Symplectic structures on free nilpotent Lie algebras

By Viviana del Barco

Departamento de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, CONICET, Av. Pellegrini 250, S2000BTP, Rosario, Sante Fe, Argentina

(Communicated by Shigefumi Mori, m.j.a., Sept. 12, 2019)

Abstract: In this short note we show a necessary and sufficient condition for the existence of symplectic structures on free nilpotent Lie algebras and their one-dimensional trivial extensions.

Key words: Free nilpotent Lie algebras; symplectic structures.

1. Introduction. A symplectic structure on a differentiable manifold of dimension $2n$ is a differential closed 2-form ω such that ω^n is non-singular. Here ω^n denotes the n-th power with respect to the wedge product. In this note we study existence of symplectic structures on *compact nilmanifolds* which are homogeneous manifolds of the form $\Gamma\backslash G$, where G is a simply connected nilpotent Lie group and Γ is a cocompact discrete subgroup of G.

On a nilmanifold, any symplectic form is cohomologous to an invariant symplectic form. In fact, since nilmanifolds are compact, the n-th power of a closed 2-form is non-singular if and only if the n-th power of any of the forms in the same cohomology class is so. Moreover, Nomizu showed that every closed form on $\Gamma\backslash G$ is cohomologous to an invariant one [13]. Therefore, ω is a symplectic form on $\Gamma\backslash G$ if and only if there is some invariant form $\omega_{\rm inv}$, in the same cohomology class, which is also symplectic.

Let $\mathfrak{g}$ denote the Lie algebra of G, then $\omega_{\rm inv}$ is an element in $\Lambda^2\mathfrak{g}^*$. Also, the property of $\omega_{\rm inv}$ being symplectic on $\Gamma\backslash G$ is equivalent to being a non-degenerate 2-form on $\mathfrak{g}$, closed under the Lie algebra differential; in this case we say that $\omega_{\rm inv}$ is a symplectic structure on $\mathfrak{g}$. As a consequence, the problem of existence of symplectic forms on compact nilmanifolds is equivalent to that of existence of symplectic structures on nilpotent Lie algebras.

There have been several approaches to the study of symplectic structures on nilpotent Lie algebras. Lie algebras of dimension ≤ 6 admitting such structures were classified by Morozov [12] and those in dimension eight which are 2-step nilpotent were studied by Goze and Remm [6]. Symplectic Lie algebras on different families of nilpotent Lie algebras were considered, for instance, in [5,11,14].

In this short note, we study the existence of symplectic structures within the family of free nilpotent Lie algebras and we achieve the following classification.

Main Theorem. *Let $\mathfrak{n}_{m,k}$ be the free k-step nilpotent Lie algebra on m generators and let $\mathfrak{g}$ be an even dimensional Lie algebra of the form*

$$\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$$

for some $t = 0, 1$. Then $\mathfrak{g}$ admits symplectic structures if and only if $t = 1$ and $(m,k) = (2,2)$ or $t = 0$ and $(m,k) = (3,2)$.

2. Free nilpotent Lie algebras. Let $\mathfrak{g}$ denote a real Lie algebra. The central descending series of $\mathfrak{g}$ is the sequence of ideals $\{C^r(\mathfrak{g})\}_{r\geq 0}$, given by

$$C^0(\mathfrak{g}) = \mathfrak{g}, \quad C^r(\mathfrak{g}) = [\mathfrak{g}, C^{r-1}(\mathfrak{g})].$$

The Lie algebra $\mathfrak{g}$ is called *k-step nilpotent* if $C^k(\mathfrak{g}) = \{0\}$ but $C^{k-1}(\mathfrak{g}) \neq \{0\}$. In this case $C^{k-1}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x,y] = 0$ for all $y \in \mathfrak{g}\}$ is the center of the Lie algebra.

