{\otimes_{H^L}} \otimes_{H^L} H_r^K(A, M) \quad \text{and} \quad E_{rs}^2 = H_s(H, H_r^K(A, M)).$$

Proof. For each $i, n \geq 0$, let $F^i(\bar{X}_n(M)) := \bigoplus_{s=0}^i \bar{X}_{n-s,s}(M)$. The chain complex $(\bar{X}_*(M), \bar{d}_*)$ is filtrated by

$$0 = F^{-1}(\bar{X}_*(M)) \subseteq F^0(\bar{X}_*(M)) \subseteq F^1(\bar{X}_*(M)) \subseteq F^2(\bar{X}_*(M)) \subseteq F^3(\bar{X}_*(M)) \subseteq \dots \quad (2.12)$$

Moreover the isomorphism $\Theta_* : (\hat{X}_*(M), \hat{d}_*) \rightarrow (\bar{X}_*(M), \bar{d}_*)$ preserves filtrations, where we consider the chain complex $(\hat{X}_*(M), \hat{d}_*)$ endowed with the filtration introduced at the beginning of [25, Subsection 3.2]. So, the spectral sequence of [25, (3.3)] coincides with the spectral sequence determined by the filtration (2.12). Thus, the formula for E_{rs}^1 follows from Theorem 2.12(1) and the formula for E_{rs}^2 follows from Theorem 2.12(2), Remark 2.13 and Proposition 1.4(1). \square

3 Hochschild cohomology of cleft extensions

Let $H, A, \rho, \chi_\rho, f, \mathcal{F}_f, E, K, M, \nu, j_\nu$ and γ be as in the previous section. Assume that the hypotheses of that section are fulfilled. In particular H is a weak Hopf algebra, A is a weak module algebra and f is convolution invertible. Let γ^{-1} be as in equality (1.11). By definition the Hochschild cohomology $H_K^*(E, M)$, of the K -algebra E with coefficients in an E -bimodule M , is the cohomology of the normalized Hochschild cochain complex $(\text{Hom}_{K^e}(\bar{E}^{\otimes^*}, M), b^*)$, where b^* is the canonical Hochschild boundary map. In [25, Section 4] a cochain complex $(\hat{X}^*(M), \hat{d}^*)$ was obtained, simpler than the canonical one, that gives the Hochschild cohomology of E with coefficients in M . In this section we prove that $(\hat{X}^*(M), \hat{d}^*)$ is isomorphic to a simpler complex $(\bar{X}^*(M), \bar{d}^*)$. When K is separable, the complex $(\bar{X}^*(M), \bar{d}^*)$ gives the absolute Hochschild cohomology of E with coefficients in M . We recall from [25, Section 4] that

$$\hat{X}^n(M) = \bigoplus_{\substack{r,s \geq 0 \\ r+s=n}} \hat{X}^{rs}(M), \quad \text{where } \hat{X}^{rs}(M) := \text{Hom}_{(A,K)}(\hat{E}^{\otimes^* r} \otimes \bar{A}^{\otimes^* s}, M),$$

and that there exist maps $\widehat{d}_l^{rs}: \widehat{X}^{r+l-1, s-l}(M) \longrightarrow \widehat{X}^{rs}(M)$ such that

$$\widehat{d}^n(\alpha) := \sum_{l=0}^{r+1} \widehat{d}_l^{r-l+1, n-r+l-1}(\alpha) \quad \text{for all } \alpha \in \widehat{X}^{r, n-r-1}(M).$$

By [24, Propositions 2.22 and 4.6], we know that M is a $(K, K \otimes_k H^L)$ -bimodule via

$$\lambda \cdot m \cdot (\lambda' \otimes_k l) := J_\nu(\lambda)\gamma(S(l)) \cdot m \cdot J_\nu(\lambda').$$

For each $r, s \geq 0$, we set $\overline{X}^{rs}(M) := \text{Hom}_{(K, K \otimes_k H^L)}(\overline{H}^{\otimes_{H^L}} \otimes_k \overline{A}^{\otimes^r}, M)$, where we consider $\overline{H}^{\otimes_{H^L}} \otimes_k \overline{A}^{\otimes^r}$ as a $(K, K \otimes_k H^L)$ -bimodule via

$$\lambda \cdot (\overline{\mathbf{h}}_{1s} \otimes_k \overline{\mathbf{a}}_{1r}) \cdot (\lambda' \otimes_k l) := \overline{\mathbf{h}}_{1s} \cdot l \otimes_k \lambda \cdot \overline{\mathbf{a}}_{1r} \cdot \lambda'.$$

Let $M^K := \{m \in M : \lambda \cdot m = m \cdot \lambda \text{ for all } \lambda \in K\}$. Since $\overline{H}^{\otimes_{H^L}} = H^L$ and $\overline{A}^{\otimes^0} = K$, we have

$$\overline{X}^{r0}(M) \simeq \text{Hom}_{K^e}(\overline{A}^{\otimes^r}, M) \quad \text{and} \quad \overline{X}^{0s}(M) \simeq \text{Hom}_{H^L}(\overline{H}^{\otimes_{H^L}}, M^K), \quad (3.1)$$

where M^K is considered as a right H^L -module via $m \cdot l := \gamma(S(l)) \cdot m$.

