THE MODULI SPACE OF LEFT-INVARIANT METRICS OF A CLASS OF SIX-DIMENSIONAL NILPOTENT LIE GROUPS

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Dedicated to Carlos Olmos on the occasion of his 60th birthday

ABSTRACT. In this paper we determine the moduli space, up to isometric automorphism, of left-invariant metrics on a 6-dimensional Lie group H, such that its Lie algebra \mathfrak{h} admits a complex structure and has first Betti number equal to four. We also investigate which of these metrics are Hermitian and classify the corresponding complex structures.

1. INTRODUCTION

The present work concerns the study of the moduli space of left-invariant metrics on nilpotent Lie groups up to diffeomorphism. As it was proved by Wolf in [Wol63], and later generalized in [Ale71, GW88] for Riemannian solvmanifolds, this is equivalent to the study of the moduli space of left-invariant metrics up to isometric automorphism. Notice that by Mal'cev criterion, every compact nilmanifold is the quotient of a simply connected nilpotent Lie group by a discrete subgroup. So, the problem we approach is closely related with the problem of determining the moduli space of invariant metrics on compact nilmanifolds, up to diffeomorphism. There is some previous work addressing the problem of determining the moduli space of left-invariant metrics on Lie groups. One can mention the work of Ha-Lee [HL09] which solves the problem in dimension 3. In [Lau03], Lauret classifies all Lie groups with only one left-invariant metric up to isometry and scaling. Later, in [KTT11], Kodama, Takahara and Tamaru give another proof of Lauret's theorem and also study the case when the moduli space up to isometry and scaling has dimension 1. In [DS12], Di Scala classifies the moduli space of left-invariant metrics up to isometric isomorphism on the Iwasawa manifold. For semi-definite metrics, some progress has also been made, for instance Vukmirović classifies in [Vuk15] the pseudo-Riemannian left-metrics metrics on Heisenberg groups. In a recent paper [KT20], Kondo and Tamaru determine the moduli space up to isometry and scaling of Lorentzian left-invariant metrics on certain nilpotent Lie groups. Some of the above results use the so-called Milnor frames, a concept that dates back to the famous paper of Milnor [Mil76], for describing the Lie bracket on 3-dimensional Lie groups in terms of an orthonomal basis. In [HTT16], it is showed a closed retationship between the moduli space of left-invariant metrics and Milnor frames by giving several examples of Milnor-type theorems.

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Geometric structures associated to low-dimensional Lie groups with left-invariant metrics have been widely studied. For the case of 6-dimensional nilpotent Lie groups, particular attention has been paid to the Iwasawa manifold $\mathcal{I} = \Gamma \setminus H$, which is a compact quotient of the 3-dimensional complex Heisenberg group H. The Hermitian geometry of \mathcal{I} , with a standard metric, was studied in [AGS97, AGS01] and [KS04]. The classification of Di Scala [DS12], of the moduli space of metrics up to isometric automorphism, relies on fixing a distinguished complex structure on the Lie algebra \mathfrak{h} of H, which allows to determine the automorphism group Aut(\mathfrak{h}) in an elegant way on the canonical basis.

In [Sal01], Salamon classified all 6-dimensional Lie algebras \mathfrak{g} which admit a complex structure. Such Lie algebras are grouped according to the first Betti number of \mathfrak{g} . In particular, 3-dimensional complex Heisenberg Lie algebra belongs to the class whose first Betti number is equal to 4. This class contains five Lie algebras that are characterized by the property that dim[$\mathfrak{g}, \mathfrak{g}$] = 2. In the notation of [Sal01], they are $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_6$ and \mathfrak{h}_9 (see Section 2.3). The complex Heisenberg Lie algebra is \mathfrak{h}_5 . All these Lie algebras are 2-step nilpotent, with the exception of \mathfrak{h}_9 which is 3-step nilpotent.

As a natural continuation of the work in [DS12], in this paper we deal with 6-dimensional, simply connected, nilpotent Lie groups H which admit a left-invariant complex structure and their Lie algebras \mathfrak{h} have first Betti number equal to 4. Our main goal is to classify the moduli space of left-invariant metrics, up to isometric automorphism, for this particular family. Left-invariant metrics on H are in a one-to-one correspondence with the inner products on \mathfrak{h} . Moreover, $\operatorname{Aut}(H)$ is isomorphic to $\operatorname{Aut}(\mathfrak{h})$ and the classification of left-invariant metrics on H up to automorphism reduces to the classification of inner products of \mathfrak{h} , up to an automorphism of \mathfrak{h} .

It is important to observe that the methods developed in [DS12] cannot be directly adapted to any of the other Lie algebras studied here. In fact, while Aut(\mathfrak{h}_5) is a complex Lie group, this is not true for any of the other Lie algebras we are dealing with. For each case, we explicitly find the automorphism group Aut(\mathfrak{h}) (for the case of \mathfrak{h}_4 and \mathfrak{h}_5 this was also done in [Mag07] by means of computational methods, and in [Saa96] for *H*-type Lie algebras). In this way, we are able to describe the moduli space $\mathcal{M}(H)/\sim$ of left-invariant metrics on *H* up to isometric automorphisms. It is interesting to notice that the only case in which $\mathcal{M}(H)/\sim$ is a differentiable manifold is when the Lie algebra of *H* is \mathfrak{h}_9 . Our classification also shows that the moduli space up to isometry and scaling of the Lie algebra \mathfrak{h}_6 has dimension 1. This extends the result in [KTT11], as the Lie algebra \mathfrak{h}_6 is not present among the examples there of Lie groups with 1-dimensional moduli space up to isometry and scaling. We also obtain the full isometry group Isom(*H*, *g*) associated to each left-invariant metric *g* on *H*.

Another interesting problem is to determine which of the metrics g are Hermitian, that is, when there exists an invariant complex structure J on H such that (g, J) is an Hermitian structure. This turns out to be a very difficult computational problem. Even though the complex structures on 6-dimensional nilpotent Lie algebras \mathfrak{h} were classified in [COUV14], it is very difficult to explicitly obtain the form of a particular complex structure J on a given basis of \mathfrak{h} . For the case of the Iwasawa manifold, the set \mathcal{C} of complex structures compatible with a standard metric and orientation of \mathcal{I} was described in [AGS01] by means of topological methods. The authors show there that \mathcal{C} is the disjoint union of the standard complex structure J_0 and a 2-sphere.

In this paper we give a complete classification of the Hermitian structures for $\mathfrak{h}_4, \mathfrak{h}_5$ and \mathfrak{h}_6 . The problem becomes wild for \mathfrak{h}_2 and \mathfrak{h}_9 , however we obtain some interesting partial results. For \mathfrak{h}_2 we prove that every left-invariant metric admits a finite number of compatible complex structures, and for \mathfrak{h}_9 we include a qualitative analysis and prove that there are left-invariant metrics which are not Hermitian.

We hope that the methods developed here will be useful to study the remaining cases in the classification of [Sal01].

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2. Preliminaries

2.1. The moduli space of left-invariant metrics. Let H be a simply connected Lie group with Lie algebra \mathfrak{h} . Every left-invariant metric on H is uniquely determined by a (positive definite) inner product on \mathfrak{h} , so, the set $\mathcal{M}(H)$ of left-invariant metrics on H is identified, after the choice of a basis of \mathfrak{h} , with the symmetric space $\operatorname{Sym}_n^+ = \operatorname{GL}_n(\mathbb{R})/\operatorname{O}(n)$, where $n = \dim H$. Recall that the group $\operatorname{Aut}(H)$ of automorphisms of Hacts on the right on $\mathcal{M}(H)$ by

$$g \cdot \varphi = \varphi_*(g), \tag{2.1}$$

for $g \in \mathcal{M}(H)$ and $\varphi \in \operatorname{Aut}(H)$, where $\varphi_*(g)(u, v) = g(d\varphi(u), d\varphi(v))$, i.e., $\varphi_*(g)$ is the pullback of g by φ . The moduli space of left-invariant metrics of H up to isometric automorphisms is $\mathcal{M}(H)/\sim$, where \sim is the equivalence relation induced by the action given in (2.1). Since H is simply connected, $\operatorname{Aut}(H)$ is isomorphic to the group $\operatorname{Aut}(\mathfrak{h})$ of automorphisms of its Lie algebra \mathfrak{h} , which we can identify with a subgroup of $\operatorname{GL}_n(\mathbb{R})$. If we think of Sym_n^+ as the set of symmetric positive definite matrices of size $n \times n$, then the action of $\operatorname{Aut}(H)$ on $\mathcal{M}(H)$ is equivalent to the action of $\operatorname{Aut}(\mathfrak{h})$ on Sym_n^+ given by

$$X \cdot A = A^T X A,$$

for $X \in \operatorname{Sym}_n^+$, $A \in \operatorname{Aut}(\mathfrak{h})$.

2.2. Complex structures. In the same spirit as in the previous paragraphs, one can identify the set $\mathcal{C}(H)$ of left-invariant complex structures on H with

$$\mathcal{C}(\mathfrak{h}) = \{ J \in \operatorname{End}_{\mathbb{R}}(\mathfrak{h}) : J^2 = -\operatorname{id}_{\mathfrak{h}} \text{ and } N_J = 0 \}$$

where N_J is the so-called Nijenhuis tensor of J, which is given for $X, Y \in \mathfrak{h}$ by

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X, JY] - [X,Y].$$

In the same manner, left-invariant abelian structures on H are identify with the subset

$$\mathcal{A}(\mathfrak{h}) = \{ J \in \mathcal{C}(\mathfrak{h}) : [JX, JY] = [X, Y] \text{ for all } X, Y \in \mathfrak{h} \}$$

of $\mathcal{C}(\mathfrak{h})$. We say that two complex (resp. abelian) structures are equivalent if they are conjugated by an element of Aut(\mathfrak{h}). It is customary to consider the left-action of Aut(\mathfrak{h}) on $\mathcal{C}(\mathfrak{h})$, which is given by

$$\varphi \cdot J = \varphi J \varphi^{-1}$$

for $J \in \mathcal{C}(\mathfrak{h})$ and $\varphi \in \operatorname{Aut}(\mathfrak{h})$. Recall however that the pullback action of $\operatorname{Aut}(H)$ on $\mathcal{C}(H)$ induces the right-action of $\operatorname{Aut}(\mathfrak{h})$ on $\mathcal{C}(\mathfrak{h})$ given by $J \cdot \varphi = \varphi^{-1} J \varphi$. These two actions have the same orbits and leave $\mathcal{A}(\mathfrak{h})$ invariant. This is not true, in general, for the left- and right-actions of $\operatorname{Aut}(\mathfrak{h})$ on Sym_n^+ .

2.3. Nilpotent Lie algebras of dimension 6. In this section we shall recall some relevant notation and useful properties of 6-dimensional nilpotent Lie algebras which will be used in the whole paper. For further details we refer the reader to [Sal01]. Let \mathfrak{h} be a 6-dimensional Lie algebra, $\mathcal{B} = \{e_1, \ldots, e_6\}$ a basis of \mathfrak{h} and $\mathcal{B}^* = \{e^1, \ldots, e^6\}$ the dual basis of \mathfrak{h}^* . For each $i = 1, \ldots, 6$, we write

$$de^k = \sum_{i < j} c^k_{ij} \, e^{ij},$$

where e^{ij} denotes the exterior product $e^i \wedge e^j$. Since \mathfrak{h} is nilpotent and 6-dimensional), one can choose \mathcal{B} in such a way that $c_{ij}^k \in \{0, 1\}$ for $i, j, k \in \{1, \ldots, 6\}$ and such that $c_{ij}^k = 0$ for i, j < k. In this way, one can completely determine \mathfrak{h} by knowing the differentials

$$de^1, de^2, \ldots, de^6$$

since this information together with the formula $d\theta(X, Y) = -\theta([X, Y])$, for $\theta \in \Lambda^1(\mathfrak{h})$, allow us to reconstruct all the Lie brackets of \mathfrak{h} . Following Salamon's notation, if $de^k = e^{i_1 j_1} + \cdots + e^{i_l j_l}$ we shall simply denote it by $i_1 j_1 + \cdots + i_l j_l$. In this way, for example, we will write

$$\mathfrak{h} = (0, 0, 0, 0, 0, 12 + 34)$$

for the Lie algebra that admits a basis \mathcal{B} such that $de^6 = e^{12} + e^{34}$, and hence on which the only non trivial brackets are $[e_1, e_2] = [e_3, e_4] = -e_6$.

As we indicated in the Introduction, we are interested on those 6-dimensional Lie algebras which admit a complex structure and has their first Betti number equal to 4. These are the Lie algebras which, in the classification of Salamon, belong to the same class of the Lie algebra of the Iwasawa manifold. With the notation presented above, there are exactly five 6-dimensional nilpotent Lie algebras with these properties:

$$\begin{aligned} \mathfrak{h}_{2} &= (0, 0, 0, 0, 12, 34) \\ \mathfrak{h}_{4} &= (0, 0, 0, 0, 12, 14 + 23) \\ \mathfrak{h}_{5} &= (0, 0, 0, 0, 13 + 42, 14 + 23) \\ \mathfrak{h}_{6} &= (0, 0, 0, 0, 12, 13) \\ \mathfrak{h}_{9} &= (0, 0, 0, 0, 12, 14 + 25). \end{aligned}$$

$$(2.2)$$

Observe that in all cases $[\mathfrak{h}, \mathfrak{h}] = \operatorname{span}\{e_5, e_6\}$. The Lie algebra \mathfrak{h}_5 corresponds to the Iwasawa manifold, which was studied in [DS12].

In order to find the moduli spaces $\mathcal{M}(H)/\sim$, for a nilpotent simply connected 6dimensional Lie group H whose Lie algebra \mathfrak{h} is one of the Lie algebras listed above, we will determine in the following sections the corresponding full automorphism groups. The following lemma picks some common behaviour present in most of these groups.

Lemma 2.1. Let \mathfrak{h} be a 2-step nilpotent Lie algebra of dimension 6 with first Betti number equal to 4. Let e_1, \ldots, e_6 be a basis of \mathfrak{h} such that $[\mathfrak{h}, \mathfrak{h}]$ is spanned by e_5, e_6 . Then there exist an algebraic subgroup $G \subset \operatorname{GL}_4(\mathbb{R})$ and a representation $\Delta : G \to \operatorname{GL}_2(\mathbb{R})$ such that $\operatorname{Aut}(\mathfrak{h}) \simeq \mathbb{R}^8 \rtimes G$. More precisely, in the above basis, every automorphism of \mathfrak{h} has the form

$$\begin{pmatrix} A & 0\\ M & \Delta(A) \end{pmatrix}$$
(2.3)

for some $A \in G$ and $M \in \mathbb{R}^{2 \times 4} \simeq \mathbb{R}^8$.

Proof. Clearly every automorphism of \mathfrak{h} leaves $[\mathfrak{h}, \mathfrak{h}]$ invariant. The group G is induced by $\operatorname{Aut}(\mathfrak{h})$ via the projection $\mathfrak{h} \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$. So we can write any automorphism as $\begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$. Since $[\mathfrak{h}, \mathfrak{h}] = \operatorname{span}_{\mathbb{R}} \{ e_5, e_6 \}$, B depends only on A, say $B = \Delta(A)$, and the group structure of $\operatorname{Aut}(\mathfrak{h})$ forces Δ to be a representation of G in \mathbb{R}^2 . Finally, it is easy to see that every linear map of the form (2.3) preserves the Lie bracket of \mathfrak{h} . Recall that with these identifications, $\mathbb{R}^{2\times 4} \simeq \mathbb{R}^8$ is an abelian normal subgroup of $\operatorname{Aut}(\mathfrak{h})$.

3. The case of
$$\mathfrak{h}_5 = (0, 0, 0, 0, 13 + 42, 14 + 23)$$

Let e_1, \ldots, e_6 be the basis of the Lie algebra \mathfrak{h}_5 such that the only non trivial Lie brackets are

$$[e_1, e_3] = [e_4, e_2] = -e_5,$$
 $[e_1, e_4] = [e_2, e_3] = -e_6.$

It was shown in [DS12] that $\operatorname{Aut}_0(\mathfrak{h}_5)$, the connected component of the identity of $\operatorname{Aut}(\mathfrak{h}_5)$, is isomorphic to a twisted (in the sense of Lemma 2.1) semi-direct product

$$\mathbb{C}^{2\times 2} \rtimes \mathrm{GL}_2(\mathbb{C})$$

and that the moduli space $\mathcal{M}(\mathfrak{h}_5)$ is homeomorphic to the product $T \times \operatorname{Sym}_2^+ / \sigma$, where T is the triangle $\{(r,s): 0 < s \leq r \leq 1\}$ and $\sigma \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} E & -F \\ -F & G \end{pmatrix}$. More precisely, in the standard basis e_1, \ldots, e_6 , every left-invariant metric is represented by a unique matrix of the form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & 0 & F & G \end{pmatrix},$$
(3.1)

where $0 < s \le r \le 1$, $EG - F^2 > 0$ and $F \ge 0$.

