Fleming-Viot selects the minimal quasi-stationary distribution: The Galton-Watson case.

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Abstract Consider N particles moving independently, each one according to a subcritical continuous-time Galton-Watson process unless it hits 0, at which time it jumps instantaneously to the position of one of the other particles chosen uniformly at random. The resulting dynamics is called Fleming-Viot process. We show that for each N there exists a unique invariant measure for the Fleming-Viot process, and that its stationary empirical distribution converges, as N goes to infinity, to the minimal quasi-stationary distribution of the Galton-Watson process conditioned on non-extinction.

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1 Introduction

The concept of *quasi-stationarity* arises in stochastic modeling of population dynamics. In 1947, Yaglom [27] considers subcritical Galton-Watson processes conditioned to survive long times. He shows that as time is sent to infinity, the conditioned process, started with one individual, converges to a law, now called a *quasi-stationary distribution.* For any Markov process, and a subset A of the state space, we denote by μT_t the law of the process at time t conditioned on not having hit A up to time t, with initial distribution μ. A probability measure on A^c is called *quasi-stationary distribution* if it is a fixed point of T_t for any $t > 0$.

In 1966, Seneta and Veres-Jones [25] realize that for subcritical Galton-Watson processes, there is a one-parameter family of quasi-stationary distributions and show that the Yaglom limit distribution has the minimal expected time of extinction among all quasi-stationary distributions. This unique *minimal* quasi-stationary distribution is denoted here ν^*_{qs}. They also show that with an initial distribution μ with finite first moment, μT_t converges to ν^*_{qs} as t goes to infinity.

In 1978, Cavender [13] shows that for Birth and Death chains on the non negative integers absorbed at 0, the set of quasi-stationary measures is either empty or is a one parameter family. In the latter case, Cavender extends the *selection principle* of Seneta and Veres-Jones. He also shows that the limit of the sequence of quasi-stationary distributions for truncated processes on $\{1, \dots, L\}$ converges to ν^*_{qs} as L is sent to infinity. This picture holds for a class of irreducible Markov processes on the non-negative integers with 0 as absorbing state, as shown in 1996 by Ferrari, Kesten, Martinez and Picco [16]. The main idea in [16] is to think of the conditioned process μT_t as a mass transport with refeeding from the absorbing state to each of the transient states with a rate proportional to the transient state mass. More precisely, denoting $\mathbb{N}$ the set of positive integers, the Kolmogorov forward equation satisfied by $\mu T_t(x)$, for each $x \in \mathbb{N}$, reads

$$\frac{\partial}{\partial t}\mu T_t(x) = \sum_{y:y\neq x} \big(q(x,y) + q(x,0)\mu T_t(y)\big)\,[\mu T_t(y) - \mu T_t(x)], \tag{1.1}$$

where $q(x,y)$ is the jump rate from x to y. The first term in the right hand side represents the displacement of mass due to the jumps of the process and the second term represents the mass going from each x to 0 and then coming instantaneously to y.

In 1996, Burdzy, Holyst, Ingerman and March [11] introduced a *genetic* particle system called *Fleming-Viot* named after models proposed in [18], which can be seen as a particle system mimicking the evolution (1.1). The particle system can be built from a process with absorption Z_t called *driving process*; the position Z_t is interpreted as a genetic trait, or fitness, of an individual at time t. In the N-particle Fleming-Viot system, each trait follows independent dynamics with the same law as Z_t except when one of them hits state 0, a lethal trait: at this moment the individual adopts the trait of one of the other individuals chosen uniformly at random. Leaving aside the genetic interpretation, the empirical distribution of the N particles at positions $\xi \in \mathbb{N}^N$ is defined as a function $m(\cdot,\xi) : \mathbb{N} \to [0,1]$ by

$$\forall x \in \mathbb{N}, \qquad m(x,\xi) := \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{\xi(i)=x\}}. \tag{1.2}$$

The generator of the Fleming-Viot process with N particles applied to bounded functions $f : \mathbb{N}^N \to \mathbb{R}$ reads

$$\mathcal{L}^N f(\xi) = \sum_{i=1}^{N}\sum_{y=1}^{\infty}\Big[q(\xi(i),y) + q(\xi(i),0)\,\tfrac{N}{N-1}\,m(y,\xi)\Big]\,[f(\xi^{i,y}) - f(\xi)], \tag{1.3}$$

where $\xi^{i,y}(i) = y$, and for $j \neq i$, $\xi^{i,y}(j) = \xi(j)$ and $q(x,y)$ are the jump rates of the driving process. Assume that the driving process has a unique quasi-stationary distribution, called ν_{qs} and that the associated N-particle Fleming-Viot system has an invariant measure λ^N. The main conjecture in [11, 12] is that assuming ξ has distribution λ^N, the law of the random measure $m(.,\xi)$ converges to the law concentrated on the constant ν_{qs}. This was proven for diffusion processes on a bounded domain of $\mathbb{R}^d$, killed at the boundary [5, 19, 20, 26], for jump processes under a Doeblin condition [17] and for finite state jump processes [1].

The subcritical Galton-Watson process has infinitely many quasi-stationary distributions. Our theorem proves that the stationary empirical distribution $m(\cdot,\xi)$ converges to ν^*_{qs}, the minimal quasi-stationary distribution. This phenomenon is a *selection principle.*

Theorem 1.1. *Consider a subcritical Galton-Watson process whose offspring law has some finite positive exponential moment. Let ν^*_{qs} be the minimal quasi-stationary distribution for* *the process conditioned on non-extinction. Then, for each $N \geq 1$, the associated N-particle Fleming-Viot system is ergodic. Furthermore, if we call its invariant measure λ^N, then*

$$\forall x \in \mathbb{N}, \qquad \lim_{N\to\infty} \int |m(x,\xi) - \nu^*_{\text{qs}}(x)|\, d\lambda^N(\xi) \;=\; 0. \tag{1.4}$$

A simple consequence is propagation of chaos. For any finite set $S \subset \mathbb{N}$,

$$\lim_{N\to\infty} \int \prod_{x\in S} m(x,\xi)\; d\lambda^N(\xi) = \prod_{x\in S} \nu^*_{\text{qs}}(x). \tag{1.5}$$

The strategy for proving Theorem 1.1 is explained in the next section, but there are two key steps in the proof. First, we control the position of the rightmost particle. Let

$$R(\xi) \;:=\; \max_{i\in\{1,\ldots,N\}} \xi(i),$$

be the position of the rightmost particle of ξ. Let ξ^ξ_t the positions at time t of the N Fleming-Viot particles, initially on ξ.

Proposition 1.2. *There is a time T and positive constants A, c_1, c_2, C and ρ, independent of N, such that for any $\xi \in \mathbb{N}^N$*

$$E\big(\exp\big(\rho R(\xi^\xi_T)\big)\big) - \exp\big(\rho R(\xi)\big) \;<\; -c_1\, e^{\rho R(\xi)} \mathbf{1}_{R(\xi)>A} + N c_2 e^{-CR(\xi)}. \tag{1.6}$$

As a consequence, for each N there is a unique invariant measure λ^N for the N-particle Fleming-Viot system. Furthermore, there is a constant $\kappa > 0$ such that for any N,

$$\int \exp(\rho R(\xi)) d\lambda^N(\xi) \;\leq\; \kappa N. \tag{1.7}$$

The second result is that the ratio between the second and the first moment of the empirical distribution plays the role of a Lyapunov functional, given that the position of the rightmost particle is not too large. For a particle configuration ξ define

$$\psi(\xi) := \frac{\sum_{1\leq i\leq N} \xi^2(i)}{\sum_{1\leq i\leq N} \xi(i)}. \tag{1.8}$$

Recall $\mathcal{L}^N$ is the Fleming-Viot generator given by (1.3).

Proposition 1.3. *There are positive constants v, C_1 and C_2 independent of N such that*

$$\mathcal{L}^N \psi(\xi) \;\leq\; -v\psi(\xi) + C_1 \frac{R^2(\xi)}{N} + C_2. \tag{1.9}$$

Propositions 1.2 and 1.3 imply that the expectation of ψ under the invariant measure λ^N is uniformly bounded in N.

Corollary 1.4. *There is a positive constant C such that for all N,*

$$\int \psi(\xi)\, d\lambda^N(\xi) \;\leq\; C. \tag{1.10}$$

There are several related works motivated by genetics. Brunet, Derrida, Mueller and Munier [9, 10] introduce a model of evolution of a population with selection. They study the genealogy of genetic traits, the empirical measure, and link the evolution of the barycenter with F-KPP equation $\partial_t u = \partial_x^2 u - u(1-u)$ introduced in 1937 by R.A. Fisher to describe the evolution of an advantageous gene in a population. These authors also discover an exactly soluble model whose genealogy is identical to those predicted by Parisi's theory of mean-field spin glasses. Durrett and Remenik [14] establish propagation of chaos for a related continuous-space and time model, and then show that the limit of the empirical measure is characterized as the solution of a free-boundary integro-differential equation. Bérard and Gouéré [3] establish a conjecture of Brunet and Derrida for the speed of the rightmost particle for still a third microscopic model of F-KPP equation introduced in [7, 8]. Maillard [21] obtains the precise behavior of the empirical measure of an approximation of the same model, building on the results of Berestycki, Berestycki and Schweinsberg [4], which establish the genealogy picture described in [7, 8].