Remark 3.1. For each $r, s \geq 0$, we have $\overline{X}^{rs}(M) \simeq \text{Hom}_{H^L}(\overline{H}^{\otimes_{H^L}}, \text{Hom}_{K^e}(\overline{A}^{\otimes^r}, M))$, where $\text{Hom}_{K^e}(\overline{A}^{\otimes^r}, M)$ is considered as right H^L -module via $(\beta \cdot l)(\overline{\mathbf{a}}_{1r}) := \gamma(S(l)) \cdot \beta(\overline{\mathbf{a}}_{1r})$.

Proposition 3.2. *For each $r, s \geq 0$ there exist maps*

$$\Theta^{rs}: \overline{X}^{rs}(M) \longrightarrow \widehat{X}^{rs}(M) \quad \text{and} \quad \Lambda^{rs}: \widehat{X}^{rs}(M) \longrightarrow \overline{X}^{rs}(M),$$

such that for

$$\mathbf{x} := J_\nu(\overline{a_1})\gamma(h_1) \otimes_A \cdots \otimes_A J_\nu(\overline{a_s})\gamma(h_s) \otimes \overline{\mathbf{a}}_{s+1, s+r} \in \widetilde{E}^{\otimes_A^s} \otimes \overline{A}^{\otimes^r} \quad \text{and} \quad \mathbf{y} := \overline{\mathbf{h}}_{1s} \otimes_k \overline{\mathbf{a}}_{1r} \in \overline{H}^{\otimes_{H^L}} \otimes_k \overline{A}^{\otimes^r},$$

we have

$$\Theta^{rs}(\beta)(\mathbf{x}) := (-1)^{rs} J_\nu(\overline{a_1})\gamma(h_1^{(1)}) \cdots J_\nu(\overline{a_s})\gamma(h_s^{(1)}) \cdot \beta(\overline{\mathbf{h}}_{1s}^{(2)} \otimes_k \overline{\mathbf{a}}_{s+1, s+r})$$

and

$$\Lambda^{rs}(\alpha)(\mathbf{y}) := (-1)^{rs} \gamma_x^{-1}(\overline{h}_{1s}^{(1)}) \cdot \alpha(\widetilde{\gamma}_A(\overline{h}_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}).$$

Moreover the maps Θ^{rs} and Λ^{rs} are inverse one of each other.

Proof. Mimic the proof of Proposition 2.8. □

For each $0 \leq l \leq s$ and $r \geq 0$ such that $r+l \geq 1$, let $\overline{d}_l^{rs}: \overline{X}^{r+l-1, s-l}(M) \rightarrow \overline{X}^{rs}(M)$ be the map $\overline{d}_l^{rs} := \Lambda^{r+l-1, s-l} \circ \widehat{d}_l^{rs} \circ \Theta^{rs}$.

Theorem 3.3. *The Hochschild cohomology of the K -algebra E is the cohomology of $(\overline{X}^*(M), \overline{d}^*)$, where*

$$\overline{X}^n(M) := \bigoplus_{r+s=n} \overline{X}^{rs}(M) \quad \text{and} \quad \overline{d}^n(\beta) := \sum_{l=0}^{r+1} \overline{d}_l^{r-l+1, n-r+l-1}(\beta) \quad \text{for all } \beta \in \overline{X}^{r, n-r-1}(M).$$

Proof. By Proposition 3.2 and the definition of $(\overline{X}_*(M), \overline{d}_*)$, the map $\Theta^*: (\overline{X}^*(M), \overline{d}^*) \longrightarrow (\widehat{X}^*(M), \widehat{d}^*)$, given by $\Theta^n := \bigoplus_{r+s=n} \Theta^{rs}$, is an isomorphism of complexes. □

By [25, Remark 4.2], if f takes its values in K , then $(\overline{X}^*(M), \overline{d}^*)$ is the total cochain complex of the double complex $(\overline{X}^{**}(M), \overline{d}_0^{**}, \overline{d}_1^{**})$; while, if $K = A$, then $(\overline{X}^*(M), \overline{d}^*) = (\overline{X}^*(M), \overline{d}_1^{0*})$.

Lemma 3.4. *Let $\beta \in \overline{X}^{rs}(M)$, $a, a_1, \dots, a_r \in A$, $h_1, \dots, h_s \in H$ and $z \in H^R$.*

- (1) *We have $\gamma_x^{-1}(\overline{h}_{1s}^{(1)}) \cdot \Theta(\beta)(\widetilde{\gamma}_A(\overline{h}_{1s}^{(2)}) \cdot J_\nu(a) \otimes \overline{\mathbf{a}}_{1r}) = (-1)^{rs} J_\nu(a) \cdot \beta(\overline{\mathbf{h}}_{1s} \otimes_k \overline{\mathbf{a}}_{1r})$.*
- (2) *For $s \geq 2$ and $1 \leq i < s$, we have*

$$\begin{aligned} & \gamma_x^{-1}(\overline{h}_{1s}^{(1)}) \cdot \Theta(\beta) \left(\widetilde{\gamma}_A(\overline{h}_{1, i-1}^{(2)}) \otimes_A \widetilde{\gamma}_A(\overline{h}_i^{(2)})\gamma(h_{i+1}^{(2)}) \otimes_A \widetilde{\gamma}_A(\overline{h}_{i+2, s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r} \right) \\ &= (-1)^{rs} \beta(\overline{\mathbf{h}}_{1, i-1} \otimes_{H^L} \overline{h_i h_{i+1}} \otimes_{H^L} \overline{\mathbf{h}}_{i+2, s} \otimes_k \overline{\mathbf{a}}_{1r}). \end{aligned}$$