In this section we will find the whole isometry group of each of the metrics (3.1). We start by recalling the following well known fact that will be used in the sequel.

Remark 3.1. Let H be a connected nilpotent Lie group endowed with a left-invariant metric g. Let us denote by \mathfrak{h} the Lie algebra de H. Then by [Wol63] (see also [Wil82]), the full isometry group of H is given by $\text{Isom}(H,g) = H \rtimes K$, where $K = \text{Aut}(\mathfrak{h}) \cap O(\mathfrak{h},g)$ under the usual identifications.

Theorem 3.2. Let g be the left-invariant metric on H_5 given in (3.1). Then the full isometry group group of g is given by

$$\operatorname{Isom}(H_5,g) = H_5 \rtimes K$$

where $K \simeq \operatorname{Aut}(\mathfrak{h}) \cap O(\mathfrak{h}, g)$. The different subgroups K, according to r, s, E, F, G are listed in Table 1.

It is important to note that the two cases when K = O(2) in Table 1 correspond to different subgroups of Aut(\mathfrak{h}_5). These inclusions will become clear in the proof of Theorem 3.2.

K	(r,s)	E, F, G
$\mathbb{Z}_2 imes \mathbb{Z}_2$	0 < s < r < 1	$F \neq 0$
$\mathbb{Z}_2 imes \mathbb{Z}_2 imes \mathbb{Z}_2$	0 < s < r < 1	F = 0
$\mathbb{Z}_2 imes \mathbb{Z}_2$	0 < s < r = 1	$F \neq 0$
$\mathbb{Z}_2 imes \mathbb{Z}_2 imes \mathbb{Z}_2$	0 < s < r = 1	$F = 0, G \neq E$
O(2)	0 < s < r = 1	F = 0, G = E
O(2)	0 < s = r < 1	$F \neq 0$
$O(2) \times \mathbb{Z}_2$	0 < s = r < 1	F = 0
$\mathrm{SU}(2) \rtimes \mathbb{Z}_2$	s = r = 1	$F \neq 0$
$(\mathrm{SU}(2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	s = r = 1	$F = 0, G \neq E$
$U(2) \rtimes \mathbb{Z}_2$	s = r = 1	F = 0, G = E

Table 1. Isotropy subgroups of $\text{Isom}(H_5, g)$.

Proof. Let g be given as in (3.1). By using Remark 3.1, in order to determine the full isometry group, we only need to compute the automorphisms of \mathfrak{h}_5 which are isometric with respect to g. Recall that from [DS12], every $\varphi \in \operatorname{Aut}_0(\mathfrak{h}_5)$ has, in the standard basis, the form

$$\varphi = \begin{pmatrix} A & 0\\ M & \Delta(A) \end{pmatrix}, \tag{3.2}$$

where, under de usual identifications, $A \in \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{GL}_4(\mathbb{R})$, $M \in \mathbb{R}^{2 \times 4}$ and $\Delta(A) = \det A \in \mathrm{GL}_1(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{R})$. Moreover, the full automorphism group of \mathfrak{h}_5 has two connected components:

$$\operatorname{Aut}(\mathfrak{h}_5) = \operatorname{Aut}_0(\mathfrak{h}_5) \cup \psi \operatorname{Aut}_0(\mathfrak{h}_5),$$

where $\psi = \text{diag}(1, -1, 1, -1, 1, -1)$.

Notice that if φ as in (3.2) preserves g, then we have that M = 0, and since $\operatorname{GL}_2(\mathbb{C})$ is connected, $A \in \operatorname{SO}_{r,s}(4)$ and $\Delta(A) \in \operatorname{SO}_{E,F,G}(2)$, where these are the orthogonal groups determined by the 4×4 and 2×2 nontrivial blocks in g. Moreover, we must have $\operatorname{det}_{\mathbb{R}} \Delta(A) = 1$ and so $\Delta(A) \in \operatorname{SO}(2) \cap \operatorname{SO}_{E,F,G}(2)$, which implies that either $\Delta(A) = \pm I_2$ or F = 0 and G = E. So, the difficult part of the proof is describing $\operatorname{GL}_2(\mathbb{C}) \cap \operatorname{SO}_{r,s}(4)$. Let us write

$$A = \begin{pmatrix} a_1 & -a_2 & b_1 & -b_2 \\ a_2 & a_1 & b_2 & b_1 \\ c_1 & -c_2 & d_1 & -d_2 \\ c_2 & c_1 & d_2 & d_1 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \in \operatorname{GL}_2(\mathbb{C})$$
(3.3)

and g' = diag(1, r, 1, s) = diag(R, S), where R = diag(1, r) and S = diag(1, s). With these identifications, we can write the orthogonality condition $A^T g' A = g'$ as

$$\begin{pmatrix} \bar{z}_1 R z_1 + \bar{z}_3 S z_3 & \bar{z}_1 R z_2 + \bar{z}_3 S z_4 \\ \bar{z}_2 R z_1 + \bar{z}_4 S z_3 & \bar{z}_2 R z_2 + \bar{z}_4 S z_4 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$$
(3.4)

After a close inspection, we notice that the last two entries on the diagonal of the left side are $b_1^2 + rb_2^2 + d_1^2 + sd_2^2$ and $rb_1^2 + b_2^2 + sd_1^2 + d_2^2$. We then equal these values to the corresponding entries on the diagonal of S in order to get that

$$(r-s)b_1^2 + (1-rs)b_2^2 + (1-s^2)d_2^2 = 0.$$
(3.5)

So we have to study several cases according to the values of r, s.

Case 0 < s < r < 1. This is the generic case and according to (3.5) we have $b_1 = b_2 = d_2 = 0$, which forces $d_1 = \pm 1$ and $z_3 = 0$. Therefore, $\bar{z}_1Rz_1 = R$, and as we noticed before, since $r \neq 1$, this implies $z_1 = \pm 1$ (i.e. $a_1 = \pm 1$ and $a_2 = 0$). Now we check for isometric automorphisms in the other connected component. Recall that these are all of the form $\psi\varphi$ with $\varphi \in \operatorname{Aut}_0(\mathfrak{h}_5)$. If we keep the notation (3.2) and call $\psi' = \operatorname{diag}(1, -1, 1, -1)$, then we find that $\psi'A$ preserves g'. Since ψ' preserves g', we conclude that A also preserves g'. Thus, if $\psi\varphi$ is an isometric automorphism then F = 0.

Case 0 < s < r = 1. We use (3.5) again in order to conclude that $z_2 = z_3 = 0$ and $z_4 = \pm 1$. Since r = 1, we must have $\bar{z}_1 z_1 = 1$, which with our identifications means that $z_1 \in SO(2)$. Also, since $\Delta(A) = \pm z_1 \in SO_{E,F,G}(2)$ we see that $F \neq 0$ or $G \neq E$ imply $z_1 = \pm 1$. When F = 0 and G = E, we trivially have $SO_{E,0,E}(2) = SO(2)$. Finally, with the same argument as in the previous case, we can find isometric automorphisms outside the connected component of the identity of Aut(\mathfrak{h}_5) if and only if F = 0. Moreover, if in addition G = E, then ψ is an isometric automorphism which lies outside the connected component of $O(2) = O_{E,0,E}(2)$.

Case 0 < s = r < 1. In this case equation (3.5) becomes $(1 - r^2)(b_2^2 + d_2^2) = 0$, which means $z_2, z_4 \in \mathbb{R}$. With the same idea we used to derive (3.5), we can also show that $z_1, z_3 \in \mathbb{R}$. Now looking back to (3.4), with R = S we get that

$$a_1^2 + c_1^2 = b_1^2 + d_1^2 = 1.$$

From this, it is not hard to see that the subgroup of $\operatorname{Aut}_0(\mathfrak{h}_4)$ preserving the metric is the intersection of $\operatorname{GL}_2(\mathbb{R}) \subset \operatorname{GL}_2(\mathbb{C}) \subset \operatorname{GL}_4(\mathbb{R})$ with O(4), which is isomorphic to O(2). Finally, we will have isometric automorphisms other that the ones in $\operatorname{Aut}_0(\mathfrak{h}_5)$ if and only if ψ is isometric, which only happens when F = 0. Notice that in this case ψ commutes with O(2), which gives us that the isotropy group of the full isometry group is isomorphic to O(2) $\times \mathbb{Z}_2$.

Case 0 < s = r = 1. This is the case with most symmetries. It is immediate that A as in (3.3) belongs to U(2) = GL₂(\mathbb{C}) \cap O(4). Since $\Delta(A) \in U(1)$, if $F \neq 0$, then $\Delta(A) = \pm 1$ and so $A \in SU(2) \rtimes \mathbb{Z}_2$. If F = 0 and $G \neq E$, we also have that $A \in SU(2) \rtimes \mathbb{Z}_2$. But ψ is an isometric automorphism, then we have two connected component for the isometric automorphisms. Finally, if F = 0 and G = E, then every automorphism in U(2) preserves the metric, and hence the isometric automorphisms are isomorphic to U(2) $\rtimes \mathbb{Z}_2$. \Box

4. The case of $\mathfrak{h}_6 = (0, 0, 0, 0, 12, 13)$

4.1. Automorphism group. Let \mathfrak{h}_6 be the 6-dimensional 2-step nilpotent real Lie algebra corresponding to (0, 0, 0, 0, 12, 13) in Salamon notation [Sal01]. That is, we have a canonical basis e_1, \ldots, e_6 such that the only non-trivial brackets are $[e_1, e_2] = -e_5$ and $[e_1, e_3] = -e_6$. Equivalently, if $d: \mathfrak{h}_6^* \to \Lambda^2(\mathfrak{h}_6^*)$ is the exterior derivative on left-invariant forms, then ker d is spanned by e^1, \ldots, e^4 and $de^5 = e^{12}$, $de^6 = e^{13}$.

It is known from [Sal01] that \mathfrak{h}_6 admits an invariant complex structure. Moreover, according to [COUV14] there is a unique invariant complex structure up to equivalence on \mathfrak{h}_6 . This means that $\operatorname{Aut}(\mathfrak{h}_6)$ acts transitively by conjugation on the set $\mathcal{C}(\mathfrak{h}_6)$ of invariant complex structures. Recall that the standard almost complex structure associated to the multiplication by $\sqrt{-1}$ via the identification $\mathfrak{h}_6 \simeq \mathbb{R}^6 \simeq \mathbb{C}^3$ is not integrable.

Lemma 4.1. The invariant almost complex structure $J : \mathfrak{h}_6 \to \mathfrak{h}_6$ determined by the equations $Je_1 = e_4$, $Je_2 = e_3$ and $Je_5 = e_6$ is integrable.

Proof. Let us denote $\Lambda^{1,0} = \Lambda^{1,0}(\mathfrak{h}_6^*)_{\mathbb{C}}$ the *i*-eigenspace of J^* on the complexification of \mathfrak{h}_6 . Notice that J^* is the transpose of J, so the equations determining J^* are $J^*e^1 = -e^4$, $J^*e^2 = -e^3$ and $J^*e^5 = -e^6$. According to [COUV14], J is integrable if and only if there exists a basis $\omega^1, \omega^2, \omega^3$ of $\Lambda^{1,0}$ such that $d\omega^1 = d\omega^2 = 0$ and

$$d\omega^3 = \omega^1 \wedge \omega^2 + \omega^1 \wedge \bar{\omega}^1 + \omega^1 \wedge \bar{\omega}^2 = \omega^1 \wedge (\bar{\omega}^1 + 2\operatorname{Re}(\omega^2)).$$
(4.1)

The standard basis of $\Lambda^{1,0}$ associated with the canonical basis of \mathfrak{h}_6 is given by

$$\begin{split} \eta^1 &= e^1 - iJ^*e^1 = e^1 + ie^4, \\ \eta^2 &= e^2 - iJ^*e^2 = e^2 + ie^3, \\ \eta^3 &= e^5 - iJ^*e^5 = e^5 + ie^6. \end{split}$$

Suppose that there exist $\omega^1, \omega^2, \omega^3$ as in (4.1). We can assume that $\omega^3 = \eta^3$, and so $d\omega^3 = e^{12} + ie^{13}$. We can further assume that ω^1, ω^2 belong to the subspace spanned by η^1, η^2 . If we write $\omega^1 = A\eta^1 + B\eta^2$ then, taking the imaginary part of both sides of equation (4.1), we get that that A = 0 and $B \neq 0$. One can also assume that B = 1 and so, $\omega^1 = \eta^2$. Let us write $\omega^2 = C\eta^1 + D\eta^2$. Then replacing it in (4.1) one gets

$$e^{12} + ie^{13} = (e^2 + ie^3) \wedge (2Ce^1 + (2D+1)e^2 - ie^3)$$
$$= -2C(e^{12} + ie^{13}) - 2i(D+1)e^{23}$$

So $C = -\frac{1}{2}$, D = -1 and $\omega^1 = \eta^2$, $\omega^2 = -\frac{1}{2}\eta^1 - \eta^2$, $\omega^3 = \eta^3$ is the basis of $\Lambda^{1,0}$ we were looking for.

Lemma 4.2. If $f \in Aut(\mathfrak{h}_6)$ then:

(1)
$$e^{1}(f(e_{j})) = 0$$
 for $j = 2, ..., 6$;
(2) $e^{2}(f(e_{j})) = 0$ for $j = 4, 5, 6$;
(3) $e^{3}(f(e_{j})) = 0$ for $j = 4, 5, 6$;
(4) $e^{4}(f(e_{j})) = 0$ for $j = 5, 6$;
(5) $e^{5}(f(e_{j})) = e^{1}(f(e_{1}))e^{2}(f(e_{j-3}))$, for $j = 5, 6$;
(6) $e^{6}(f(e_{j})) = e^{1}(f(e_{1}))e^{3}(f(e_{j-3}))$, for $j = 5, 6$.

Proof. Since the center of \mathfrak{h}_6 is spanned by e_4, e_5, e_6 and f leaves the center invariant, we get that $e^k(f(e_j)) = 0$ for all k = 1, 2, 3 and j = 4, 5, 6. Also, since dim(ker ad_{e_j}) is preserved under automorphisms, $e^1(f(e_j)) = 0$ if $j \ge 2$. These two observations together prove (1), (2) and (3). Part (4) follows from Lemma 2.1. For parts (5) and (6) recall that $e_5 = -[e_1, e_2]$. Then

$$f(e_5) = -[f(e_1), f(e_2)]$$

= -[e¹(f(e_1))e_1, e²(f(e_2))e_2 + e³(f(e_2))e_3]
= e¹(f(e_1))e²(f(e_2))e_5 + e¹(f(e_1))e³(f(e_2))e_6

With the same argument we can see that

$$f(e_6) = e^1(f(e_1))e^2(f(e_3))e_5 + e^1(f(e_1))e^3(f(e_3))e_6.$$

Lemma 4.3. Let J be the complex structure of Lemma 4.1. Then the isotropy subgroup at J of $Aut(\mathfrak{h}_6)$ is isomorphic to

$$\mathbb{R}^4 \rtimes_{\varphi} (\mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{C})),$$

where $\varphi : \operatorname{GL}_1(\mathbb{R}) \times \operatorname{GL}_1(\mathbb{C}) \to \operatorname{GL}_4(\mathbb{R})$ is the representation given by

$$\varphi(r, a + ib) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & r & 0 & 0 \\ -b & 0 & a & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

for $r \neq 0$ and $a^2 + b^2 \neq 0$.