We now mention two open problems. The first is to solve the analogous to Theorem 1.1 for a random walk with a constant drift toward the origin. The second is to obtain propagation of chaos directly on the stationary empirical measure, with a bound of order $1/N$.

In the next section, we describe our model, sketch the proof of our main result and describe the organization of the paper.

2 Notation and Strategy

Let $\sigma > 0$ and p be a probability distribution on $\mathbb{N} \cup \{0\}$ such that

$$\sum_{\ell \geq 0} p(\ell)\, e^{\sigma \ell} \; < \; \infty. \tag{2.1}$$

Consider a Galton-Watson process $Z_t \in \mathbb{N} \cup \{0\}$ with offspring law p. Each individual lives an exponential time of parameter 1, and then gives birth to a random number of children with law p. We assume that Galton-Watson is subcritical, that is we ask p to satisfy

$$-v \; := \sum_{\ell \geq -1} \ell p(\ell+1) \; < \; 0. \tag{2.2}$$

In other words, the drift when $Z_t = x$ is $-vx < 0$. For distinct $x, y \in \mathbb{N} \cup \{0\}$, the rates of jump are given by

$$q(x,y) := \begin{cases} xp(0), & \text{if } y = x-1 \geq 0, \\ xp(y-x+1), & \text{if } y > x \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

The Galton-Watson process starting at x is denoted Z_t^x. For a distribution μ on $\mathbb{N}$, the law of the process starting with μ conditioned on non-absorption until time t is given by

$$\mu T_t(y) := \frac{\sum_{x \in \mathbb{N}} \mu(x) p_t(x,y)}{\sum_{x,z \in \mathbb{N}} \mu(x) p_t(x,z)}, \tag{2.4}$$

where $p_t(x,y) = P(Z_t^x = y)$.

Recall that ξ_t^ξ denotes the Fleming-Viot system with generator (1.3) and initial state ξ; $\xi_t(i)$ denotes the position of the i-th particle at time t. For a real $\alpha > 0$ define $K(\alpha)$ as the subset of distributions on $\mathbb{N}$ given by

$$K(\alpha) := \left\{ \mu : \frac{\sum_{x\in\mathbb{N}} x^2 \mu(x)}{\sum_{x\in\mathbb{N}} x\mu(x)} \le \alpha \right\}. \tag{2.5}$$

Observe that $\mu \in K(\alpha)$ implies $\sum x\mu(x) \le \alpha$.

Proof of Theorem 1.1. The existence of the unique invariant measure λ^N for Fleming-Viot is given in Proposition 1.2.

To show (1.4) we use the invariance of λ^N and perform the following decomposition.

$$\begin{aligned}
\int \big|m(x,\xi) - \nu_{\rm qs}^*(x)\big|\, d\lambda^N(\xi) \;&=\; \int E\big|m(x,\xi_t^\xi) - \nu_{\rm qs}^*(x)\big|\, d\lambda^N(\xi) \\
&\le\; \lambda^N(\psi > \alpha) + \int_{\psi\le\alpha} E\big|m(x,\xi_t^\xi) - \nu_{\rm qs}^*(x)\big|\, d\lambda^N(\xi) \\
&\le\; \lambda^N(\psi > \alpha) + \int_{\psi\le\alpha} E\big|m(x,\xi_t^\xi) - m(\cdot,\xi)T_t(x)\big|\, d\lambda^N(\xi) + \int_{\psi\le\alpha} \big|m(\cdot,\xi)T_t(x) - \nu_{\rm qs}^*(x)\big|\, d\lambda^N(\xi) \\
&\le\; \lambda^N(\psi > \alpha) + \sup_{\xi:\psi(\xi)\le\alpha} \big|m(\cdot,\xi)T_t(x) - \nu_{\rm qs}^*(x)\big| + \sup_{\xi:\psi(\xi)\le\alpha} E\big|m(x,\xi_t^\xi) - m(\cdot,\xi)T_t(x)\big|,
\end{aligned} \tag{2.6}$$

where ψ is defined in (1.8). We bound the three terms of the last line of (2.6).

First term. Corollary 1.4 and Markov inequality imply that there is a constant $C > 0$ such that for any $\alpha > 0$

$$\lambda^N(\psi > \alpha) \le \frac{C}{\alpha}. \tag{2.7}$$

Second term. Note that $\psi(\xi) < \alpha$ if and only if $m(\cdot,\xi) \in K(\alpha)$. The Yaglom limit converges to the minimal quasi-stationary distribution $\nu_{\rm qs}^*$, uniformly in $K(\alpha)$ as we show later in Proposition 7.2:

$$\lim_{t\to\infty} \sup_{\mu\in K(\alpha)} |\mu T_t(x) - \nu_{\rm qs}^*(x)| = 0. \tag{2.8}$$

Third term. We perform the decomposition

$$E\big|m(x,\xi_t^\xi) - m(\cdot,\xi)T_t(x)\big| \;\le\; E\big|m(x,\xi_t^\xi) - Em(x,\xi_t^\xi)\big| + \big|Em(x,\xi_t^\xi) - m(\cdot,\xi)T_t(x)\big|, \tag{2.9}$$

and show that there exist positive constants C_1 and C_2 such that

$$\sup_{\xi\in\mathbb{N}^N} E\big|m(x,\xi_t^\xi) - Em(x,\xi_t^\xi)\big| \;\le\; \frac{C_1 e^{C_2 t}}{\sqrt{N}} \tag{2.10}$$

and

$$\sup_{\xi\in\mathbb{N}^N} \big|Em(x,\xi_t^\xi) - m(\cdot,\xi)T_t(x)\big| \;\le\; \frac{C_1 e^{C_2 t}}{N}, \tag{2.11}$$

for all N, see Proposition 8.1 later. The issue here is a uniform bound for the correlations of the empirical distribution of Fleming-Viot at sites $x, y \in \mathbb{N}$ at fixed time t. This was carried out in [1].

To show (1.4), it suffices to bound the three terms in the bottom line of (2.6). Choose α large and use (2.7) to make the first term small (uniform in N). Use (2.8) to choose t large to make the second term small. For this fixed time, take N large and use (2.9), (2.10) and (2.11) to make the third term small. ∎

The rest of the paper is organized as follows. In Section 3, we perform the graphical construction of Fleming-Viot jointly with a Multitype Branching Markov Chain. In Section 4 we obtain large deviation estimates for the Galton-Watson process. In Section 5 we obtain large deviation estimates for the rightmost particle of the Fleming-Viot system. In Section 6 we study the Lyapunov-like functional and prove Proposition 1.3 and Corollary 1.4. Convergence of the conditional evolution uniformly on $K(\alpha)$ is proved in Section 7. Finally, (2.10)-(2.11) are handled in Proposition 8.1 of Section 8.

3 Embedding Fleming-Viot on a multitype branching Markov process.

In this section we construct a coupling between the Fleming-Viot system and an auxiliary multitype branching Markov process (hereafter, the branching process). We call *particles* the Fleming-Viot positions and *individuals* the branching positions. Each individual has a a type in $\{1, \dots, N\}$ and a position in $\mathbb{N}$.

When a particle chooses the position of another particle and jumps to it, the process builds correlations making difficult to control the position of the rightmost particle. In our coupling when a particle jumps, either an individual jumps at the same time or a branching occurs at the site where the particle arrives. In this way the particles always stay at sites occupied by individuals and the maximum particle position is dominated by the position of the rightmost individual (if this is so at time zero). This, in turn, is dominated by the sum of the individual positions which we control.

The coupling relies on the Harris construction of Markov processes: the state of the process at time t is defined as a function of the initial configuration and a family of independent Poisson processes in the time interval $[0, t]$. The coupling holds when the driving process is a Markov process with rates $\{q(x, y), x, y \in \mathbb{N} \cup \{0\}\}$ with 0 being the absorbing state and $\bar{q} := \sup_x q(x, 0) < \infty$.

There are two types of jumps of the Fleming-Viot particle i. Those due to the spatial evolution at rate $\tilde{q}$ and those due to "jumps to zero and then to the position of particle j chosen uniformly at random" at rate $q(x, 0)/(N-1)$.

Spatial evolution. Each individual has a position in $\mathbb{N}$ which evolves independently with transition rates $(\tilde{q}(x, y), x, y \in \mathbb{N})$ defined by $\tilde{q}(x, y) := q(x, y)\mathbf{1}_{\{y \neq 0\}}$ so that there are no jumps to zero. The spatial evolution of new individuals born at branching times are independent and with the same rates $\tilde{q}$. Under our coupling, each spatial jump performed by the i-particle is also performed by some i-individual.

The refeeding and branching. At rate $\bar{q}/(N-1)$, each j-individual branches into two new individuals, one of type j and one of type i; each new born i-individual takes the position of the corresponding j-individual and then evolves independently with rates $\tilde{q}$. If the i-particle is at x, at rate $q(x,0)/(N-1)$ it jumps to the position of the j-particle. Under our coupling, each time particle i chooses particle j, each j-individual branches into an i and a j-individual. In this way, the i-particle occupies always the site of some i-individual.