- (3) *We have $\gamma_x^{-1}(\overline{h}_{1s}^{(1)})\gamma(z) \cdot \Theta(\beta)(\widetilde{\gamma}_A(\overline{h}_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}) = (-1)^{rs} \beta(\overline{\Pi}^R(z) \cdot \overline{\mathbf{h}}_{1s} \otimes_k \overline{\mathbf{a}}_{1r})$.*

Proof. Mimic the proof of Lemma 2.10. □

Notation 3.5. For each k -subalgebra R of A and $0 \leq u \leq r$, we set $\overline{X}_u^{rs}(R, M) := \Lambda(\widehat{X}_u^{rs}(R, M))$, where $\widehat{X}_u^{rs}(R, M)$ is as in [25, Notation 4.4]

Theorem 3.6. *Let $\mathbf{y} := \overline{\mathbf{h}}_{1s} \otimes_k \overline{\mathbf{a}}_{1r}$, where $h_1, \dots, h_s \in H$ and $a_1, \dots, a_r \in A$. The following assertions hold:*

(1) For $r \geq 1$ and $s \geq 0$, we have

$$d_0(\beta)(\mathbf{y}) = j_\nu(a_1)\beta(\bar{\mathbf{h}}_{1s} \otimes_k \bar{\mathbf{a}}_{2r}) + \sum_{i=1}^{r-1} (-1)^i \beta(\bar{\mathbf{h}}_{1s} \otimes_k (\bar{\mathbf{a}}_{1,i-1} \otimes_{\bar{a}_i a_{i+1}} \bar{\mathbf{a}}_{i+2,r})) + \beta(\bar{\mathbf{h}}_{1s} \otimes_k \bar{\mathbf{a}}_{1,r-1}) \cdot j_\nu(a_r).$$

(2) For $r \geq 0$ and $s = 1$, we have

$$\bar{d}^1(\beta)(\mathbf{y}) = (-1)^r \left(\gamma(\Pi^R(h_1)) \cdot \beta(\bar{\mathbf{a}}_{1r}) - \gamma^{-1}(h_1^{(1)}) \cdot \beta(h_1^{(2)} \cdot \bar{\mathbf{a}}_{1r}) \cdot \gamma(h_1^{(3)}) \right), \quad (3.2)$$

while for $r \geq 0$ and $s > 1$, we have

$$\begin{aligned} \bar{d}_1(\beta)(\mathbf{y}) &= (-1)^r \beta((\bar{\Pi}^R(h_1)h_2 \otimes_{HL} \bar{\mathbf{h}}_{3s}) \otimes_k \bar{\mathbf{a}}_{1r}) \\ &\quad + \sum_{i=1}^{s-1} (-1)^{r+i} \beta((\bar{\mathbf{h}}_{1,i-1} \otimes_{HL} \overline{h_i h_{i+1}} \otimes_{HL} \bar{\mathbf{h}}_{i+2,s}) \otimes_k \bar{\mathbf{a}}_{1r}) \\ &\quad + (-1)^{r+s} \gamma^{-1}(h_s^{(1)}) \cdot \beta(\bar{\mathbf{h}}_{1,s-1} \otimes_k h_s^{(2)} \cdot \bar{\mathbf{a}}_{1r}) \cdot \gamma(h_s^{(3)}). \end{aligned}$$

(3) For $r \geq 0$ and $s \geq 2$, we have

$$\bar{d}_2(\beta)(\mathbf{y}) = -\gamma^{-1}(h_s^{(1)})\gamma^{-1}(h_{s-1}^{(1)}) \cdot \beta(\bar{\mathbf{h}}_{1,s-2} \otimes_k \mathfrak{T}(h_{s-1}^{(2)}, h_s^{(2)}, \bar{\mathbf{a}}_{1r})) \cdot \gamma(h_{s-1}^{(3)} h_s^{(3)}),$$

where $\mathfrak{T}(h_{s-1}, h_s, \bar{\mathbf{a}}_{1r})$ is as in Theorem 2.12(3).

(4) Let R be a k -subalgebra of A . If R is stable under ρ and f takes its values in R , then

$$\bar{d}_l(\bar{X}^{r+l-1, s-l}(M)) \subseteq \bar{X}_{l-1}^{rs}(R, M),$$

for each $r \geq 0$ and $1 < l \leq s$.

Proof. Mimic the proof of Theorem 2.12. □

Remark 3.7. By the second equality in (1.3), we have

$$(\beta \cdot \bar{\Pi}^R(h))(\bar{\mathbf{a}}_{1r}) = \gamma(\Pi^R(h_1)) \cdot \beta(\bar{\mathbf{a}}_{1r}) \quad \text{for all } h \in H, \beta \in \text{Hom}_{K^e}(\bar{A}^{\otimes r}, M) \text{ and } a_1, \dots, a_r \in A.$$

This gives an alternative formula for $\bar{d}^1(\beta)(\mathbf{y})$ in (3.2). We will use this fact in the proof of Proposition 3.10.