Proof. Let $f \in Aut(\mathfrak{h}_6)$ and identify it with its matrix (a_{ij}) in the basis e_1, \ldots, e_6 . From Lemma 4.2 we must have

$$a_{1j} = 0 \text{ for } j \ge 2, \quad a_{2j} = a_{3j} = 0 \text{ for } j \ge 4, \quad a_{45} = a_{46} = 0,$$

$$a_{55} = a_{11}a_{22}, \qquad a_{56} = a_{11}a_{23}, \qquad a_{65} = a_{11}a_{32}, \quad a_{66} = a_{11}a_{33}.$$
(4.2)

If, in addition, we ask f to be in the isotropy of J, then f must commute with the matrix associated to J, and thus it has the form

$$f = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & -a_{23} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & 0 & 0 \\ a_{51} & a_{52} & -a_{62} & -a_{61} & a_{11}a_{22} & a_{11}a_{23} \\ a_{61} & a_{62} & a_{52} & a_{51} & -a_{11}a_{23} & a_{11}a_{22} \end{pmatrix}.$$
 (4.3)

with $a_{11} \neq 0$ and $a_{22}^2 + a_{23}^2 \neq 0$. Moreover, every linear map of the form (4.3) is an automorphism of \mathfrak{h}_6 . In order to see this, we can show that $f^* \circ d = d \circ f^*$. Since ker d is spanned by e^1, e^2, e^3, e^4 , clearly $f^*(de^j) = d(f^*(e^j))$ for $1 \leq j \leq 4$. Also,

$$d(f^*(e^5)) = a_{11}a_{22}e^{12} + a_{11}a_{23}e^{13}$$

= $a_{11}e^1 \wedge (a_{22}e^2 + a_{23}e^3)$
= $f^*(e^1) \wedge f^*(e^2)$
= $f^*(e^{12}) = f^*(de^5)$

and similarly $d(f^*(e^6)) = f^*(de^6)$. Hence f commutes with J if and only it has the form (4.3).

Finally, notice that $\operatorname{Aut}(\mathfrak{h}_6)_J = K \rtimes H$ is the inner semi-direct product of the normal subgroup

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_{51} & a_{52} & -a_{62} & -a_{61} & 1 & 0 \\ a_{61} & a_{62} & a_{52} & a_{51} & 0 & 1 \end{pmatrix} : a_{51}, a_{52}, a_{61}, a_{62} \in \mathbb{R} \right\} \simeq \mathbb{R}^4$$

and the subgroup

$$H = \left\{ \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & -a_{23} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11}a_{22} & a_{11}a_{23} \\ 0 & 0 & 0 & 0 & -a_{11}a_{23} & a_{11}a_{22} \end{pmatrix} : a_{11} \neq 0, a_{22}^2 + a_{23}^2 \neq 0 \right\}$$
$$\simeq \operatorname{GL}_1(\mathbb{R}) \times \operatorname{GL}_1(\mathbb{C}).$$

Now one can easily check that $\operatorname{Aut}(\mathfrak{h}_6)_J \simeq \mathbb{R}^4 \rtimes_{\varphi} (\operatorname{GL}_1(\mathbb{R}) \times \operatorname{GL}_1(\mathbb{C}))$ as stated. \Box

It is convenient to introduce some notation before stating the main result of this section. Let us consider the presentation of the 5-dimensional Heisenberg Lie group $\text{Heis}_2 = \{(x, y, z) : x, y \in \mathbb{R}^2, z \in \mathbb{R}\}$ with the multiplication given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + y \cdot x')$$

and let G be the subgroup of $GL_4(\mathbb{R})$ consisting of all the matrices in block form

$$A = \begin{pmatrix} r & & \\ x & \tilde{A} & \\ z & y^T & s \end{pmatrix}$$
(4.4)

where $r, s \in \mathbb{R} - \{0\}$, $\tilde{A} \in \mathrm{GL}_2(\mathbb{R})$, $x, y \in \mathbb{R}^2 \simeq \mathbb{R}^{2 \times 1}$ and $z \in \mathbb{R}$. It is not hard to see that

 $G \simeq (\operatorname{Heis}_{2} \rtimes_{\varphi_{1}} \operatorname{GL}_{2}(\mathbb{R})) \rtimes_{\varphi_{2}} (\operatorname{GL}_{1}(\mathbb{R}) \times \operatorname{GL}_{1}(\mathbb{R}))$ (4.5)

where $\varphi_1 : \operatorname{GL}_2(\mathbb{R}) \to \operatorname{Aut}(\operatorname{Heis}_2)$ and $\varphi_2 : \operatorname{GL}_1(\mathbb{R}) \times \operatorname{GL}_1(\mathbb{R}) \to \operatorname{Aut}(\operatorname{Heis}_2 \rtimes_{\varphi_1} \operatorname{GL}_2(\mathbb{R}))$ are the Lie groups morphisms given by

$$\varphi_1(\tilde{A})(x, y, z) = (\tilde{A}x, (\tilde{A}^{-1})^T y, z)$$
$$\varphi_2(r, s)(x, y, z, \tilde{A}) = \left(\frac{x}{r}, s(\tilde{A}^{-1})^T y, \frac{sz}{r}, \tilde{A}\right)$$

Let us also consider the Lie groups epimorphism $\Delta: G \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\Delta(A) = r\hat{A}.\tag{4.6}$$

Recall that after the identification of G given by (4.5), the kernel of Δ is a normal subgroup isomorphic to $\operatorname{Heis}_2 \rtimes \operatorname{GL}_1(\mathbb{R})$.

Theorem 4.4. Let G be the Lie subgroup of $GL_4(\mathbb{R})$ defined in (4.4). There exists an isomorphism of Lie groups

$$\operatorname{Aut}(\mathfrak{h}_6) \simeq \mathbb{R}^{2 \times 4} \rtimes_{\varphi} G,$$

where $\mathbb{R}^{2\times 4}$ is the abelian Lie group of 2×4 matrices and $\varphi : G \to \mathrm{GL}(\mathbb{R}^{2\times 4})$ is given by $\varphi(A)M = \Delta(A)MA^{-1}$, being Δ defined as in (4.6). Moreover, every automorphism of \mathfrak{h}_6 is represented in the canonical basis by a matrix of the form

$$\begin{pmatrix} A & 0\\ M & \Delta(A) \end{pmatrix} \tag{4.7}$$

where $A \in G$ and $M \in \mathbb{R}^{2 \times 4}$.

Proof. Let G the subgroup consisting of all the matrices of the form (4.7). Recall that this subgroup agrees with the one defined by the equations (4.2). So, from Lemma 4.2 and the above paragraphs,

$$\operatorname{Aut}(\mathfrak{h}_6) \subset \tilde{G} \simeq \mathbb{R}^{2 \times 4} \rtimes_{\varphi} G.$$

Now, it follows from [Sal01] that $\mathcal{C}(\mathfrak{h}_6)$ has real dimension 12. Since $\operatorname{Aut}(\mathfrak{h}_6)$ is transitive on $\mathcal{C}(\mathfrak{h}_6)$, it follows from Lemma 4.3 that dim $\operatorname{Aut}(\mathfrak{h}_6) = \dim \tilde{G} = 19$. So, the identity components of $\operatorname{Aut}(\mathfrak{h}_6)$ and \tilde{G} coincide. In order to see $\operatorname{Aut}(\mathfrak{h}_6) = \tilde{G}$, it is enough to see that there is an automorphism of \mathfrak{h}_6 in each of the other seven connected components of \tilde{G} . Let us choose the following representatives for the connected components of \tilde{G} :

$$\begin{aligned} f_1 &= I_6 & f_5 &= \operatorname{diag}(-1, 1, 1, 1, -1, -1), \\ f_2 &= \operatorname{diag}(1, 1, -1, 1, 1), & f_6 &= \operatorname{diag}(-1, 1, 1, -1, -1, -1), \\ f_3 &= \operatorname{diag}(1, 1, -1, 1, 1, -1), & f_7 &= \operatorname{diag}(-1, 1, -1, 1, -1, 1), \\ f_4 &= \operatorname{diag}(1, 1, -1, -1, 1, -1), & f_8 &= \operatorname{diag}(-1, 1, -1, -1, -1, 1). \end{aligned}$$

Since the f_j 's form a subgroup of \tilde{G} and every f_j but f_1 has order 2, it is enough to show that three out of f_2, \ldots, f_8 are in Aut(\mathfrak{h}_6). Moreover, from Lemma 4.3, $f_6 \in Aut(\mathfrak{h}_6)$. Let us verify that also $f_2, f_3 \in Aut(\mathfrak{h}_6)$. Reasoning as in the proof of Lemma 4.3,

$$\begin{aligned} d(f_2^*(e^5)) &= de^5 = e^{12} = f_2^*(e^{12}) = f_2^*(de^5) \\ d(f_2^*(e^6)) &= de^6 = e^{13} = f_2^*(e^{13}) = f_2^*(de^6) \\ d(f_3^*(e^5)) &= de^5 = e^{12} = f_3^*(e^{12}) = f_3^*(de^5) \\ d(f_3^*(e^6)) &= -de^6 = -e^{13} = f_3^*(e^{13}) = f_3^*(de^6) \end{aligned}$$

This concludes the proof of the theorem.

The following result is an immediate consequence of the proof of Theorem 4.4.

Corollary 4.5. Aut (\mathfrak{h}_6) has 8 connected components. Moreover,

 $\operatorname{Aut}(\mathfrak{h}_6)/\operatorname{Aut}_0(\mathfrak{h}_6)\simeq \mathbb{Z}_2\oplus \mathbb{Z}_2\oplus \mathbb{Z}_2.$

4.2. Left-invariant metrics. Consider an inner product g on \mathfrak{h}_6 . Then in the canonical basis, g can be represented by a symmetric positive definite matrix of the form

$$g = \begin{pmatrix} B & C^T \\ C & D \end{pmatrix}$$
(4.8)

with $B \in \text{Sym}_4^+$, $D \in \text{Sym}_2^+$ and $C \in \mathbb{R}^{2 \times 4}$.

Lemma 4.6. Let G be the subgroup of $GL_4(\mathbb{R})$ defined in (4.4). Then:

- (1) G acts transitively on Sym_4^+ .
- (2) Any metric g on \mathfrak{h}_6 is equivalent, by an automorphism in $G \subset \operatorname{Aut}(\mathfrak{h}_6)$, to a metric of the form $\begin{pmatrix} I_4 & \tilde{C}^T \\ \tilde{C} & \tilde{D} \end{pmatrix}$.
- (3) Any metric g on \mathfrak{h}_6 is equivalent, by an automorphism in $\mathbb{R}^8 \subset \operatorname{Aut}(\mathfrak{h}_6)$ to a metric of the form $\begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{D} \end{pmatrix}$.

Recall that the inclusions $G \subset \operatorname{Aut}(\mathfrak{h}_6)$ and $\mathbb{R}^8 \simeq \mathbb{R}^{2 \times 4} \subset \operatorname{Aut}(\mathfrak{h}_6)$ are the ones provided by Theorem 4.4.

Proof. Observe that any element of Sym_4^+ can be written as $X^T X$, where X is a lowertriangular matrix. Since the set of lower-triangular matrices is contained in G, we conclude that any element of Sym_4^+ is in the orbit of the identity. This proves part (1). Item (2) follows from the first property. In fact, if g has the form given in equation (4.8), one only needs to choose an element $A \in G \subset \text{Aut}(\mathfrak{h}_6)$ such that $A^T B A = I_4$. Finally, for (3), let $M \in \mathbb{R}^{2 \times 4} \simeq \mathbb{R}^8 \subset \text{Aut}(\mathfrak{h}_6)$. Then if g is as in equation (4.8),

$$M^T g M = \begin{pmatrix} \tilde{B} & \tilde{C}^T \\ \tilde{C} & D \end{pmatrix}$$

where $\tilde{B} = B + C^T M + M^T C + M^T D M$ and $\tilde{C} = C + D M$. Choosing $M = -D^{-1}C$ we get the desired result.

Corollary 4.7. Any inner product g on \mathfrak{h}_6 is equivalent via an element of $\operatorname{Aut}(\mathfrak{h}_6)$ to one of the form $\begin{pmatrix} I_4 & 0 \\ 0 & \tilde{D} \end{pmatrix}$, with $\tilde{D} \in \operatorname{Sym}_2^+$.

Remark 4.8. Observe that any two inner products g, g' given by matrices D, D' as in Corollary 4.7 are equivalent by an automorphism of \mathfrak{h}_6 if and only if the matrices \tilde{D} and \tilde{D}' are conjugated by an element of $O(2) \subset G \subset \operatorname{Aut}(\mathfrak{h}_6)$. Since $\operatorname{Sym}_2^+ = \operatorname{GL}_2(\mathbb{R})/O(2)$, each family of equivalent metrics can be identified with an orbit of the isotropy action in this symmetric space.

Theorem 4.9. Let H_6 be the simply connected Lie group with Lie algebra \mathfrak{h}_6 . Each left-invariant metric on H_6 is equivalent by an automorphism to a metric of the form

$$g = \sum_{i=1}^{4} e^i \otimes e^i + ae^5 \otimes e^5 + be^6 \otimes e^6, \qquad (4.9)$$

with a, b > 0. Moreover, the moduli space $\mathcal{M}(H_6)/\sim$ is homeomorphic to

$$\{(a, b) \in \mathbb{R}^2 : 0 < a \le b\}.$$

Proof. By Remark 4.8, we only need to find a section to the orbits of the O(2)-action on the symmetric space $\operatorname{Sym}_2^+ = \operatorname{GL}_2(\mathbb{R})/\operatorname{O}(2)$. Observe that $\mathfrak{gl}_2(\mathbb{R}) = \mathfrak{so}(2) + \operatorname{Sym}_2$ is a Cartan decomposition of $\mathfrak{gl}_2(\mathbb{R})$, where Sym_2 denote the subspace of symmetric matrices. So, a section of the O(2)-action on Sym_2^+ is the exponential of a maximal abelian subalgebra of Sym_2 , which is given by the 2×2 diagonal matrices. This proves the first assertion. The second one is a consequence of the fact that conjugation by $J_0 \in \operatorname{O}(2)$ of a diagonal matrix interchanges the diagonal entries, where J_0 denotes the multiplication by $\sqrt{-1}$ in $\mathbb{C} \simeq \mathbb{R}^2$.

Corollary 4.10. Let $g_{a,b}$ be the left-invariant metric on H_6 given by (4.9). Then the full isometry group of $g_{a,b}$ is given by

$$\operatorname{Isom}(H_6, g_{a,b}) = \begin{cases} H_6 \rtimes (\operatorname{O}(2) \times \mathbb{Z}_2 \times \mathbb{Z}_2), & a = b \\ H_6 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) & a \neq b \end{cases}$$

Proof. According to Remark 3.1, we only need to compute the isometric automorphisms of \mathfrak{h}_6 . From Theorem 5.3, an automorphism f of \mathfrak{h}_6 has the form

$$\begin{pmatrix} r & & \\ x & A & & \\ z & y^T & s & \\ M_1 & M_2 & M_3 & rA \end{pmatrix}$$

in the canonical basis, where $r, s \in \mathbb{R} - \{0\}$, $z \in \mathbb{R}$, $x, y, M_1, M_3 \in \mathbb{R}^{2 \times 1}$, $A \in \mathrm{GL}_2(\mathbb{R})$ and $M_2 \in \mathbb{R}^{2 \times 2}$. If f leaves $g_{a,b}$ invariant, then x, y, z, M_1, M_2, M_2 all vanish, $r, s \in \{\pm 1\}$ and $A \in O(2)$. Moreover, if $a \neq b$ then $A = \pm I_2$. This implies the result. \Box

5. The case of $\mathfrak{h}_4 = (0, 0, 0, 0, 12, 14 + 23)$

5.1. Automorphism group. Let us consider the basis of \mathfrak{h}_4 whose only non vanishing are differentials are $de^5 = e^{12}$ and $de^6 = e^{14} + e^{23}$. In terms of the Lie bracket, we can assume that the only non trivial brackets in the above basis are

$$[e_1, e_2] = -e_5,$$
 $[e_1, e_4] = [e_2, e_3] = -e_6.$

Lemma 5.1. Let $f \in Aut(\mathfrak{h}_4)$, then:

- (1) $e^{i}(f(e_{j})) = 0$ for i = 1, 2 and j = 3, 4;
- (2) $e^i(f(e_5)) = 0$ for i = 1, ..., 4;
- (3) $e^i(f(e_6)) = 0$ for $i = 1, \dots, 5$.

Proof. Since f is an automorphism, it leaves invariant dim(ker ad_x) for all $x \in \mathfrak{h}_4$. In particular, if j = 3, 4, then dim(ker ad_{f(e_j)}) = 1 and hence $e^1(f(e_j)) = e^2(f(e_j)) = 0$. Now if j = 5, 6 then $e^i(f(e_j)) = 0$ for $i = 1, \ldots 4$, since f leaves the center of \mathfrak{h}_4 invariant. Moreover, since $f(e_6) = -[f(e_1), f(e_4)]$ and the e_1 - and e_2 -components of $f(e_4)$ are zero, it follows that $e^5(f(e_6)) = 0$.