The branching process has state space

$$\mathcal{B} := \Big\{ \zeta \in \mathbb{N}^{\{1,\dots,N\}\times\mathbb{N}} : \sum_{i=1}^{N} \sum_{x\in\mathbb{N}} \zeta(i,x) < \infty \Big\}$$

For $i \in \{1,\dots,N\}$, $x \in \mathbb{N}$, $\zeta_t(i,x)$ indicates the number of individuals of type i at site x at time t. Let $\delta_{(i,x)} \in \mathcal{B}$ be the delta function on (i,x) defined by $\delta_{(i,x)}(i,x) = 1$ and $\delta_{(i,x)}(j,y) = 0$ for $(j,y) \neq (i,x)$. The rates corresponding to the (independent) spatial evolution of the individuals at x are

$$b(\zeta, \zeta + \delta_{(i,y)} - \delta_{(i,x)}) = \zeta(i,x) q(x,y), \qquad i \in \{1,\dots,N\},\ x,y \in \mathbb{N},$$

and those corresponding to the branching of all j-individuals into an individual of type j and an individual of type i are

$$b\Big(\zeta, \zeta + \sum_{x\in\mathbb{N}} \zeta(j,x)\delta_{(i,x)}\Big) = \frac{\bar{q}}{N-1}, \qquad i \neq j \in \{1,\dots,N\}$$

Note that the new born i-individuals get the spatial position of the corresponding j-individual.

Harris construction of the branching process Let $(\mathcal{N}(i,x,y,k),\ i \in \{1,\dots,N\}, x,y \in \mathbb{N}, k \in \mathbb{N})$ be a family of Poisson processes with rates $k\tilde{q}(x,y)$ such that $\mathcal{N}(i,x,y,k) \subset \mathcal{N}(i,x,y,k+1)$ for all k; we think a Poisson process as a random subset of $\mathbb{R}$. The process $\mathcal{N}(i,x,y,k)$ is used to produce a jump of an i-individual from x to y when there are k i-individuals at site x. The families $(\mathcal{N}(i,x,y,k), k \geq 1)$ are taken independent. Let $(\mathcal{N}(i,j),\ i \neq j)$, be a family of independent Poisson processes of rate $\bar{q}/(N-1)$, these processes are used to branch all j-individuals into an i-individual and a j-individual. The two families are taken independent.

Fix $\zeta_0 = \zeta \in \mathcal{B}$, assume the process is defined until time $s \geq 0$ and proceed by recurrence.

1. Define $\tau(\zeta_s, s) := \inf\{t > s : t \in \cup_{i,x,y}\mathcal{N}(i,x,y,\zeta_s(i,x)) \cup \cup_{i,j}\mathcal{N}(i,j)\}$.
2. For $t \in [s,\tau)$ define $\zeta_t = \zeta_s$.
3. If $\tau \in \mathcal{N}\big(i,x,y,\zeta_s(i,x)\big)$ then set $\zeta_\tau = \zeta_s + \delta_{(i,y)} - \delta_{(i,x)}$.
4. If $\tau \in \mathcal{N}(i,j)$ then set $\zeta_\tau = \zeta_s + \sum_{x\in\mathbb{N}} \zeta_s(j,x)\delta_{(i,x)}$.

The process is then defined until time τ. Put $s = \tau$ and iterate to define ζ_t for all $t \geq 0$. Denote ζ_t^ζ the process with initial state ζ. We leave the reader to prove that ζ_t^ζ so defined is the branching process, that is, a Markov process with rates b and initial state ζ.

Let $|\zeta| := \sum_{i,x} \zeta(i,x)$ be the total number of individuals in ζ. Let

$$R(\zeta) := \max\Big\{x : \sum_i \zeta(x,i) > 0\Big\}.$$

Let $\widetilde{Z}_t^z$ be the process on $\mathbb{N}$ with rates $\tilde{q}$ and initial position $z \in \mathbb{N}$.

Lemma 3.1. $E|\zeta_t^\zeta| \;=\; |\zeta|\, e^{\bar{q}t}$.

Proof. $E|\zeta_t|$ satisfies the equation

$$\frac{d}{dt}E|\zeta_t| \;=\; \frac{\bar{q}}{N-1}E\Big(\sum_i \sum_{j:j\neq i} \sum_x \zeta_t(j,x)\Big) \;=\; \frac{\bar{q}}{N-1}\,(N-1)\,E|\zeta_t| \;=\; \bar{q}E|\zeta_t|, \tag{3.1}$$

with initial condition $E|\zeta_0| = |\zeta|$. ∎

Lemma 3.2. *Let $g : \mathbb{N} \to \mathbb{R}^+$ be non decreasing. Then*

$$Eg(R(\zeta_t^\zeta)) \;\le\; E|\zeta_t^\zeta|\, Eg(\widetilde{Z}_t^{R(\zeta)}). \tag{3.2}$$

Proof. Consider the following partial order on $\mathcal{B}$:

$$\zeta \prec \zeta' \quad \text{if and only if} \quad \sum_{y\ge x} \zeta(i,y) \le \sum_{y\ge x} \zeta'(i,y), \quad \text{for all } i,x. \tag{3.3}$$

The branching process is attractive: the Harris construction with initial configurations $\zeta \prec \zeta'$ gives $\zeta_t^\zeta \prec \zeta_t^{\zeta'}$ almost surely; we leave the proof to the reader. Let $\zeta' := \sum_{i,x} \zeta(i,x)\delta_{(i,R(\zeta))}$ be the configuration having the same number of individuals of type i as ζ for all i, but all are located at $r := R(\zeta)$. Hence $\zeta \prec \zeta'$ and

$$Eg(R(\zeta_t^\zeta)) \;\le\; \sum_{i,x} g(x)E\zeta_t^\zeta(i,x) \;\le\; \sum_x g(x) \sum_i E\zeta_t^{\zeta'}(i,x), \tag{3.4}$$

because g is non-decreasing. Fix i and x and define

$$b_t(r,x) := \sum_i E\zeta_t^{\zeta'}(i,x), \quad a_t := E|\zeta_t^\zeta|, \quad \tilde{p}_t(r,x) := P(\widetilde{Z}_t^r = x).$$

Since $b_t(r,x)$ and $a_t\tilde{p}_t(r,x)$ satisfy the same Kolmogorov backwards equations and have the same initial condition, the right hand side of (3.4) is the same as the right hand side of (3.2). This can be seen as an application of the one-to-many lemma, see [2]. ∎

Harris construction of Fleming-Viot Let $\mathcal{N}(i,j,x) \subset \mathcal{N}(i,j)$ be the Poisson process obtained by independently including each $\tau \in \mathcal{N}(i,j)$ into $\mathcal{N}(i,j,x)$ with probability $q(x,0)/\bar{q}$ (≤ 1, by definition of $\bar{q}$). The processes $(\mathcal{N}(i,j,x), i,j \in \{1,\dots,N\}, x \in \mathbb{N})$ are independent Poisson processes of rate $q(x,0)/(N-1)$.

Fix $\xi_0 = \xi \in \mathbb{N}^{\{1,\dots,N\}}$, assume the process is defined until time $s \ge 0$ and proceed iteratively from $s = 0$ as follows.

1. Define $\tau(\xi_s, s) = \inf\{t > s\, :\, t \in \cup_{i,y}\mathcal{N}(i, \xi_s(i), y, 1) \cup \cup_{i,j}\mathcal{N}(i, j, \xi_s(i))\}$
2. For $t \in [s, \tau)$ define $\xi_t = \xi_s$.
3. If $\tau \in \mathcal{N}(i, \xi_s(i), y, 1)$, then set $\xi_\tau(i) = y$ and for $i' \neq i$ set $\xi_\tau(i') = \xi_s(i')$.
4. If $\tau \in \mathcal{N}(i, j, \xi_s(i))$, then set $\xi_\tau(i) = \xi_s(j)$ and for $i' \neq i$ set $\xi_\tau(i') = \xi_s(i')$.

The process is then defined until time τ. Put $s = \tau$ and iterate to define ξ_t for all $t \geq 0$. We leave the reader to prove that ξ_t^ξ is a Markov process with generator $\mathcal{L}^N$ and initial configuration ξ and the following lemma.

Lemma 3.3. *The Fleming-Viot i-particle coincides with the position of a branching i-individual at time t if this happens at time zero for all i. More precisely,*

$$\zeta_0(i, \xi_0(i)) \geq 1 \textit{ for all } i \textit{ implies } \zeta_t(i, \xi_t(i)) \geq 1 \textit{ for all } i, \quad a.s.. \tag{3.5}$$

Corollary 3.4. *Assume $\zeta_0(i, \xi_0(i)) \geq 1$ for all i. Then,*

$$R(\xi_t) \leq R(\zeta_t), \quad a.s. \tag{3.6}$$

4 Galton-Watson estimates

We show now that for ρ small enough the functions $e^{\rho\cdot}$ belong to the domain of the generator of Galton-Watson, that is, the Kolmogorov equations hold for these functions. The total number of births of the Galton-Watson process Z_t^x is a random variable $H^x := x + \sum_{t>0}(Z_t^x - Z_{t-}^x)^+$. Theorem 2 in [22] says that (2.1)-(2.2) are equivalent to the existence of a $\sigma' > 0$ such that

$$E(\exp(\sigma' H^1)) \;<\; \infty. \tag{4.1}$$

Clearly $\sigma' \leq \sigma$. Let

$$\mathbf{F} \;:=\; \Big\{f : \mathbb{N} \cup \{0\} \to \mathbb{R} \,:\, \sum_{\ell \geq 0} e^{-\rho\ell}|f(\ell)| \;<\; \infty \text{ for some } \rho < \sigma'\}. \tag{4.2}$$

Note that if $f \in \mathbf{F}$, then there exist $\rho < \sigma'$ and $C > 0$ such that $|f(\ell)| \leq Ce^{\rho\ell}$, $\ell \geq 0$. For $f \in \mathbf{F}$ define the Galton-Watson semigroup by

$$S_t f(x) := E(f(Z_t^x)) \;<\; \infty, \tag{4.3}$$

because $Z_t^x \leq H^x$ for all $t \geq 0$. The generator Q of Galton-Watson applied on functions f is given by

$$Qf(x) := \sum_{\ell=-1}^{\infty} xp(\ell + 1)\big(f(x + \ell) - f(x)\big), \qquad x \geq 0, \tag{4.4}$$

if the right hand side is well defined.