Proposition 3.8. For each $h \in H$ the map $F_h^*: (\text{Hom}_{K^e}(\bar{A}^{\otimes r}, M), b^*) \rightarrow (\text{Hom}_{K^e}(\bar{A}^{\otimes r}, M), b^*)$, defined by

$$F_h^r(\beta)(\bar{\mathbf{a}}_{1r}) := \gamma^{-1}(h^{(1)}) \cdot \beta(h^{(2)} \cdot \bar{\mathbf{a}}_{1r}) \cdot \gamma(h^{(3)}),$$

is a morphism of complexes. Moreover the following facts hold

- (1) For each $h, l \in H$, the endomorphisms of $\mathbf{H}_K^*(A, M)$ induced by $F_l^* \circ F_h^*$ and F_{hl}^* coincide.
- (2) $F_l^r(\beta)(\bar{\mathbf{a}}_{1r}) = (\beta \cdot l)(\bar{\mathbf{a}}_{1r})$, for all $l \in H^L$ (see Remark 3.1). In particular F_1^* is the identity map.

Consequently $\mathbf{H}_K^*(A, M)$ is a right H -module.

Proof. Mimic the proof of Proposition 2.14. □

Example 3.9. If $A = K$, then $\mathbf{H}_K^*(E, M) = \mathbf{H}^*(H, M^K)$, where M^K is considered as a right H -module via $m \cdot h := \gamma(h^{(2)}) \cdot m \cdot \gamma^{-1}(h^{(1)})$ (for the well definition use the first identity in (1.8) and Lemma 2.2(4)).

Proposition 3.10. The spectral sequence of [25, (4.3)] satisfies

$$E_1^{rs} = \text{Hom}_{HL}(\bar{H}^{\otimes_{HL} s}, \mathbf{H}_K^r(A, M)) \quad \text{and} \quad E_2^{rs} = \mathbf{H}^s(H, \mathbf{H}_K^r(A, M)).$$

Proof. For each $s, n \geq 0$, let $F_i(\bar{X}^n(M)) := \bigoplus_{s=i}^n \bar{X}^{n-s, s}(M)$. The cochain complex $(\bar{X}^*(M), \bar{d}^*)$ is filtrated by

$$F_0(\bar{X}^*(M)) \supseteq F_1(\bar{X}^*(M)) \supseteq F_2(\bar{X}^*(M)) \supseteq F_2(\bar{X}^*(M)) \supseteq \dots \quad (3.3)$$

Since the isomorphism $\Theta^*: (\bar{X}^*(M), \bar{d}^*) \rightarrow (\hat{X}^*(M), \hat{d}^*)$ preserves filtrations, the spectral sequence of [25, (4.3)] coincides with the one determined by the filtration (3.3). Thus, the formula for E_1^{rs} follows from Theorem 3.6(1) and the formula for E_2^{rs} follows from Theorem 3.6(2), Remark 3.7 and Proposition 1.4(2). □

4 The cup and cap products for cleft extensions

Let $H, A, \rho, f, E, K, M, \nu, \gamma$ and γ^{-1} be as in Section 2. Thus H is a weak Hopf algebra, A is a weak module algebra and f is convolution invertible. In this section we obtain formulas involving the complexes $(\overline{X}^*(E), \overline{d}^*)$ and $(\overline{X}_*(M), \overline{d}_*)$ that induce the cup product of $\mathrm{HH}_K^*(E)$ and the cap product of $\mathrm{H}_*^K(E, M)$. We will use freely the operators \bullet and \diamond introduced in [25, Section 5].

Notation 4.1. Given $h_1, \dots, h_s \in H$ and $a_1, \dots, a_r \in A$ we set $h_{1s} \cdot \overline{a}_{1r} := h_1 \cdot (h_2 \cdot (\dots \cdot (h_s \cdot \overline{a}_{1r}))) \dots$.

Definition 4.2. Let $\beta \in \overline{X}^{rs}(E)$ and $\beta' \in \overline{X}^{r's'}(E)$. We define $\beta \cdot \beta' \in \overline{X}^{r''s''}(E)$ by

$$(\beta \cdot \beta')(\mathbf{y}) := (-1)^{r's} \gamma_x^{-1}(h_{s+1, s''}^{(1)}) \beta(\overline{\mathbf{h}}_{1s}^{(1)} \otimes_k h_{s+1, s''}^{(2)} \cdot \overline{\mathbf{a}}_{1r}) \gamma_x(h_{s+1, s''}^{(3)}) \beta'(\overline{\mathbf{h}}_{s+1, s''}^{(4)} \otimes_k \overline{\mathbf{a}}_{r+1, r''}),$$

where $r'' := r + r', s'' := s + s', h_1, \dots, h_{s''} \in H, a_1, \dots, a_{r''} \in A$ and $\mathbf{y} := \overline{\mathbf{h}}_{1s''} \otimes_k \overline{\mathbf{a}}_{1r''}$.

Proposition 4.3. *If f takes its values in K , then the cup product in $\mathrm{HH}_K^*(E)$ is induced by \cdot .*

Proof. By [25, Corollary 5.3] it suffices to prove that

$$\Theta(\beta \cdot \beta') = \Theta(\beta) \bullet \Theta(\beta') \quad \text{for each } \beta \in \overline{X}^{rs}(E) \text{ and } \beta' \in \overline{X}^{r's'}(E).$$

