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Cosmic microwave background bispectrum and slow roll inflation

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ABSTRACT

Recent tentative findings of non-Gaussian structure in the COBE-DMR dataset have triggered renewed attention to candidate models from which such intrinsic signature could arise. In the framework of slow roll inflation with built-in non linearities in the inflaton field evolution we present expressions for both the cosmic microwave background (CMB) skewness and the full angular bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$ in terms of the slow roll parameters. We use an estimator for the angular bispectrum recently proposed in the literature and calculate its variance for an arbitrary ℓ_i multipole combination. We stress that a real detection of non-Gaussianity in the CMB would imply that an important component of the anisotropies arises from processes *other* than primordial quantum fluctuations. We further investigate the behavior of the signal-to-(theoretical) noise ratio and demonstrate for generic inflationary models that it decreases in the limited range of small-ℓ's considered for increasing multipole ℓ while the opposite applies for the standard $\mathcal{C}_\ell$'s.

Key words: cosmic microwave background - methods: analytical - cosmology: theory - large scale structures of Universe - early Universe.

1 INTRODUCTION

The theory of inflation provides an elegant means to solve the usual problems (horizon and flatness problems) of the standard model of Cosmology (Guth 1981). It consists in assuming that a phase of accelerated expansion took place in the very early Universe at the GUT energy scale. In the most simple models of inflation, this phase of accelerated expansion is driven by a scalar field. The physical origin of this field is still an open problem and Physics at GUT scale could be much more complicated than assumed in these simple models. However, it should be stressed that the very concept of inflation lies in the fact that the second cosmic time derivative of the scale factor was positive in the early Universe and is, in this sense, independent of any model-building provided that the effective inflaton potential is flat enough to allow an inflationary expansion of at least ≈ 70e-folds. Therefore *slow roll* inflation can be viewed as a generic framework which permits to implement concretely the concept of inflation and allows to perform simple analytical calculations.

The beauty of the inflationary scenario is that, combined with Quantum Mechanics, it also provides a natural explanation of the origin of the large scale structures and of the cosmic microwave background (CMB) radiation anisotropies observed in our Universe (Guth & Pi 1982; Starobinsky 1982; Hawking 1982; Bardeen, Steinhardt & Turner 1983). In this explanation, the quantum character of the inflaton field plays a crucial role since the seeds of these perturbations are the unavoidable quantum fluctuations present at the beginning of the inflationary epoch. Then these fluctuations are parametrically amplified during the accelerated phase of expansion (Grishchuk 1974). Therefore the properties of the initial spectrum of perturbations depend on the initial quantum state in which the fluctuations were placed and on the behaviour of the scale factor during inflation. Observationally, we have access to the initial spectrum when one looks at the CMB anisotropy multipole moments corresponding to the largest angular scales on the celestial sphere. Indeed these multipoles are dominated by modes whose wavelengths are comparable to the size of the horizon today. This means that after their creation these modes spent most of their time outside the Hubble radius and as a consequence were not contaminated by astrophysical processes: in a certain sense, they can be viewed as a pure relic of the very early Universe.

Among the many features of the perturbations, the statistical properties are certainly of a big importance. In the theory of cosmological perturbations of quantum mechanical origin, it is assumed that the initial state is the vacuum. This seems to be the most natural choice although it was already noticed that it could be difficult to understand why the fluctuations of a field which is initially out of equilibrium would be placed in this state (Unruh 1998). It was recently shown (Martin, Riazuelo & Sakellariadou 1999) that if one tries to start the evolution from a non vacuum initial state, then observations require that this state be close to the vacuum. This seems to indicate that the vacuum is indeed a reasonable choice. Since each mode of the perturbations can be viewed as an oscillator, one immediately reaches the conclusion that the corresponding statistical properties must be Gaussian (recall that the ground-state wavefunction of an harmonic oscillator is a Gaussian function). This constitutes an important and generic prediction of inflation.

Another source of cosmological relevant density inhomogeneities arises in models with topological defects of the vacuum, like cosmic strings and textures. These would leave different imprints on the CMB both at recombination and later, during the photon travel from the surface of last scattering to the present, on various angular scales (Allen et al. 1997; Magueijo et al. 1996; Battye et al. 1998; Contaldi et al. 1999; Pogosian & Vachaspati 1999; Durrer et al. 1996; Pen et al. 1997). There is the hope that future balloon-borne (e.g., MAXIMA, Lee et al. 1999; BOOMERanG, Lange et al. 1999) and satellite (MAP and Planck surveyor, Bersanelli et al. 1996) missions will allow a clean distinction among these different classes of models by mapping the CMB with unprecedent precision.

A number of authors (e.g., Smoot et al. 1994; Torres et al. 1995; Hinshaw et al. 1994; Hinshaw et al. 1995; Kogut et al. 1996) have analysed the COBE-DMR sky maps with a variety of test (like three-point statistics, genus, and extrema correlation function) and found perfect agreement with a Gaussian distribution. Recently however, three groups (Ferreira et al. 1998; Pando et al. 1998; Bromley & Tegmark 1999) have analysed the COBE-DMR four-year dataset and reported detections of non-Gaussianity casting doubts on these early findings. Banday et al. 1999 have further analyzed the same data, finding that the non-Gaussian signal is driven by the 53 GHz sky maps. They concluded that this frequency dependence strongly indicates that the signal is not of primordial origin. Despite all this, it seems now that due to limited signal-to-noise, sky coverage, and uncertainty in foreground substraction, present day experiments cannot conclusively exclude non-Gaussianity to a satisfactory confidence level.

The above remarks regarding the Gaussian character of the primordial perturbations have been established within the framework of the linear theory of cosmological pertubations. It is clear that generic higher order, in particular quadratic, terms are present and will produce a non vanishing signal even for inflationary models (Linde & Mukhanov 1997). The predictions for the three-point correlation function on large angular scales due to nonlinearities in the inflaton evolution were considered in the past (Falk et al. 1993; Gangui et al. 1994; Gangui 1994).

