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WEAK TYPE (1,1) ESTIMATES FOR CAFFARELLI-CALDERÓN GENERALIZED MAXIMAL OPERATORS FOR SEMIGROUPS ASSOCIATED WITH BESSEL AND LAGUERRE OPERATORS

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(Communicated by Alexander Iosevich)

Dedicated to the memory of our friend Professor Pablo González Vera

Abstract. In this paper we prove that the generalized (in the sense of Caffarelli and Calderón) maximal operators associated with heat semigroups for Bessel and Laguerre operators are weak type $(1,1)$. Our results include other known ones, and our proofs are simpler than the ones for the known special cases.

1. Introduction

Stein investigated in [16] harmonic analysis associated to diffusion semigroups of operators. If $\{T_t\}_{t>0}$ is a diffusion semigroup in the measure space (Ω, μ), in [16, p. 73] it was proved that the maximal operator T_* defined by

$$T_* f = \sup_{t>0} |T_t f|$$

is bounded from $L^p(\Omega, \mu)$ into itself, for every $1 < p < \infty$. As far as we know there is not a result showing the behavior of T_* on $L^1(\Omega, \mu)$ for every diffusion semigroup $\{T_t\}_{t>0}$. The behavior of T_* on $L^1(\Omega, \mu)$ must be established by taking into account the intrinsic properties of $\{T_t\}_{t>0}$. The usual result says that T_* is bounded from $L^1(\Omega, \mu)$ into $L^{1,\infty}(\Omega, \mu)$, but not bounded from $L^1(\Omega, \mu)$ into $L^1(\Omega, \mu)$. In order to analyze T_* in $L^1(\Omega, \mu)$, in many cases this maximal operator is controlled by a Hardy-Littlewood type maximal operator, and also, the vector valued Calderón-Zygmund theory ([14]) can be used. These procedures have been employed to study the maximal operators associated to the classical heat semigroup [17, p. 57], to Hermite operators ([10], [15] and [20]), to Laguerre operators ([8], [9], [10], [13] and [19]), to Bessel operators ([1], [2], [3], [11] and [18]) and to Jacobi operators ([11] and [12]), amongst others.

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Our objective in this paper is to study the L^p-boundedness properties, $1 \leq p \leq \infty$, for the generalized (in the sense of Caffarelli and Calderón [5]) maximal operators associated to the multidimensional Bessel and Laguerre operators.

Our results (see the theorems below) extend the others known for the Bessel operators ([3, Theorem 2.1] and [1, Theorem 1.1]) and for the Laguerre operators ([13, Theorem 1.1]). Moreover, by exploiting ideas developed by Caffarelli and Calderón ([5] and [6]), we are able to prove our result in a much simpler way than the one followed in [1], [3] and [13].

We now recall some definitions and properties in the Bessel and Laguerre settings which allow us to state our results.

We consider for $\lambda > -1/2$ the Bessel operator Δ_λ defined by

$$\Delta_\lambda = -x^{-2\lambda}\frac{d}{dx}x^{2\lambda}\frac{d}{dx} = -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}, \quad \text{on } (0,\infty),$$

and, if J_ν represents the Bessel function of the first kind and order ν, the Hankel transformation h_λ is given by

$$h_\lambda(f)(x) = \int_0^\infty (xy)^{-\lambda+1/2} J_{\lambda-1/2}(xy) f(y) y^{2\lambda} dy, \quad x \in (0,\infty),$$

for every $f \in L^1((0,\infty), x^{2\lambda}dx)$. h_λ can be extended to $L^2((0,\infty), x^{2\lambda}dx)$ as an isometry in $L^2((0,\infty), x^{2\lambda}dx)$ and $h_\lambda^{-1} = h_\lambda$. If $f \in C_c^\infty(0,\infty)$ we have that

$$h_\lambda(\Delta_\lambda f)(x) = x^2 h_\lambda(f)(x), \quad x \in (0,\infty).$$

This property suggests extending the definition of Δ_λ as follows:

$$\Delta_\lambda f = h_\lambda(x^2 h_\lambda(f)), \quad f \in D(\Delta_\lambda),$$

where

$$D(\Delta_\lambda) = \{f \in L^2((0,\infty), x^{2\lambda}dx) : x^2 h_\lambda(f) \in L^2((0,\infty), x^{2\lambda}dx)\}.$$