Let $\mathfrak{f}_m$ denote the free Lie algebra on m generators, with $m \geq 2$ (notice that a unique element spans an abelian Lie algebra). The quotient Lie algebra $\mathfrak{n}_{m,k} = \mathfrak{f}_m/C^k(\mathfrak{f}_m)$ is the *free k-step nilpotent Lie algebra on m generators*. The image of a generator set of $\mathfrak{f}_m$ by the quotient map is a *generator set* of $\mathfrak{n}_{m,k}$.

Given an ordered generator set of $\mathfrak{n}_{m,k}$, a Hall basis of $\mathfrak{n}_{m,k}$ is a basis of the Lie algebra constituted

2010 Mathematics Subject Classification. Primary 53D05; Secondary 17B01, 17B30, 22E25.

doi: 10.3792/pjaa.95.88

by monomials in the generator set [7,9]. The *length* of an element in the Hall basis is the degree of the corresponding monomial. Denote by $\mathfrak{p}(m,s)$ the subspace of $\mathfrak{n}_{m,k}$ spanned by the elements of the Hall basis of length s. Then $\mathfrak{n}_{m,k}$ is a graded Lie algebra since

$$\mathfrak{n}_{m,k} = \bigoplus_{s=1}^{k} \mathfrak{p}(m,s).$$

Moreover, the central descending series satisfies

$$C^r(\mathfrak{n}_{m,k}) = \oplus_{s=r+1}^{k} \mathfrak{p}(m,s).$$

Denote by $d_m(s)$ the dimension of $\mathfrak{p}(m,s)$. Inductively one gets [16]

$$(1) \qquad s \cdot d_m(s) = m^s - \sum_{r|s, r<s} r \cdot d_m(r), \quad s \geq 1.$$

The center $\mathfrak{z}(\mathfrak{n}_{m,k})$ of $\mathfrak{n}_{m,k}$ is the last term of the central descending series, namely $\mathfrak{p}(m,k)$. In particular, $\dim \mathfrak{z}(\mathfrak{n}_{m,2}) = d_m(2) = m(m-1)/2$ and $\dim \mathfrak{z}(\mathfrak{n}_{m,3}) = d_m(3) = m(m^2-1)/3$.

3. Free nilpotent Lie algebras and symplectic structures. We shall stress that the Lie algebras $\mathbf{R} \oplus \mathfrak{n}_{2,2}$ and $\mathfrak{n}_{3,2}$, appearing in the Main Theorem, possess symplectic structures.

The free Lie algebra $\mathfrak{n}_{2,2}$ is the so called Heisenberg Lie algebra of dimension three. The free 2-step nilpotent Lie algebra on three generators $\mathfrak{n}_{3,2}$ is six dimensional and has a basis $\{e_1, \dots, e_6\}$ with non-zero bracket relations

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6.$$

It is easy to check that both $\mathbf{R} \oplus \mathfrak{n}_{2,2}$ and $\mathfrak{n}_{3,2}$ are symplectic (or see, for instance, [3]).

The following is a well known obstruction to the existence of symplectic structures on nilpotent Lie algebras [2,8]. This is a particular instance of Lemma 2.1 in [4] (see also [1]).

Lemma 3.1. *Let ω be a symplectic structure on a nilpotent Lie algebra $\mathfrak{g}$, then*

$$(2) \qquad \dim \mathfrak{z}(\mathfrak{g}) \leq \dim(\mathfrak{g}/C^1(\mathfrak{g})).$$

Remark. Condition (2) is not sufficient for the existence of symplectic structures, in general. Indeed, any filiform Lie algebra satisfies (2), whilst there exist filiform Lie algebras admitting no symplectic structures (see [3]). Nevertheless, Pouseele and Tirao showed that (2) implies the existence of symplectic structures if $\mathfrak{g}$ is a 2-step nilpotent Lie algebra associated to a graph [14].

We apply Lemma 3.1 to free nilpotent Lie algebras $\mathfrak{n}_{m,k}$ and their trivial extensions.