For this, take $\mathbf{x} := \tilde{\gamma}_A(h_{1s''}) \otimes \overline{\mathbf{a}}_{1r''}$, where $r'' := r + r', s'' := s + s', h_1, \dots, h_{s''} \in H$ and $a_1, \dots, a_{r''} \in A$. By Lemma 2.6 and the definitions of Θ, \bullet and \cdot ,

$$\begin{aligned} (\Theta(\beta) \bullet \Theta(\beta'))(\mathbf{x}) &= (-1)^{rs'} \Theta(\beta)(\tilde{\gamma}_A(h_{1s}) \otimes h_{s+1, s''}^{(1)} \cdot \overline{\mathbf{a}}_{1r}) \Theta(\beta')(\tilde{\gamma}_A(h_{s+1, s''}^{(2)}) \otimes \overline{\mathbf{a}}_{r+1, r''}) \\ &= (-1)^{rs'+rs+r's'} \gamma_x(h_{1s}^{(1)}) \beta(\overline{\mathbf{h}}_{1s}^{(2)} \otimes_k h_{s+1, s''}^{(1)} \cdot \overline{\mathbf{a}}_{1r}) \gamma_x(h_{s+1, s''}^{(2)}) \beta(\overline{\mathbf{h}}_{s+1, s''}^{(3)} \otimes_k \overline{\mathbf{a}}_{r+1, r''}) \\ &= (-1)^{r''s''} \gamma_x(h_{1s''}^{(1)}) (\beta \cdot \beta')(\overline{\mathbf{h}}_{1s''}^{(2)} \otimes_k \overline{\mathbf{a}}_{1r''}) \\ &= \Theta(\beta \cdot \beta')(\mathbf{x}), \end{aligned}$$

as desired. \square

Definition 4.4. Let $\beta \in \overline{X}^{r's'}(E)$ and let $\mathbf{y} := \overline{\mathbf{h}}_{1s} \otimes_{HL} [m \otimes \overline{\mathbf{a}}_{1r}] \in \overline{X}_{rs}(M)$, where $m \in M, a_1, \dots, a_r \in A$ and $h_1, \dots, h_s \in H$. Assume that $r \geq r'$ and $s \geq s'$. We define $\mathbf{y} * \beta$ by

$$\mathbf{y} * \beta := (-1)^{rs'+r's'} \overline{\mathbf{h}}_{s'+1, s}^{(4)} \otimes_{HL} \left[m \cdot \gamma_x^{-1}(h_{s'+1, s}^{(1)}) \beta(\overline{\mathbf{h}}_{1s'}^{(1)} \otimes_k h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \gamma_x(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r} \right].$$

If $r' > r$ or $s' > s$, then we set $\mathbf{y} * \beta = 0$.

Proposition 4.5. *If f takes its values in K , then in terms of the complexes $(\overline{X}_*(M), \overline{d}_*)$ and $(\overline{X}^*(E), \overline{d}^*)$, the cap product is induced by the operation $*$.*

Proof. By [25, Corollary 5.7] it suffices to prove that

$$\Lambda(\mathbf{y} * \beta) = \Lambda(\mathbf{y}) \diamond \Theta(\beta) \quad \text{for each } \mathbf{y} \in \overline{X}_{rs}(M) \text{ and } \beta \in \overline{X}^{r's'}(E).$$

If $r' > r$ or $s' > s$, then both sides of this equality are zero. Then we can assume that $r' \leq r, s' \leq s$ and \mathbf{y} is as in Definition 4.4. We have

$$\begin{aligned} \Lambda(\mathbf{y}) \diamond \Theta(\beta) &= (-1)^{rs} [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \overline{\mathbf{a}}_{1r}] \diamond \Theta(\beta) \\ &= (-1)^{rs+r'(s-s')} [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \Theta(\beta)(\tilde{\gamma}_A(h_{1s'}^{(2)}) \otimes h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \otimes_A \tilde{\gamma}_A(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r}] \\ &= (-1)^{rs+r's} [m \cdot \gamma_x^{-1}(h_{1s}^{(1)}) \gamma_x(h_{1s'}^{(2)}) \beta(\overline{\mathbf{h}}_{1s'}^{(3)} \otimes_k h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \otimes_A \tilde{\gamma}_A(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r}] \\ &= (-1)^{rs+r's} [m \cdot \gamma_x^{-1}(h_{s'+1, s}^{(1)}) \gamma(1^{(1)}) \beta(\overline{\mathbf{h}}_{1s'}^{(1)} \cdot 1^{(2)} \otimes_k h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \otimes_A \tilde{\gamma}_A(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r}] \\ &= (-1)^{rs+r's} [m \cdot \gamma_x^{-1}(h_{s'+1, s}^{(1)}) \beta(\overline{\mathbf{h}}_{1s'}^{(1)} \otimes_k h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \otimes_A \tilde{\gamma}_A(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r}] \\ &= (-1)^{rs'+r's'} \Lambda\left(\overline{\mathbf{h}}_{s'+1, s}^{(4)} \otimes_{HL} \left[m \cdot \gamma_x^{-1}(h_{s'+1, s}^{(1)}) \beta(\overline{\mathbf{h}}_{1s'}^{(1)} \otimes_k h_{s'+1, s}^{(2)} \cdot \overline{\mathbf{a}}_{1r'}) \gamma_x(h_{s'+1, s}^{(3)}) \otimes \overline{\mathbf{a}}_{r'+1, r} \right]\right) \\ &= \Lambda(\mathbf{y} * \beta), \end{aligned}$$

as desired. Here the first equality holds by the definition of Λ ; the second one, by the definition of \diamond ; the third one, by the definition of $\Theta(\beta)$; the fourth one, by Lemma 2.7; the fifth one, since β is a right H^L -module morphism and $\gamma(1^{(1)})\gamma(S(1^{(2)})) = 1_E$; the fifth one, by the definition of Λ and Lemma 2.6; and the last one, by the definition of $*$. \square