Thus, Δ_λ is a positive and selfadjoint operator. Moreover, $-\Delta_\lambda$ generates a semigroup of operators $\{W_t^\lambda\}_{t>0}$ in $L^2((0,\infty), x^{2\lambda}dx)$ where

$$W_t^\lambda(f) = h_\lambda\left(e^{-ty^2} h_\lambda(f)\right), \quad f \in L^2((0,\infty), x^{2\lambda}dx) \text{ and } t > 0. \tag{1}$$

According to [21, p. 395 (1)] we can write, for $f \in L^2((0,\infty), x^{2\lambda}dx)$,

$$W_t^\lambda(f)(x) = \int_0^\infty W_t^\lambda(x,y) f(y) y^{2\lambda} dy, \quad x,t \in (0,\infty), \tag{2}$$

where the Hankel heat kernel semigroup $W_t^\lambda(x,y)$ is defined by

$$W_t^\lambda(x,y) = \frac{(xy)^{-\lambda+1/2}}{2t} I_{\lambda-1/2}\left(\frac{xy}{2t}\right) e^{-(x^2+y^2)/(4t)}, \quad x,y,t \in (0,\infty), \tag{3}$$

and I_ν denotes the modified Bessel function of the first kind and order ν.

Since $\int_0^\infty W_t^\lambda(x,y) y^{2\lambda} dy = 1$, $x,t \in (0,\infty)$, $\{W_t^\lambda\}_{t>0}$ defined by (2) is a diffusion semigroup in $L^p((0,\infty), x^{2\lambda}dx)$, $1 \leq p \leq \infty$.

Suppose now that $\lambda = (\lambda_1, \dots, \lambda_n) \in (-1/2, \infty)^n$. We define the n-dimensional Bessel operator Δ_λ by

$$\Delta_\lambda = \sum_{j=1}^{n} \Delta_{\lambda_j, x_j}.$$

The operator $-\Delta_\lambda$ generates the diffusion semigroup $\{\mathbb{W}_t^\lambda\}_{t>0}$ in $L^p((0,\infty)^n, d\mu_\lambda)$, $1 \le p \le \infty$, where $d\mu_\lambda(x) = \prod_{j=1}^{n} x_j^{2\lambda_j} dx_j$, $x = (x_1, \dots, x_n) \in (0,\infty)^n$, and

$$\mathbb{W}_t^\lambda(f)(x) = \int_{(0,\infty)^n} \mathbb{W}_t^\lambda(x,y) f(y) d\mu_\lambda(y), \quad f \in L^p((0,\infty)^n, d\mu_\lambda) \text{ and } x, t \in (0,\infty),$$

being

$$\mathbb{W}_t^\lambda(x,y) = \prod_{j=1}^{n} W_t^{\lambda_j}(x_j, y_j), \quad x, y \in (0,\infty)^n \text{ and } t > 0.$$

The maximal operator $\mathbb{W}_*^\lambda$ associated with $\{\mathbb{W}_t^\lambda\}_{t>0}$ is defined by

$$\mathbb{W}_*^\lambda(f) = \sup_{t>0} |\mathbb{W}_t^\lambda(f)|.$$

In [1, Theorem 1.1] (also in [2, Theorem 2.1] when $\lambda \in (0,\infty)^n$ and in [3, Theorem 2.1] for $n = 1$) it was proved that $\mathbb{W}_*^\lambda$ is a bounded operator from $L^1((0,\infty)^n, d\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda)$. Note that since $\{\mathbb{W}_t^\lambda\}_{t>0}$ is a diffusion semigroup, $\mathbb{W}_*^\lambda$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda)$ into itself, for every $1 < p \le \infty$ (see [16, p. 73]).

