

On the quantization of sectorially Hamiltonian dissipative systems.

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Abstract

We present a theoretical discussion showing that, although some dissipative systems may have a sectorial Hamiltonian description, this description does not allow for canonical quantization. However, a quantum Liouville counterpart of these systems may be possible, although it may not be unique.

1 Introduction.

In a recent paper [1], we have shown that on phase space, one dimensional systems satisfying an equation of motion of the form $\ddot{x} + F(x, \dot{x}) = 0$ is sectorially Hamiltonian. This means that the phase space can be divided into disjoint sectors such that the behavior of this dissipative system can be obtained from a Hamiltonian function defined in each sector. These sector may change with a change of variables, but this fact is not essential in our discussion. The methods we presented in [1] were valid for one dimensional systems one the configuration space, or two dimensional on phase space. However, these methods could not be extended in general for more dimensions because they were based on the existence of integrating factors for some Pfaffian equations. This investigation had its origin in a previous

paper of our group in which the existence of local constants of motion for the Sinai billiard was investigated [2].

The next step in our research should be double. On one side, we should investigate if these results can be extended to arbitrary dimensions without making use of integrating factor, which restricts the number of cases in which our results can be valid.

On the other side, we should investigate when these type of systems admit quantization. This paper is one step in this direction. Previous studies have been study the quantization of the one dimensional oscillator with friction [3, 4]. More recently, the problem of quantizing dissipative systems have been studied in [5, 6].

A rather recent work [7] shows that the proper framework for canonical quantization of dissipative systems could be the Liouvillian formulation of quantum mechanics. In the present paper, we show that this idea is initially true, although quantization in the Liouville space should not be unique.

This paper is organized as follows: In Section 2 we show that sectorially defined Hamiltonians do not admit in general canonical quantization, since this eventual quantization does not lead to self adjoint Hamiltonians. In Section 3, we give a rigorous argument on why this quantization should be possible on the Liouville formulation of quantum Mechanics and why we do not expect uniqueness.

2 Classical sectorially defined Hamiltonians does not give observables through canonical quantization.

The notion of sectorially defined Hamiltonians, although probably not new, has been introduced by the authors in [1]. In principle, this notion was introduced for two dimensional systems on phase space, or one dimensional systems on configuration space although generalization to higher dimensions is possible.

To illustrate the idea, let us consider the free particle with friction. Classically, it obeys to an equation of motion of the following form

$$\ddot{x}(t) + a\dot{x}(t) = 0. \tag{1}$$

equation (1) is obviously equivalent to the following system

$$\dot{x} = p \quad ; \quad \dot{p} = -ap. \tag{2}$$

It has been shown that (2) obey some sort of Hamilton equations in certain sectors of the phase space (and for this reason, we say that (2) is sectorially Hamiltonian), with sectorial Hamiltonians given by [1]

$$H(x, y) := \begin{cases} H_+(x, p) = p + ax & \text{if } p > 0 \\ H_-(x, p) = -p - ax & \text{if } p < 0 \end{cases}. \quad (3)$$

The points of the straight line $p = 0$ are fixed points. This line coincides with the configuration axis. Note that the Hamiltonian is different in both half planes.

Now, let us consider $H(x, y)$ in (3) as a whole and ask the question: Does $H(x, y)$ admit canonical quantization? The traditional answer is no because the energy of the system is not preserved. We are going to give here another answer more based in arguments of mathematical nature.

Assume that canonical quantization be valid for $H(x, y)$ as in (3). Then, we shall obtain a quantum Hamiltonian which is given by H_+ if $p > 0$ and by H_- if $p < 0$. Let us check whether this Hamiltonian is self adjoint. For this, it is more convenient to work in the momentum representation. Note that $H(x, p)$ is formally symmetric in its arguments and therefore its formal adjoint has the same expression. Let us calculate its deficiency indices.