The post recombination Rees-Sciama effect, due to the mildly non-linear evolution of the perturbations also contributes to the signal (Luo & Schramm 1993; Mollerach et al. 1995;

Munshi et al. 1995). Evolving networks of topological defects continuously seed perturbations on the CMB that, by the very nature of the sources, are predicted to be highly non-Gaussian (Bouchet et al. 1988; Avelino et al. 1999; Gangui & Perivolaropoulos 1995; Gangui & Mollerach 1996). Further secondary effects contribute to produce non-Gaussianities at smaller scales (Aghanim & Forni 1999) and would be characterized by detectable correlations between gravitational lensing and Sunyaev-Zel'dovich maps (Spergel & Goldberg 1999; Goldberg & Spergel 1999).

We here present a general discussion of non-Gaussian features arising in the framework of slow roll inflation. Our article borrows some definitions and formulas for the CMB three-point correlation function (specially in section 2) from (Gangui et al 1994). Our main aim here is to present explicitly the derivations of the non-Gaussian estimators as a function of the inflationary slow roll parameters in the Legendre space, namely the full angular bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$. Then, in the third section, we present for the first time the analytical expression for the variance of an estimator recently proposed for the bispectrum (Ferreira et al. 1998) in the mildly non-Gaussian approximation. This allows us, in the last section, to compare the behaviour of both quantities for various multipoles and conclude with the by now established result supporting the view that the recently observed non-Gaussianity cannot be explained in the framework of slow roll inflation.

2 NON-GAUSSIAN SIGNAL IN REAL AND LEGENDRE SPACES

In the framework of the theory of cosmological perturbations of quantum mechanical origin, $\Delta T/T(\vec{x},\hat{\gamma})$ is an operator. The corresponding statistical properties are then calculated by "sandwiching" this operator (or a combination of these operators) with the quantum state $|\Psi\rangle$ in which the quantum perturbations are placed. However, it has been shown in (Grishchuk & Martin 1997) that one can think to $\Delta T/T(\vec{x},\hat{\gamma})$ as a classical stochastic process. This stochastic process can be expanded in spherical harmonics

$$\frac{\Delta T}{T}(\vec{x},\hat{\gamma}) = \sum_{\ell,m} a_\ell^m(\vec{x})\mathcal{W}_\ell Y_\ell^m(\hat{\gamma}), \tag{1}$$

where $\mathcal{W}_l$ represents the window function of the particular experiment. The coefficients $a_\ell^m(\vec{x})$ are random variables and are in principle different for different observers at positions $\vec{x}$. The statistical properties of $\Delta T/T(\vec{x},\hat{\gamma})$ are completely specified if the probability density function (pdf) of the $a_\ell^m(\vec{x})$'s is known. Choosing the initial state to be the vacuum, i.e.

$|\Psi\rangle = |0\rangle$, and considering only linear terms is equivalent to saying that the pdf of the $a_\ell^m(\vec{x})$'s is a Gaussian distribution (Grishchuk & Martin 1997). This means that

$$\langle a_\ell^m(\vec{x})\rangle = 0, \quad \langle a_{\ell_1}^{m_1}(\vec{x}) a_{\ell_2}^{m_2 *}(\vec{x})\rangle = \mathcal{C}_{\ell_1}\delta_{\ell_1\ell_2}\delta_{m_1m_2}, \tag{2}$$

where brackets $\langle\cdot\rangle$ stands for an average over the ensemble of possible universes in the sense explained above. The variance is rotationally invariant, i.e. depends only on ℓ, signaling statistical isotropy. For Gaussian fields the previous equations are sufficient since this kind of fields are completely characterized by giving their two-point correlation function or, equivalently, their (angular) power spectrum.

We now take into account the non linearities. This means that the pdf of the $a_\ell^m(\vec{x})$'s is no longer a Gaussian distribution. The two first moments will still be given by Eq. (2) and the difference will show up at the level of the third order moment. Predictions from different models usually come as expressions for the ensemble average $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3}\rangle$ which can be written in full analogy with Eq. (2) in terms of the angular bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$ as follows:

$$\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3}\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathcal{C}_{\ell_1\ell_2\ell_3}, \tag{3}$$

where now the proportionality factor is a Wigner 3-j symbol. This is non-zero only if the indices ℓ_i, m_i $(i = 1, 2, 3)$ fulfill the relations: $|\ell_j - \ell_k| \leq \ell_i \leq |\ell_j + \ell_k|$ and $m_1 + m_2 + m_3 = 0$. There is an additional "selection rule" in this equation that arises from demanding that $\langle \Delta T(\hat{\gamma}_1)\Delta T(\hat{\gamma}_2)\Delta T(\hat{\gamma}_3)\rangle$ be invariant under spatial inversions. One then obtains $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3}\rangle = 0$ for $\ell_1 + \ell_2 + \ell_3 = odd$ (Luo 1994).

For a Gaussian model we clearly have $\mathcal{C}_{\ell_1\ell_2\ell_3} = 0$ whereas the non linear evolution of the perturbations will induce a $\mathcal{C}_{\ell_1\ell_2\ell_3} \neq 0$. Therefore in order to probe the Gaussian character of the stochastic process $\Delta T/T(\vec{x}, \hat{\gamma})$, it is certainly convenient to study quantities related to the third moment. *A priori* a large choice is allowed. Here below we will concentrate on the CMB collapsed three-point correlation function, defined in real space. The skewness will just be the particular case of the collapsed function at zero lag.