Lemma 4.1. *Under the assumption* (2.1), *for* $f \in \mathbf{F}$, $Qf(x)$ *is well defined and the Kolmogorov equations hold:*

$$\frac{d}{dt} S_t f = Q S_t f = S_t Q f. \tag{4.5}$$

Proof. Since $|f(x)| \leq C \exp(\rho x)$ for all $x \in \mathbb{N}$,

$$|Qf(x)| \;\leq\; Cxe^{\rho x}\Big(\sum_{\ell \geq -1} p(\ell+1)\, e^{\rho \ell} + 1\Big)\Big). \tag{4.6}$$

This shows the first part of the lemma. Consider $f \in \mathbf{F}$ and define the local martingale (see [24, Section IV-20, pp. 30-37])

$$M_t^x \;:=\; f(Z_t^x) - f(x) - \int_0^t Qf(Z_s^x) ds.$$

Using (4.6), for all $s \leq t$

$$|M_s^1| \;\leq\; e^{\rho} + \exp(\rho H^1) + tCH^1 \exp(\rho H^1) \;\leq\; \tilde{C} \exp(\tilde{\rho} H^1),$$

with $\rho < \tilde{\rho} < \sigma'$. Hence $E \sup_{s \in [0,t]} |M_s^1| < \infty$ and M_t^1 is a martingale by dominated convergence. Since for $\rho \leq \sigma'$, $E \exp(\rho H^x) = (E \exp(\rho H^1))^x$, the same reasoning shows that M_t^x is a martingale and

$$Ef(Z_t^x) = f(x) + E \int Qf(Z_s^x) ds,$$

which is equivalent to (4.5) for $f \in \mathbf{F}$. ∎

The generator of the reflected Galton-Watson process $\widetilde{Z}_t$ reads

$$\widetilde{Q}f(x) := \sum_{\ell=-1}^{\infty} xp(\ell+1) \mathbf{1}_{\{x+\ell \geq 1\}} \big(f(x+\ell) - f(x)\big), \qquad x \in \mathbb{N}, \tag{4.7}$$

if the right hand side is well defined. The reflected process can be thought of as an absorbed process regenerated at position 1 each time it gets extinct. Since the absorbed process can terminate only when it is at state 1 and jumps to 0 at rate $p(0)$, the number of regenerations until time t is dominated by a Poisson random variable $\mathcal{N}_t$ of mean $tp(0)$ and

$$E\big(\exp(\rho \widetilde{Z}_t^1)\big) \;\leq\; E \exp\Big(\rho \sum_{n=1}^{\mathcal{N}_t} H_n^1\Big),$$

where H_n^1 are i.i.d random variables with the same distribution as H^1 and $\mathcal{N}_t$ is independent of $(H_n^1, n \geq 1)$. Hence,

$$E\big(\exp(\rho \widetilde{Z}_t^1)\big) \;\leq\; \exp\big(tp(0)\, C(\rho)\big).$$

Let $\widetilde{S}_t$ be the semigroup of the reflected Galton-Watson process. Using the same reasoning as before, we obtain

Corollary 4.2. *Any* $f \in \mathbf{F}$ *satisfies the Kolmogorov equations for* $\widetilde{Q}$*:*

$$\frac{d}{dt} \widetilde{S}_t f = \widetilde{Q} \widetilde{S}_t f = \widetilde{S}_t \widetilde{Q} f. \tag{4.8}$$

Large deviations We study $\widetilde{Z}_t$, the reflected Galton-Watson process with generator $\widetilde{Q}$ given by (4.7). Since p satisfies (2.1), for $\rho < \sigma' \leq \sigma$,

$$\Gamma(\rho) \;:=\; p(0) + \sum_{\ell=1}^{\infty} p(\ell+1)\ell^2 e^{\rho\ell} \;<\; \infty. \tag{4.9}$$

Recall that v is defined in (2.2) and define β as

$$\beta = \sup\{\rho > 0 \colon \rho\Gamma(\rho) \leq v\}, \tag{4.10}$$

which is well defined thanks to the exponential moment of p.

Lemma 4.3. *For any $\rho < \min\{\beta, \sigma'\}$, and $x \in \mathbb{N}$,*

$$E\exp(\rho\widetilde{Z}_t^x) \;\leq\; e^{-\frac{\rho v}{2}t}e^{\rho x} + te^{\rho}. \tag{4.11}$$

Proof. Since $\rho < \sigma' \leq \sigma$, the reflected Galton-Watson generator (4.7) applied to $e^{\rho\cdot}$ is well defined and gives

$$\begin{aligned}
\widetilde{Q}(e^{\rho\cdot})(x) \;&=\; \sum_{\ell=-1}^{\infty} xp(\ell+1)e^{\rho x}\big(e^{\rho\ell}-1\big) - p(0)\mathbf{1}_{\{x=1\}}\big(1-e^{\rho}\big)\\
&=\; xe^{\rho x}\Big(-\rho v + \sum_{\ell=-1}^{\infty} p(\ell+1)\big(e^{\rho\ell}-1-\rho\ell\big)\Big) + p(0)\mathbf{1}_{\{x=1\}}\big(e^{\rho}-1\big).
\end{aligned}$$

Using that for $a \geq 0$, $e^a - (1+a) \leq \frac{a^2}{2}e^a$,

$$\begin{aligned}
\widetilde{Q}(e^{\rho\cdot})(x) \;&\leq\; \rho xe^{\rho x}\Big(-v + \frac{\rho}{2}\Gamma(\rho)\Big) + p(0)\mathbf{1}_{\{x=1\}}e^{\rho}\\
&\leq\; -\frac{v\rho}{2}e^{\rho x} \;+\; e^{\rho},
\end{aligned} \tag{4.12}$$

using $\rho < \beta$ and $\beta\Gamma(\beta) \leq v$. Since $\rho < \sigma'$, Corollary 4.2 and Gronwall's inequality give (4.11). ∎

We obtain now a Large Deviation estimate.

Proposition 4.4. *Let $T \geq \frac{1}{16v}$ and $\delta \geq \max\{1, 4Tp(0)\}$. Then, there is a constant κ, independent of x, such that*

$$P\Big(\sup_{s<T}\big(\widetilde{Z}_s^x - e^{-vs}x\big) \geq \delta\Big) \;\leq\; \exp\Big(-\frac{\kappa}{T}\frac{\delta^2}{\max\{x,\delta\}}\Big). \tag{4.13}$$

Proof. Set $z_t^x = e^{-vt}x$ and introduce the process

$$\begin{aligned}
\epsilon_t^x \;&:= \widetilde{Z}_t^x - x + v\int_0^t \widetilde{Z}_s^x ds\\
&= \big(\widetilde{Z}_t^x - z_t^x\big) + v\int_0^t \big(\widetilde{Z}_s^x - z_s^x\big)ds.
\end{aligned} \tag{4.14}$$

To stop $\widetilde{Z}^x_t$ when it crosses $2\max\{x,\delta\}$ define

$$\tau := \inf\Big\{t \geq 0 :\; \widetilde{Z}^x_t \geq 2\max\{x,\delta\}\Big\}. \tag{4.15}$$

Note that if $\tau < \infty$, then $\widetilde{Z}^x_\tau - z^x_\tau \geq 2\max\{x,\delta\} - x \geq \delta$. Thus,

$$\Big\{\widetilde{Z}^x_t - z^x_t \geq \delta\Big\} \subset \Big\{\widetilde{Z}^x_{t\wedge\tau} - z^x_{t\wedge\tau} \geq \delta\Big\}. \tag{4.16}$$

For functions $g_1,\, g_2 : \mathbb{R} \to \mathbb{R}$ verifying

$$g_1(t) = g_2(t) + v\int_0^t g_2(s)ds, \qquad v \geq 0,$$

it holds

$$\sup_{t\leq T}|g_1(t)| \leq \frac{\delta}{2} \implies \sup_{t\leq T}|g_2(t)| \leq \delta.$$

Hence,

$$\left\{\sup_{t\leq T}\left|\widetilde{Z}^x_{t\wedge\tau} - z^x_{t\wedge\tau}\right| \geq \delta\right\} \;\subset\; \left\{\sup_{t\leq T}|\epsilon^x_{t\wedge\tau}| \geq \frac{\delta}{2}\right\}. \tag{4.17}$$

Note that

$$\left\{\sup_{t\leq T}|\epsilon^x_{t\wedge\tau}| \geq \frac{\delta}{2}\right\} \;=\; \left\{\sup_{t\leq T}\epsilon^x_{t\wedge\tau} \geq \frac{\delta}{2}\right\} \cup \left\{\inf_{t\leq T}\epsilon^x_{t\wedge\tau} \leq -\frac{\delta}{2}\right\}.$$