For $p > 0$, the equations $(H_+^\dagger \pm iI)\phi = 0$ in momentum representation are:

$$(p - ia \frac{\partial}{\partial p} \pm i) \phi(p) = 0 \iff (p \pm i) \phi(p) = ia\phi'(p). \quad (4)$$

The solution of (4) is given by

$$\phi(p) = C \exp\{-\frac{ip^2}{2a}\} e^{\pm p/a}, \quad (p > 0), \quad (5)$$

where C is an irrelevant integration constant. For $p < 0$, the equations $(H_-^\dagger \pm iI)\phi = 0$ in momentum representation take the form:

$$(p + ia \frac{\partial}{\partial p} \pm i) \phi(p) = 0 \iff (p \pm i) \phi(p) = -ia\phi'(p), \quad (6)$$

where the general solution is given by

$$\phi(p) = C \exp\{\frac{ip^2}{2a}\} e^{\mp p/a}, \quad (p < 0). \quad (7)$$

Note that the solution in (5) with sign $+$ and the solution of (7) with sign $-$ both makes the solution of $(H^\dagger + iI)\phi(p) = 0$ for $p > 0$ and $p < 0$

respectively. Note that this solution is not square integrable. On the other hand, the solution in (5) with sign $-$ and the solution of (7) with sign $+$ describe the solution of $(H^\dagger - iI)\phi(p) = 0$ for $p > 0$ and $p < 0$ respectively. This solution is obviously square integrable. Therefore the deficiency indices for our Hamiltonian are $n_+ = 0$ and $n_- = 1$, which are different and therefore H does not admit self adjoint extensions and cannot define any quantum observable. In spite of it is formally Hermitian (symmetric).

A second illustrative example is the following: Let us consider an one dimensional dissipative system composed by one particle of mass $1/2$ subject to a friction force. In this case, the Newton equations are given by the following elementary relations:

$$m\ddot{x}(t) = -\alpha, \quad \text{if } \dot{x}(t) > 0; \quad m\ddot{x}(t) = +\alpha, \quad \text{if } \dot{x}(t) < 0. \quad (8)$$

For simplicity, we shall henceforth assume that $m = 1/2$. Then, this system is sectorially Hamiltonian and this Hamiltonian can be expressed in the following form:

$$H := \begin{cases} H_+ = p^2 + \alpha x & \text{si } p > 0 \\ H_- = p^2 - \alpha x & \text{si } p < 0. \end{cases} \quad (9)$$

As in the previous case, H in (9) is formally symmetric. To see that it is not self adjoint, we solve a pair of differential equations similar to those solved in the previous case.

For $p < 0$, these equations are

$$i\alpha\psi'(p) + p^2\psi(p) = \pm i\psi(p). \quad (10)$$

and their respective solutions are:

$$\psi(p) = C \exp\left\{\frac{ip^3}{3\alpha}\right\} e^{\pm p/\alpha}. \quad (11)$$

For $p > 0$, the equations are

$$-i\alpha\psi'(p) + p^2\psi(p) = \pm i\psi(p), \quad (12)$$

which solutions are given by

$$\psi(p) = C \exp\left\{\frac{-ip^3}{3\alpha}\right\} e^{\mp p/\alpha}. \quad (13)$$

Note that (11) and (13) are quite similar to (5) and (7) respectively. The conclusion is the same with respect to the square integrability of the solutions. Again, our Hamiltonian has different deficiency indices given by 1 and 0 and therefore, it does not admit self adjoint extensions. Or on other words, in spite that H is formally Hermitian, it does not define any quantum observable.

We have seen two examples of very simple classical systems with friction having a sectorial Hamiltonian description that do not admit canonical quantization in the usual sense because the resulting Hamiltonian are not self adjoint in spite of being formally Hermitian.