The three–point correlation function for points at three arbitrary angular separations α, β and γ is given by the average product of temperature fluctuations in all possible three directions with those angular separations among them (Gangui et al. 1994). The collapsed case corresponds to the choice $\alpha = \beta$ and $\gamma = 0$ and reads

$$C_3(\alpha) \equiv \int \frac{d\Omega_{\hat{\gamma}_1}}{4\pi} \int \frac{d\Omega_{\hat{\gamma}_2}}{2\pi} \frac{\Delta T}{T}(\hat{\gamma}_1) \frac{\Delta T}{T}^2(\hat{\gamma}_2) \delta(\hat{\gamma}_1 \cdot \hat{\gamma}_2 - \cos\alpha). \tag{4}$$

For $\alpha = 0$, we recover the well-known expression for the skewness, $C_3(0)$. Using the spherical harmonics expansion (1) the last equation can be rewritten as:

$$C_3(\alpha) = \frac{1}{4\pi} \sum_{\ell_1,\ell_2,\ell_3,m_1,m_2,m_3} P_{\ell_1}(\cos\alpha) a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \mathcal{W}_{\ell_1}\mathcal{W}_{\ell_2}\mathcal{W}_{\ell_3} \bar{\mathcal{H}}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3} \ , \tag{5}$$

where we have defined the coefficients $\bar{\mathcal{H}}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}$ by

$$\bar{\mathcal{H}}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3} \equiv \int d\Omega_{\hat\gamma} Y_{\ell_1}^{m_1}(\hat\gamma) Y_{\ell_2}^{m_2}(\hat\gamma) Y_{\ell_3}^{m_3}(\hat\gamma) \ , \tag{6}$$

which has a simple expression in terms of Wigner 3-j symbols (Messiah 1976):

$$\bar{\mathcal{H}}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \ . \tag{7}$$

We see here that the condition $\ell_1 + \ell_2 + \ell_3 = even$ is enforced by the presence of the first Wigner 3-j symbol. Substitution of Eq. (3) together with Eq. (7) into Eq. (5), where we have taken the ensemble average, yields for the mean collapsed three-point correlation function

$$\langle C_3(\alpha)\rangle = \sum_{\ell_1,\ell_2,\ell_3} \sqrt{\frac{2\ell_1+1}{4\pi}} \sqrt{\frac{2\ell_2+1}{4\pi}} \sqrt{\frac{2\ell_3+1}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{W}_{\ell_1}\mathcal{W}_{\ell_2}\mathcal{W}_{\ell_3} \mathcal{C}_{\ell_1\ell_2\ell_3} P_{\ell_1}(\cos\alpha). \tag{8}$$

We see from this that all terms in the sum satisfying $\ell_1 + \ell_2 + \ell_3 = even$ and the triangle inequalities *but otherwise arbitrary* will contribute to the value of the collapsed three-point function and hence also to the skewness. In general a complete probe of the three-point function will require the knowledge of all the coefficients $\mathcal{C}_{\ell_1\ell_2\ell_3}$ and not just the "diagonal" ones, $\mathcal{C}_{\ell\ell\ell}$.

To estimate the amplitude of the non–Gaussian character of the fluctuations one usually considers the "dimensionless" skewness $\mathcal{S}_1 \equiv \langle C_3(0)\rangle / \langle C_2(0)\rangle^{3/2}$. Alternatively, if we want our results to be independent of the normalisation, we may also define the ratio $\mathcal{S}_2 \equiv \langle C_3(0)\rangle / \langle C_2(0)\rangle^2$. Both quantities will be known once the bispectrum has been calculated. Therefore our aim is now to compute $\mathcal{C}_{\ell_1\ell_2\ell_3}$ in the context of slow roll inflation, i.e. in terms of the inflationary slow roll parameters.

In the framework of the stochastic approach to inflation (Starobinski 1986; Goncharov, Linde & Mukhanov 1987), the calculations reported in (Gangui et al 1994) valid for models satisfying slow roll dynamics yield

$$\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3}\rangle = \frac{15}{48\pi}\left[X^2 - 4m_{\rm Pl}X'\right]\left[\mathcal{C}_{\ell_1}\mathcal{C}_{\ell_2} + \mathcal{C}_{\ell_2}\mathcal{C}_{\ell_3} + \mathcal{C}_{\ell_3}\mathcal{C}_{\ell_1}\right]\bar{\mathcal{H}}_{\ell_1\ell_2\ell_3}^{m_1m_2m_3} \ , \tag{9}$$

where in general one requires that the inflaton potential $V(\phi)$ be a smooth function of its argument, which translates into requiring well defined values for the steepness of the potential $X \equiv m_{\rm Pl} V'/V$ (here $' \equiv {\rm d}/{\rm d}\phi$ and $m_{\rm Pl}$ is Planck mass) and its derivatives throughout the range of relevant scales (Turner 1993). Another way of expressing this result is in terms of the standard slow roll parameters (Dodelson et al. 1997; see also Stewart & Lyth 1993; Liddle et al. 1994)

$$\epsilon \equiv \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \quad \text{and} \quad \eta \equiv \frac{m_{\rm Pl}^2}{8\pi}\left[\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right]. \tag{10}$$

For the very large scales we focus on here and for standard chaotic initial conditions in the inflaton field, both parameters satisfy $\epsilon, \eta \ll 1$. In terms of these, a comparison of Eqs. (3) and (9) leads to:

$$\mathcal{C}_{\ell_1\ell_2\ell_3} = \frac{5}{2\sqrt{\pi}}(3\epsilon - 2\eta)\sqrt{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}[\mathcal{C}_{\ell_1}\mathcal{C}_{\ell_2} + \mathcal{C}_{\ell_2}\mathcal{C}_{\ell_3} + \mathcal{C}_{\ell_3}\mathcal{C}_{\ell_1}] \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

This equation shows explicitly the full angular bispectrum arising from generic inflationary models in terms of the slow roll parameters. To be specific in the sequel we will take a potential $V(\phi) \propto \phi^p$, with $p > 1$. Then, after evaluation of the slow roll parameters at Hubble radius crossing for the relevant very large scales one has $(\epsilon, \eta) = (p/(p{+}200), (p{-}2)/(p{+}200))$ corresponding to a scalar spectral index $n = 1-(2p+4)/(p+200)$. The calculations reported in the following sections will be performed for a quadratic potential, namely, $p = 2$.