In order to compute the full automorphism group of \mathfrak{h}_4 , it is easier to determine first the connected component of the identity. Recall that the Lie algebra of $\operatorname{Aut}(\mathfrak{h}_4)$ is given by the derivations of \mathfrak{h}_4 ,

$$Der(\mathfrak{h}_4) = \{ D \in \mathfrak{gl}(\mathfrak{h}_4) : D[X, Y] = [DX, Y] + [X, DY] \text{ for all } X, Y \in \mathfrak{h}_4 \}.$$

Identifying, as usual, D with its matrix in the basis e_1, \ldots, e_6 , the conditions $D[e_i, e_j] = [De_i, e_j] + [e_i, De_j]$, for i < j, define a linear system in the entries of D. A straight-forward computation, together with Lemma 5.1, allows us to prove the following fact.

Lemma 5.2. The Lie algebra $\text{Der}(\mathfrak{h}_4)$, after the usual identification, is given by the Lie subalgebra of $\mathfrak{gl}_6(\mathbb{R})$ which consists of the matrices of the following form

$$D = \begin{pmatrix} d_{11} & d_{12} & 0 & 0 & 0 & 0 \\ d_{21} & d_{22} & 0 & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{11} + x & -d_{12} & 0 & 0 \\ d_{41} & d_{42} & -d_{21} & d_{22} + x & 0 & 0 \\ d_{51} & d_{52} & d_{53} & d_{54} & d_{11} + d_{22} & 0 \\ d_{61} & d_{62} & d_{63} & d_{64} & -d_{31} + d_{42} & d_{11} + d_{22} + x \end{pmatrix},$$
(5.1)

where $x, d_{ij} \in \mathbb{R}$. In particular, dim Aut $(\mathfrak{h}_4) = 17$.

In order to describe the full automorphism group, we introduce the following notation. Let $\sigma : \operatorname{GL}_2(\mathbb{R}) \to \operatorname{GL}_2(\mathbb{R})$ be the Lie involution given by

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

and let (\cdot, \cdot) the semi-definite inner product on $\mathfrak{gl}_2(\mathbb{R})$ defined as

$$(A, B) = a_{11}b_{22} - a_{12}b_{21} + a_{22}b_{12} - a_{22}b_{11}.$$
(5.2)

Consider the closed Lie subgroup $G \subset GL_4(\mathbb{R})$ of matrices of the form

$$\begin{pmatrix} A & 0\\ B & x\sigma(A) \end{pmatrix}$$
(5.3)

where $A \in GL_2(\mathbb{R})$, $B \in \mathbb{R}^{2 \times 2}$ and $x \in \mathbb{R} - \{0\}$. It follows that G is isomorphic to the semi-direct product

$$G \simeq \mathbb{R}^{2 \times 2} \rtimes_{\varphi_1} (\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R}))$$

where

$$\varphi_1(A, x)B = x\sigma(A)BA^{-1}$$

Finally, consider the representation $\Delta: G \to \operatorname{GL}_2(\mathbb{R})$ given by

$$\Delta \begin{pmatrix} A & 0 \\ B & x\sigma(A) \end{pmatrix} = \begin{pmatrix} \det A & 0 \\ (A, B) & x \det A \end{pmatrix}.$$
 (5.4)

Theorem 5.3. Let G be the Lie subgroup of $GL_4(\mathbb{R})$ defined in (5.3). There exists an isomorphism of Lie groups

$$\operatorname{Aut}(\mathfrak{h}_4) \simeq \mathbb{R}^{2 \times 4} \rtimes_{\varphi} G,$$

where $\mathbb{R}^{2\times 4}$ is the abelian Lie group of 2×4 real matrices and $\varphi : G \to \mathrm{GL}(\mathbb{R}^{2\times 4})$ is the representation given by $\varphi(A)M = \Delta(A)MA^{-1}$, with Δ defined as in (5.4). Moreover, any automorphism of \mathfrak{h}_4 is represented in the canonical basis by a matrix of the form

$$\begin{pmatrix} A & 0\\ M & \Delta(A) \end{pmatrix},\tag{5.5}$$

where $A \in G$ and $M \in \mathbb{R}^{2 \times 4}$.

Proof. Let \tilde{G} the Lie subgroup of $\operatorname{GL}_6(\mathbb{R})$ which consists of all the matrices of the form (5.5). It follows from Lemma 5.2 that $\tilde{G} \subset \operatorname{Aut}(\mathfrak{h}_4)$. Moreover, since these two groups have dimension 17, their connected components coincide. It only remains to show that $\operatorname{Aut}(\mathfrak{h}_4)$ has no other connected components apart from the ones given by \tilde{G} . From Lemma 5.1, we know that any $f \in \operatorname{Aut}(\mathfrak{h}_4)$, with matrix (a_{ij}) in the basis e_1, \ldots, e_6 , is such that

$$a_{1j} = a_{2j} = 0$$
 for $j \ge 3$, $a_{3j} = a_{4j} = 0$ for $j \ge 5$, $a_{56} = 0$. (5.6)

Of course, some of these parameters are dependent on the others. Lets call

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}, \qquad C = \begin{pmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{pmatrix}.$$

Since $[f(e_1), f(e_2)] = -f(e_5)$, we easily check that $a_{55} = \det A$ and $a_{65} = (A, B)$ where (\cdot, \cdot) is the bilinear form defined in (5.2). Using that $[f(e_1), f(e_4)] = [f(e_2), f(e_3)] = -f(e_6)$ and $[f(e_1), f(e_3)] = [f(e_2), f(e_4)] = 0$, we obtain the following equations:

$$a_{11}a_{43} + a_{21}a_{33} = a_{22}a_{34} + a_{12}a_{44} = 0$$

$$a_{11}a_{44} + a_{21}a_{34} = a_{22}a_{33} + a_{12}a_{43} = -a_{66}$$

We can rewrite the above system as

$$\operatorname{adj}(\sigma(C))A = -a_{66}I_2$$

and hence the only possible solution is

$$C = x\sigma(A),$$
 $a_{66} = x \det(A)$

for some $x \neq 0$, as we wanted to show.

Corollary 5.4. Aut (\mathfrak{h}_4) has 4 connected components. Moreover,

$$\operatorname{Aut}(\mathfrak{h}_4) / \operatorname{Aut}_0(\mathfrak{h}_4) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Recall that $\mathcal{C}(\mathfrak{h}_4)$ is an algebraic variety and according to [Sal01], dim $\mathcal{C}(\mathfrak{h}_4) = 12$.

Corollary 5.5. $\mathcal{A}(\mathfrak{h}_4)$ is a 9-dimensional smooth manifold.

Proof. According to [ABD11], Aut(\mathfrak{h}_4) is transitive on $\mathcal{C}(\mathfrak{h}_4)$. Moreover, every abelian structure on \mathfrak{h}_4 is conjugated by an automorphism to the one given, in the canonical basis, by the matrix

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let f be an automorphism of \mathfrak{h}_4 . Assume that f is represented in the canonical basis by the matrix

$$\begin{pmatrix} A & 0 & 0 \\ B & x\sigma(A) & 0 \\ M_1 & M_2 & \Delta(A) \end{pmatrix}$$

with $A \in \operatorname{GL}_2(\mathbb{R})$, $B, M_1, M_2 \in \mathbb{R}^{2 \times 2}$ and $x \neq 0$. Notice that we are keeping the notation of Theorem 5.3 but replacing the matrix M by the square matrices M_1 and M_2 . It follows that f commutes with J if and only if, after the usual identifications, x = 1, $A \in \operatorname{GL}_1(\mathbb{C}) \subset \operatorname{GL}_2(\mathbb{R})$, and $B, M_1, M_2 \in \mathbb{C} \subset \mathbb{R}^{2 \times 2}$. In particular,

$$\mathcal{A}(\mathfrak{h}_4) \simeq \frac{\mathbb{R}^8 \rtimes (\mathbb{R}^4 \rtimes (\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_1(\mathbb{R})))}{\mathbb{C}^2 \rtimes (\mathbb{C} \rtimes \mathrm{GL}_1(\mathbb{C}))}$$

and the corollary follows.

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5.2. Left-invariant metrics. We have seen that $\operatorname{Aut}(\mathfrak{h}_4) = \mathbb{R}^{2 \times 4} \rtimes G$, where G is the subgroup of $\operatorname{GL}_4(\mathbb{R})$ defined in (5.3). We shall study first the action of G on the symmetric space $\operatorname{Sym}_4^+ = \operatorname{GL}_4(\mathbb{R})/\operatorname{O}(4)$. Recall, as usual, that a generic element $g \in \operatorname{Sym}_4^+$ has the form

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}$$

where $P, R \in \text{Sym}_2^+$. On the other hand, any element $\varphi \in G$ has the form

$$\begin{pmatrix} A & 0 \\ B & x\sigma(A) \end{pmatrix}$$
 (5.7)

where $A \in \operatorname{GL}_2(\mathbb{R})$, $B \in \mathbb{R}^{2 \times 2}$ and $x \in \mathbb{R} - \{0\}$. The action of φ on g is the restriction of the right-action of $\operatorname{GL}_4(\mathbb{R})$, which is given by $g \cdot \varphi = \varphi^T g \varphi$. In particular, if we take $A = I_2$ and x = 1 in (5.7), there exists a unique $B \in \mathbb{R}^{2 \times 2}$ such that $g \cdot \varphi$ is block diagonal. More precisely, $B = -R^{-1}Q^T$.

We can now make an element of G with B = 0 act on a block diagonal representative of g and, since $\operatorname{GL}_2(\mathbb{R})$ is transitive on Sym_2^+ , we conclude that every orbit of G meets an element of the form

$$\begin{pmatrix} I_2 & 0\\ 0 & R \end{pmatrix}$$

with $R \in \text{Sym}_2^+$. Now, to fully determine the action of G on Sym_4^+ we only need to look at the action of $O(2) \times \text{GL}_1(\mathbb{R})$ on Sym_2^+ , since any element of G which leaves a representative of g of the previous form must have $A \in O(2)$. Note that this action is the one given as follows

$$R \cdot (A, x) = x^2 A^T R A = x^2 A^{-1} R A.$$

On the other hand, R is conjugated by an orthogonal matrix to a diagonal matrix. So one can choose x in such a way that the first diagonal element of the conjugated matrix is 1. Hence we obtain the following result.

Lemma 5.6. Every orbit of G on Sym_4^+ intersects exactly once the subset

$$\{ \text{diag}(1, 1, 1, r) : 0 < r \le 1 \}.$$

Proof. Let r > 0 and denote $g_r = \text{diag}(1, 1, 1, r)$. We have seen that each orbit has an element of the form g_r . Suppose that there exists $\varphi \in G$ and r' > 0 such that $g_r \cdot \varphi = g_{r'}$. Assuming that φ has the form (5.7), a simple calculation shows that

$$g_r \cdot \varphi = \begin{pmatrix} A^T A + B^T \operatorname{diag}(1, r) B & x B^T \operatorname{diag}(1, r) \sigma(A) \\ x \sigma(A)^T \operatorname{diag}(1, r) B & x^2 \sigma(A)^T \operatorname{diag}(1, r) \sigma(A) \end{pmatrix}$$

It follows that B = 0 and $A \in O(2)$. Without loss of generality, we can assume that $A \in SO(2)$ rotates an angle θ around the origin. Now

$$x^{2}\sigma(A)^{T}\begin{pmatrix}1&0\\0&r\end{pmatrix}\sigma(A) = x^{2}\begin{pmatrix}\cos^{2}\theta + r\sin^{2}\theta&(1-r)\sin\theta\cos\theta\\(1-r)\sin\theta\cos\theta&\sin^{2}\theta + r\cos^{2}\theta\end{pmatrix}$$

So, the only possibilities for $g_r \cdot \varphi = g_{r'}$ are r = 1, which implies r' = 1; $\sin \theta = 0$, which implies r' = r; and $\cos \theta = 0$ which implies r' = 1/r. From this the lemma follows. \Box

Remark 5.7. Recall that in the symmetric space $\operatorname{Sym}_4^+ = \operatorname{GL}_4(\mathbb{R})/\operatorname{O}(4)$ the symmetry, at an element $p \in \operatorname{Sym}_4^+$ is given by $s_p(q) = pq^{-1}p$. Set $S' = \{\operatorname{diag}(1,1,1,r) : r \in \mathbb{R}^+\}$. Then it is straightforward to see that $s_p(S') = S'$ for each $p \in S'$ and hence, S' is a totally geodesic submanifold. In fact, if α denotes the second fundamental form of S', for each $p \in S'$ and $v, w \in S'$ we have that $-\alpha(v, w) = (ds_p)_p(\alpha(v, w)) = \alpha((ds_p)_p v, (ds_p)_p w) = \alpha(v, w)$. So $\alpha \equiv 0$.

Theorem 5.8. The moduli space $\mathcal{M}(H_4)/\sim$ of left-invariant metrics on H_4 up to isometric automorphism is homeomorphic to the space

 $(0,1] \times \operatorname{Sym}_2^+ / \mathbb{Z}_2,$

where \mathbb{Z}_2 is the subgroup of Isom(Sym₂⁺) generated by $\sigma \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & -b \\ -b & c \end{pmatrix}$. Moreover, every left-invariant metric is conjugated by an automorphism to a unique metric of the form

$$g = \sum_{i=1}^{3} e^{i} \otimes e^{i} + re^{4} \otimes e^{4} + ae^{5} \otimes e^{5} + 2be^{5} \otimes e^{6} + ce^{6} \otimes e^{6}$$
(5.8)

where $0 < r \le 1$, $a, b, c \ge 0$ and $ac - b^2 > 0$.

Proof. Let g be a left-invariant metric on H_4 . Identify g with the inner product on \mathfrak{h}_4 which in the canonical basis is represented by the matrix $\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}$ where $P \in \operatorname{Sym}_4^+$, $R \in \operatorname{Sym}_2^+$ and $Q \in \mathbb{R}^{4 \times 2}$. With a similar argument as the one given for Lemma 5.6 one can assume that Q = 0 and $P = \operatorname{diag}(1, 1, 1, r)$ with $0 < r \leq 1$. Denote $g = g_{r,R}$ to indicate that, up to automorphism, g only depends on $0 < r \leq 1$ and $R \in \operatorname{Sym}_2^+$. Let φ be an automorphism of \mathfrak{h}_4 and let us write φ in the canonical basis as

$$\begin{pmatrix} A & 0 & 0 \\ B & x\sigma(A) & 0 \\ M_1 & M_2 & \Delta(A, B, x) \end{pmatrix}$$
(5.9)

(see Theorem 5.3). As it follows from the proof of Lemma 5.6, $g_{r,R} \cdot \varphi = g_{r',R'}$ if and only if $A \in O(2)$, $x = \pm 1$ and $B = M_1 = M_2 = 0$. So, $\Delta(A, B, x) \in \{I_2, \operatorname{diag}(1, -1)\} \cup \{-I_2, \operatorname{diag}(-1, 1)\}$. Since $-I_2$ acts trivially on Sym_2^+ , we can assume that $\Delta(A, B, x) \in \{I_2, \operatorname{diag}(1, -1)\} \simeq \mathbb{Z}_2$. Since conjugation by $\operatorname{diag}(1, -1)$ acts as the involution σ , we conclude that any left-invariant metric is equivalent to one of the form $g_{r,R}$, and such a metric is unique if we require $0 < r \leq 1$ and that all the entries on R are non negative. \Box

Corollary 5.9. Let $g_{r,a,b,c}$ be the left-invariant metric on H_4 given in (5.8). Then the full isometry group of $g_{r,a,b,c}$ is given by

$$\operatorname{Isom}(H_4, g_{r,a,b,c}) = \begin{cases} H_4 \rtimes (\mathcal{O}(2) \rtimes \mathbb{Z}_2) & r = 1 \text{ and } b = 0\\ H_4 \rtimes \mathcal{O}(2) & r = 1 \text{ and } b \neq 0\\ H_4 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2) & r \neq 1 \text{ and } b = 0\\ H_4 \rtimes \mathbb{Z}_2 & r \neq 1 \text{ and } b \neq 0 \end{cases}$$

Proof. We use the same argument as in the proof of Corollary 4.10. A generic automorphism φ of Aut(\mathfrak{h}_4) can be written as (5.9). Recall that if φ preserves the metric then B, M_1 and M_2 must vanish and $A \in O(2)$. This implies that

$$\Delta(A, B, x) = \Delta(A, x) = \begin{pmatrix} \varepsilon & 0\\ 0 & x\varepsilon \end{pmatrix}$$

with $\varepsilon \in \{\pm 1\}$. Hence |x| = 1 and x = -1 is only possible if b = 0. Since $\sigma : \operatorname{GL}_2(\mathbb{R}) \to \operatorname{GL}_2(\mathbb{R})$ leaves O(2) invariant, if $r \neq 1$, then $A \in \{\pm I_2\}$. From the previous comments the corollary follows.