5 Cyclic homology of cleft extensions

Let $H, A, \rho, f, E, K, M, \nu, \gamma$ and γ^{-1} be as in Section 2. Thus H is a weak Hopf algebra, A is a weak module algebra and f is convolution invertible. In this section we prove that the mixed complex $(\widehat{X}_*(E), \widehat{d}_*, \widehat{D}_*)$ of [25, Section 6] is isomorphic to a simpler mixed complex $(\overline{X}_*(E), \overline{d}_*, \overline{D}_*)$. Let

$$\Theta_*: (\widehat{X}_*(E), \widehat{d}_*) \longrightarrow (\overline{X}_*(E), \overline{d}_*) \quad \text{and} \quad \Lambda_*: (\overline{X}_*(E), \overline{d}_*) \longrightarrow (\widehat{X}_*(E), \widehat{d}_*),$$

be the maps introduced in the proof of Theorem 2.9. For each $n \geq 0$, let $\overline{D}_n: \overline{X}_n(E) \rightarrow \overline{X}_{n+1}(E)$ be the map defined by $\overline{D}_n := \Theta_{n+1} \circ \widehat{D}_n \circ \Lambda_n$.

Theorem 5.1. *The triple $(\overline{X}_*(E), \overline{d}_*, \overline{D}_*)$ is a mixed complex that gives the Hochschild and that cyclic type homologies of the K -algebra E . More precisely, the mixed complexes $(\overline{X}_*(E), \overline{d}_*, \overline{D}_*)$ and $(E \otimes \overline{E}^{\otimes *}, b_*, B_*)$ are homotopically equivalent.*

Proof. Since Θ_* and Λ_* are inverse one of each other it is clear that $(\overline{X}_*(E), \overline{d}_*, \overline{D}_*)$ is a mixed complex and that $\Theta_*: (\widehat{X}_*(E), \widehat{d}_*, \widehat{D}_*) \rightarrow (\overline{X}_*(E), \overline{d}_*, \overline{D}_*)$ is an isomorphism of mixed complexes. Consequently the result follows from [25, Theorem 6.3]. \square

Definition 5.2. For each $r, s \geq 0$, let $\widehat{D}_{rs}^0: \widehat{X}_{rs}(E) \rightarrow \widehat{X}_{r,s+1}(E)$ and $\widehat{D}_{rs}^1: \widehat{X}_{rs}(E) \rightarrow \widehat{X}_{r+1,s}(E)$ be the maps introduced in [25, Definition 6.5]. We define $\overline{D}_{rs}^0: \overline{X}_{rs}(E) \rightarrow \overline{X}_{r,s+1}(E)$ and $\overline{D}_{rs}^1: \overline{X}_{rs}(E) \rightarrow \overline{X}_{r+1,s}(E)$ by $\overline{D}_{rs}^0 := \Theta_{r,s+1} \circ \widehat{D}_{rs}^0 \circ \Lambda_{rs}$ and $\overline{D}_{rs}^1 := \Theta_{r+1,s} \circ \widehat{D}_{rs}^1 \circ \Lambda_{rs}$, respectively.

Proposition 5.3. *let R be a subalgebra of A and let $\mathbf{y} := \overline{\mathbf{h}}_{1s} \otimes_{HL} [a_0 \cdot \gamma(h_0) \otimes \overline{\mathbf{a}}_{1r}] \in \overline{X}_{rs}(E)$. If R is stable under ρ and f takes its values in R , then $\overline{D}(\mathbf{y}) = \overline{D}^0(\mathbf{y}) + \overline{D}^1(\mathbf{y})$ module $\bigoplus_{s=0}^i \overline{X}_{s,n+1-s}^1(R, M)$.*

Proof. This follows by Remark 1.13 and [25, Proposition 6.6(1)]. \square

Corollary 5.4. *If f takes its values in K , then $\overline{D} = \overline{D}^0 + \overline{D}^1$.*

We next compute the maps \overline{D}^0 and \overline{D}^1 . We will need the following proposition.

Notation 5.5. Given $h_1, \dots, h_s \in S$ and $1 \leq j \leq s$, we set $S_x(h_{1s}) := S(h_s)S(h_{s-1}) \cdots S(h_1)$.

Proposition 5.6. *For each $s \in \mathbb{N}$ there exists a map $T_s: H^{\otimes_k^{s+1}} \rightarrow A$ such that*

$$\gamma(h_0)\gamma_x^{-1}(h_{1s}) = j_\nu(T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)}))\gamma(h_0^{(2)}S_x(h_{1s}^{(1)})) \quad \text{and} \quad T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)})(h_0^{(2)}S_x(h_{1s}^{(1)}) \cdot 1_A) = T_s(h_{0s}).$$

Proof. We will proceed by an inductive argument. Assume first that $s = 1$. By equality (1.11), the first identity in (1.8) and Theorem 1.5(d), we have

$$\begin{aligned} \gamma(h_0)\gamma^{-1}(h_1) &= \gamma(h_0)j_\nu(f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)}))\gamma(S(h_1^{(1)})) \\ &= j_\nu(h_0^{(1)} \cdot f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)}))\gamma(h_0^{(2)})\gamma(S(h_1^{(1)})) \\ &= j_\nu\left(\left(h_0^{(1)} \cdot f^{-1}(S(h_1^{(3)}) \otimes_k h_1^{(4)})\right)f(h_0^{(2)} \otimes_k S(h_1^{(2)}))\right)\gamma(h_0^{(3)}S(h_1^{(1)})). \end{aligned}$$