With this result in mind, we can now re-express the three-point correlation function and skewness. We first need to evaluate the multipole moments $\mathcal{C}_\ell$. For large scales we have $P_\Phi(k) \propto k^{n-4}$ with n corresponding to the primordial index of density fluctuations (e.g., $n = 1$ is the Harrison-Zel'dovich, scale invariant case) in which case (Bond & Efstathiou 1987; Fabbri, Lucchin & Matarrese 1987)

$$\mathcal{C}_\ell = \mathcal{C}_2 \frac{\Gamma(\ell + n/2 - 1/2)\Gamma(9/2 - n/2)}{\Gamma(\ell + 5/2 - n/2)\Gamma(3/2 + n/2)} \equiv \mathcal{C}_2\, \tilde{\mathcal{C}}_\ell, \tag{12}$$

with $\mathcal{C}_2$ related to the quadrupole power spectrum normalization $Q_{\rm rms-PS} = T_0(5\mathcal{C}_2/4\pi)^{1/2}$. Sometimes it turns out to be convenient to factorise the quadrupole amplitude out by using $\tilde{\mathcal{C}}_\ell$. In general, the quadrupole depends on the spectral index n. Thus, normalization analyses of datasets yield the estimate of the pair $(n, Q_{\rm rms-PS})$. For example, the maximum likelihood analysis of the COBE-DMR dataset performed in (Bunn & White 1997) yields $(n, Q_{\rm rms-PS}) = (1.2, 16.2\mu K)$ while their best fit scale invariant normalization that we will use for the numerics in the next section is $Q_{\rm rms-PS} = 18.7\mu K$ (same as in Gorski et al. 1996).

It is now easy to obtain $\mathcal{S}_1$ and $\mathcal{S}_2$ in terms of the slow roll parameters. They are given by

$$\mathcal{S}_1 = \frac{\sqrt{\pi}}{m_{\rm Pl}^2}\sqrt{\frac{3V}{\epsilon}}(3\epsilon-2\eta)\left(\frac{\Gamma(3-n)\Gamma(3/2+n/2)}{\Gamma^2(2-n/2)\Gamma(9/2-n/2)}\right)^{1/2}\mathcal{I}_{3/2}(n), \tag{13}$$

and

$$\mathcal{S}_2 = 15\,(3\epsilon-2\eta)\mathcal{I}_2(n), \tag{14}$$

where the normalization dependence (in $\mathcal{S}_1$) is made explicit by defining the spectral index-dependent geometrical factor

$$\mathcal{I}_q(n) \equiv \frac{\frac{1}{3}\sum_{\ell_1,\ell_2,\ell_3}(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)[\tilde{\mathcal{C}}_{\ell_1}\tilde{\mathcal{C}}_{\ell_2}+\tilde{\mathcal{C}}_{\ell_2}\tilde{\mathcal{C}}_{\ell_3}+\tilde{\mathcal{C}}_{\ell_3}\tilde{\mathcal{C}}_{\ell_1}]\mathcal{W}_{\ell_1}\mathcal{W}_{\ell_2}\mathcal{W}_{\ell_3}\mathcal{F}_{\ell_1\ell_2\ell_3}}{\left[\sum_\ell(2l+1)\tilde{\mathcal{C}}_\ell\mathcal{W}_\ell^2\right]^q}. \tag{15}$$

The exponent q in the denominator takes values 3/2 and 2 for $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The coefficients $\mathcal{F}_{\ell_1\ell_2\ell_3} \equiv (4\pi)^{-2}\int d\Omega_{\hat{\gamma}}\int d\Omega_{\hat{\gamma}'}P_{\ell_1}(\hat{\gamma}\cdot\hat{\gamma}')P_{\ell_2}(\hat{\gamma}\cdot\hat{\gamma}')P_{\ell_3}(\hat{\gamma}\cdot\hat{\gamma}')$ may be suitably expressed in terms of products of factorials of ℓ_1, ℓ_2 and ℓ_3, using standard relations for Wigner 3-j symbols: in fact we have $\mathcal{F}_{k\ell m} = \begin{pmatrix} k & \ell & m \\ 0 & 0 & 0\end{pmatrix}^2$. For the COBE-DMR window function, the numerical factors $\mathcal{I}_q(n)$ in Eq. (15) are of order one for all interesting values of the primordial scalar spectral index. Eqs. (13) to (15) were already presented in a different form in (Gangui et al 1994). Particular cases of these equations have also been displayed in (Kamionkowski & Kosowsky 1999; Verde et al. 1999).

3 BISPECTRUM ESTIMATOR AND ITS VARIANCE

When one particular mechanism for the generation of CMB non–Gaussian features is specified, it is a direct procedure to compute the analytical angular bispectrum. One such example was shown in the previous section in the case of slow roll inflation. However, when dealing with just one realization of a stochastic process, as is the case for the CMB, all computed quantities come with theoretical error bars (Scaramella & Vittorio 1991; Srednicki 1993). Even though we can analytically compute mean values, when an actual observation is made there is a non-vanishing probability that it will fall within a value $\pm\sigma$ apart from the mean. This problem has been dubbed "cosmic variance". To deal with it, one has to introduce an estimator $\hat{E}$ of the quantity e we seek, i.e. a random variable such that $\langle\hat{E}\rangle = e$. In this case the estimator is said to be unbiased. Then one should compute the variance of the estimator, $\sigma_{\hat{E}}$, and try to minimize it. If it turns out that $\sigma_{\hat{E}} = 0$ then we can find e with the help of one realization only (in fact because each realization gives e). In general, we have $\sigma_{\hat{E}} \neq 0$, and one can show that this is linked to the fact that a stochastic process cannot be ergodic on a (celestial) sphere (Grishchuk & Martin 1997). In that case $\sigma_{\hat{E}}$ will express the unavoidable error made when one estimates the mean of a stochastic process from one realization.