6. The case of $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$

6.1. Automorphism group. Let e_1, \ldots, e_6 be the basis of \mathfrak{h}_2 such that the only non trivial brackets are

$$[e_1, e_2] = -e_5, \qquad [e_3, e_4] = -e_6.$$

Clearly, \mathfrak{h}_2 is isomorphic to the direct product of two copies of the 3-dimensional Heisenberg Lie algebra \mathfrak{heis}_1 . Recall that the only ideals of \mathfrak{h}_2 isomorphic to \mathfrak{heis}_1 are the ones corresponding to factors in the decomposition

$$\mathfrak{h}_2 \simeq \mathfrak{heis}_1 \oplus \mathfrak{heis}_1 \tag{6.1}$$

modulo $[\mathfrak{h}_2, \mathfrak{h}_2]$. More precisely, \mathfrak{k} is such an ideal if and only if

$$\mathfrak{k} = \operatorname{span}_{\mathbb{R}} \{ e_1 + Z_1, e_2 + Z_2, e_5 \} \quad \text{or} \quad \mathfrak{k} = \operatorname{span}_{\mathbb{R}} \{ e_3 + Z_1, e_4 + Z_2, e_6 \}$$

where Z_1, Z_2 are two fixed elements in $[\mathfrak{h}_2, \mathfrak{h}_2]$. In fact, let \mathfrak{k} be an ideal of \mathfrak{h}_2 isomorphic to \mathfrak{heis}_1 . There must exist at least one element $x \in \mathfrak{k}$ such that $e^i(x) \neq 0$ for some $i = 1, \ldots, 4$. This implies that either e_5 or e_6 belongs to \mathfrak{k} . Assume first that $e_5 \in \mathfrak{k}$. Since the center of \mathfrak{k} is one dimensional, $e_6 \notin \mathfrak{k}$ and so $e^3(\mathfrak{k}) = e^4(\mathfrak{k}) = 0$. The other case is analogous. Now if $\varphi : \mathfrak{h}_2 \to \mathfrak{h}_2$ is an automorphism, then the induced linear map $\tilde{\varphi} : \mathfrak{h}_2/[\mathfrak{h}_2, \mathfrak{h}_2] \to \mathfrak{h}_2/[\mathfrak{h}_2, \mathfrak{h}_2]$ either preserves or swaps the factors of the decomposition $\mathfrak{h}_2/[\mathfrak{h}_2, \mathfrak{h}_2] \simeq \operatorname{span}_{\mathbb{R}}\{e_1, e_2\} \oplus \operatorname{span}_{\mathbb{R}}\{e_3, e_4\}$. Notice that the involution $\varphi_0 : \mathfrak{h}_2 \to \mathfrak{h}_2$, which is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$
(6.2)

in the canonical basis, is an automorphism of \mathfrak{h}_2 that reverses the decomposition (6.1).

Theorem 6.1. There exists an isomorphism of Lie groups

$$\operatorname{Aut}(\mathfrak{h}_2) \simeq \mathbb{R}^8 \rtimes ((\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R})) \rtimes \mathbb{Z}_2).$$
(6.3)

More precisely, every automorphism of \mathfrak{h}_2 can be represented in the canonical basis by a matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ M_1 & M_2 & \Delta(A, B) \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & A & 0 \\ B & 0 & 0 \\ M_1 & M_2 & \Delta'(A, B) \end{pmatrix}$$
(6.4)

where $A, B \in GL_2(\mathbb{R}), M_1, M_2 \in \mathbb{R}^{2 \times 2}$, and

$$\Delta(A,B) = \begin{pmatrix} \det A & 0 \\ 0 & \det B \end{pmatrix}, \qquad \Delta'(A,B) = \begin{pmatrix} 0 & \det A \\ \det B & 0 \end{pmatrix}.$$

In particular, $Aut(\mathfrak{h}_2)$ has 8 connected components and

$$\operatorname{Aut}(\mathfrak{h}_2)/\operatorname{Aut}_0(\mathfrak{h}_2) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \simeq \mathbb{Z}_4 \rtimes \mathbb{Z}_2$$

is isomorphic to the dihedral group D_4 .

Recall the following identifications. The subgroup \mathbb{Z}_2 is identified with the subgroup of Aut(\mathfrak{h}_2) generated by φ_0 in (6.2). The subgroups isomorphic to $\operatorname{GL}_2(\mathbb{R})$ correspond to the automorphisms with $M_1 = M_2 = 0$ and $B = I_2$ or $A = I_2$. Finally the normal subgroup isomorphic to \mathbb{R}^8 is obtained in the connected component of the identity with $A = B = I_2$.

Proof of Theorem 6.1. It follows from the discussion at the beginning of this subsection. In fact, any Lie algebra automorphism $\mathfrak{h}_2 \to \mathfrak{h}_2$, which preserves or swaps the factors of the decomposition (6.1) modulo $[\mathfrak{h}_2, \mathfrak{h}_2]$, has one of the forms described in (6.4) and it is easy to verify that all of these maps are automorphisms.

6.2. Left-invariant metrics. We follow the same approach as in the previous cases, so let us first study the action of $(\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R})) \rtimes \mathbb{Z}_2$ on $\operatorname{Sym}_4^+ = \operatorname{GL}_4(\mathbb{R})/O(4)$. We do not lose generality by considering the action of the diagonal subgroup $\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R}) \subset$ $\operatorname{GL}_4(\mathbb{R})$. Recall that if we write a generic element $g \in \operatorname{Sym}_4^+$ as

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix},$$

with $P, R \in \text{Sym}_2^+$ and $Q \in \mathbb{R}^{2 \times 2}$, then the above action is given by

$$g \cdot (A, B) = \begin{pmatrix} A^T P A & A^T Q B \\ B^T Q^T A & B^T R B \end{pmatrix}$$

and so the orbit of every g meets an element of the form $g = \begin{pmatrix} I_2 & Q \\ Q^T & I_2 \end{pmatrix}$. Therefore we can restrict our attention to the action of $O(2) \times O(2)$ on $\mathbb{R}^{2 \times 2}$ given by

$$Q \cdot (A, B) = A^T Q B. \tag{6.5}$$

Recall that the positive definite inner product

$$\langle Q_1, Q_2 \rangle = \frac{1}{2} \operatorname{tr}(Q_1 Q_2^T) \tag{6.6}$$

makes $\mathbb{R}^{2\times 2}$ an Euclidean space and moreover, the action (6.5) is isometric.

Lemma 6.2. Let $Q \in \mathbb{R}^{2 \times 2}$ and let \mathcal{O}_Q be the orbit of Q under the action of $O(2) \times O(2)$ given by $Q \cdot (A, B) = A^T Q B$. Then:

- (1) \mathcal{O}_Q intersect the subspace of diagonal matrices.
- (2) Moreover, \mathcal{O}_Q contains exactly one element of the form diag(a, b) with $0 \le a \le b$.

Proof. The decomposition $\mathbb{R}^{2\times 2} = \mathfrak{so}(2) \oplus \operatorname{Sym}_2$ is orthogonal with respect to the metric given in (6.6). Let us consider first the isometric action of SO(2) on $\mathbb{R}^{2\times 2}$ given by the restriction to the connected component of the first factor: $Q \cdot A = A^T Q$ and let \mathcal{O}'_Q be the orbit of Q under this action. It follows that $\mathcal{O}'_Q \cap \operatorname{Sym}_2 \neq \emptyset$. In fact, we can assume, by multiplying by a multiple of I_2 that ||Q|| = 1. Hence \mathcal{O}'_Q is a great circle in the unit sphere $S^3 \subset \mathbb{R}^{2\times 2}$ and it must intersect the 3-dimensional subspace Sym_2 . Item (1) follows by noticing that if $Q \in \operatorname{Sym}_2$, then there is $A \in O(2)$ such that $A^T Q A$ is diagonal.

Now suppose that diag $(a, b) \in \mathcal{O}_Q$. This implies that diag $(\varepsilon_1 a, \varepsilon_2 b)$ and diag $(\varepsilon_1 b, \varepsilon_2 a)$ also belong to \mathcal{O}_Q , for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. So, \mathcal{O}_Q has an element of the form diag(a, b)with $0 \leq a \leq b$. Suppose that there exists $A, B \in O(2)$ such that $A^T \operatorname{diag}(a, b)B =$ diag(a', b'), for some $0 \leq a' \leq b'$. Since the action is isometric we can assume that $a^2+b^2=(a')^2+(b')^2=1.$ Moreover, we do not lose generality by assuming that $A,B\in \mathrm{SO}(2),$ say

$$A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \qquad B = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}, \qquad (6.7)$$

which yields to the equations

$$a \cos t = a' \cos s,$$

 $a \sin t = b' \sin s,$
 $b \sin t = a' \sin s,$
 $b \cos t = b' \cos s.$

We can further assume that $a, b, a', b', \cos t, \sin t, \cos s, \sin s$ are all non zero, otherwise the result holds trivially. This implies ab' = ba' and aa' = bb'. From this it is easy to see that a = b, and hence a' = b', which proves (2).

Remark 6.3. Observe that the right $(O(2) \times O(2))$ -action on $\mathfrak{gl}_2(\mathbb{R})$ of Lemma 6.2 coincides with the isotropy representation of the symmetric space $O(2, 2)/(O(2) \times O(2))$, i.e., the Grassmannian of positive definite 2-planes in \mathbb{R}^4 with the metric of signature 2. This readily implies that there must always exist a section given by the diagonal matrices.

Remark 6.4. We can also give a geometric argument for the proof of part (2) of Lemma 6.2. Consider the geodesics $\gamma(t) = A^T \operatorname{diag}(a, b)$ and $\beta(s) = \operatorname{diag}(a', b')B^T$ of $S^3 \subset \mathbb{R}^{2\times 2}$, where A and B are as in (6.7). The image of $\gamma(t)$ is the intersection of S^3 with the plane $\pi_1 \subset \mathbb{R}^{2\times 2}$ generated by $\operatorname{diag}(a, b)$ and $\begin{pmatrix} 0 & b \\ -a & 0 \end{pmatrix}$. Similarly, the image of $\beta(s)$ is the intersection of S^3 with the plane π_2 generated by $\operatorname{diag}(a', b')$ and $\begin{pmatrix} 0 & a' \\ -b' & 0 \end{pmatrix}$. So, assuming $0 \le a \le b, 0 \le a' \le b'$ and $a^2 + b^2 = (a')^2 + (b')^2 = 1$, the condition $(a, b) \ne (a', b')$

implies $\pi_1 \cap \pi_2 = \{0\}$ and therefore $A^T \operatorname{diag}(a, b) \neq \operatorname{diag}(a', b') B^T$ for all $A, B \in O(2)$.

Theorem 6.5. Let H_2 be the simply connected Lie group with Lie algebra \mathfrak{h}_2 . The moduli space $\mathcal{M}(H_2)/\sim$ of left-invariant metrics on H_2 up to isometric automorphism is homeomorphic to the space

$$\{(a,b) \in \mathbb{R}^2 : 0 \le a \le b < 1\} \times \operatorname{Sym}_2^+ / \mathbb{Z}_2,$$

where \mathbb{Z}_2 is the subgroup of $\text{Isom}(\text{Sym}_2^+)$ generated by the involution

$$\sigma \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} E & -F \\ -F & G \end{pmatrix}.$$

Moreover, every left-invariant metric on H_2 is conjugated by an automorphism to a unique metric of the form

$$g = \sum_{i=1}^{4} e^{i} \otimes e^{i} + 2ae^{1} \otimes e^{3} + 2be^{2} \otimes e^{4} + Ee^{5} \otimes e^{5} + 2Fe^{5} \otimes e^{6} + Ge^{6} \otimes e^{6}, \qquad (6.8)$$

where $0 \le a \le b$, $E, F, G \ge 0$ and $EG - F^2 > 0$.

Proof. From Theorem 6.1, $\operatorname{Aut}_0(\mathfrak{h}_2) \simeq \mathbb{R}^8 \rtimes (\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R}))_0$, and hence we can use a similar argument as in Lemma 4.6 to conclude that any left-invariant metric on H_2 is equivalent to a block diagonal metric $g = \operatorname{diag}(g_1, g_2)$ with $g_1 \in \operatorname{Sym}_4^+$ and $g_2 \in \operatorname{Sym}_2^+$. From the discussion at the beginning of this subsection and Lemma 6.2 we know that there exist unique $0 \le a \le b < 1$ such that g_1 is equivalent under the action of $\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R})$ to

$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \\ a & 0 & 1 & 0 \\ 0 & b & 0 & 1 \end{pmatrix} .$$
 (6.9)

Recall that we need to impose the additional condition b < 1 in order to get that the matrix (6.9) is positive definite. To complete the proof of the theorem, one only needs to note that the isotropy of $(\operatorname{GL}_2(\mathbb{R}) \times \operatorname{GL}_2(\mathbb{R}))_0$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. This group induces, after making the action effective, the action of \mathbb{Z}_2 on Sym_2^+ given by the involution σ . \Box

Corollary 6.6. Let $g = g_{a,b,E,F,G}$ be the left-invariant metric on H_2 defined in (6.8). Then the full isometry group of g is given by

$$\operatorname{Isom}(H_2,g) = \begin{cases} H_2 \rtimes ((\operatorname{O}(2) \times \operatorname{O}(2)) \rtimes \mathbb{Z}_2), & a = b = 0, \ F = 0, \ E = G \\ H_2 \rtimes (\operatorname{O}(2) \times \operatorname{O}(2)), & a = b = 0, \ F = 0, \ E \neq G \\ H_2 \rtimes (\operatorname{S}(\operatorname{O}(2) \times \operatorname{O}(2)) \rtimes \mathbb{Z}_2), & a = b = 0, \ F \neq 0, \ E = G \\ H_2 \rtimes \operatorname{S}(\operatorname{O}(2) \times \operatorname{O}(2)), & a = b = 0, \ F \neq 0, \ E \neq G \\ H_2 \rtimes (\operatorname{diag}(\operatorname{O}(2) \times \operatorname{O}(2)) \rtimes \mathbb{Z}_2), & a = b \neq 0, \ E = G \\ H_2 \rtimes \operatorname{diag}(\operatorname{O}(2) \times \operatorname{O}(2)), & a = b \neq 0, \ E \neq G \\ H_2 \rtimes \operatorname{diag}(\operatorname{O}(2) \times \operatorname{O}(2)), & a = b \neq 0, \ E \neq G \\ H_2 \rtimes \operatorname{diag}(\operatorname{O}(2) \times \operatorname{O}(2)), & a = b \neq 0, \ E \neq G \\ H_2 \rtimes \operatorname{diag}(\operatorname{O}(2) \times \operatorname{O}(2)), & a = b \neq 0, \ E \neq G \\ H_2 \rtimes \operatorname{O}_4, & 0 \leq a < b, \ E = G \\ H_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2), & 0 \leq a < b, \ E \neq G. \end{cases}$$

Proof. From Theorem 6.5, we can identify the left-invariant metric $g = g_{a,b,E,F,G}$ with the symmetric positive definite matrix

$$g = \begin{pmatrix} 1 & 0 & a & 0 & 0 & 0 \\ 0 & 1 & 0 & b & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & 0 & F & G \end{pmatrix}$$

.

On the other hand, from Theorem 6.1, the discrete group $\operatorname{Aut}(\mathfrak{h}_2)/\operatorname{Aut}_0(\mathfrak{h}_2)$ is isomorphic to the dihedral group $D_4 \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where each \mathbb{Z}_2 it is generated by the projection of the involutive automorphisms given by

$$\varphi_{1} = \operatorname{diag}(-1, 1, 1, 1, -1, 1)$$
$$\varphi_{2} = \operatorname{diag}(1, 1, -1, 1, 1, -1)$$
$$\varphi_{3} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Moreover, φ_1 and φ_2 generate the two connected components on each factor of O(2) × O(2), after their natural inclusion into Aut(\mathfrak{h}_2), and φ_3 gives the bijection between the block diagonal and anti-diagonal automorphisms described in Theorem 6.1.

The following facts are easy to verify:

(1) $g \cdot (\varphi_1 \varphi_2) = g \cdot (\varphi_2 \varphi_1) = g;$

(2) $g \cdot \varphi_1 = g$ if and only if a = 0 and F = 0;

(3) $g \cdot \varphi_2 = g$ if and only if a = 0 and F = 0;

(4) $g \cdot \varphi_3 = g$ if and only if E = G.