3 Liouville description.

Let us consider a quantum system with formal Hamiltonian given by H , defined in the Hilbert space \mathcal{H} . The Liouvillian is an operator on the Hilbert space $\mathcal{H} \otimes \mathcal{H}^\times$, where \otimes denotes tensor product and \mathcal{H}^\times is the dual space of \mathcal{H} (obviously isomorphic to \mathcal{H}), defined as

$$\mathcal{L}\rho = -i[H, \rho]. \quad (14)$$

Here, ρ is a general mixed state and the brackets stand for the commutator. The properties of \mathcal{L} come after the properties of the Hamiltonian. For instance, if H is self adjoint on the Hilbert space of Hilbert-Schmidt operators, so is \mathcal{L} . The spectrum of \mathcal{L} can be derived from the spectrum of H , etc. However, the Hamiltonian and Liouvillian descriptions of a quantum systems are not equivalent, contrary to an old belief [8]. In particular, there are systems that admit a description in terms of a Liouvillian, but not of a Hamiltonian. There are two kinds of such systems:

1.- There are self adjoint operators H on a Hilbert space with a purely continuous singular spectrum such that its corresponding Liouvillian $\mathcal{L} := H \otimes I - I \otimes H$, which is self adjoint, has a purely absolutely continuous spectrum. This means that the Liouvillian description admits scattering states and that the original Hamiltonian description does not [9].

2.- Symmetric non self-adjoint operators still admit a Liouville operator, which is self adjoint. However, there are infinitely many inequivalent ways of finding the Liouvillian in this case. This is the situation that concerns us.

The idea is based in the following result that has been already published [10]. For completeness, we include the proof in an Appendix. The result is the following:

Let H be a maximal symmetric operator with different deficiency indices. Then, its corresponding Liouvillian has equal deficiency indices equal to ∞ .

This means that the Liouvillian has infinite self adjoint extensions, i.e., there are an infinite number of solutions to the canonical quantization of the above classical problems with friction in the Liouville space, nonetheless there is no solutions on the the standard Hamiltonian formalism.

Let us illustrate this situation for the case of the sectorial Hamiltonian given in (9). The Hilbert space for the Liouville space is the tensor product of two copies of $L^2(\mathbb{R})$ and its functions are linear combinations of square integrable functions in two variables and limits thereof.

In order to obtain the self adjoint extensions of the Liouvillian, we first consider the equations of the form

$$\mathcal{L}\psi(p, q) \pm i\psi(p, q) = 0, \quad (15)$$

in the momentum space (note that p and q are both momenta). Note that (15) is the equation that give us the deficiency indices for \mathcal{L} .

Assume first that $p > 0$, $q > 0$. In this sector, $H = H_+$ and (15) has the form

$$p^2\psi(p, q) - i\alpha\psi_p(p, q) - q^2\psi(p, q) + i\alpha\psi_q(p, q) \pm i\psi(p, q) = 0. \quad (16)$$

Since (16) is a linear equation, we try factorized solutions of the form $\psi(p, q) = P(p)Q(q)$. We obtain separation of variables:

$$\frac{p^2P(p) - i\alpha P'(p) \pm P(p)}{P(p)} = \frac{q^2Q(q) - i\alpha Q'(q)}{Q(q)}. \quad (17)$$

Since both terms in (17) depend on independent variables, both terms are equal to a constant, which is complex in our case. Note that (17) is not really one equation but two: one for each sign. Thus, we have that

$$\frac{p^2P_{\pm}(p) - i\alpha P'_{\pm}(p) \pm P_{\pm}(p)}{P_{\pm}(p)} = \lambda_{\pm} \quad (18)$$

$$\frac{q^2Q_{\pm}(q) - i\alpha Q'_{\pm}(q)}{Q_{\pm}(q)} = \lambda_{\pm}. \quad (19)$$

Integration of these equations give respectively:

$$P_{\pm}(p) = C e^{-\frac{1}{\alpha}(\frac{ip^3}{3} - i\lambda_{\pm}p \mp p)}, \quad p > 0 \quad (20)$$

$$Q_{\pm}(q) = C' e^{-\frac{1}{\alpha}(\frac{iq^3}{3} - i\lambda_{\pm}q)}, \quad q > 0 \quad (21)$$