For the standard angular spectrum $\mathcal{C}_\ell$ the best unbiased estimator is (Grishchuk & Martin 1997; Tegmark 1997)

$$\hat{f}_\ell = \frac{1}{2\ell+1}\sum_m a_\ell^m a_\ell^{m*} \tag{16}$$

and it is easy to check that it is unbiased, namely $\langle \hat{f}_\ell \rangle = \mathcal{C}_\ell$. Its variance, the smallest one amongst all possible estimators variances, is given by

$$\sigma_{\hat{f}_\ell} = \sqrt{\frac{2}{2\ell+1}}\mathcal{C}_\ell. \tag{17}$$

It is clear that such an optimal strategy should be followed for the bispectrum as well (and for the higher order moments). The following expression

$$\hat{f}_{\ell_1\ell_2\ell_3} = \sum_{m_1,m_2,m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \tag{18}$$

is an unbiased estimator of the bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$ since we easily check that $\langle \hat{f}_{\ell_1\ell_2\ell_3} \rangle = \mathcal{C}_{\ell_1\ell_2\ell_3}$. Its variance (squared)

$$\sigma^2_{\hat{f}_{\ell_1\ell_2\ell_3}} = \langle \hat{f}^2_{\ell_1\ell_2\ell_3} \rangle - \langle \hat{f}_{\ell_1\ell_2\ell_3} \rangle^2 \tag{19}$$

will give us a first indication of the theoretical uncertainties we have to deal with. However, it should be clear that at this level nothing tells us that the one given in Eq. (18) is the *best* unbiased estimator for the bispectrum. As a consequence, working with it might be not the best choice and its variance might well be not the smallest one. Finding the best estimator for the bispectrum is not a trivial task and is presently under investigation (Martin & Gangui 1999). Foregrounds, detector noise, sample variance in the cut sky are among the additional issues that need be mastered before claiming a real non-Gaussian detection.

Recently, similar analyses for the computation of the variance for the estimator of $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \rangle$ were presented (Luo 1994; Heavens 1998). If compared with the analysis of (Luo 1994) note that we are not estimating $a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3}$ but the statistically isotropic combination of Eq. (18) and there the presence of the 3-j symbol makes the whole difference. Positive detection of intrinsic non-Gaussianity in the COBE-DMR four-year dataset was recently suggested (Ferreira et al. 1998; Pando et al. 1998; Bromley & Tegmark 1999). In particular, Ferreira and collaborators, in the attempt to unveil an eventually obscured non-Gaussian signal in real space, worked in the Legendre space and made use of an estimator in the lines of Eq. (18) above but with $\ell_1 = \ell_2 = \ell_3 \equiv \ell$. They also normalized it by dividing $\hat{f}_{\ell\ell\ell}$ by the estimator of $\mathcal{C}_\ell$ [$\hat{f}_\ell$ in the notation of Eq. (16)] to the power 3/2.

We expect departures from Gaussianity to be weak and hence neglect the contribution of $\langle \hat{f}_{\ell_1\ell_2\ell_3}\rangle^2$ to Eq. (19). In this mildly non-Gaussian approximation and after some straight algebra we obtain

$$\sigma^2_{\hat{f}_{\ell_1\ell_2\ell_3}} = \mathcal{C}_{\ell_1}\mathcal{C}_{\ell_2}\mathcal{C}_{\ell_3}(1 + \delta_{\ell_1\ell_2} + \delta_{\ell_2\ell_3} + \delta_{\ell_3\ell_1} + 2\ \delta_{\ell_1\ell_2}\delta_{\ell_2\ell_3}), \tag{20}$$

where we demanded $\ell_1 + \ell_2 + \ell_3 = even$ (otherwise $\mathcal{C}_{\ell_1\ell_2\ell_3} = 0$) and $\ell_i \neq 0$, what considerably simplified the resulting expression. We note in passing that for the above computation it is useful to recall the identity (Mollerach et al 1995)

$$\sum_{m=-\ell}^{\ell} (-1)^m \begin{pmatrix} \ell & \ell & 2k \\ -m & m & 0 \end{pmatrix} = (-1)^\ell \sqrt{2\ell+1}\ \delta_{k,0}. \tag{21}$$

We are now in a position to compare the signal, i.e. the bispectrum given in Eq. (11), with the theoretical noise characterized by (20).

4 DISCUSSION AND CONCLUSIONS

In the previous sections we have computed both the expression for the angular bispectrum, as obtained generically in the framework of slow roll inflation whenever one goes beyond the linear order, and the variance associated with an unbiased estimator, assuming a mildly non-Gaussian process. We can now compare these results for an arbitrary configuration of ℓ_i multipoles. As a representative example, and given the fact that this was actually the case considered in the literature, we consider $\ell_1 = \ell_2 = \ell_3 \equiv \ell = even$. We then have the bispectrum

$$\mathcal{C}_{\ell\ell\ell} = \frac{15}{2\sqrt{\pi}}(3\epsilon - 2\eta)(2\ell+1)^{3/2}\ \mathcal{C}_\ell^2 \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

while the variance is now given by

$$\sigma_{\hat{f}_{\ell\ell\ell}} = \sqrt{6}\ \mathcal{C}_\ell^{3/2}. \tag{23}$$

We show the relative amplitudes in Fig.1. The plot allows us to judge how plausible it is for generic one-field inflationary models to reproduce any possible non–Gaussian structure found on large angular scales, in particular on the COBE-DMR dataset. Single different values (correlations) for the indices ℓ_1, ℓ_2, ℓ_3 (satisfying $\ell_1 + \ell_2 + \ell_3 = even$ and the triangle inequalities) can be tried with similar result. Leaving aside for the time being the possibility