So, in order to compute all the isometric automorphisms we can restrict our attention to the action of the connected component $SO(2) \times SO(2)$. This raises three possibilities. Firstly, if a = b = 0, then it is clear that $g \cdot \varphi = g$ for all $\varphi \in SO(2) \times SO(2)$. Secondly, if $a = b \neq 0$, and $\varphi = (A, B)$ is such that $g \cdot \varphi = A^{-1}gB$, then A = B. Finally, if a < b, the only $\varphi \in SO(2) \times SO(2)$ such that $g \cdot \varphi$ are $\varphi = \pm I_6$. This completes the proof of the corollary.

7. The case of $\mathfrak{h}_9 = (0, 0, 0, 0, 12, 14 + 25)$

This is the most difficult case to describe, since as we will see, $\operatorname{Aut}(\mathfrak{h}_9)$ does not admit a normal abelian subgroup such that $\operatorname{Aut}(\mathfrak{h}_9)$ is the semi-direct product of this subgroup and an algebraic subgroup which descends down to the quotient $\mathfrak{h}_9/[\mathfrak{h}_9, \mathfrak{h}_9]$. According to our notation \mathfrak{h}_9 , has a basis e_1, \ldots, e_6 such that $de^1 = \cdots = de^4 = 0$, $de^5 = e^{12}$ and $de^6 = e^{14} + e^{25}$, or equivalently, the non trivial brackets are

$$[e_1, e_2] = -e_5 \qquad [e_1, e_4] = [e_2, e_5] = -e_6.$$

In particular, \mathfrak{h}_9 is 3-step nilpotent. We find it convenient change to the basis $\hat{e}_1 = e_2$, $\hat{e}_2 = e_1$, $\hat{e}_3 = e_4$, $\hat{e}_4 = e_3$, $\hat{e}_5 = e_5$ and $\hat{e}_6 = e_6$ where the non trivial brackets are

$$[\hat{e}_1, \hat{e}_2] = \hat{e}_5$$
 $[\hat{e}_1, \hat{e}_5] = [\hat{e}_2, \hat{e}_3] = -\hat{e}_6.$ (7.1)

Notice that with respect to this basis we have $\mathfrak{z}(\mathfrak{h}_9) = \operatorname{span}_{\mathbb{R}}\{\hat{e}_4, \hat{e}_6\}$ and $[\mathfrak{h}_9, \mathfrak{h}_9] = \operatorname{span}_{\mathbb{R}}\{\hat{e}_5, \hat{e}_6\}$.

7.1. Automorphism group.

 $[\hat{e}$

Theorem 7.1. With respect to the basis $\hat{e}_1, \ldots, \hat{e}_6$ every automorphism $\varphi \in Aut(\mathfrak{h}_9)$ has the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}^2 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & -a_{11}a_{21} & 0 & a_{11}a_{22} & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{22}a_{31} - a_{21}a_{32} - a_{11}a_{52} & a_{11}^2a_{22} \end{pmatrix}.$$

$$(7.2)$$

In particular, $\operatorname{Aut}(\mathfrak{h}_9)$ is a 15-dimensional solvable Lie group, which has 8 connected components and $\operatorname{Aut}(\mathfrak{h}_9)/\operatorname{Aut}_0(\mathfrak{h}_9) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. We show first that the matrix (a_{ij}) of φ in the given basis is lower triangular. It follows from (7.1) that $\hat{e}^1(\varphi(\hat{e}_2)) = 0$. In fact, $(\mathrm{ad}_{\varphi(\hat{e}_2)})^2 = (\mathrm{ad}_{\hat{e}_2})^2 = 0$ and so the \hat{e}_1 -component of $\varphi(\hat{e}_2)$ must vanish. Similarly, since dim(ker $\mathrm{ad}_{\varphi(\hat{e}_3)}) = \mathrm{dim}(\mathrm{ker} \, \mathrm{ad}_{\hat{e}_i}) = 5$, then $\hat{e}^1(\varphi(\hat{e}_3)) = \hat{e}^2(\varphi(\hat{e}_3)) = 0$. Recall that φ preserves the subalgebras $\mathfrak{z}(\mathfrak{h}_9)$, $[\mathfrak{h}_9, \mathfrak{h}_9]$, $\mathfrak{z}(\mathfrak{h}_9) + [\mathfrak{h}_9, \mathfrak{h}_9]$ and $\mathfrak{z}(\mathfrak{h}_9) \cap [\mathfrak{h}_9, \mathfrak{h}_9]$. This implies that $\hat{e}^i(\varphi(\hat{e}_j)) = 0$ for all $i = 1, 2, 3, j = 4, 5, 6; \hat{e}^4(\varphi(\hat{e}_j)) = 0$ for j = 5, 6, and $\hat{e}^5(\varphi(\hat{e}_6)) = 0$. Moreover, since $\hat{e}^4 \in \mathfrak{z}(\mathfrak{h}_9)$, one has $\hat{e}^5(\varphi(\hat{e}_4)) = 0$. On the other side,

$$\hat{e}^{5}(\varphi(\hat{e}_{5})) = \hat{e}^{5}([\varphi(\hat{e}_{1}),\varphi(\hat{e}_{2})]) = a_{11}a_{22}$$
$$\hat{e}^{6}(\varphi(\hat{e}_{5})) = \hat{e}^{6}([\varphi(\hat{e}_{1}),\varphi(\hat{e}_{2})]) = a_{22}a_{31} - a_{21}a_{32} - a_{11}a_{52}$$

and since $a_{11}^2 a_{22} = [\varphi(\hat{e}_1), \varphi(\hat{e}_5)] = [\varphi(\hat{e}_2), \varphi(\hat{e}_3)] = -\varphi(\hat{e}_6)$ we conclude that

$$\hat{e}^3(\varphi(\hat{e}_3)) = a_{11}^2, \qquad \hat{e}^5(\varphi(\hat{e}_3)) = -a_{11}a_{21}, \qquad \hat{e}^6(\varphi(\hat{e}_6)) = a_{11}^2a_{22}.$$

This proves that φ has the form (7.2). It is easy to see that any linear map of this form is an automorphism of \mathfrak{h}_9 . (For instance, one can compute the dimension of $\operatorname{Der}(\mathfrak{h}_9)$ and check that it equals 15. The automorphisms with $a_{11} = a_{22} = a_{44} = 1$ are the ones in the exponential of the nilradical of $\operatorname{Der}(\mathfrak{h}_9)$.)

Remark 7.2. Observe that from the previous theorem, $Aut(\mathfrak{h}_9)$ is not the semi-direct product of an abelian normal subgroup.

7.2. Left-invariant metrics. Let T_6 be the subgroup of $\operatorname{GL}_6(\mathbb{R})$ of lower triangular matrices, and denote by T_6^+ the normal subgroup of T_6 of matrices whose diagonal entries are all positive. It is known that T_6^+ acts simply transitively on Sym_6^+ with the restriction of the action of $\operatorname{GL}_6(\mathbb{R})$. Moreover, this action is proper since it is equivalent to the action by right multiplication of T_6^+ on itself.

Theorem 7.3. Let H_9 be the simply connected Lie group with Lie algebra \mathfrak{h}_9 . The moduli space $\mathcal{M}(H_9)/\sim$ is a 6-dimensional smooth manifold. Moreover, $\mathcal{M}(H_9)/\sim$ is diffeomorphic to the homogeneous manifold $T_6^+/\operatorname{Aut}_0(\mathfrak{h}_9)$ and every left-invariant metric on H_9 is equivalent to a unique metric of the form

$$g = \hat{e}^{1} \otimes \hat{e}^{1} + \hat{e}^{2} \otimes \hat{e}^{2} + (A^{2} + D^{2})\hat{e}^{3} \otimes \hat{e}^{3} + DE\hat{e}^{3} \otimes \hat{e}^{4} + BD\hat{e}^{3} \otimes \hat{e}^{5} + (E^{2} + 1)\hat{e}^{4} \otimes \hat{e}^{4} + BE\hat{e}^{4} \otimes \hat{e}^{5} + (B^{2} + F^{2})\hat{e}^{5} \otimes \hat{e}^{5} + CF\hat{e}^{5} \otimes \hat{e}^{6} + C^{2}\hat{e}^{6} \otimes \hat{e}^{6},$$
(7.3)

where $A, B, C > 0, D, E, F \in \mathbb{R}$, and $\hat{e}^1, \ldots, \hat{e}^6$ is the dual basis of $\hat{e}_1, \ldots, \hat{e}_6$.

Proof. Since $\mathcal{M}(H_9) \simeq \operatorname{Sym}_6^+$ is connected, it is enough to consider the orbits of $\operatorname{Aut}_0(\mathfrak{h}_9)$ in Sym_6^+ . Let $\Phi: T_6^+ \to \operatorname{Sym}_6^+$ given by $\Phi(X) = X^t X$. As we mention above, the action of $\operatorname{Aut}_0(\mathfrak{h}_9)$ on Sym_6^+ is equivalent via Φ to the action of $\operatorname{Aut}_0(\mathfrak{h}_9)$ by right multiplication. Note that the submanifold Σ of T_6^+ given by

$$\Sigma = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & D & E & B & 0 \\ 0 & 0 & 0 & 0 & F & C \end{pmatrix} : A, B, C > 0, D, E, F \in \mathbb{R} \right\}$$
(7.4)

is a slice for the action of $\operatorname{Aut}_0(\mathfrak{h}_6)$. Moreover, the map $\Sigma \times \operatorname{Aut}_0(\mathfrak{h}_9) \to T_6^+$ given by $(S, \varphi) \mapsto S\varphi$ is a diffeomorphism. In fact, lets see that for any $X \in T_6^+$, there exist unique $S \in \Sigma$ and $\varphi \in \operatorname{Aut}_0(\mathfrak{h}_9)$ such that $S\varphi = X$. Denote $X = (x_{ij})$ and assume that φ and S are as in (7.2) and (7.4) respectively. It is clear that A, B, C and the elements on the diagonal of φ are uniquely determined by X and so we can assume that A, B, C and all the elements on the diagonals of X and φ are equal to 1. Moreover, since the principal

 4×4 block of $S\varphi$ coincides with the one of φ , we can assume further that $a_{ij} = x_{ij} = 0$ for all $2 \le i \le 4$ and $1 \le j \le 3$. The equation $S\varphi = X$ has now the form

/	1	0	0	0	0	0\		/ 1	0	0	0	0	0\
	0	1	0	0	0	0		0	1	0	0	0	0
	0	0	1	0	0	0		0	0	1	0	0	0
	0	0	0	1	0	0	_	0	0	0	1	0	0
	a_{51}	a_{52}	D	E	1	0		x_{51}	x_{52}	x_{53}	x_{54}	1	0
$\backslash Fa_5$	$_1 + a_{61}$	$Fa_{52} + a_{62}$	a_{63}	a_{64}	$F - a_{52}$	1/		$\setminus x_{61}$	x_{62}	x_{63}	x_{64}	x_{65}	1/

which clearly has a unique solution in S, φ .

So, $\Phi(\Sigma)$ is a full slice in $\mathcal{M}(H_9)$ for the action of the action of the automorphism group of \mathfrak{h}_9 and every left-invariant metric on H_9 is equivalent to a unique metric which in the basis given by (7.1) is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^2 + D^2 & DE & BD & 0 \\ 0 & 0 & DE & E^2 + 1 & BE & 0 \\ 0 & 0 & BD & BE & B^2 + F^2 & CF \\ 0 & 0 & 0 & 0 & CF & C^2 \end{pmatrix},$$
(7.5)

with A, B, C > 0, as we wanted to show.

Corollary 7.4. Let g be the left-invariant metric on H_9 given by (7.3). Then the full isometry group of g is given by

$$\operatorname{Isom}(H_9,g) \simeq H_9 \rtimes \mathbb{Z}_2^k$$

where k is the number of null parameters among D, E and F.

Proof. Let φ be an automorphism of \mathfrak{h}_9 and let $\varphi = N + S$ be Jordan-Chevalley decomposition of φ , that is, N is nilpotent and S is semisimple such that NS = SN. Since N and S can be obtained as polynomials on φ , it follows from Theorem 7.1 that, in the basis $\hat{e}_1, \ldots, \hat{e}_6$, N is a strictly lower triangular matrix and S is a lower triangular matrix, such that its diagonal elements coincide with the diagonal elements of φ . Since the isometric automorphisms of H_9 (which are induced by isometric automorphisms of \mathfrak{h}_9) constitute a compact subgroup of the isometry group of H_9 , the matrix S must have all its diagonal entrances equal to ± 1 .

Assume first that $S = I_6$. Since the coefficients $(N^k)_{i+1,i} = 0$ for all $k \ge 2, i = 1, ..., 5$ and

$$\varphi^n = \sum_{k=0}^n \binom{n}{k} N^k,$$

we get that $(\varphi^n)_{i+1,i} = nN_{i+1,i}$. Since φ^n is an isometric automorphism (which must be bounded), we conclude that $N_{i+1,i} = 0$ for all i = 1, ..., 5. Applying the same argument one can prove that N = 0.

If $S \neq I_6$, we consider the isometric automorphism $\varphi^2 = S^2 + 2SN + N^2$. Since all the diagonal entrances of φ^2 are equal to 1, its Jordan-Chevalley decomposition is given by $\varphi^2 = N' + I_6$. We can apply the previous reasoning to φ^2 and conclude that N' = 0. On the other hand, S^2 is semisimple and $2SN + N^2$ is nilpotent. Hence $S^2 = I_6$ and

$$2SN + N^2 = N' = 0. (7.6)$$

But SN and N^2 have minimal polynomials of different degrees, unless N = 0. So, equation (7.6) holds only if N = 0.

We conclude that an isometric automorphism of $\mathfrak{h}_{\mathfrak{g}}$ must be a subgroup of

$$\mathbb{Z}_2^3 = \{ \operatorname{diag}(\varepsilon_1, \varepsilon_2, 1, \varepsilon_3, \varepsilon_1 \varepsilon_2, \varepsilon_2) : \varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1 \}.$$

Now a straightforward computation concludes the proof.

8. Hermitian metrics

8.1. Hermitian structures on \mathfrak{h}_5 . Recall that, from [DS12], every left-invariant metric on H_5 is equivalent by an automorphism to one and only one metric which is represented in the standard basis e_1, \ldots, e_6 by the symmetric positive definite matrix

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & 0 & F & G \end{pmatrix},$$
(8.1)

where $0 < s \leq r \leq 1, 0 < E, 0 \leq F, 0 < G$ and $0 < EG - F^2$. It is not difficult to see that the left-invariant almost Hermitian structures, with respect to g, which preserve the orientation induced by the standard basis, are parameterized by two 2-spheres. More precisely, any orientation preserving, left-invariant almost Hermitian structure (in the standard basis) has either the form

$$J_{1} = \begin{pmatrix} 0 & -a\sqrt{r} & -b & -c\sqrt{s} & 0 & 0\\ \frac{a}{\sqrt{r}} & 0 & -\frac{c}{\sqrt{r}} & \frac{b\sqrt{s}}{\sqrt{r}} & 0 & 0\\ b & c\sqrt{r} & 0 & -a\sqrt{s} & 0 & 0\\ \frac{c}{\sqrt{s}} & -\frac{b\sqrt{r}}{\sqrt{s}} & \frac{a}{\sqrt{s}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{F}{\sqrt{-F^{2}+EG}} & -\frac{G}{\sqrt{-F^{2}+EG}}\\ 0 & 0 & 0 & 0 & \frac{E}{\sqrt{-F^{2}+EG}} & \frac{F}{\sqrt{-F^{2}+EG}} \end{pmatrix}$$
(8.2)

or the form

$$J_{2} = \begin{pmatrix} 0 & -a\sqrt{r} & -b & -c\sqrt{s} & 0 & 0 \\ \frac{a}{\sqrt{r}} & 0 & \frac{c}{\sqrt{r}} & -\frac{b\sqrt{s}}{\sqrt{r}} & 0 & 0 \\ b & -c\sqrt{r} & 0 & a\sqrt{s} & 0 & 0 \\ \frac{c}{\sqrt{s}} & \frac{b\sqrt{r}}{\sqrt{s}} & -\frac{a}{\sqrt{s}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{F}{\sqrt{-F^{2} + EG}} & \frac{G}{\sqrt{-F^{2} + EG}} \\ 0 & 0 & 0 & 0 & -\frac{E}{\sqrt{-F^{2} + EG}} & -\frac{F}{\sqrt{-F^{2} + EG}} \end{pmatrix},$$
(8.3)

where $(a, b, c) \in S^2$. So, we distinguish two cases in order to determine when J_1 and J_2 are integrable.