The treatment of the two terms on the right hand side of the previous formula is similar, and we only give the simple argument for the first of them. For $\rho < \sigma'$, the following functional is a local martingale (see [15, page 66]).

$$\mathcal{M}_t := \exp\Big(\rho\widetilde{Z}^x_t - \rho x - \int_0^t \big(e^{-\rho\cdot}\widetilde{Q}(e^{\rho\cdot})\big)(\widetilde{Z}^x_s)ds\Big). \tag{4.18}$$

Using the bounds of Lemma 4.1 we obtain that $\mathcal{M}_t$ is in fact a martingale. Observe that

$$\begin{aligned} e^{-\rho x}\widetilde{Q}(e^{\rho\cdot})(x) \;&=\; x\sum_{\ell=-1}^{\infty} p(\ell+1)\big(e^{\rho\ell} - 1\big) + p(0)\mathbf{1}_{\{x=1\}}\big(e^{\rho} - 1\big) \\ &\leq\; -\rho v x + \rho\, p(0) + \frac{\rho^2}{2}\big(x\Delta(\rho) + p(0)e^{\rho}\big), \end{aligned} \tag{4.19}$$

with,

$$\Delta(\rho) := \frac{2}{\rho^2}\sum_{\ell=-1}^{\infty} p(\ell+1)\big(e^{\rho\ell} - 1 - \rho\ell\big) \geq 0. \tag{4.20}$$

We have already seen that $\Delta(\rho) \leq \Gamma(\rho)$. Then, we bound the martingale $\mathcal{M}_t$ as follows.

$$\begin{aligned} \mathcal{M}_t \;&\geq\; \exp\Big(\rho\big(\widetilde{Z}^x_t - x\big) - \big(-\rho v + \frac{\rho^2}{2}\Delta(\rho)\big)\int_0^t \widetilde{Z}^x_s ds - \rho p(0)t - \frac{\rho^2}{2}te^{\rho}\Big) \\ &\geq\; \exp\Big(\rho\epsilon^x_t - \rho p(0)t - \frac{\rho^2}{2}\Gamma(\rho)\int_0^t \widetilde{Z}^x_s ds - \frac{\rho^2}{2}te^{\rho}\Big). \end{aligned} \tag{4.21}$$

By stopping the process at τ, and using that $\delta \geq 1$, we obtain for $t \leq T$

$$\exp\big(\rho\epsilon^x_{t\wedge\tau}\big) \leq \mathcal{M}_{t\wedge\tau} \exp\big(\rho p(0)T + \rho^2 \max\{x, \delta\} T\Gamma(\rho)\big). \tag{4.22}$$

Using (4.16), (4.17) and (4.22), and the bound $p(0)T \leq \delta/4$, we obtain for any $\rho > 0$

$$\begin{aligned} P\Big(\sup_{s\leq T}\big(\widetilde{Z}^x_s - z^x_s\big) \geq \delta\Big) \;&\leq\; P\Big(\sup_{s\leq T} \mathcal{M}_{s\wedge\tau} \geq \exp\big(\frac{\rho\delta}{4} - \rho^2 \max\{x,\delta\} T\Gamma(\rho)\big)\Big) \\ &\leq\; \exp\Big(-\frac{\rho\delta}{8} + \rho^2 \max\{x,\delta\} T \sup_{\rho<\beta} \Gamma(\rho)\Big), \end{aligned} \tag{4.23}$$

by Doob's martingale inequality and for $\rho < \beta$. Optimize over $0 < \rho < \beta$ (recalling that $16T\beta\Gamma(\beta) > 1$), and choose ρ^*

$$\rho^* := \frac{1}{16T} \frac{\delta}{\max\{x,\delta\}} \frac{1}{\Gamma(\beta)} < \beta. \tag{4.24}$$

The result follows now from (4.23) and (4.24). ∎

5 Bounds for the rightmost Fleming-Viot-particle

In this section, we bound small exponential moments of the rightmost Fleming-Viot-particle. We first define a threshold A, such that with very small probability, the rightmost particle's position does not decrease when it is initially larger than A. Define

$$\gamma := \frac{1}{2}\Big(1 - \exp\Big(-\frac{v}{4p(0)}\Big)\Big) \in (0,1).$$

Choose

$$\rho_0 := \frac{\min\{\beta, \sigma', \gamma\kappa p(0)\}}{4}$$

where κ is the constant given by Proposition 4.4. Define

$$A := \frac{2\kappa p(0)}{\rho_0} > 1. \tag{5.1}$$

Define the time and the error δ entering in the large deviation estimate of Proposition 4.4 as follows. For an arbitrary initial condition ξ,

$$T := \frac{1}{4p(0)}, \quad \text{and} \quad \delta := \max\Big\{1, \frac{R(\xi)}{A}\Big\}, \tag{5.2}$$

recall here that $R(\xi) = \max_{i\leq N} \xi(i)$, and set $V_L(\xi) = \exp(\rho \min(R(\xi), L))$ for $L > A$ which will be taken to infinity later. We use the notation $[F(\xi_t)]_0^T := F(\xi_T) - F(\xi_0)$.

Proof of Proposition 1.2. We use the construction in Section 3 to couple the Fleming-Viot process ξ_t^ξ and the branching process ζ_t^ζ with $\zeta = \sum_i \delta_{(i,\xi(i))}$, so that $\zeta(i,\xi(i)) = 1$ for all i. Then, by (3.6) $R(\xi_t) \le R(\zeta_t)$ and it is sufficient to prove an inequality like (1.6) for $R(\zeta_t)$. Notice that for the initial configurations ξ and ζ, $R(\xi) = R(\zeta)$. We drop the superscripts ξ and ζ in the remainder of this proof.

Define the event

$$\mathcal{G} = \mathcal{G}(\xi, T) := \left\{ R(\zeta_T) - e^{-vT} R(\xi) \le \delta \right\}, \tag{5.3}$$

and for a positive real c, we define the set

$$K_c := \{\xi \colon R(\xi) \le c\}.$$

On K_A^c, $\delta = R/A < R$, and on $K_A^c \cap \mathcal{G}$,

$$R(\zeta_T) \;\le\; \Big(\frac{1}{A} + e^{-vT}\Big) R(\xi) \;\le\; (1-\gamma) R(\xi). \tag{5.4}$$

Hence,

$$\mathbf{1}_{K_A^c \cap \mathcal{G}} \left[V_L(\zeta_t)\right]_0^T \;\le\; V_L(\xi)\left(e^{-\gamma\rho R(\xi)} - 1\right) \mathbf{1}_{K_A^c \cap K_L \cap \mathcal{G}} \;\le\; -V_L(\xi)\left(1 - e^{-\gamma\rho A}\right) \mathbf{1}_{K_A^c \cap K_L \cap \mathcal{G}}. \tag{5.5}$$

Since $A > 1$, on $K_A \cap \mathcal{G}$, $R(\zeta_T) \le A e^{-vT} + 1 \le 2A$ so that

$$\mathbf{1}_{K_A \cap \mathcal{G}} \left[e^{\rho R(\zeta_t)}\right]_0^T \le e^{2\rho A} \mathbf{1}_{K_A \cap \mathcal{G}}.$$

Thus

$$\begin{aligned} \left[V_L(\zeta_t)\right]_0^T &\le -\big(1 - e^{-\gamma\rho A}\big) e^{\rho R(\xi)} \mathbf{1}_{K_A^c \cap K_L \cap \mathcal{G}} + e^{2\rho A} \mathbf{1}_{K_A \cap \mathcal{G}} + \left[e^{\rho R(\zeta)}\right]_0^T \mathbf{1}_{\mathcal{G}^c} \\ &\le -\big(1 - e^{-\gamma\rho A}\big) V_L(\xi) \mathbf{1}_{K_A^c \cap K_L} + e^{2\rho A} \mathbf{1}_{K_A} + 2 e^{\rho R(\zeta_T)} \mathbf{1}_{\mathcal{G}^c}, \end{aligned} \tag{5.6}$$

where we used that

$$\mathbf{1}_{K_A^c \cap K_L} - \mathbf{1}_{K_A^c \cap K_L \cap \mathcal{G}} \le \mathbf{1}_{\mathcal{G}^c}.$$

Choose $\rho := \min(\rho_0, \frac{\kappa}{4TA^2})$ and observe that by Lemma 3.2,

$$E\left[e^{2\rho R(\zeta_T)}\right] \le E|\zeta_T|\, E\left[\exp\Big(2\rho \widetilde{Z}_T^{R(\xi)}\Big)\right] \le N e^{p(0)T}\left(e^{-2\rho v T} e^{2\rho R(\xi)} + T e^{2\rho}\right).$$

by Lemma 3.1 for the bound of the first factor and Lemma 4.3 for the bound of the second factor. Also, Lemma 4.4 implies

$$P(\mathcal{G}^c) \le E|\zeta_T| P\Big(\sup_{s<T} \big(\widetilde{Z}_s^{R(\xi)} - e^{-vs} R(\xi)\big) > \delta\Big) \;\le\; N e^{p(0)T} (e^{-\frac{\kappa}{TA}} \mathbf{1}_{R(\xi) \le A} + e^{-\frac{\kappa R(\xi)}{TA^2}} \mathbf{1}_{R(\xi) > A}).$$