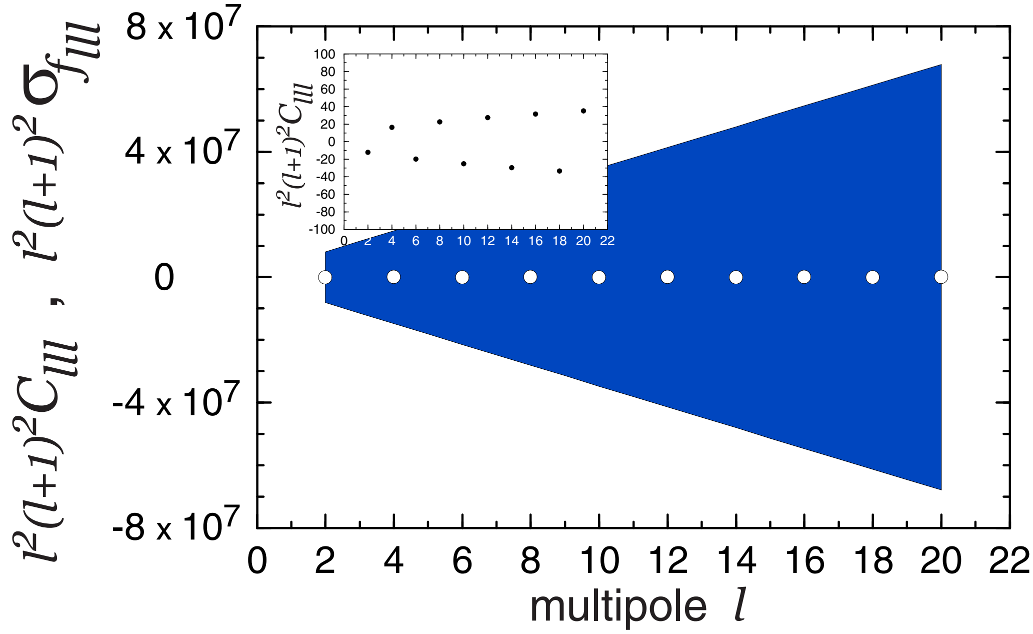

Figure 1. Normalised angular bispectrum $\mathcal{C}_{\ell\ell\ell}$ as predicted by a generic slow roll inflation model, in units of $\mathcal{C}_2^2$, with $\mathcal{C}_2 = 1.18 \times 10^{-10}$ related to the quadrupole power spectrum normalization $Q_{\mathrm{rms-PS}} = 18.7\mu K$, as a function of the multipole index ℓ for all even values up to $\ell = 20$ (white dots in main plot). Grey band corresponds to the normalized variance $\sigma_{\hat{f}_{\ell\ell\ell}}$ (also in units of $\mathcal{C}_2^2$) associated to the estimator of Eq. (18). In the inset we zoom up $\ell^2(\ell+1)^2\mathcal{C}_{\ell\ell\ell}$ in the same units, which permits to see the alternating sign of the normalised bispectrum and its actual smooth increase in amplitude with increasing ℓ.

of foreground contamination and assuming any non–Gaussian signal is intrinsic to the CMB, we should conclude that the presently considered class of models cannot explain it.

In order to be more specific, we now turn to the study of the "signal-to-noise" ratio defined by the following expressions

$$\left(\frac{S}{N}\right)_{2,\ell} \equiv \frac{\mathcal{C}_\ell}{\sigma_{\hat{f}_\ell}}, \qquad \left(\frac{S}{N}\right)_{3,\ell} \equiv \frac{\mathcal{C}_{\ell\ell\ell}}{\sigma_{\hat{f}_{\ell\ell\ell}}}, \tag{24}$$

for the angular spectrum and bispectrum respectively. For the first one [neglecting specificities of the particular experiment, detector sensitivity, pixelization, etc. (Knox 1995)] the following well-known behaviour is found

$$\left(\frac{S}{N}\right)_{2,\ell} = \sqrt{\frac{2\ell+1}{2}}. \tag{25}$$

This means that the signal emerges from the noise while going towards big values of ℓ, accounting for the fact that the cosmic variance is important at large scales only. For the bispectrum, one has

$$\left(\frac{S}{N}\right)_{3,\ell} = \frac{15}{2\sqrt{6\pi}}(3\epsilon - 2\eta)(2\ell+1)^{3/2}\mathcal{C}_\ell^{1/2}\begin{pmatrix}\ell & \ell & \ell\\ 0 & 0 & 0\end{pmatrix}, \tag{26}$$

and we see that, contrary to the previous case, the signal-to-noise ratio depends on $\mathcal{C}_\ell$ and on the slow roll parameters. The behaviour of $|(S/N)_{3,\ell}|$ is displayed in Fig. 2 where we see