Motivated by [5] we consider a function $r = (r_1, \dots, r_n)$ where, for every $j = 1, \dots, n$, $r_j : [0,\infty) \longrightarrow [0,\infty)$ is continuous and increasing, $r_j(0) = 0$ and $\lim_{t\to+\infty} r_j(t) = +\infty$, and we define the maximal operator

$$\mathbb{W}_{r,*}^\lambda(f) = \sup_{t>0} |\mathbb{W}_{r(t)}^\lambda(f)|,$$

where

$$\mathbb{W}_{r(t)}^\lambda(f)(x) = \int_{(0,\infty)^n} \mathbb{W}_{r(t)}^\lambda(x,y) f(y) d\mu_\lambda(y), \quad f \in L^p((0,\infty)^n, d\mu_\lambda),\ 1 \le p \le \infty,$$

and

$$\mathbb{W}_{r(t)}^\lambda(x,y) = \prod_{j=1}^{n} W_{r_j(t)}^{\lambda_j}(x_j, y_j), \quad x, y \in (0,\infty)^n \text{ and } t > 0.$$

It is clear that if $r_j(t) = t$, $t \ge 0$, $j = 1, \dots, n$, then $\mathbb{W}_{r,*}^\lambda = \mathbb{W}_*^\lambda$.

Since $\{\mathbb{W}_t^\lambda\}_{t>0}$ is a diffusion semigroup, it can be seen that $\mathbb{W}_{r,*}^\lambda$ is a bounded operator from $L^p((0,\infty)^n, d\mu_\lambda)$ into itself, for every $1 < p \le \infty$. The weak type (1,1) inequality is established in the following result.

Theorem 1.1. *Suppose that $\lambda \in (-1/2, \infty)^n$ and r is a function as above. Then, the maximal operator $\mathbb{W}_{r,*}^\lambda$ is bounded from $L^1((0,\infty)^n, d\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda)$.*

An immediate consequence of Theorem 1.1 is the next convergence result.

Corollary 1.2. *Let $\lambda \in (-1/2,\infty)^n$ and r be a function as above. Then, for every $f \in L^p((0,\infty)^n, d\mu_\lambda)$, $1 \le p < \infty$,*

$$\lim_{t\to 0^+} \mathbb{W}^\lambda_{r(t)}(f)(x) = f(x), \quad a.e.\ x \in (0,\infty)^n.$$

We now consider the Laguerre operator $\mathcal{L}_\lambda$, $\lambda > -1/2$, defined by

$$\mathcal{L}_\lambda = \Delta_\lambda + \frac{x^2}{4}, \quad \text{on } (0,\infty).$$

Also, for every $k \in \mathbb{N}$, we define the k-th Laguerre function ψ^λ_k by

$$\psi^\lambda_k(x) = 2^{-(2\lambda-1)/4}\left(\frac{k!}{\Gamma(k+\lambda+1/2)}\right)^{1/2} L_k^{\lambda-1/2}\left(\frac{x^2}{2}\right)e^{-x^2/4}, \quad x \in (0,\infty),$$

where L^α_k denotes the k-th Laguerre polynomial with parameter $\alpha > -1$. The system $\{\psi^\lambda_k\}_{k\in\mathbb{N}}$ is a complete orthonormal family in $L^2((0,\infty), x^{2\lambda}dx)$. Moreover,

$$\mathcal{L}_\lambda(\psi^\lambda_k) = (2k+\lambda+1/2)\psi^\lambda_k, \quad k \in \mathbb{N}.$$

We extend the definition of the operator $\mathcal{L}_\lambda$ as follows:

$$\mathcal{L}_\lambda(f) = \sum_{k=0}^\infty (2k+\lambda+1/2)\langle f, \psi^\lambda_k\rangle \psi^\lambda_k, \quad f \in D(\mathcal{L}_\lambda),$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2((0,\infty), x^{2\lambda}dx)$, and

$$D(\mathcal{L}_\lambda) = \{f \in L^2((0,\infty), x^{2\lambda}dx) : \sum_{k=0}^\infty (2k+\lambda+1/2)^2|\langle f,\psi^\lambda_k\rangle|^2 < \infty\}.$$

Thus, $\mathcal{L}_\lambda$ is positive and selfadjoint in $L^2((0,\infty), x^{2\lambda}dx)$. Moreover, $-\mathcal{L}_\lambda$ generates a diffusion semigroup $\{L^\lambda_t\}_{t>0}$ on $L^2((0,\infty), x^{2\lambda}dx)$ where, for every $t > 0$,

$$(4) \quad L^\lambda_t(f)(x) = \int_0^\infty L^\lambda_t(x,y)f(y)y^{2\lambda}dy, \quad f \in L^2((0,\infty), x^{2\lambda}dx),\ x,t \in (0,\infty),$$

being

(5)
$$L^\lambda_t(x,y) = \frac{e^{-t}}{1-e^{-2t}}(xy)^{-\lambda+1/2} I_{\lambda-1/2}\left(\frac{e^{-t}xy}{1-e^{-2t}}\right)\exp\left(-\frac{1}{4}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)\right),$$
$$x,y,t \in (0,\infty).$$

Moreover, (4) also defines a diffusion semigroup in $L^p((0,\infty), x^{2\lambda}dx)$, $1 \le p \le \infty$.