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8.1.1. The case of J_1 . In order to simplify some calculations we denote

$$\Delta = EG - F^2, \qquad \qquad \alpha = \frac{\sqrt{r} + \sqrt{s}}{1 + \sqrt{rs}}.$$

With much patience and after long computations one can see that the non-trivial equations on the integrability of J_1 are given by

$$a = \frac{(1-b^2)\sqrt{\Delta}}{G\alpha},\tag{8.4}$$

$$a = \frac{(1 - c^2)\alpha\sqrt{\Delta}}{E},\tag{8.5}$$

$$Fa = bc\sqrt{\Delta},\tag{8.6}$$

$$Fb = \frac{Ec}{\alpha} - ac\sqrt{\Delta},\tag{8.7}$$

$$Fc = G\alpha b - ab\sqrt{\Delta}.$$
(8.8)

It follows from (8.4) and (8.5) that a > 0 and, in the generic case $F \neq 0$, (8.6) says that b and c must have the same sign. Moreover, it is not hard to see that (8.6) follows from (8.4) and (8.5) by multiplying these two equations. With a similar argument, one can see that (8.7) and (8.8) also follow from (8.4) and (8.5). Now we can combine (8.4), (8.5) and the condition $a^2 + b^2 + c^2 = 1$ to obtain the following quadratic equation in a:

$$a^{2} - a\frac{1}{\sqrt{\Delta}}\left(\frac{E}{\alpha} + G\alpha\right) + 1 = 0.$$
(8.9)

With a little algebra we see that the condition on (8.9) for having real roots is equivalent to the tautology $\left(\frac{E}{\alpha} - G\alpha\right)^2 + 4F^2 \ge 0$. So the solution between 0 and 1 is

$$a = \frac{1}{2} \left(\frac{1}{\sqrt{\Delta}} \left(\frac{E}{\alpha} + G\alpha \right) - \sqrt{\frac{1}{\Delta} \left(\frac{E}{\alpha} + G\alpha \right)^2 - 4} \right), \tag{8.10}$$

which gives the following two values for b and c:

$$b = \pm \sqrt{1 - \frac{Ga\alpha}{\sqrt{\Delta}}}, \qquad c = \pm \sqrt{1 - \frac{Ea}{\alpha\sqrt{\Delta}}}.$$
 (8.11)

Finally, notice that when F = 0 equations (8.6), (8.7) and (8.8) reduces to

$$bc = \left(\frac{\sqrt{E}}{\sqrt{G}\alpha} - a\right)c = \left(\frac{\sqrt{G}\alpha}{\sqrt{E}} - a\right)b = 0.$$
(8.12)

Hence, if F = 0 we get

$$(a, b, c) = \begin{cases} \left(\frac{\sqrt{E}}{\sqrt{G}\alpha}, 0, \pm \sqrt{1 - \frac{E}{G\alpha^2}}\right), & \text{if } \frac{E}{G} \le \alpha^2, \\ \left(\frac{\sqrt{G}\alpha}{\sqrt{E}}, \pm \sqrt{1 - \frac{G\alpha^2}{E}}, 0\right), & \text{if } \alpha^2 \le \frac{E}{G}. \end{cases}$$
(8.13)

8.1.2. The case of J_2 . The equations for the integrability of J_2 are somewhat more delicate as they behave differently depending on the values of r, s. The general form for such equations is

$$a(\sqrt{r} - \sqrt{s}) = -\frac{(1 - b^2)(1 - \sqrt{rs})\sqrt{\Delta}}{G},$$
(8.14)

$$a(1 - \sqrt{rs}) = -\frac{(1 - c^2)(\sqrt{r} - \sqrt{s})\sqrt{\Delta}}{E},$$
 (8.15)

$$Fa(\sqrt{r} - \sqrt{s}) = bc(\sqrt{r} - \sqrt{s})\sqrt{\Delta}, \qquad (8.16)$$

$$Fa(1 - \sqrt{rs}) = bc(1 - \sqrt{rs})\sqrt{\Delta}, \qquad (8.17)$$

$$Fc(1 - \sqrt{rs}) = -Gb(\sqrt{r} - \sqrt{s}) - ab(1 - \sqrt{rs})\sqrt{\Delta}, \qquad (8.18)$$

$$Fb(\sqrt{r} - \sqrt{s}) = -Ec(1 - \sqrt{rs}) - ac(\sqrt{r} - \sqrt{s})\sqrt{\Delta}, \qquad (8.19)$$

where $\Delta = EG - F^2$ as in the previous case.

Recall that when s = r = 1 all the equations hold trivially, which means that J_2 is a complex structure for all $(a, b, c) \in S^2$. This is a result already known (see [AGS97, AGS01]). If r = s < 1, equations (8.14) to (8.19) reduce to

$$a = 0,$$
 $b = \pm 1,$ $c = 0.$ (8.20)

So, we can assume that 0 < s < r < 1. Let us denote

$$\beta = \frac{\sqrt{r} - \sqrt{s}}{1 - \sqrt{rs}} \tag{8.21}$$

and notice that β is always positive. Now we can rewrite equations (8.14) to (8.19) as

$$a = -\frac{(1-b^2)\sqrt{\Delta}}{G\beta},\tag{8.22}$$

$$a = -\frac{(1-c^2)\beta\sqrt{\Delta}}{E},\tag{8.23}$$

$$Fa = bc\sqrt{\Delta},\tag{8.24}$$

$$Fb = -\frac{Ec}{\beta} - ac\sqrt{\Delta},\tag{8.25}$$

$$Fc = -G\beta b - ab\sqrt{\Delta}.$$
(8.26)

Notice that equations (8.22) to (8.26) are formally equal to equations (8.4) to (8.8) if we replace α by $-\beta$. The only difference is that in this case a < 0 and b, c have opposite signs. Since we never actually use the value of α when solving (8.4) to (8.8) we can conclude that when $F \neq 0$

$$a = \frac{1}{2} \left(\frac{-1}{\sqrt{\Delta}} \left(\frac{E}{\beta} + G\beta \right) + \sqrt{\frac{1}{\Delta} \left(\frac{E}{\beta} + G\beta \right)^2 - 4} \right), \tag{8.27}$$

$$b = \pm \sqrt{1 + \frac{Ga\beta}{\sqrt{\Delta}}},\tag{8.28}$$

$$c = \mp \sqrt{1 + \frac{Ea}{\beta\sqrt{\Delta}}} \tag{8.29}$$

and if F = 0,

$$(a,b,c) = \begin{cases} \left(-\frac{\sqrt{E}}{\sqrt{G}\beta}, 0, \pm \sqrt{1-\frac{E}{G\beta^2}}\right), & \text{if } \frac{E}{G} \le \beta^2\\ \left(-\frac{\sqrt{G}\beta}{\sqrt{E}}, \pm \sqrt{1-\frac{G\beta^2}{E}}, 0\right), & \text{if } \beta^2 \le \frac{E}{G} \end{cases}$$
(8.30)

Summarizing, we obtained the following result.

Theorem 8.1. Consider in H_5 the left-invariant metric g given in (8.1). Then (g, J) is an Hermitian structure on H_5 if and only if

- (1) $J = \pm J_1$, as in (8.2) with a, b, c given as in Table 2 or
- (2) $J = \pm J_2$, as in (8.3) with a, b, c given as in Table 3.

In particular, every left-invariant metric on H_5 is Hermitian with respect to some leftinvariant complex structure.

Table 2. Case \mathfrak{h}_5 : J_1 where $\Delta = EG - F^2$, $\alpha = \frac{\sqrt{r} + \sqrt{s}}{1 + \sqrt{rs}}$, $\gamma = \frac{E}{\alpha} + G\alpha$

a	b	С	F	(r,s)
$\frac{\gamma - \sqrt{\gamma^2 - 4\Delta}}{2\sqrt{\Delta}}$	$\pm \sqrt{1 - \frac{Ga\alpha}{\sqrt{\Delta}}}$	$\pm \sqrt{1 - \frac{Ea}{\alpha\sqrt{\Delta}}}$	> 0	any
$\frac{\sqrt{E}}{\sqrt{G}\alpha}$	0	$\pm \sqrt{1 - \frac{E}{G\alpha^2}}$	= 0	$\frac{E}{G} \le \alpha^2$
$\frac{\sqrt{G}\alpha}{\sqrt{E}}$	$\pm \sqrt{1 - \frac{G\alpha^2}{E}}$	0	= 0	$\alpha^2 \leq \frac{E}{G}$

Proof. It follows from the above discussion. Notice that we introduce the \pm sign in the statement of the theorem so that our classification also includes the Hermitian structures which reverse the orientation.

8.2. Hermitian structures on \mathfrak{h}_4 . We follow the same approach as in the previous case. Remember that for H_4 , the left-invariant metrics, up to isometric automorphism,

a	b	С	F	(r,s)
	$a^2 + b^2 + c^2 =$	any	s = r = 1	
0	±1	0	any	s = r < 1
$-rac{\delta-\sqrt{\delta^2-4\Delta}}{2\sqrt{\Delta}}$	$\pm\sqrt{1+\frac{Ga\beta}{\sqrt{\Delta}}}$	$\mp \sqrt{1 + \frac{Ea}{\beta\sqrt{\Delta}}}$	> 0	s < r < 1
$-rac{\sqrt{E}}{\sqrt{G}eta}$	0	$\pm \sqrt{1 - \frac{E}{G\beta^2}}$	= 0	$s < r < 1$ and $\frac{E}{G} \le \beta^2$
$-\frac{\sqrt{G}\beta}{\sqrt{E}}$	$\pm \sqrt{1 - \frac{G\beta^2}{E}}$	0	= 0	$s < r < 1$ and $\beta^2 \leq \frac{E}{G}$

Table 3. Case \mathfrak{h}_5 : J_2 where $\Delta = EG - F^2$, $\beta = \frac{\sqrt{r} - \sqrt{s}}{1 - \sqrt{rs}}$, $\delta = \frac{E}{\beta} + G\beta$

are given, in the standard basis e_1, \ldots, e_6 defined in Section 5, by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & E & F \\ 0 & 0 & 0 & 0 & F & G \end{pmatrix}$$
(8.31)

where $0 < r \leq 1, E, G > 0, F \geq 0$ and $\Delta = EG - F^2 > 0$. In this case the orientation preserving almost Hermitian structures, with respect to this metric, are given by

$$J_{1} = \begin{pmatrix} 0 & -a & -b & -c\sqrt{r} & 0 & 0\\ a & 0 & -c & b\sqrt{r} & 0 & 0\\ b & c & 0 & -a\sqrt{r} & 0 & 0\\ \frac{c}{\sqrt{r}} & -\frac{b}{\sqrt{r}} & \frac{a}{\sqrt{r}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{F}{\sqrt{\Delta}} & -\frac{G}{\sqrt{\Delta}}\\ 0 & 0 & 0 & 0 & \frac{E}{\sqrt{\Delta}} & \frac{F}{\sqrt{\Delta}} \end{pmatrix}$$
(8.32)

and

$$J_{2} = \begin{pmatrix} 0 & -a & -b & -c\sqrt{r} & 0 & 0\\ a & 0 & c & -b\sqrt{r} & 0 & 0\\ b & -c & 0 & a\sqrt{r} & 0 & 0\\ \frac{c}{\sqrt{r}} & \frac{b}{\sqrt{r}} & -\frac{a}{\sqrt{r}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{F}{\sqrt{\Delta}} & \frac{G}{\sqrt{\Delta}}\\ 0 & 0 & 0 & 0 & -\frac{E}{\sqrt{\Delta}} & -\frac{F}{\sqrt{\Delta}} \end{pmatrix}$$
(8.33)

where

$$a^2 + b^2 + c^2 = 1. ag{8.34}$$

With the same ideas as in Subsection 8.1, we can find conditions on a, b, c for the integrability of J_1, J_2 . For the sake of completeness we present the non-trivial equations but we omit the calculations that are too similar to the ones in the case of \mathfrak{h}_5 .

8.2.1. The case of J_1 . Let us denote

$$\alpha = \frac{1 + \sqrt{r}}{\sqrt{r}}.\tag{8.35}$$

Then, the non-trivial equation for the vanishing of Nijenhuis tensor of J_1 are

$$b = -\frac{(1-a^2)\sqrt{\Delta}}{G\alpha},$$

$$b = -\frac{(1-c^2)\alpha\sqrt{\Delta}}{E},$$

$$Fb = -ac\sqrt{\Delta},$$

$$Fa = \frac{Ec}{\alpha} + bc\sqrt{\Delta},$$

$$Fc = G\alpha a + ab\sqrt{\Delta},$$

which are the same equations as (8.4) to (8.8) under the symmetry $(a, b, c) \mapsto (-b, a, c)$. 8.2.2. The case of J_2 . The general form for the meaningful equations for the integrability of J_2 is

$$b(1 - \sqrt{r}) = -\frac{(1 - a^2)\sqrt{r}\sqrt{\Delta}}{G},$$

$$b = -\frac{(1 - c^2)(1 - \sqrt{r})\sqrt{\Delta}}{E\sqrt{r}},$$

$$Fb = -ac\sqrt{\Delta},$$

$$Fa(1 - \sqrt{r}) = Ec\sqrt{r} + bc(1 - \sqrt{r})\sqrt{\Delta},$$

$$Fc = \frac{Ga(1 - \sqrt{r})}{\sqrt{r}} + ab\sqrt{\Delta},$$

which resembles equations (8.14) to (8.19). Moreover, if $r \neq 1$ and we denote

$$\beta = \frac{1 - \sqrt{r}}{\sqrt{r}},$$

then these equations simplify to

$$b = -\frac{(1-a^2)\sqrt{\Delta}}{G\beta},$$

$$b = -\frac{(1-c^2)\beta\sqrt{\Delta}}{E},$$

$$Fb = -ac\sqrt{\Delta},$$

$$Fa = \frac{Ec}{\beta} + bc\sqrt{\Delta},$$

$$Fc = G\beta a + ab\sqrt{\Delta}.$$

Again, these equations behave exactly as in the previous case, after replacing α with β . So, with no extra effort we achieve the following classification of the left-invariant Hermitian structures on H_4 .

Theorem 8.2. Consider in H_4 the left-invariant metric induced by g, given as in (8.31). Then (g, J) is an Hermitian structure on H_4 if and only if

- (1) $J = \pm J_1$, as in (8.32) with a, b, c given as in Table 4 or
- (2) $J = \pm J_2$, as in (8.33) with a, b, c given as in Table 5.

In particular, every left-invariant metric on H_4 is Hermitian with respect to some leftinvariant complex structure.

Table 4. Case \mathfrak{h}_4 : J_1 where $\Delta = EG - F^2$, $\alpha = \frac{1+\sqrt{r}}{\sqrt{r}}$, $\gamma = \frac{E}{\alpha} + G\alpha$

a	b	С	F	r
$\pm \sqrt{1 + \frac{Gb\alpha}{\sqrt{\Delta}}}$	$-\frac{\gamma{-}\sqrt{\gamma^2{-}4\Delta}}{2\sqrt{\Delta}}$	$\pm \sqrt{1 - \frac{Eb}{\alpha\sqrt{\Delta}}}$	> 0	any
0	$-rac{\sqrt{E}}{\sqrt{G}lpha}$	$\pm \sqrt{1 - \frac{E}{G\alpha^2}}$	= 0	$\frac{E}{G} \le \alpha^2$
$\pm \sqrt{1 - \frac{G\alpha^2}{E}}$	$-\frac{\sqrt{G}\alpha}{\sqrt{E}}$	0	= 0	$\alpha^2 \le \frac{E}{G}$

Table 5. Case \mathfrak{h}_4 : J_2 where $\Delta = EG - F^2$, $\beta = \frac{1 - \sqrt{r}}{\sqrt{r}}$, $\delta = \frac{E}{\beta} + G\beta$

a	b	С	F	r
±1	0	0	any	r = 1
$\pm\sqrt{1+\frac{Gb\beta}{\sqrt{\Delta}}}$	$-\frac{\delta-\sqrt{\delta^2-4\Delta}}{2\sqrt{\Delta}}$	$\pm \sqrt{1 + \frac{Eb}{\beta\sqrt{\Delta}}}$	> 0	0 < r < 1
0	$-rac{\sqrt{E}}{\sqrt{G}eta}$	$\pm \sqrt{1 - \frac{E}{G\beta^2}}$	= 0	$0 < r < 1$ and $\frac{E}{G} \le \beta^2$
$\pm \sqrt{1 - \frac{G\beta^2}{E}}$	$-rac{\sqrt{G}eta}{\sqrt{E}}$	0	= 0	$0 < r < 1$ and $\beta^2 \leq \frac{E}{G}$

Remark 8.3. Recall that in H_4 there is, up to automorphism, a unique left-invariant abelian structure, the one given in the proof of Corollary 5.5, and it can be obtained as J_2 from (8.33) with a = 1, b = c = 0, F = 0 and G = E.