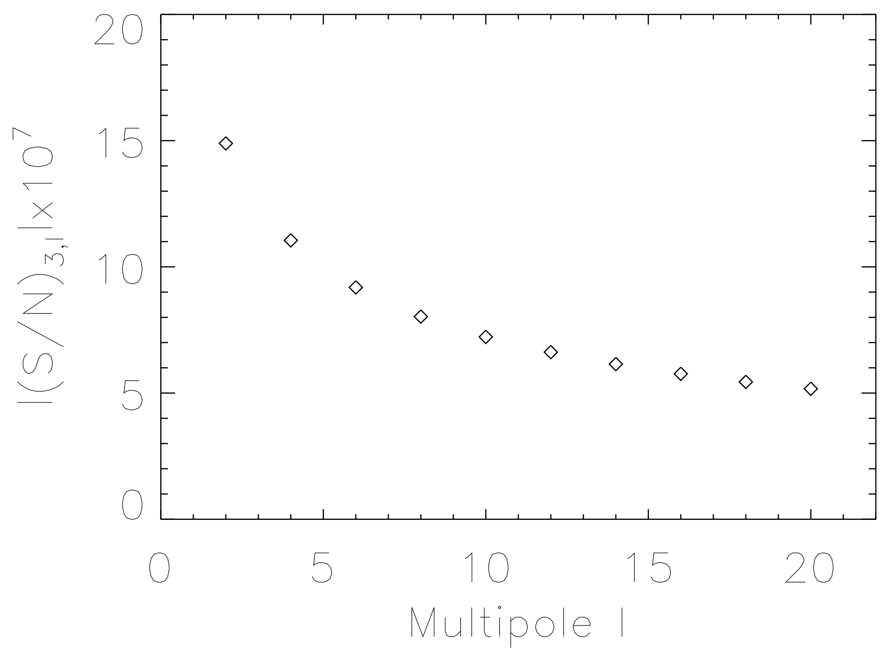

Figure 2. Absolute value of the signal-to-noise ratio for the bispectrum $(S/N)_{3,\ell}$ versus the multipole index ℓ.

that it diminishes in absolute value with increasing ℓ. This behaviour is easily understood once we look at the hierarchical form of $\mathcal{C}_{\ell_1\ell_2\ell_3}$ and compare with the $\mathcal{C}_\ell$ dependence of the variance. Again, even restricting ourselves to small ℓ's, we see that the presence of the Wigner 3-j symbol is the responsible for this particular behaviour. Hence, approaching the largest multipoles in the COBE-DMR data set (in particular $\ell = 16$) the situation gets worse. In fact, from Eq. (26) we roughly have $\mathcal{C}_\ell \propto \ell^{-1}(\ell+1)^{-1}$ and $\ell \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}$ almost constant with ℓ (see Fig. 3), and then $|(S/N)_{3,\ell}| \propto \ell^{-1/2}$, while $= |(S/N)_{2,\ell}| \propto \ell^{1/2}$ in the same range of validity. Note that the above analysis should be supplemented by a similar one wherein the behaviour of the bispectrum coming from the post recombination Rees-Sciama effect is also considered. However, given the smallness of the non-Gaussian signal this should not modify our conclusions very much.

To conclude, let us emphasize again the main results obtained in this article. We have stressed that a real detection of non-Gaussianity in the CMB would imply that an important component of the anisotropies arises from processes *other* than quantum fluctuations during an early inflationary epoch. This notwithstanding, inflation predicts an actual generic form for the bispectrum, Eq.(11), and we here showed it explicitly in terms of the slow roll

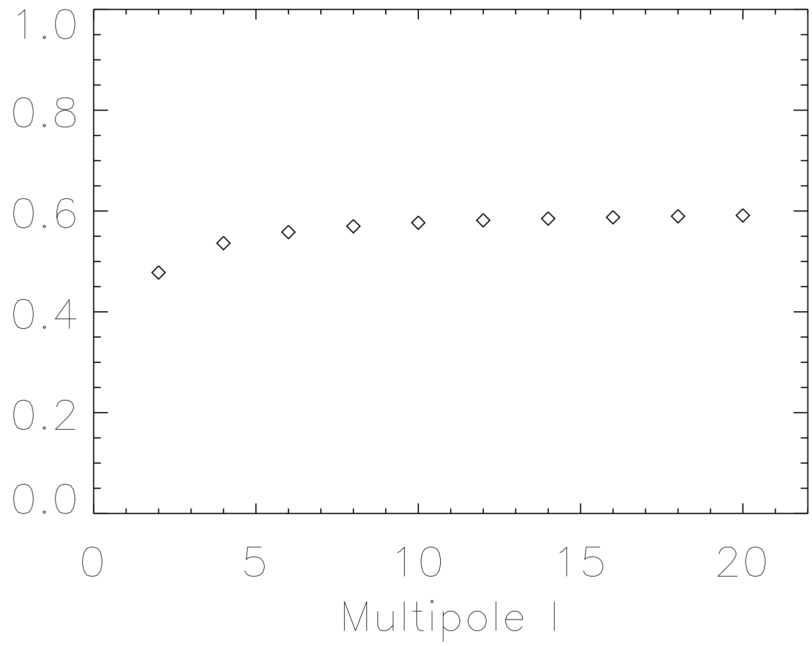

Figure 3. Absolute value of the product $\ell \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}$ for different ℓ's. Note that the product vanish for all $\ell = odd$.

parameters. We also computed for the first time the variance of one candidate estimator for the bispectrum often employed in the literature and showed that the signal is drowned in it. Contrary to the standard spectrum case, one cannot hope to palliate at least somewhat this problem by going to higher values of ℓ (always within the small-ℓ region) since the signal-to-noise ratio decreases with ℓ like $\ell^{-1/2}$. However, it should also be stressed that this conclusion might well be weaken by the finding of the actual best unbiased estimator. We hope to address this question elsewhere.

Note added: After the submission of this paper, a preprint by Wang and Kamionkowski (`astro-ph/9907431`) appeared in which similar conclusions were reached.

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REFERENCES

Aghanim N. and Forni O., preprint `astro-ph/9905124`.

Allen B. et al., Phys. Rev. Lett. **79**, 2624 (1997).

Avelino P. P. et al, Astrophys. J. Lett., preprint `astro-ph/9803120`.

Banday A. J., Zaroubi S. and Górski K. M., preprint `astro-ph/9908070`.