Suppose now that $\lambda \in (-1/2,\infty)^n$. The n-dimensional heat Laguerre semigroup $\{\mathbb{L}^\lambda_t\}_{t>0}$ is defined as follows. For every $t > 0$, $f \in L^p((0,\infty)^n, d\mu_\lambda)$, $1 \le p \le \infty$, we write

$$\mathbb{L}^\lambda_t(f)(x) = \int_{(0,\infty)^n} \mathbb{L}^\lambda_t(x,y)f(y)d\mu_\lambda(y), \quad x \in (0,\infty)^n,$$

being

$$\mathbb{L}^\lambda_t(x,y) = \prod_{j=1}^n L^{\lambda_j}_t(x_j,y_j), \quad x,y \in (0,\infty)^n,\ t > 0.$$

In [13, Theorem 1.1] it was shown that the maximal operator $\mathbb{L}_*^\lambda$, defined by

$$\mathbb{L}_*^\lambda(f) = \sup_{t>0} |\mathbb{L}_t^\lambda(f)|,$$

is bounded from $L^1((0,\infty)^n, d\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda)$ by employing an ingenious but long and not easy procedure.

Assume that the function $r : [0,\infty) \longrightarrow [0,\infty)^n$ is as in Theorem 1.1. We define the maximal operator $\mathbb{L}_{r,*}^\lambda$ by

$$\mathbb{L}_{r,*}^\lambda(f) = \sup_{t>0} |\mathbb{L}_{r(t)}^\lambda(f)|,$$

where

$$\mathbb{L}_{r(t)}^\lambda(f)(x) = \int_{(0,\infty)^n} \mathbb{L}_{r(t)}^\lambda(x,y) f(y) d\mu_\lambda(y), \quad x \in (0,\infty)^n,\ t>0,$$

being

$$\mathbb{L}_{r(t)}^\lambda(x,y) = \prod_{j=1}^n L_{r_j(t)}^{\lambda_j}(x_j,y_j), \quad x,y \in (0,\infty)^n,\ t>0.$$

We have that $|L_t^\lambda(f)| \le W_t^\lambda(|f|)$, $t>0$ ([7, (6.2)]). This inequality can be deduced from (3) and (5) by using that

$$\frac{2te^{-t}}{1-e^{-2t}} \le 1 \quad \text{and} \quad \frac{t(1+e^{-2t})}{1-e^{-2t}} \ge 1,\ \ t\in(0,\infty).$$

Then, from Theorem 1.1 and the comments just before it, we deduce the following result, which includes, as a special case, [13, Theorem 1.1].

Theorem 1.3. *Suppose that $\lambda \in (-1/2,\infty)^n$ and r is as in Theorem 1.1. Then, the maximal operator $\mathbb{L}_{r,*}^\lambda$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda)$ into itself, for every $1<p\le\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda)$.*

If we denote, for every $k=(k_1,\dots,k_n)\in\mathbb{N}^n$ and $\lambda\in(-1/2,\infty)^n$, $\psi_k^\lambda(x) = \prod_{j=1}^n \psi_{k_j}^{\lambda_j}(x_j)$, $x\in(0,\infty)^n$, the subspace $\operatorname{span}\{\psi_k^\lambda\}_{k\in\mathbb{N}^n}$ is dense in $L^p((0,\infty)^n, d\mu_\lambda)$, $1\le p<\infty$. For every $f\in\operatorname{span}\{\psi_k^\lambda\}_{k\in\mathbb{N}^n}$, we have that

$$\mathbb{L}_{r(t)}^\lambda(f) = \sum_{k\in\mathbb{N}^n} e^{-\sum_{j=1}^n r_j(t)(2k_j+\lambda_j+1/2)} \langle f,\psi_k^\lambda\rangle \psi_k^\lambda.$$

Since this last sum has at most a finite number of terms, it is clear that $\lim_{t\to0^+}\mathbb{L}_{r(t)}^\lambda(f)(x) = f(x)$, $x\in(0,\infty)^n$, for every $f\in\operatorname{span}\{\psi_k^\lambda\}_{k\in\mathbb{N}^n}$. Then, standard arguments allow us to deduce the following convergence result.