Now, let us write λ_{\pm} in terms of its real and imaginary parts as $\lambda_{\pm} = a_{\pm} + ib_{\pm}$. Note that $P_{+}(p)$ and $P_{-}(p)$ are square integrable if and only if $1 < b_{+}$ and $-1 < b_{-}$ respectively. Analogously, $Q_{+}(p)$ and $Q_{-}(p)$ are square integrable if and only if $b_{+} > 0$ and $b_{-} > 0$ respectively. Therefore in the sector $p > 0, q > 0$, $\psi_{+}(p, q) = P_{+}(p)Q_{+}(q)$ is square integrable if and only if $b_{+} > 1$ and $\psi_{-}(p, q) = P_{-}(p)Q_{-}(q)$ is square integrable if and only if $b_{-} > 0$.

Let us now consider the case, $p < 0, q < 0$. In this sector, we have $H = H_{-}$. The differential equation (15) is given by

$$p^2\psi(p, q) + i\alpha\psi_p(p, q) - q^2\psi(p, q) - i\alpha\psi_q(p, q) \pm i\psi(p, q) = 0, \quad (22)$$

which gives with the factorization $\psi_{\pm}(p, q) = R_{\pm}(p)S_{\pm}(q)$:

$$\frac{p^2R_{\pm}(p) + i\alpha R'_{\pm}(p) \pm R_{\pm}(p)}{R_{\pm}(p)} = \frac{q^2S_{\pm}(q) + i\alpha S'_{\pm}(q)}{S_{\pm}(q)} = \mu_{\pm}, \quad (23)$$

where again μ_{\pm} are complex numbers. Equations (23) have the following solutions respectively,

$$R_{\pm}(p) = C e^{\frac{1}{\alpha}(\frac{ip^3}{3} - i\mu_{\pm}p \mp p)}, \quad p < 0. \quad (24)$$

$$S_{\pm}(q) = C' e^{\frac{1}{\alpha}(\frac{iq^3}{3} - i\mu_{\pm}q)}, \quad q < 0. \quad (25)$$

Equations (24) and (25) are simultaneously square integrable if and only if $d_{+} > 1$ and $d_{-} > 0$, where d_{\pm} is the imaginary part of μ_{\pm} .

Since any of the exponential functions in (20), (21), (24), (25) are linearly independent if the constants λ_{\pm} or μ_{\pm} are different, we conclude that equations (15) have an infinite number of linearly independent solutions, which shows that both deficiency indices are infinite. The question is now how we can construct the infinite self adjoint extensions of the Liouvillian for this case. This is not an easy task and we shall give a partial answer in the next subsection.

3.1 Self adjoint extensions of the Liouvillian.

The fact that the deficiency indices of \mathcal{L} are infinite makes the search for explicit self adjoint extensions of \mathcal{L} (and henceforth for explicit quantizations of our system) a very complicated mathematical task. Nevertheless, we shall outline here the guidelines for obtaining these extensions and propose an explicit example.

A general results states the following [11, 12, 13]: Let A be a symmetric operator with domain \mathcal{D}_A on a separable infinite dimensional Hilbert space \mathcal{H} . The operator A has nonzero deficiency subspaces K_+ and K_- of equal dimension either finite or infinite. For each unitary operator U from K_+ onto K_- , we can construct a unique self adjoint extension A_U of A , with domain

$$\mathcal{D}_U := \{\varphi + \varphi_+ + U\varphi_+ \mid \forall \varphi \in \mathcal{D}_A; \forall \varphi_+ \in K_+\}. \quad (26)$$

Then, on this domain,

$$A_U(\varphi + \varphi_+ + U\varphi_+) := A\varphi + i\varphi_+ - iU\varphi_+, \quad (27)$$

which is the self adjoint extension of A associated to the unitary operator U .