Taking expectation on (5.6) we bound the last term as follows. For constants $C_1, C_2, \tilde{C}_1$, and $\tilde{C}_2$

$$\begin{aligned} E\left[e^{\rho R(\zeta_T)} \mathbf{1}_{\{\mathcal{G}^c\}}\right] &\le \left(P(\mathcal{G}^c) E\left[e^{2\rho R(\zeta_T)}\right]\right)^{1/2} \\ &\le N e^{p(0)T}\left(C_1 \mathbf{1}_{K_A} + C_2 \exp\left(-\frac{\kappa R(\xi)}{2TA^2}\right) \mathbf{1}_{K_A^c}\right)^{1/2} \\ &\le \tilde{C}_1 N \mathbf{1}_{K_A} + \tilde{C}_2 N \exp\Big(-\frac{\kappa R(\xi)}{4TA^2}\Big) \mathbf{1}_{K_A^c}. \end{aligned} \tag{5.7}$$

Gathering (5.7) and (5.6) we obtain, for any $L > A$,

$$\begin{aligned} EV_L(\xi_T^\xi) - V_L(\xi) &< -c_1 V_L(\xi)\mathbf{1}_{L>R(\xi)>A} + C_1 N \mathbf{1}_{R(\xi)\le A} + C_2 N e^{-\rho \tilde{c}_2 R(\xi)} \\ &\le -c_1 V_L(\xi)\mathbf{1}_{L>R(\xi)>A} + CNe^{-c_2 R(\xi)} \end{aligned} \tag{5.8}$$

which completes the first part of the proof, inequality (1.6), as one takes L to infinity in (5.8).

For the second part, take $C > 0$ and observe that the set of ξ such that the right hand side of (5.8) is larger than $-C$ is finite. Foster's criteria, [23, Theorems 8.6 and 8.13] implies that both the chain $(\xi_0, \xi_T, \xi_{2T}, \cdots)$ and the process ξ_t are ergodic with the same invariant measure that we call λ^N.

Now, consider again (5.8) for a fixed L. Note that V_L is bounded, so that by integrating (5.8) with this invariant measure, and then taking L to infinity, we obtain (1.7). ∎

6 The empirical moments of Fleming-Viot

In this section we prove Corollary 1.4. Introduce the occupation numbers $\eta : \mathbb{N} \times \mathbb{N}^N \to \mathbb{N}$ defined as

$$\eta(x, \xi) := \sum_{i=1}^N \mathbf{1}_{\xi(i)=x},$$

for which we often drop the coordinate ξ. Notice that $m(x, \xi) = \eta(x, \xi)/N$.

For any integer k, define the k-th moment of the N particles' positions as

$$M_k(\xi) := \sum_{i=1}^N \xi^k(i) = \sum_{x=1}^\infty x^k \eta(x, \xi).$$

As there are only N particles, M_k is well defined. Instead of working with the barycenter M_1/N, we consider $\psi := M_2/M_1$. Note the inequalities

$$1 \le \frac{M_1(\xi)}{N} \le \psi(\xi) \le R(\xi). \tag{6.1}$$

The function ψ is not compactly supported (nor bounded). Even though $\mathcal{L}^N\psi$ is well defined, we need to use later that $\int \mathcal{L}^N \psi d\lambda^N = 0$. We do so by approximating ψ by a compactly supported function ψ^L for which we have

$$\int \mathcal{L}^N \psi^L d\lambda^N = 0, \quad \text{and} \quad \lim_{L\to\infty} \mathcal{L}^N \psi^L = \mathcal{L}^N \psi \quad \text{pointwise.} \tag{6.2}$$

We approximate the unbounded test function ψ by the following one

$$\psi^L(\xi) = \frac{M_2^L(\xi)}{M_1^L(\xi)}, \quad \text{with} \quad M_k^L(\xi) = \sum_{i=1}^N \min(\xi^k(i), L^k) = \sum_{x=1}^L x^k \eta(x, \xi) + L^k \sum_{x>L} \eta(x, \xi). \tag{6.3}$$

As N is fixed, $M_k^L = L^k N - \sum_{x=1}^{L}(L^k - x^k)\eta(x)$, and has compact support. It is easy, and we omit the proof, to see that there exist a positive constant C such that

$$\left|\mathcal{L}^N\psi - \mathcal{L}^N\psi^L\right| \le \left|\mathcal{L}^N\psi\right| + \left|\mathcal{L}^N\psi^L\right| \le C\psi \le C\,R, \tag{6.4}$$

where we recall that $R(\xi) = \max_i \xi(i)$. We have established in Proposition 1.2 that $R(\xi)$ is integrable with respect to λ^N, so that (6.2) implies that

$$\int \mathcal{L}^N\psi\, d\lambda^N = 0. \tag{6.5}$$

The main result of this section is the following.

Lemma 6.1. *There are positive constants C_1, C_2 such that for any integer N large enough,*

$$\int \psi\, d\lambda^N \;\le\; C_1 + \frac{C_2}{N}\int R^2 d\lambda^N. \tag{6.6}$$

Proof of Lemma 6.1. We decompose the generator (1.3) into two generators, one governing the refeed part and the other the spatial evolution of the particles: $\mathcal{L}^N = \mathcal{L}^N_{\text{drift}} + \mathcal{L}^N_{\text{refeed}}$, which applied to functions depending on ξ only through $\eta(\cdot,\xi)$, read

$$\mathcal{L}^N_{\text{refeed}} = p(0)\eta(1)\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\left(A_1^- A_x^+ - \mathbf{1}\right), \quad \text{with} \quad A_x^{\pm}(\eta)(y) = \begin{cases}\eta(y) & y \ne x,\\ \eta(x)\pm 1 & y = x,\end{cases} \tag{6.7}$$

$$\mathcal{L}^N_{\text{drift}} = \sum_{x=2}^{\infty} x\eta(x)p(0)(A_x^- A_{x-1}^+ - \mathbf{1}) + \sum_{x=1}^{\infty} x\eta(x,\xi)\sum_{i=1}^{\infty} p(i+1)(A_x^- A_{x+i}^+ - \mathbf{1}). \tag{6.8}$$

It is convenient to introduce a boundary term

$$B = -\eta(1)p(0)(A_1^- A_0^+ - \mathbf{1}) \quad \text{and call} \quad \mathcal{L}^N_0 = \mathcal{L}^N_{\text{drift}} - B, \tag{6.9}$$

which applied on ψ yield

$$B\psi \;=\; -p(0)\eta(1)\left(\frac{M_2 - M_1}{M_1(M_1-1)}\right); \tag{6.10}$$

$$\begin{aligned}
\mathcal{L}^N_0\psi \;&=\; \sum_{x=1}^{\infty} x\eta(x)\sum_{i=-1}^{\infty} p(i+1)\left(\frac{M_2 + 2ix + i^2}{M_1 + i} - \frac{M_2}{M_1}\right)\\
&=\; \sum_{x=1}^{\infty} x\eta(x)\left\{\sum_{i=-1}^{\infty} ip(i+1)\left(\frac{2xM_1 - M_2 + iM_1}{M_1(M_1+i)}\right)\right\}\\
&=\; -p(0)\frac{M_2 - M_1}{M_1 - 1} + \left(\sum_{i=1}^{\infty} p(i+1)i\frac{M_1}{M_1+i}\right)\times\frac{M_2}{M_1} + \sum_{i=1}^{\infty} p(i+1)i^2\frac{M_1}{M_1+i}\\
&\le\; -v\psi + p(0)\frac{M_1}{M_1-1} + \sum_{i=1}^{\infty} p(i+1)i^2 \;\le\; -v\psi + C_0,
\end{aligned} \tag{6.11}$$

for some positive constant C_0. Finally, for the jump term

$$\begin{aligned}
\mathcal{L}^N_{\text{refeed}}\psi &= p(0)\eta(1)\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\left(\frac{M_2+x^2-1}{M_1+x-1}-\frac{M_2}{M_1}\right) \\
&= p(0)\eta(1)\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\frac{M_1(x^2-1)-M_2(x-1)}{M_1(M_1-1)}\times\frac{1}{1+\frac{x}{M_1-1}}.
\end{aligned} \tag{6.12}$$

If we set $\Delta(x)=1/(1+x)-(1-x)$, for $x\in[0,1]$, then

$$\Delta(x)=\frac{x^2}{1+x},\quad \text{and}\quad 0\le\Delta(x)\le x^2. \tag{6.13}$$

We apply (6.13) to expand the last term in (6.12), with $x/(M_1-1)\le 1$ for $x\le R(\xi)$, and obtain

$$\mathcal{L}^N_{\text{refeed}}\psi = p(0)\eta(1)\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\frac{M_1(x^2-1)-M_2(x-1)}{M_1(M_1-1)}\times\left(1-\frac{x}{M_1-1}+\Delta(\frac{x}{M_1-1})\right). \tag{6.14}$$

Note that

$$\sum_{x=1}^{\infty}\eta(x)\big(M_1x^2-M_2x\big)=0,\quad\text{and}\quad \sum_{x=1}^{\infty}\eta(x)\big(M_1x^2-M_2x\big)(-x)=-M_3M_1+(M_2)^2.$$