Corollary 1.4. *Let $\lambda\in(-1/2,\infty)^n$ and r be as in Theorem 1.1. Then, for every $f\in L^p((0,\infty)^n, d\mu_\lambda)$, $1\le p<\infty$,*

$$\lim_{t\to0^+} \mathbb{L}_{r(t)}^\lambda(f)(x) = f(x), \quad a.e.\ x\in(0,\infty)^n.$$

In the next section we present the proofs of Theorems 1.1 and Corollary 1.2.

Throughout the paper, by C and c we denote positive constants that can change from one line to the other.

2. Proof of the results

In order to prove Theorem 1.1 we need some properties of the Bessel heat kernel $W_r^\lambda(x,y)$, $r,x,y\in(0,\infty)$, $\lambda>-1/2$.

By proceeding as in the proof of [3, Lemma 3.1] we can show the following result.

Lemma 2.1. *Let $\lambda>-1/2$. Then, for every $r,x,y\in(0,\infty)$,*

$$W_r^\lambda(x,y)\leq C\begin{cases} x^{-2\lambda-1}e^{-cx^2/r}, & 0<y\leq x/2; \quad (6)\\ x^{-2\lambda-1}e^{-cx^2/r}+\dfrac{(xy)^{-\lambda}}{\sqrt{r}}e^{-(x-y)^2/(4r)}, & x/2<y<2x; \quad (7)\\ y^{-2\lambda-1}e^{-cy^2/r}, & 0<2x\leq y. \quad (8)\end{cases}$$

According to [21, Chapter VI, Section 6.15], if $\nu>-1/2$, we can write

$$I_\nu(z)=\frac{z^\nu}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)}\int_{-1}^1 e^{-zs}(1-s^2)^{\nu-1/2}ds,\quad z\in(0,\infty).$$

Moreover, $I_\nu(z)=2(\nu+1)I_{\nu+1}(z)/z+I_{\nu+2}(z)$, $z\in(0,\infty)$ and $\nu>-1$ ([21, Chapter III, Section 3.71]). Hence, if $\lambda>-1/2$ we obtain, for every $z\in(0,\infty)$,

$$\begin{aligned}I_{\lambda-1/2}(z)=&\frac{2\lambda+1}{z}I_{\lambda+1/2}(z)+I_{\lambda+3/2}(z)\\ =&\frac{(2\lambda+1)z^{\lambda-1/2}}{\sqrt{\pi}2^{\lambda+1/2}\Gamma(\lambda+1)}\int_{-1}^1 e^{-zs}(1-s^2)^\lambda ds\\ &+\frac{z^{\lambda+3/2}}{\sqrt{\pi}2^{\lambda+3/2}\Gamma(\lambda+2)}\int_{-1}^1 e^{-zs}(1-s^2)^{\lambda+1}ds.\end{aligned}$$

Then, the Bessel heat kernel can be written as

$$\begin{aligned}W_r^\lambda(x,y)=&\frac{1}{\sqrt{\pi}2^{2\lambda+1}\Gamma(\lambda+1)}\Big(\frac{2\lambda+1}{r^{\lambda+1/2}}\int_{-1}^1 e^{-(x^2+y^2+2xys)/(4r)}(1-s^2)^\lambda ds\\ &+\frac{(xy)^2}{2^3(\lambda+1)r^{\lambda+5/2}}\int_{-1}^1 e^{-(x^2+y^2+2xys)/(4r)}(1-s^2)^{\lambda+1}ds\Big),\\ &\qquad r,x,y\in(0,\infty),\end{aligned}\tag{9}$$

where $\lambda>-1/2$.

The key result to show Theorem 1.1 is the following.

Proposition 2.2. *Let $\lambda>-1/2$. Then, there exist $C,c>0$ such that*

$$W_r^\lambda(x,y)\leq C\sum_{k=0}^\infty\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}\chi_{I_k(x,r)}(y),\quad r,x,y\in(0,\infty),$$

where $I_k(x,r)=[x-2^k\sqrt{r},x+2^k\sqrt{r}]\cap(0,\infty)$, $r,x\in(0,\infty)$ and $k\in\mathbb{N}$.