Corollary 3.2. *Let $\mathfrak{g}$ be an even dimensional Lie algebra of the form $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$ for some $t = 0, 1$. If $\mathfrak{g}$ admits a symplectic structure then*

$$(3) \qquad \dim \mathfrak{z}(\mathfrak{n}_{m,k}) \leq m.$$

Proof. Since $\mathfrak{g}$ is a trivial extension of $\mathfrak{n}_{m,k}$, it is clear that $C^1(\mathfrak{g}) = C^1(\mathfrak{n}_{m,k})$, $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{n}_{m,k}) \oplus \mathbf{R}^t$ and $\dim(\mathfrak{g}/C^1(\mathfrak{g})) = \dim(\mathfrak{n}_{m,k}/C^1(\mathfrak{n}_{m,k})) + t$. Therefore (2) applied to $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$ reads

$$\dim \mathfrak{z}(\mathfrak{n}_{m,k}) + t \leq \dim(\mathfrak{n}_{m,k}/C^1(\mathfrak{n}_{m,k})) + t,$$

which clearly implies (3). □

The proof of the Main Theorem is obtained by showing that (3) is too restrictive for free nilpotent Lie algebras: except for some small m and k, the dimension of $\mathfrak{z}(\mathfrak{n}_{m,k})$ is much bigger than the size of the generator set.

Proof of the Main Theorem. Let $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$ be an even dimensional Lie algebra. We have already mentioned that $\mathfrak{g}$ admits symplectic structures if $\mathfrak{g}$ is either $\mathbf{R} \oplus \mathfrak{n}_{2,2}$ or $\mathfrak{n}_{3,2}$. It remains to prove that $\mathfrak{g}$ has no symplectic structures in the other cases.

Assume $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$ with $k \geq 4$, then it is possible to give a lower bound of $\dim \mathfrak{z}(\mathfrak{n}_{m,k})$ by constructing different elements of length k in a Hall basis $\mathcal{B}$ of $\mathfrak{n}_{m,k}$. Indeed, let $\{e_1, \dots, e_m\}$ be a set of generators of $\mathfrak{n}_{m,k}$, then the elements

$$[[[e_2, e_1], e_i], \cdots], e_i], \quad i = 1, \dots, m$$
$$[[[[e_2, e_1], e_1], e_m], \cdots], e_m]$$

are $m+1$ linearly independent vectors of length k in the Hall basis of $\mathfrak{n}_{m,k}$. In particular, these are in $C^{k-1}(\mathfrak{n}_{m,k})$ so we get

$$\dim \mathfrak{z}(\mathfrak{n}_{m,k}) \geq m + 1.$$

Then $\mathfrak{g}$ has no symplectic structures because of Corollary 3.2.

Suppose now that $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,3}$. The dimension of the center of $\mathfrak{n}_{m,3}$ is $m(m^2-1)/3$ (see Section 2), which is clearly bigger than m whenever $m \geq 3$ so $\mathfrak{g}$ possesses no symplectic structures in this case as consequence of Corollary 3.2. For $m = 2$, the Lie algebra $\mathfrak{g} = \mathbf{R} \oplus \mathfrak{n}_{2,3}$ is of dimension six and neither possesses symplectic structures [3].

Finally, consider $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,2}$. In this case, $\dim(\mathfrak{z}_{m,2}) = m(m-1)/2$ so $\dim(\mathfrak{z}_{m,2}) > m$ as soon as

$m \geq 4$. Thus, from Corollary 3.2 we get that $\mathfrak{g}$ is not symplectic if $m \geq 4$. □

Remark. Our Main Theorem here extends the results in [5, Example 4.9]. In fact, they have proved the non-existence of symplectic structures on 2-step free nilpotent Lie algebras, using a different technique. Those do not apply for every degree of nilpotency.