So, we have $T_1(h_{01}) = (h_0^{(1)} \cdot f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)}))f(h_0^{(2)} \otimes_k S(h_1^{(1)}))$. The first equality in the statement follows immediately from the equality in Theorem 1.5(1). Assume now that $s \geq 1$ and there exists T_s as in the statement. Then

$$\begin{aligned} \gamma(h_0)\gamma_x^{-1}(h_{1,s+1}) &= j_\nu(T_s(h_0^{(1)} \otimes_k h_{2,s+1}^{(2)}))\gamma(h_0^{(2)}S_x(h_{2,s+1}^{(1)}))j_\nu(f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)}))\gamma(S(h_1^{(1)})) \\ &= j_\nu\left(T_s(h_0^{(1)} \otimes_k h_{2,s+1}^{(3)})h_0^{(2)}S_x(h_{2,s+1}^{(2)}) \cdot f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)})\right)\gamma(h_0^{(3)}S_x(h_{2,s+1}^{(1)}))\gamma(S(h_1^{(1)})) \\ &= j_\nu\left(T_s(h_0^{(1)} \otimes_k h_{2,s+1}^{(4)})h_0^{(2)}S_x(h_{2,s+1}^{(3)}) \cdot f^{-1}(S(h_1^{(2)}) \otimes_k h_1^{(3)})f(h_0^{(3)}S_x(h_{2,s+1}^{(2)}) \otimes_k S(h_1^{(1)}))\right)\gamma(h_0^{(4)}S_x(h_{1,s+1}^{(1)})), \end{aligned}$$

which provides the recursive step. The same argument as above proves the second equality in the statement. \square

Remark 5.7. By Theorem 1.5(d), Propositions 1.9(3) and 5.6, and [24, Propositions 4.5 and 5.19],

$$\begin{aligned} \gamma(h_0^{(1)})\gamma_x^{-1}(h_{1s}^{(2)})\gamma^{-1}(h_0^{(2)}S_x(h_{1s}^{(1)})) &= j_\nu(T_s(h_0^{(1)} \otimes_k h_{1s}^{(3)}))\gamma(h_0^{(2)}S_x(h_{1s}^{(2)}))\gamma^{-1}(h_0^{(3)}S_x(h_{1s}^{(1)})) \\ &= (-1)^{jr+r}(T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)}))\gamma(\Pi^L(h_0^{(2)}S_x(h_{1s}^{(1)}))) \\ &= j_\nu(T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)}))j_\nu(h_0^{(2)}S_x(h_{1s}^{(1)}) \cdot 1_A) \\ &= j_\nu(T_s(h_0 \otimes_k h_{1s})). \end{aligned}$$

Since $(A \otimes \epsilon) \circ j_\nu = \text{id}_A$, this implies that

$$T_s(h_0 \otimes_k h_{1s}) = (A \otimes \epsilon)\left(\gamma(h_0^{(1)})\gamma_x^{-1}(h_{1s}^{(2)})\gamma^{-1}(h_0^{(2)}S_x(h_{1s}^{(1)}))\right).$$

Proposition 5.8. For $\mathbf{y} := \bar{\mathbf{h}}_{1s} \otimes_{HL} [J_\nu(a_0)\gamma(h_0) \otimes \bar{\mathbf{a}}_{1r}] \in \bar{X}_{rs}(E)$, we have

$$\bar{D}^0(\mathbf{y}) = \sum_{j=0}^s \alpha_{jrs} (\bar{\mathbf{h}}_{j+1,s}^{(5)} \otimes_{HL} \overline{h_0^{(2)} S_x(h_{1s}^{(1)})} \otimes_{HL} \mathbf{h}_{1j}^{(2)}) \otimes_{HL} [\gamma_x(h_{j+1,s}^{(4)}) J_\nu(a_0)\gamma(h_0^{(1)})\gamma_x^{-1}(h_{j+1,s}^{(2)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}]$$

and

$$\bar{D}^1(\mathbf{y}) = \sum_{j=0}^s \beta_{jrs} \bar{\mathbf{h}}_{1s}^{(6)} \otimes_{HL} \left[\gamma(h_0^{(3)} S_x(h_{1s}^{(1)})) \gamma_x(h_{1s}^{(5)}) \otimes \bar{\mathbf{a}}_{j+1,r} \otimes \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(3)})} \otimes (h_0^{(2)} S_x(h_{1s}^{(2)})) \cdot (h_{1s}^{(4)} \cdot \bar{\mathbf{a}}_{1j}) \right],$$

where $\alpha_{jrs} := (-1)^{js+r+s}$ and $\beta_{jrs} := (-1)^{jr+r}$.