Proof. Let $r, x \in (0,\infty)$. We consider different cases.

Suppose that $x \leq \sqrt{r}$. Then, $I_0(x,r) = [0, x+\sqrt{r}]$ and

$$\mu_\lambda(I_0(x,r)) = \frac{(x+\sqrt{r})^{2\lambda+1}}{2\lambda+1} \leq Cr^{\lambda+1/2}.$$

Since $x^2+y^2+2xys = (x-y)^2+2xy(1+s) \geq 0$, $y \in (0,\infty)$ and $s \in (-1,1)$, from (9) we deduce that

$$W_r^\lambda(x,y) \leq \frac{C}{r^{\lambda+1/2}}\left(1+\left(\frac{xy}{r}\right)^2\right) \leq \frac{C}{r^{\lambda+1/2}}\left(1+\left(\frac{x(x+\sqrt{r})}{r}\right)^2\right) \leq \frac{C}{r^{\lambda+1/2}}$$

$$\leq \frac{C}{\mu_\lambda(I_0(x,r))}, \quad y \in I_0(x,r). \tag{10}$$

Assume now that $x > \sqrt{r}$. Then, $I_0(x,r) = [x-\sqrt{r}, x+\sqrt{r}]$ and

$$\mu_\lambda(I_0(x,r)) = \frac{1}{2\lambda+1}\left((x+\sqrt{r})^{2\lambda+1} - (x-\sqrt{r})^{2\lambda+1}\right).$$

The mean value theorem leads to $\mu_\lambda(I_0(x,r)) = 2\sqrt{r}u^{2\lambda}$ for a certain $u \in (x-\sqrt{r}, x+\sqrt{r})$. If $\lambda \geq 0$, it follows that $\mu_\lambda(I_0(x,r)) \leq 2\sqrt{r}(x+\sqrt{r})^{2\lambda}$. On the other hand, if $-1/2 < \lambda < 0$, we distinguish two cases.

- If $x \in (\sqrt{r}, 3\sqrt{r})$, then

$$\mu_\lambda(I_0(x,r)) \leq \int_0^{x+\sqrt{r}} y^{2\lambda}dy \leq C(x+\sqrt{r})^{2\lambda+1} \leq C\sqrt{r}(x+\sqrt{r})^{2\lambda}.$$

- If $x \geq 3\sqrt{r}$, then

$$\mu_\lambda(I_0(x,r)) \leq C\sqrt{r}(x-\sqrt{r})^{2\lambda} \leq C\sqrt{r}\left(\frac{x+\sqrt{r}}{2}\right)^{2\lambda}.$$

Hence, we conclude that $\mu_\lambda(I_0(x,r)) \leq C\sqrt{r}x^{2\lambda} \leq Cx^{2\lambda+1}$ in either case. By keeping in mind Lemma 2.1, in order to estimate $W_r^\lambda(x,y)$ we distinguish three regions. First, by (6) it follows that

$$W_r^\lambda(x,y) \leq Cx^{-2\lambda-1} \leq \frac{C}{\mu_\lambda(I_0(x,r))}, \quad 0 < y \leq x/2,$$

and from (8) we deduce that

$$W_r^\lambda(x,y) \leq Cy^{-2\lambda-1} \leq Cx^{-2\lambda-1} \leq \frac{C}{\mu_\lambda(I_0(x,r))}, \quad 2x \leq y.$$

Moreover, (7) implies that

$$W_r^\lambda(x,y) \leq C\left(x^{-2\lambda-1} + \frac{x^{-2\lambda}}{\sqrt{r}}\right) \leq \frac{C}{\mu_\lambda(I_0(x,r))}, \quad x/2 < y < 2x.$$

We obtain that

$$W_r^\lambda(x,y) \leq \frac{C}{\mu_\lambda(I_0(x,r))}, \quad y \in (0,\infty). \tag{11}$$

Suppose now that $k \in \mathbb{N} \setminus \{0\}$. We define $C_k(x,r) = \{y \in (0,\infty) : 2^{k-1}\sqrt{r} < |x-y| \le 2^k\sqrt{r}\}$. It is clear that $C_k(x,r) \subset I_k(x,r)$.