Also,

$$\begin{aligned}
\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\big(M_2-M_1\big)\left(1-\frac{x}{M_1-1}\right) &= \left(N-\frac{M_1}{M_1-1}\right)\frac{\big(M_2-M_1\big)}{N-1} \\
&= \left(1-\frac{1}{(N-1)(M_1-1)}\right)\big(M_2-M_1\big) \\
&= \big(M_2-M_1\big)-\frac{M_2-M_1}{(N-1)(M_1-1)}.
\end{aligned}$$

Thus

$$\mathcal{L}^N_{\text{refeed}}(\psi)=-p(0)\frac{\eta(1)}{N-1}\frac{M_3M_1-(M_2)^2}{M_1(M_1-1)^2}+p(0)\eta(1)\frac{M_2-M_1}{M_1(M_1-1)}+\text{Rest},$$

where

$$\text{Rest} = -\frac{p(0)\eta(1)(M_2-M_1)}{(N-1)(M_1-1)}+p(0)\eta(1)\sum_{x=1}^{\infty}\frac{\eta(x)}{N-1}\frac{M_1(x^2-1)-M_2(x-1)}{M_1(M_1-1)}\times\Delta(\frac{x}{M_1-1}). \tag{6.15}$$

Using that $M_2-M_1\ge 0$,

$$\text{Rest} \le p(0)\frac{\eta(1)}{N-1}\sum_{x=1}^{\infty}\eta(x)\times\frac{x^2}{(M_1-1)^2}\Big(\frac{M_1x^2+M_2x}{(M_1-1)^2}+\frac{M_2-M_1}{(M_1-1)^2}\Big) \tag{6.16}$$

$$\le 3p(0)\Big(\frac{M_1}{M_1-1}\Big)^4\frac{\eta(1)}{N-1}\frac{M_2}{(M_1)^2}\frac{R^2}{M_1} \le 24p(0)\frac{R^2}{N}. \tag{6.17}$$

Thus, we reach that for C_0 independent of N and L,

$$\mathcal{L}^N(\psi) \;\leq\; -v\,\psi \;+\; 24\,p(0)\,\frac{R^2}{N} + C_0. \tag{6.18}$$

We now integrate (6.18) with respect to the invariant measure, and use that $\int \mathcal{L}^N \psi d\lambda^N = 0$ to obtain for constants C_1, and C_2 (independent of N)

$$\int \psi d\lambda^N \leq C_1 + C_2 \frac{\int R^2\, d\lambda^N}{N}. \tag{6.19}$$

∎

7 Uniform convergence to the Yaglom limit

Define the generating function of a distribution μ on $\mathbb{N}$ by

$$G(\mu; z) := \sum_{x\in\mathbb{N}} \mu(x) z^x, \qquad z \in \mathbb{R},\ |z| < 1. \tag{7.1}$$

In this section we show a uniform convergence for $\mu \in K(\alpha)$ of the generating functions of μT_t to the generating function of the QSD ν. We invoke a key result of Yaglom [27]. The continuous time version can be found in Zolotarev [28].

Lemma 7.1. *[Yaglom 1947, Zolotarev 1957]. There is a probability measure ν such that*

$$\lim_{t\to\infty} G(\delta_1 T_t; z) = G(\nu; z), \tag{7.2}$$

and the generating function of ν is given by

$$G(\nu; z) = 1 - \exp\Big(- v \int_0^z \frac{du}{\sum_{\ell\geq 0} p(\ell) u^\ell - z} \Big), \qquad z \in [0,1). \tag{7.3}$$

The measure ν is in fact ν^*_{qs}, the minimal QSD. We do not use the explicit expression (7.3) of the generating function of ν; we only use (7.2). Recall that μT_t is the law of Z_t with initial distribution μ conditioned on survival until t and that $K(\alpha)$ is defined in (2.5). The next result says that the Yaglom limit holds uniformly for all initial measures in $K(\alpha)$.

Proposition 7.2. *For any $\alpha > 0$*

$$\lim_{N\to\infty} \sup_{\mu\in K(\alpha)} \big| G(\mu T_t; z) - G(\nu^*_{\text{qs}}; z) \big| \;=\; 0. \tag{7.4}$$

As a consequence, for each $x \in \mathbb{N}$,

$$\lim_{t\to\infty} \sup_{\mu\in K(\alpha)} |\mu T_t(x) - \nu^*_{\text{qs}}(x)| = 0. \tag{7.5}$$

Proof of Proposition 7.2. Recall that S_t is the semigroup of the Galton-Watson process and observe that for any $\ell \in \mathbb{N}$, $G(\delta_\ell S_t; z) = G^\ell(\delta_1 S_t; z)$. We set, for simplicity,

$$g(z) \;:=\; 1 - G(\delta_1 S_t; z) \;\in\; [0,1],$$

for $z \in [0,1]$. The following inequalities are useful. For $z \in [0,1]$,

$$1 - \ell g(z) \;\le\; (1-g(z))^\ell \;\le\; 1 - \ell g(z) + \ell^2 g^2(z). \tag{7.6}$$

The generating function of μT_t reads (the sums run on $\ell \in \mathbb{N}$)

$$\begin{aligned} G(\mu T_t; z) &= \frac{G(\mu S_t; z) - G(\mu S_t; 0)}{1 - G(\mu S_t; 0)} \\ &= \frac{\sum_\ell \mu(\ell)\big(G(\delta_\ell S_t; z) - G(\delta_\ell S_t; 0)\big)}{\sum_\ell \mu(\ell)\big(1 - G(\delta_\ell S_t; 0)\big)} \\ &= \frac{\sum_\ell \mu(\ell)\Big((1-g(z))^\ell - (1-g(0))^\ell\Big)}{\sum_\ell \mu(\ell)\big(1-(1-g(0))^\ell\big)}. \end{aligned} \tag{7.7}$$

Also,

$$1 - G(\delta_1 T_t; z) \;=\; \frac{1 - G(\delta_1 S_t; z)}{1 - G(\delta_1 S_t; 0)} \;=\; \frac{g(z)}{g(0)}.$$

We now produce upper and lower bounds for $G(\mu T_t; z) - G(\nu^*_{\mathrm{qs}}; z)$. We start with the upper bound. Using first (7.7) and then (7.6),

$$\begin{aligned} G(\mu T_t; z) - G(\nu^*_{\mathrm{qs}}; z) &= \frac{\sum_\ell \mu(\ell)\Big((1-g(z))^\ell - 1 + \big(1-(1-g(0))^\ell\big)(1 - G(\nu^*_{\mathrm{qs}}; z))\Big)}{\sum_\ell \mu(\ell)\big(1-(1-g(0))^\ell\big)} \\ &\le \frac{\sum_\ell \ell\mu(\ell)\Big(-g(z) + \ell g^2(z) + g(0)(1 - G(\nu^*_{\mathrm{qs}}; z))\Big)}{\sum_\ell \ell\mu(\ell)\Big(g(0) - \ell g^2(0)\Big)} \\ &\le \frac{\sum_\ell \ell\mu(\ell)\Big((1 - G(\nu^*_{\mathrm{qs}}; z)) - \frac{g(z)}{g(0)}\Big) + \sum_\ell \ell^2\mu(\ell)\frac{g(z)}{g(0)}g(z)}{\sum_\ell \ell\mu(\ell)\big(1 - \ell g(0)\big)} \\ &\le \frac{G(\delta_1 T_t; z) - G(\nu^*_{\mathrm{qs}}; z) + \frac{M_2(\mu)}{M_1(\mu)}(1 - G(\delta_1 T_t; z))g(z)}{1 - \frac{M_2(\mu)}{M_1(\mu)}g(0)}, \end{aligned} \tag{7.8}$$

where $M_k(\mu) := \sum_\ell \ell^k \mu(\ell)$, $k \in \mathbb{N}$. Thus,

$$\sup_{\mu\in K(\alpha)} G(\mu T_t; z) - G(\nu^*_{\mathrm{qs}}; z) \;\le\; \frac{|G(\delta_1 T_t; z) - G(\nu^*_{\mathrm{qs}}; z)| + (1 - G(\delta_1 T_t; z))g(z)\alpha}{1 - \alpha g(0)}. \tag{7.9}$$

Now, for the lower bound, we use similar arguments to reach

$$\begin{aligned} G(\mu T_t; z) - G(\nu^*_{\mathrm{qs}}; z) &\geq \frac{\sum_\ell \ell\mu(\ell)\Big(- g(z) + g(0)(1 - G(\nu^*_{\mathrm{qs}}; z)) - \ell g^2(0)(1 - G(\nu^*_{\mathrm{qs}}; z))\Big)}{\sum_\ell \ell\mu(\ell) g(0)} \\ &\geq G(\delta_1 T_t; z) - G(\nu^*_{\mathrm{qs}}; z) - \frac{M_2(\mu)}{M_1(\mu)} g(0)(1 - G(\nu^*_{\mathrm{qs}}; z)). \end{aligned} \tag{7.10}$$

Thus,

$$\inf_{\mu\in K(\alpha)} G(\mu T_t; z) - G(\nu^*_{\mathrm{qs}}; z) \;\geq\; -|G(\delta_1 T_t; z) - G(\nu^*_{\mathrm{qs}}; z)| - \alpha g(0)(1 - G(\nu^*_{\mathrm{qs}}; z)). \tag{7.11}$$

Since $g(z)$ goes to 0 as the implicit t goes to infinity, both (7.9) and (7.11) go to 0. This proves (7.4). The proof of (2.8) follows from (7.4) and Lemma 7.3 below on convergence of probability measures. ∎