8.3. Hermitian structures on \mathfrak{h}_6 . The case of \mathfrak{h}_6 can be treated in the same way as \mathfrak{h}_5 and \mathfrak{h}_4 . And as a matter of fact, calculations are much simpler for \mathfrak{h}_6 . We do no repeat such calculations but only state the theorem of classification of Hermitian structures on \mathfrak{h}_6 . Recall from Theorem 4.9 that any left-invariant metric on H_6 is equivalent to one, and only one, of the form

$$g = \text{diag}(1, 1, 1, 1, E, G), \tag{8.36}$$

where $0 < E \leq G$ and with respect to the standard basis e_1, \ldots, e_6 given at the beginning of Section 4.

Theorem 8.4. Consider in H_6 the left-invariant metric induced by g, as in (8.36) and let us denote $\alpha = \sqrt{E/G}$. Then (g, J) is a Hermitian structure on H_6 if and only if, in

the standard basis, $J = \pm J_1^{\pm}$ or $J = \pm J_2^{\pm}$ where

$$J_1^{\pm} = \begin{pmatrix} 0 & 0 & \pm\sqrt{1-\alpha^2} & -\alpha & 0 & 0\\ 0 & 0 & -\alpha & \mp\sqrt{1-\alpha^2} & 0 & 0\\ \mp\sqrt{1-\alpha^2} & \alpha & 0 & 0 & 0 & 0\\ \alpha & \pm\sqrt{1-\alpha^2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha}\\ 0 & 0 & 0 & 0 & \alpha & 0 \end{pmatrix}$$

and

$$J_2^{\pm} = \begin{pmatrix} 0 & 0 & \pm\sqrt{1-\alpha^2} & -\alpha & 0 & 0\\ 0 & 0 & \alpha & \pm\sqrt{1-\alpha^2} & 0 & 0\\ \mp\sqrt{1-\alpha^2} & -\alpha & 0 & 0 & 0 & 0\\ \alpha & \mp\sqrt{1-\alpha^2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha}\\ 0 & 0 & 0 & 0 & -\alpha & 0 \end{pmatrix}$$

In particular, $J_1^+ = J_1^-$ and $J_2^+ = J_2^-$ if and only if E = G.

Corollary 8.5. Every left-invariant metric on H_6 is Hermitian.

8.4. Hermitian structures on \mathfrak{h}_2 . The cases of \mathfrak{h}_2 and \mathfrak{h}_9 are significantly harder to treat by the above methods. In fact, the generic metrics in our classification for both Lie algebras are not diagonal, with respect to the standard basis, when restricted to the orthogonal complement of the commutator. So the polynomial equations describing the integrability of an arbitrary almost Hermitian structure, by means of the Cholesky decomposition, become wild. However, one can still recover some information from these polynomials in order to estimate the amount of Hermitian structures.

In order to simplify the exposition, let us change slightly the notation for the metrics computed in Theorem 6.5 for the Lie group H_2 . We denote such metrics by

$$g = \sum_{i=1}^{4} e^{i} \otimes e^{i} + 2Ae^{1} \otimes e^{3} + 2Be^{2} \otimes e^{4} + Ee^{5} \otimes e^{5} + 2Fe^{5} \otimes e^{6} + Ge^{6} \otimes e^{6}, \quad (8.37)$$

where $0 \leq A \leq B < 1$, $E, F, G \geq 0$ and $EG - F^2 > 0$. As in the previous cases, we denote $\Delta = EG - F^2$ and in addition we introduce the notations

$$\alpha = \sqrt{1 - A^2}, \qquad \qquad \beta = \sqrt{1 - B^2}.$$

Observe that these parameters satisfy

$$A^2 + \alpha^2 = B^2 + \beta^2 = 1.$$

In order to simplify some calculations, we further define

$$\phi = B\alpha - A\beta, \qquad \qquad \psi = AB + \alpha\beta.$$

It is easy to see that ψ is always positive and ϕ is non negative, and positive if $A \neq B$.

The orientation-preserving almost Hermitian structures with respect to (8.37) are homeomorphic to the disjoint union of two 2-spheres. Since the computation in the general case become very complicated, we illustrate the procedure only for the connected component given by

$$J = \begin{pmatrix} -\frac{Ab}{\alpha} & -\frac{a\alpha + Ac}{\alpha} & -\frac{b}{\alpha} & -\frac{a\phi + c\psi}{\alpha} & 0 & 0\\ \frac{a\beta - Bc}{\beta} & \frac{Bb}{\beta} & -\frac{a\phi + c\psi}{\beta} & \frac{b}{\beta} & 0 & 0\\ \frac{b}{\alpha} & \frac{c}{\alpha} & \frac{Ab}{\alpha} & -\frac{a\beta - Bc}{\alpha} & 0 & 0\\ \frac{c}{\beta} & -\frac{b}{\beta} & \frac{a\alpha + Ac}{\beta} & -\frac{Bb}{\beta} & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{F}{\sqrt{\Delta}} & -\frac{G}{\sqrt{\Delta}}\\ 0 & 0 & 0 & 0 & \frac{E}{\sqrt{\Delta}} & \frac{F}{\sqrt{\Delta}} \end{pmatrix}$$
(8.38)

where

$$a^2 + b^2 + c^2 = 1. ag{8.39}$$

After some long computations, which were verify using the computer software Sage-Math, we obtain that J as in (8.38) is integrable if the following equations hold:

$$0 = -a^2\beta\phi + b^2A + c^2B\psi + ac(B\phi - \beta\psi) - \frac{b(F\alpha + G\beta)}{\sqrt{\Delta}},$$
(8.40)

$$0 = ab\sqrt{\Delta} + aE\phi + c(E\psi + F), \qquad (8.41)$$

$$0 = (1 - a^2)\sqrt{\Delta} + bE\phi, \qquad (8.42)$$

$$0 = (a\phi + c\psi)^2 + b^2 + \frac{Gb\phi}{\sqrt{\Delta}},$$
(8.43)

$$0 = ac\alpha + (1 - a^2)A - \frac{b(F\alpha + E\beta)}{\sqrt{\Delta}},$$
(8.44)

$$0 = ac\phi + (1 - a^2)\psi - \frac{bF\phi}{\sqrt{\Delta}},\tag{8.45}$$

$$0 = (c\phi - a\psi)b\sqrt{\Delta} + aF\phi + c(F\psi + G), \qquad (8.46)$$

$$0 = a^{2}\alpha\phi + b^{2}B + c^{2}A\psi + ac(A\phi + \alpha\psi) + \frac{b(G\alpha + F\beta)}{\sqrt{\Delta}},$$
(8.47)

$$0 = ac\beta - (1 - a^2)B - \frac{b(E\alpha + F\beta)}{\sqrt{\Delta}}.$$
(8.48)

We can fully solve the following important particular case.

Proposition 8.6. If A = B, or equivalently $\phi = 0$, then J is integrable if and only if $a = \pm 1$ and b = c = 0. Moreover, in this case J is abelian.

Remark 8.7. We can approach the abelian case from the classification given in [ABD11] using a similar argument as the one we will use in the next subsection. In fact, one can prove that if an abelian structure is hermitian with respect to a metric of the form (8.37), then A = B.

One can argue that the same approach mentioned in the above remark could be use in the non-abelian case by using the classification of [COUV14], but due to the complexity of the problem, the calculations in this case turn extremely difficult to be solve explicitly. Instead, to treat the case $\phi \neq 0$ we note that from (8.42),

$$b = -\frac{(1-a^2)\sqrt{\Delta}}{E\phi}.$$

Observe that, since $a \neq \pm 1$, we get that b is negative and from (8.45) we get

$$c = \frac{1}{a\alpha} \left(\frac{Fb\phi}{\sqrt{\Delta}} - (1 - a^2)\psi \right)$$
$$= -\frac{1 - a^2}{a\alpha} \left(\frac{F}{E\phi} + \psi \right).$$

By replacing these values of b and c in (8.39) we obtain a quartic equation for a, which is in fact quadratic in a^2 . Disregarding the complex solutions we get the following result.

Proposition 8.8. There exists at most two different values of (a, b, c) such that J is a complex structure.

Note that one still needs to check that the almost complex structure given by this construction satisfies the integrability equations other that (8.42) and (8.45). However, it is our belief that these equations always admit a solution, as we could check by solving them numerically for random parameters.

8.5. Hermitian structures on \mathfrak{h}_9 . As in the above case, the problem of determining the Hermitian metrics on H_9 is very hard, since the moduli space of left-invariant metrics is described by six real parameters. However, we can perform a qualitative analysis by using an appropriate decomposition of a subgroup of automorphisms which preserve the metrics given in (7.3). At some point, calculations become extremely tedious and we use SageMath to check some computations.

Recall that in the basis $\hat{e}_1, \ldots, \hat{e}_6$, every left-invariant metric on H_9 is equivalent, via an automorphism, to one in the slice $\Phi(\Sigma)$, which consist of all the metrics induced by the matrices given in (7.5), where A, B, C > 0 and $D, E, F \in \mathbb{R}$. Notice also that form [ABD11] there exists a unique complex structure (which is also abelian) on \mathfrak{h}_9 up to automorphism. After rearranging the basis, we get that such complex structure, say J_0 , is given by the relations

$$J_0 \hat{e}_1 = -\hat{e}_2, \qquad \qquad J_0 \hat{e}_3 = \hat{e}_5, \qquad \qquad J_0 \hat{e}_4 = -\hat{e}_6.$$

Given a complex structure J on H_9 and $\varphi \in \operatorname{Aut}(\mathfrak{h}_9)$ we denote by $\varphi \cdot J = \varphi J \varphi^{-1}$ the standard action of the automorphism group on the space on complex structures. Let

$$\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma} = \{ \varphi \in \operatorname{Aut}_0(\mathfrak{h}_9) : (g, \varphi \cdot J_0) \text{ is an Hermitian structure for some } g \in \Phi(\Sigma) \}.$$

As we will see later, $\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ is not a subgroup of $\operatorname{Aut}_0(\mathfrak{h}_9)$, but this set is relevant to understand the problem of the existence of Hermitian metrics. In fact, since every complex structure has the form $\varphi \cdot J_0$, if g is a Hermitian metric then there must exist $g' \in \Phi(\sigma)$ and $\varphi' \in \operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ such that g' is isometric to g and $(g', \varphi' \cdot J_0)$ is a Hermitian structure.

Lets keep the notation of (7.2) for a generic $\varphi \in \operatorname{Aut}_0(\mathfrak{h}_9)$. Even though $\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ is not a group, it is not hard to see that it is contained in

$$G = \{ \varphi \in \operatorname{Aut}_0(\mathfrak{h}_9) : a_{21} = 0, a_{22} = a_{11} \},\$$

which is a closed subgroup of the automorphism group. The group structure is easier to understand than the one of the full automorphism group. In fact, consider the following subgroups of G:

$$G_{1} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 1 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & -a_{52} & 1 \end{pmatrix} \right\},\$$

$$G_{2} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_{31} & 1 \end{pmatrix} \right\},\$$

$$G_{3} = \{ \operatorname{diag}(a_{11}, a_{11}, a_{11}^{2}, a_{44}, a_{11}^{2}, a_{11}^{3}) \}.$$

With a routine calculation, one can see that G_1 is a normal subgroup of G. The group G_2 is not a normal subgroup of G, however, $G_1 \rtimes G_2$ is a normal subgroup of G and therefore G is the triple semi-direct product

$$G = (G_1 \rtimes G_2) \rtimes G_3.$$

Let us denote

$$G_i^{\Sigma} = G_i \cap \operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$$

for i = 1, 2, 3 and

$$\Phi(\Sigma_i) = \{ g \in \Phi(\Sigma) : (g, \varphi \cdot J_0) \text{ is a Hermitian structure for some } \varphi \in G_i \}.$$

The following results are straightforward.

is a subgroup of G. Moreover, if $\varphi \in G_1^{\Sigma}$, then $\varphi \cdot J_0$ is Hermitian with respect to any metric in $\Phi(\Sigma_1)$ such that $a_{63} = \frac{AE}{\sqrt{E^2+1}}$.

Lemma 8.10. (1)

$$\Phi(\Sigma_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A^2 + F^2 & F \\ 0 & 0 & 0 & 0 & F & 1 \end{pmatrix} : A > 0, F \in \mathbb{R} \right\}.$$

(2)

$$G_2^{\Sigma} = \left\{ \varphi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : a_{43} \in \mathbb{R} \right\}$$

is a subgroup of G. Moreover, if $\varphi \in G_2^{\Sigma}$, then $\varphi \cdot J_0$ is Hermitian with respect to any metric in $\Phi(\Sigma_2)$ such that $a_{43} = -F$.

Lemma 8.11. (1) $\Phi(\Sigma_3) = \{ \text{diag}(1, 1, A^2, 1, A^2, C^2) : A, C > 0 \}.$ (2) $G_3^{\Sigma} = G_3$. Moreover, if $\varphi \in G_3$ then $\varphi \cdot J_0$ is Hermitian with respect to any metric in $\Phi(\Sigma_3)$ such that $C = \frac{a_{44}}{a_{11}^3}$.

Remark 8.12. It follows from Lemmas 8.9 and 8.10 that G_1^{Σ} commutes with G_2^{Σ} and their product is normalized by G_3 . Moreover, the four dimensional Lie group

$$G' = (G_1^{\Sigma} \times G_2^{\Sigma}) \rtimes G_3$$

is contained in $\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$. In fact, an arbitrary element in G' has the form

$$\varphi' = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11}^2 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11}^2 & 0 \\ 0 & 0 & a_{63} & 0 & 0 & a_{11}^3 \end{pmatrix}$$

where $a_{11}, a_{44} > 0$ and $a_{43}, a_{63} \in \mathbb{R}$, and after long computations one can check that $(g', \varphi' \cdot J_0)$ is a Hermitian structure for the metric $g' \in \Phi(\Sigma)$ given by

$$B = \frac{A^2 a_{11}^5}{\sqrt{A^2 a_{11}^{10} - a_{44}^2 a_{63}^2}}, \qquad C = \frac{A a_{11}^2 a_{44}}{\sqrt{A^2 a_{11}^{10} - a_{44}^2 a_{63}^2}}, \qquad D = 0,$$

$$E = \frac{a_{44} a_{63}}{\sqrt{A^2 a_{11}^{10} - a_{44}^2 a_{63}^2}}, \qquad F = -\frac{A a_{11}^3 a_{43}}{\sqrt{A^2 a_{11}^{10} - a_{44}^2 a_{63}^2}}$$

for sufficiently large values of A. However, notice that $G' \neq \operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$. In order to prove this we can note that the automorphisms in

$$G'' = \left\{ \varphi_{s,t} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & 0 \\ t & -t & 0 & 1 & 0 & 0 \\ s & 0 & 0 & 0 & 1 & 0 \\ t & t & 0 & 0 & 0 & 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

form an abelian group which is contained in $\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ such that $G'' \cap G' = \{I_6\}$. In fact, it is easy to see that $\varphi_{s,t} \cdot J_0$ is hermitian with respect to any metric of the form $g = \operatorname{diag}(1, 1, A, 1, A, 1)$ (recall that all of these metrics belong to $\Phi(\Sigma_3)$, in the notation of Lemma 8.11).

Remark 8.13. $\operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ is not a subgroup of $\operatorname{Aut}_0(\mathfrak{h}_9)$. In fact, one can check by a direct calculation that if $\varphi_2 \in G_2^{\Sigma}$ is not the identity, then $\varphi_{s,t}\varphi_2 \in \operatorname{Aut}_0(\mathfrak{h}_9)^{\Sigma}$ if and only if s = 0 (we are keeping the notation of Remark 8.12).

Finally, using the previous results, we were able to detect a family of left-invariant metrics on H_9 which are not Hermitian.

Proposition 8.14. Let us consider the left-invariant metrics on H_9 given by

$$g_{A,B} = \hat{e}^1 \otimes \hat{e}^1 + \hat{e}^2 \otimes \hat{e}^2 + A^2 \hat{e}^3 \otimes \hat{e}^3 + \hat{e}^4 \otimes \hat{e}^4 + B^2 \hat{e}^5 \otimes \hat{e}^5 + \hat{e}^6 \otimes \hat{e}^6$$

with A, B > 0. Then $g_{A,B}$ is Hermitian if and only if B = A.

Proof. It follows form a direct computation. Notice that one only needs to deal with complex structures obtained by conjugating J_0 by an element of the group G.

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