Assume that $x \le 2^k\sqrt{r}$. Then, $I_k(x,r) = [0, x+2^k\sqrt{r}]$ and $\mu_\lambda(I_k(x,r)) \le C(2^k\sqrt{r})^{2\lambda+1}$. According to (9), since $x^2+y^2+2xys = (x-y)^2 + 2xy(1+s)$, $y \in (0,\infty)$ and $s \in (-1,1)$, we have that

$$(12)\qquad \begin{aligned} W_r^\lambda(x,y) &\le C\frac{e^{-c2^{2k}}}{r^{\lambda+1/2}}\left(1+\left(\frac{x(x+2^k\sqrt{r})}{r}\right)^2\right) \le C\frac{2^{4k}e^{-c2^{2k}}}{r^{\lambda+1/2}} \\ &\le C\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}, \quad y \in C_k(x,r). \end{aligned}$$

We now take $x > 2^k\sqrt{r}$. Then $I_k(x,r) = [x-2^k\sqrt{r}, x+2^k\sqrt{r}]$, and by proceeding as above we get $\mu_\lambda(I_k(x,r)) \le C2^k\sqrt{r}x^{2\lambda} \le Cx^{2\lambda+1}$. We again distinguish three cases. If $0 < y \le x/2$ and $y \in C_k(x,r)$ we have that $2^{k-1}\sqrt{r} \le x \le 2^{k+1}\sqrt{r}$. Then (6) implies that

$$W_r^\lambda(x,y) \le C\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}, \quad 0 < y \le x/2.$$

Also, from (8) we deduce

$$W_r^\lambda(x,y) \le C\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}, \quad 2x \le y.$$

Finally, by (7) if follows that

$$\begin{aligned} W_r^\lambda(x,y) &\le Ce^{-c2^{2k}}\left(x^{-2\lambda-1} + \frac{x^{-2\lambda}}{\sqrt{r}}\right) \\ &\le C\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}, \quad x/2 < y < 2x \text{ and } y \in C_k(x,r). \end{aligned}$$

Hence, we get

$$(13)\qquad W_r^\lambda(x,y) \le C\frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}, \quad y \in C_k(x,r).$$

By combining (10), (11), (12) and (13) we obtain

$$\begin{aligned} W_r^\lambda(x,y) &= W_r^\lambda(x,y)\chi_{I_0(x,r)}(y) + \sum_{k=1}^\infty W_r^\lambda(x,y)\chi_{C_k(x,r)}(y) \\ &\le C\left(\frac{\chi_{I_0(x,r)}(y)}{\mu_\lambda(I_0(x,r))} + \sum_{k=1}^\infty \frac{e^{-c2^{2k}}\chi_{C_k(x,r)}(y)}{\mu_\lambda(I_k(x,r))}\right) \\ &\le C\sum_{k=0}^\infty \frac{e^{-c2^{2k}}}{\mu_\lambda(I_k(x,r))}\chi_{I_k(x,r)}(y), \quad y \in (0,\infty). \end{aligned}$$

□

2.1. **Proof of Theorem 1.1.** According to Proposition 2.2 we have that

$$|\mathbb{W}^\lambda_{r(t)}(f)(x)| \le \int_{(0,\infty)^n} \prod_{j=1}^n W^{\lambda_j}_{r_j(t)}(x_j,y_j)|f(y)|d\mu_\lambda(y)$$
$$\le K \sum_{k\in\mathbb{N}^n} \prod_{j=1}^n e^{-c2^{2k_j}} \frac{1}{\mu_\lambda(R_k(x,r(t)))} \int_{R_k(x,r(t))} |f(y)|d\mu_\lambda(y),$$
$$x \in (0,\infty)^n \text{ and } t>0,$$

where $R_k(x,r(t)) = \prod_{j=1}^n I_{k_j}(x_j, r_j(t))$ and $K>0$.