Proof. By the definition of Λ and the first equality in Proposition 5.6,

$$\begin{aligned} \Lambda(\mathbf{y}) &= (-1)^{rs} [J_\nu(a_0)\gamma(h_0)\gamma_x^{-1}(h_{1s}^{(1)}) \otimes_A \tilde{\gamma}_A(h_{1s}^{(2)}) \otimes \bar{\mathbf{a}}_{1r}] \\ &= (-1)^{rs} [J_\nu(a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)})) \gamma(h_0^{(2)} S_x(h_{1s}^{(1)})) \otimes_A \tilde{\gamma}_A(h_{1s}^{(3)}) \otimes \bar{\mathbf{a}}_{1r}]. \end{aligned}$$

So

$$\hat{D}^0(\Lambda(\mathbf{y})) = \sum_{j=0}^s \alpha'_{jrs} \left[1_E \otimes_A \tilde{\gamma}_A(h_{j+1,s}^{(4)}) \otimes_{AJ_\nu} \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)})} \gamma(h_0^{(2)} S_x(h_{1s}^{(1)})) \otimes_A \tilde{\gamma}_A(h_{1j}^{(3)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r} \right]$$

and

$$\hat{D}^1(\Lambda(\mathbf{y})) = \sum_{j=0}^r \beta'_{jrs} \left[\gamma(h_0^{(3)} S_x(h_{1s}^{(1)})) \otimes_A \tilde{\gamma}_A(h_{1s}^{(5)}) \otimes \bar{\mathbf{a}}_{j+1,r} \otimes \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(3)})} \otimes (h_0^{(2)} S_x(h_{1s}^{(2)})) \cdot (h_{1s}^{(4)} \cdot \bar{\mathbf{a}}_{1j}) \right],$$

where $\alpha'_{jrs} := (-1)^{rs+js+s}$ and $\beta'_{jrs} := (-1)^{rs+jr+r+s}$. Now, we have

$$\begin{aligned} &(-1)^{rs+r} \Theta \left([1_E \otimes_A \tilde{\gamma}_A(h_{j+1,s}^{(4)}) \otimes_{AJ_\nu} \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(2)})} \gamma(h_0^{(2)} S_x(h_{1s}^{(1)})) \otimes_A \tilde{\gamma}_A(h_{1j}^{(3)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}] \right) \\ &= (\bar{\mathbf{h}}_{j+1,s}^{(5)} \otimes_{HL} \overline{h_0^{(2)} S_x(h_{1s}^{(1)})} \otimes_{HL} \bar{\mathbf{h}}_{1j}^{(4)}) \otimes_{HL} [\gamma_x(h_{j+1,s}^{(4)}) J_\nu(a_0)\gamma(h_0^{(1)})\gamma_x^{-1}(h_{1s}^{(2)})\gamma_x(h_{1j}^{(3)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}] \\ &= (\bar{\mathbf{h}}_{j+1,s}^{(5)} \otimes_{HL} \overline{h_0^{(2)} S_x(h_{1s}^{(1)})} \otimes_{HL} \mathbf{h}_{1j}^{(2)} \cdot 1^{(2)}) \otimes_{HL} [\gamma_x(h_{j+1,s}^{(4)}) J_\nu(a_0)\gamma(h_0^{(1)})\gamma_x^{-1}(h_{j+1,s}^{(2)})\gamma(1^{(1)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}] \\ &= (\bar{\mathbf{h}}_{j+1,s}^{(5)} \otimes_{HL} \overline{h_0^{(2)} S_x(h_{1s}^{(1)})} \otimes_{HL} \mathbf{h}_{1j}^{(2)}) \otimes_{HL} [\gamma_x(h_{j+1,s}^{(4)}) J_\nu(a_0)\gamma(h_0^{(1)})\gamma_x^{-1}(h_{j+1,s}^{(2)})\gamma(1^{(1)} S(1^{(2)})) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}] \\ &= (\bar{\mathbf{h}}_{j+1,s}^{(5)} \otimes_{HL} \overline{h_0^{(2)} S_x(h_{1s}^{(1)})} \otimes_{HL} \mathbf{h}_{1j}^{(2)}) \otimes_{HL} [\gamma_x(h_{j+1,s}^{(4)}) J_\nu(a_0)\gamma(h_0^{(1)})\gamma_x^{-1}(h_{j+1,s}^{(2)}) \otimes \mathbf{h}_{j+1,s}^{(3)} \cdot \bar{\mathbf{a}}_{1r}], \end{aligned}$$

where the first equality holds by the definition on Θ and the first identity in Proposition 5.6; the second one, by Lemma 2.7; the third one, by the definition of the action in (2.4) and [24, Proposition 2.22(2)]; and the fourth one, since $\gamma(1^{(1)} S(1^{(2)})) = \gamma(1) = 1$. Similarly

$$\begin{aligned} &(-1)^{rs+s} \Theta \left(\left[\gamma(h_0^{(3)} S_x(h_{1s}^{(1)})) \otimes_A \tilde{\gamma}_A(h_{1s}^{(5)}) \otimes \bar{\mathbf{a}}_{j+1,r} \otimes \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(3)})} \otimes (h_0^{(2)} S_x(h_{1s}^{(2)})) \cdot (h_{1s}^{(4)} \cdot \bar{\mathbf{a}}_{1j}) \right] \right) \\ &= \bar{\mathbf{h}}_{1s}^{(6)} \otimes_{HL} \left[\gamma(h_0^{(3)} S_x(h_{1s}^{(1)})) \gamma_x(h_{1s}^{(5)}) \otimes \bar{\mathbf{a}}_{j+1,r} \otimes \overline{a_0 T_s(h_0^{(1)} \otimes_k h_{1s}^{(3)})} \otimes (h_0^{(2)} S_x(h_{1s}^{(2)})) \cdot (h_{1s}^{(4)} \cdot \bar{\mathbf{a}}_{1j}) \right]. \end{aligned}$$

From these facts it follows that the formulas in the statement are true. \square

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