Denote by $N_{m,k}$ the simply connected Lie group corresponding to $\mathfrak{n}_{m,k}$. The structure constants of $\mathfrak{n}_{m,k}$ relative to a Hall basis are rational [15], therefore $N_{m,k}$ admits a cocompact discrete subgroup Γ [10]. Let $M = S^t \times \Gamma\backslash N_{m,k}$ be the even dimensional manifold with S^t being the circle for $t = 1$ or a point for $t = 0$. This is a nilmanifold with corresponding Lie algebra $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$. As stated in the introduction, $N_{m,k}$ admits a symplectic structure if and only if $\mathfrak{g} = \mathbf{R}^t \oplus \mathfrak{n}_{m,k}$ is symplectic. Therefore, the Main Theorem can be stated in terms of nilmanifolds.

Theorem. *Let $N_{m,k}$ be the simply connected nilpotent Lie group corresponding to the free k-step nilpotent Lie algebra on m generators and let Γ be a cocompact subgroup. The even dimensional compact manifold $M = S^t \times \Gamma\backslash N_{m,k}$ with $t = 0, 1$ admits symplectic structures if and only if $t = 1$ and $(m, k) = (2, 2)$, or $t = 0$ and $(m, k) = (3, 2)$.*

Acknowledgements. Part of this paper is from the author's Ph.D. Thesis. The author is grateful to Prof. Isabel Dotti for all her supervision work and to Prof. Gabriela Ovando for useful comments and suggestions. V. del Barco was supported by CONICET-PhD Fellowship and by FAPESP grant 2017/13725-4 during the preparation of the final version of the manuscript.

References

[1] O. Baues and V. Cortés, Symplectic Lie groups I–III, arXiv:1307.1629.

[2] C. Benson and C. S. Gordon, Kähler and symplectic structures on nilmanifolds, Topology **27** (1988), no. 4, 513–518.

[3] M. Goze and A. Bouyakoub, Sur les algèbres de Lie munies d'une forme symplectique, Rend. Sem. Fac. Sci. Univ. Cagliari **57** (1987), no. 1, 85–97.

[4] L. Cagliero and V. del Barco, Nilradicals of parabolic subalgebras admitting symplectic structures, Differential Geom. Appl. **46** (2016), 1–13.

[5] I. Dotti and P. Tirao, Symplectic structures on Heisenberg-type nilmanifolds, Manuscripta Math. **102** (2000), no. 3, 383–401.

[6] M. Goze and E. Remm, Symplectic structures on 2-step nilpotent Lie algebras, arXiv:1510.08212.

[7] M. Grayson and R. Grossman, Models for free nilpotent Lie algebras, J. Algebra **135** (1990), no. 1, 177–191.

[8] Z.-D. Guan, Toward a classification of compact nilmanifolds with symplectic structures, Int. Math. Res. Not. IMRN **2010**, no. 22, 4377–4384.

[9] M. Hall, A basis for free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc. **1** (1950), 575–581.

[10] A. I. Malcev, On a class of homogeneous spaces, Amer. Math. Soc. Translation **1951** (1951), no. 39.

[11] D. V. Millionschikov, Graded filiform Lie algebras and symplectic nilmanifolds, in *Geometry, topology, and mathematical physics*, Amer. Math. Soc. Transl. Ser. 2, 212, Adv. Math. Sci., 55, Amer. Math. Soc., Providence, RI, 2004, pp. 259–279.

[12] V. V. Morozov, Classification of nilpotent Lie algebras of sixth order, Izv. Vysš. Učebn. Zaved. Matematika **1958**, no. 4 (5), 161–171.

[13] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. of Math. (2) **59** (1954), 531–538.

[14] H. Pouseele and P. Tirao, Compact symplectic nilmanifolds associated with graphs, J. Pure Appl. Algebra **213** (2009), no. 9, 1788–1794.

[15] C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, 7, The Clarendon Press, Oxford University Press, New York, 1993.

[16] J. P. Serre, *Lie algebras and Lie groups, 1964 lectures given at Harvard University*, 2nd ed., Lecture Notes in Mathematics, 1500, Springer-Verlag, Berlin, 1992.