Then, it follows that

$$|\mathbb{W}^\lambda_{r,*}(f)(x)| \le K \sum_{k\in\mathbb{N}^n} \left(\prod_{j=1}^n e^{-c2^{2k_j}}\right) \mathcal{M}^\lambda_{r,k}(f)(x), \quad x\in(0,\infty)^n, \tag{14}$$

where $\mathcal{M}^\lambda_{r,k}$ represents the maximal function defined by

$$\mathcal{M}^\lambda_{r,k}(f)(x) = \sup_{t>0} \frac{1}{\mu_\lambda(R_k(x,r(t)))} \int_{R_k(x,r(t))} |f(y)|d\mu_\lambda(y), \quad x\in(0,\infty)^n.$$

By [5, Theorem 1], for every $k\in\mathbb{N}^n$ and $\gamma>0$, we get

$$\mu_\lambda\left(\{x\in(0,\infty)^n : \mathcal{M}^\lambda_{r,k}(f)(x) > \gamma\}\right) \le \frac{6^n n!}{\gamma}\|f\|_{L^1((0,\infty)^n,d\mu_\lambda)}, \tag{15}$$
$$f\in L^1((0,\infty)^n, d\mu_\lambda).$$

Since

$$\sum_{k\in\mathbb{N}^n}\prod_{j=1}^n e^{-\alpha 2^{2k_j}} = \left(\sum_{m=0}^\infty e^{-\alpha 2^{2m}}\right)^n < \infty, \quad \text{when } \alpha>0,$$

by defining

$$Q_k = \left(2K\sum_{\ell\in\mathbb{N}^n}\prod_{j=1}^n e^{-c2^{2\ell_j-1}}\right)^{-1} \prod_{j=1}^n e^{c2^{2k_j-1}}, \quad k\in\mathbb{N}^n,$$

we have that

$$\left\{x\in(0,\infty)^n : |\mathbb{W}^\lambda_{r,*}(f)(x)|>\gamma\right\} \subset \bigcup_{k\in\mathbb{N}^n}\left\{x\in(0,\infty)^n : \mathcal{M}^\lambda_{r,k}(f)(x) > \gamma Q_k\right\}.$$

Hence, from (15) we deduce that

$$\mu_\lambda\left(\{x\in(0,\infty)^n : |\mathbb{W}^\lambda_{r,*}(f)(x)|>\gamma\}\right) \le \sum_{k\in\mathbb{N}^n}\mu_\lambda\left(\left\{x\in(0,\infty)^n : \mathcal{M}^\lambda_{r,k}(f)(x)>\gamma Q_k\right\}\right)$$
$$\le 2K\frac{6^n n!}{\gamma}\left(\sum_{k\in\mathbb{N}^n}\prod_{j=1}^n e^{-c2^{2k_j-1}}\right)\left(\sum_{\ell\in\mathbb{N}^n}\prod_{j=1}^n e^{-c2^{2\ell_j-1}}\right)\|f\|_{L^1((0,\infty)^n,d\mu_\lambda)},\ \gamma>0.$$

Thus we prove that $\mathbb{W}^\lambda_{r,*}$ is bounded from $L^1((0,\infty)^n, d\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda)$. □

2.2. **Proof of Corollary 1.2.** In order to show this theorem it is sufficient to see that for every $f \in C_c^\infty((0,\infty)^n)$ (the space of smooth functions with compact support on $(0,\infty)^n$) we have that

$$\lim_{t\to 0^+} \mathbb{W}^\lambda_{r(t)}(f)(x) = f(x), \quad x \in (0,\infty)^n.$$

Let $f \in C_c^\infty((0,\infty)^n)$. The Hankel transform $h_\lambda(f)$ of f is defined by

$$h_\lambda(f)(x) = \int_{(0,\infty)^n} \prod_{j=1}^n (x_j y_j)^{-\lambda_j+1/2} J_{\lambda_j-1/2}(x_j y_j) f(y) d\mu_\lambda(y), \quad x \in (0,\infty)^n.$$

According to (1) we deduce that

$$\mathbb{W}^\lambda_{r(t)}(f)(x) = h_\lambda\left(\prod_{j=1}^n e^{-y_j^2 r_j(t)} h_\lambda(f)(y)\right)(x), \quad x \in (0,\infty)^n.$$

By using the dominated convergence theorem we conclude that

$$\lim_{t\to 0^+} \mathbb{W}^\lambda_{r(t)}(f)(x) = h_\lambda(h_\lambda(f))(x), \quad x \in (0,\infty)^n,$$

and the proof is completed because $h_\lambda^{-1} = h_\lambda$ in $L^2((0,\infty)^n, d\mu_\lambda)$ (see [4, p. 125]). □

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