Lemma 7.3. *Let $\{\mu_n^\gamma, n \in \mathbb{N}, \gamma \in \Gamma\}$ be a family of probability measures. Assume that for each $z \in [0,1]$ we have*

$$\lim_{n\to\infty} \sup_{\gamma\in\Gamma} |G(\mu_n^\gamma, z) - G(\nu, z)| = 0. \tag{7.12}$$

Then, for each $x \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \sup_{\gamma\in\Gamma} |\mu_n^\gamma(x) - \nu(x)| = 0.$$

Proof. Let $f = \mathbf{1}_{\{x\}}$. We consider the one-point compactification of $\mathbb{N}$, which we denote $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and extend $f\colon \bar{\mathbb{N}} \to \mathbb{R}$ by $f(\infty) = 0$. Since f is continuous function on $\bar{\mathbb{N}}$, the Stone-Weierstrass approximation theorem yields a function h, which is a linear combination of functions of the form $\{y \mapsto a^y,\ 0 \leq a \leq 1\}$ (finite linear combinations of these functions form an algebra that separates points and contains the constants), such that for any $\varepsilon > 0$, $\sup_{y\in\mathbb{N}} |f(y) - h(y)| < \varepsilon$. Then

$$\sup_\gamma |\mu_n^\gamma(x) - \nu(x)| \;=\; \sup_\gamma |\mu_n^\gamma f - \nu f| \;\leq\; \sup_\gamma |\mu_n^\gamma f - \mu_n^\gamma h| \;+\; \sup_\gamma |\mu_n^\gamma h - \nu h| \;+\; |\nu h - \nu f|.$$

The first and the third term on the r.h.s. are smaller than ε while the second one goes to zero as n goes to infinity by assumption. ∎

8 Closeness of the two semi-groups

In this section we show how propagation of chaos implies the closeness of $Em(x, \xi_t^\xi)$ and $m(\cdot, \xi)T_t$ uniformly in $\xi \in \Lambda^N$. The arguments are similar to those used in [17, 1]. The key is a control of the correlations that we state below. For a signed measure μ in $\mathbb{N}$ we will need to work with the ℓ_2 norm given by $\|\mu\|^2 = \sum_{x\in\mathbb{N}} (\mu(x))^2$.

Proposition 8.1. *There exist constants c and C such that,*

$$\sup_{\xi\in\mathbb{N}^N} \|E[m(x,\xi_t^\xi)] - m(\cdot,\xi)T_t\| \;\leq \frac{Ce^{ct}}{N}. \tag{8.1}$$

As a consequence,

$$\sup_{\xi\in\mathbb{N}^N} |E[m(x,\xi_t^\xi)] - m(\cdot,\xi)T_t(x)| \;\leq \frac{Ce^{ct}}{N}. \qquad x\in\mathbb{N}. \tag{8.2}$$

Furthermore

$$\sup_{\xi\in\mathbb{N}^N} E\big[m(x,\xi_t^\xi) - m(\cdot,\xi)T_t\big]^2 \leq \frac{Ce^{ct}}{N}, \qquad x\in\mathbb{N}. \tag{8.3}$$

Proposition 8.2 (Proposition 2 of [1])**.** *For each $t>0$, and any $x,y\in\mathbb{N}$*

$$\sup_{\xi\in\mathbb{N}^N} \big|E[m(x,\xi_t^\xi)m(y,\xi_t^\xi)] - E[m(y,\xi_t^\xi)]\,E[m(x,\xi_t^\xi)]\big| \;\leq\; \frac{2p(0)e^{2p(0)t}}{N}. \tag{8.4}$$

The paper [1] proves this proposition for processes with bounded rates, but the extension to our case is straightforward.

Proof of Proposition 8.1. Fix $\xi\in\mathbb{N}^N$ and introduce the simplifying notations

$$u(t,x) := Em(x,\xi_t^\xi) \quad\text{and}\quad v(t,x) := m(\cdot,\xi)T_t(x). \tag{8.5}$$

Define $\delta(t,x) = u(t,x) - v(t,x)$. We want to show that for any $t>0$,

$$\frac{\partial}{\partial t}\|\delta(t)\|^2 \;\leq\; \frac{5}{2}\|\delta(t)\|^2 + \frac{4p(0)e^{2p(0)t}}{N}. \tag{8.6}$$

Recall the definition (2.3) of the rates q and the evolution equations satisfied by $v(t,x)$ and $u(t,x)$:

$$\frac{\partial}{\partial t}v(t,x) = \sum_{z\neq x, z>0} q(z,x)v(t,z) - \Big(\sum_{z\neq x} q(x,z)\Big)v(t,x) + p(0)v(t,1)v(t,x), \tag{8.7}$$

$$\frac{\partial}{\partial t}u(t,x) = \sum_{z\neq x, z>0} q(z,x)u(t,z) - \Big(\sum_{z\neq x} q(x,z)\Big)u(t,x) + p(0)u(t,1)u(t,x) + W(\xi;t,x). \tag{8.8}$$

Here,

$$W(\xi;t,x) = p(0)\Big(\frac{N}{N-1}E[m(x,\xi_t^\xi)m(1,\xi_t^\xi)] - E[m(1,\xi_t^\xi)]\,E[m(x,\xi_t^\xi)]\Big). \tag{8.9}$$

Proposition 8.2 implies that

$$\sup_{\xi} |W(\xi;t,x)| \leq \frac{2p(0)e^{2p(0)t}}{N}. \tag{8.10}$$

Observe two simple facts. First, set $D = \{(x,z):\ x \geq 1, z \geq 1,\ x \neq z\}$, and for any function $f : \mathbb{N} \to \mathbb{R}$

$$\sum_{(x,z)\in D} \big(q(x,z) + q(z,x)\big) f^2(x) - 2 \sum_{(x,z)\in D} q(x,z) f(x) f(z) = \sum_{(x,z)\in D} q(z,x)(f(x) - f(z))^2. \tag{8.11}$$

The second observation is specific to our rates. For $x > 0$

$$\sum_{z\neq x} q(z,x) \leq \sum_{z\neq x} q(x,z) + p(0). \tag{8.12}$$

Observation (8.11) is obvious and we omit its proof. Observation (8.12) is done in details.

$$\begin{aligned} \sum_{z\neq x} q(z,x) &= \sum_{z\geq 0, z\neq x} z p(x-z+1) = x \sum_{z\geq 0, z\neq x} p(x-z+1) + \sum_{z\geq 0, z\neq x} (z-x) p(x-z+1) \\ &= x\big(p(0) + p(1) + \cdots + p(x+1)\big) + \big(p(0) - p(2) - \cdots - x p(x+1)\big) \\ &\leq x \sum_{i\geq 0} p(i) + p(0) = \sum_{z\neq x} q(x,z) + p(0). \end{aligned} \tag{8.13}$$

Now, we have

$$\begin{aligned} \sum_{x>0} \delta(t,x) \frac{\partial}{\partial t} \delta(t,x) &= \sum_{(x,z)\in D} \big(q(z,x)\delta(t,x)\delta(t,z) - q(x,z)\delta^2(t,x)\big) \\ &+ p(0) \sum_{x>0} \big(u(t,x)u(t,1) - v(t,x)v(t,1)\big)\delta(t,x) + \sum_{x>0} \delta(t,x) W(\xi; t, x). \end{aligned} \tag{8.14}$$

Let us deal with each term of the right hand side of (8.14). For the first term we use (8.11) and (8.12).

$$\begin{aligned} &\sum_{(x,z)\in D} \big(q(z,x)\delta(t,x)\delta(t,z) - q(x,z)\delta^2(t,x)\big) \\ &\leq \sum_{(x,z)\in D} q(z,x)\delta(t,x)\delta(t,z) - \frac{1}{2} \sum_{x>0} \big(\sum_{z\neq x} q(x,z) + \sum_{z\neq x} q(z,x) - p(0)\big)\delta^2(t,x) \\ &\leq -\frac{1}{2} \sum_{(x,z)\in D} q(z,x)(\delta(t,x) - \delta(t,z))^2 + \frac{p(0)}{2} \|\delta(t)\|^2 \\ &\leq \frac{p(0)}{2} \|\delta(t)\|^2. \end{aligned} \tag{8.15}$$

To deal with the second term, first note that

$$\sup_{x>0} \big|\delta(t,x)\big| \ \leq\ \sqrt{\sum_{x>0} \delta^2(t,x)} \ =\ \|\delta(t)\|.$$

Then,

$$\begin{aligned}\sum_{x>0}\big(u(t,x)u(t,1)-v(t,x)v(t,1)\big)\delta(t,x) &\le \sum_{x>0}\big(\delta(t,x)u(t,1)+v(t,x)\delta(t,1)\big)\delta(t,x)\\ &\le \sum_{x>0}\delta^2(t,x)+|\delta(t,1)|\sup_{x>0}\big|\delta(t,x)\big|\sum_{x>0}v(t,x)\le 2\|\delta(t)\|^2.\end{aligned}\tag{8.16}$$

For the last term, we have

$$|\sum_{x>0}\delta(t,x)W(\xi;t,x)|\le\sup_{x>0}|W(\xi;t,x)|\times\sum_{x>0}|\delta(t,x)|\le 2\sup_{x>0}|W(\xi;t,x)|.\tag{8.17}$$

Thus, we obtain (8.6). Gronwall's inequality allows to conclude. ∎

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