Bardeen J. M., Steinhardt P. J. and Turner M. S., Phys. Rev. D **28**, 679 (1983).

Battye R. A., Robinson J. and Albrecht A., Phys. Rev. Lett. **80**, 4847 (1998).

Bersanelli M. et al., Planck red book (1996), WWW page:

`http://astro.estec.esa.nl/SA-general/Projects/Planck/`

Bond J. R. and Efstathiou G., MNRAS **226**, 655 (1987).

Bouchet F. R., Bennett D. P. and Stebbins A., Nature **335**, 410 (1988).

Bromley B. C. and Tegmark M., preprint `astro-ph/9904254`.

Bunn E. F. and White M., Astrophys. J **480**, 6 (1997).

Dodelson S., Kinney W. H. and Kolb E. W., Phys. Rev. D **56**, 3207 (1997).

Durrer R., Gangui A. and Sakellariadou M., Phys. Rev. Lett. **76**, 579 (1996).

Contaldi C., Hindmarsh M. and Magueijo J., Phys. Rev. Lett. **82**, 2034 (1999).

Fabbri R., Lucchin F. and Matarrese S., Astrophys. J **315**, 1 (1987).

Falk T., Rangarajan R. and Srednicki M., Astrophys. J. **403**, L1 (1993).

Ferreira P. G., Magueijo J. and Górski K. M., Astrophys. J **503**, L1 (1998).

Gangui A., Phys. Rev. D. **50**, 3684 (1994).

Gangui A., Lucchin F., Matarrese S. and Mollerach S., Astrophys. J **430**, 447 (1994).

Gangui A. and Mollerach S., Phys. Rev. D. **54**, 4750 (1996).

Gangui A. and Perivolaropoulos L., Astrophys. J. **447**, 1 (1995).

Goldberg D. and Spergel D., Phys.Rev. D **59**, 103002 (1999).

Goncharov A. S., Linde A. D. and Mukhanov V. F., Int. J. Mod. Phys. A **2**, 561 (1987).

Gorski K. M. et al, Astrophys. J. Lett. **464**, 11 (1996).

Grishchuk L. P., Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [Sov. Phys. JETP **40**, 409 1975)].

Grishchuk L. P. and Martin J., Phys. Rev. D **56**, 1924 (1997).

Guth A., Phys. Rev. D **23**, 347 (1981).

Guth A. and Pi S. Y., Phys. Rev. Lett. **49**, 1110 (1982).

Hawking S. W., Phys. Lett. B **115**, 295 (1982).

Heavens A. F., MNRAS **299**, 805 (1998).

Hinshaw G. et al., Astrophys. J **431**, 431 (1994).

Hinshaw G. et al., Astrophys. J **446**, L67 (1995).

Kamionkowski M. and Kosowsky A., preprint `astro-ph/9904108`.

Knox L., Phys. Rev. D **52**, 4307 (1995).

Kogut A. et al, Astrophys. J **464**, L29 (1996).

Lange L. et al., BOOMERanG WWW page:
`http://astro.caltech.edu/~lgg/boom/boom.html`

Lee A. T. et al, preprint astro-ph/9903249, MAXIMA WWW page:
`http://cfpa.berkeley.edu/group/cmb/gen.html`

Liddle A. R., Parsons P. and Barrow J. D, Phys. Rev. D **50**, 7222 (1994).

Linde A. and Mukhanov V., Phys. Rev. D **56**, 535 (1997).

Luo X., Astrophys. J **427**, L71 (1994).

Luo X. and Schramm D.N., Phys. Rev. Lett. **71**, 1124 (1993).

Magueijo J., Albrecht A., Coulson D. and Ferreira P., Phys. Rev. Lett. **76**, 2617 (1996).

MAP WWW page: `http://map.gsfc.nasa.gov/`

Martin J. and Gangui A., in preparation.

Martin J., Riazuelo A. and Sakellariadou M., preprint `astro-ph/9904167`.

Mather J. C. et al., Astrophys. J **420**, 439 (1994).

Messiah A., *Quantum Mechanics*, Vol. 2, Amsterdam, North-Holland.

Mollerach S., Gangui A., Lucchin F. and Matarrese S., Astrophys. J **453**, 1 (1995).

Munshi D., Souradeep T. and Starobinsky A. A., Astrophys. J. **454**, 552 (1995).

Pando J., Valls-Gabaud D. and Fang L., Phys. Rev. Lett. **81**, 4568 (1998).

Pen U., Seljak U. and Turok N., 1997, Phys. Rev. Lett. **79**, 1611.

Pogosian L. and Vachaspati T., Phys. Rev. D **60**, 083504 (1999).

Scaramella R. and Vittorio N., Astrophys. J. **375**, 439 (1991).

Smoot G. F. et al., Astrophys. J **437**, 1 (1994).

Spergel D. and Goldberg D., Phys.Rev. D **59**, 103001 (1999).

Srednicki M., Astrophys. J. **416**, L1 (1993).

Starobinsky A. A., Phys. Lett. B **117**, 175 (1982).

Starobinsky A. A, *Field Theory, Quantum Gravity and Strings*, ed. H. J. de Vega & N. Sanchez (Lectures Notes in Physics, Vol. 246, Berlin Springer Verlag) (1986).

Stewart E. W. and Lyth D. H, Phys. Lett. B **302**, 171 (1993).

Tegmark M., Phys. Rev. D **55**, 5895 (1997).

Torres S., Cayón L., Martínez-González E. and Sanz J.L., MNRAS **274**, 853 (1995).

Turner M. S., Phys. Rev. D **48**, 3502 (1993); ibid 5539.

Unruh W., preprint `astro-ph/9802323`.

Verde L., Wang L., Heavens A. and Kamionkowski M., preprint `astro-ph/9906301`.