In order to apply this result to our case, we need to find out the domain of H , \mathcal{D}_H in (9). To this end, let us consider the subspace of square integrable functions $\varphi(p)$, in the momentum representation, such that

1. The function $p^2\varphi(p)$ is square integrable.
2. The function $\varphi(p)$ is absolutely continuous and its derivative (which exists almost elsewhere), $\varphi'(p)$, is square integrable.
3. The following limits exist: $\varphi(-\infty) = \varphi(\infty) = 0$.
4. Finally, the functions in our subspace verify that $\varphi(0) = 0$.

A simple calculation shows that for any $\psi(p)$ and $\varphi(p)$ with the above properties, one has that $\langle \varphi | H\psi \rangle = \langle H\varphi | \psi \rangle$, so that H is a symmetric operator on this subspace and therefore closable. Let \overline{H} the closure of H . Then, we take as \mathcal{D}_H the domain of \overline{H} , which is the minimal closed extension of H . Without any substantial loss, we shall henceforth identify H with \overline{H} .

Now, let us recall that $\mathcal{L} = H \otimes I - I \otimes H$. Then a possible domain for \mathcal{L} is $\mathcal{D}_H \otimes \mathcal{D}_H$. This is the space of linear combinations of tensor products of the

form $\psi \otimes \varphi$ with $\psi(p), \varphi(p) \in \mathcal{D}_H$ [14]. Then, \mathcal{L} is symmetric on $\mathcal{D}_H \otimes \mathcal{D}_H$ but in general not closed. Let $\mathcal{D}_{\mathcal{L}}$ be the domain of the minimal closed extension of \mathcal{L} , which is also a symmetric operator that we can identify with \mathcal{L} without any loss.

In any case, for any $\rho \in \mathcal{D}_H \otimes \mathcal{D}_H$, the Liouville equation holds:

$$\mathcal{L}\rho = -i\{(H \otimes I)\rho - (I \otimes H)\rho\} = -i[H, \rho]. \quad (28)$$

From the above comments, we know that there is an one to one correspondence between unitary operators from K_+ onto K_- and self adjoint extensions of \mathcal{L} . Then, if $\rho \in \mathcal{D}_{\mathcal{L}}$ and $\rho_+ \in K_+$, the self adjoint extension of \mathcal{L} associated to U is given by

$$\mathcal{L}(\rho + \rho_+ + U\rho_+) = \mathcal{L}\rho + i\rho_+ - iU\rho_+. \quad (29)$$

If instead of having $\rho \in \mathcal{D}_{\mathcal{L}}$, we choose $\rho \in \mathcal{D}_H \otimes \mathcal{D}_H \subset \mathcal{D}_{\mathcal{L}}$, then we have for any $\rho_+ \in K_+$:

$$\mathcal{L}(\rho + \rho_+ + U\rho_+) = -i[H, \rho] + B\rho_+, \quad (30)$$

with $B\rho_+ = i\rho_+ - iU\rho_+$. Note that (30) resembles an equation of Lindblad type. With domain

$$\{\rho + \rho_+ + U\rho_+ \mid \forall \rho \in \mathcal{D}_H \otimes \mathcal{D}_H; \forall \rho_+ \in K_+\}, \quad (31)$$

\mathcal{L} is essentially self adjoint and therefore it uniquely determines the self adjoint extension of \mathcal{L} associated to U .

3.1.1 A self adjoint extension of \mathcal{L} .

Let us go back to (20) and (21). Take the function

$$P_+(p)Q_+(q) = e^{-\frac{1}{\alpha}(\frac{ip^3}{3} - i\lambda_+p - p)} e^{-\frac{1}{\alpha}(\frac{iq^3}{3} - i\lambda_+q)} \quad (32)$$

and recall that this function is square integrable if and only if the imaginary part of λ_+ , b_+ , has the property $b_+ > 1$. Therefore, we consider only complex values of λ_+ with this property. Also recall that we are in the sector given by $p > 0$, $q > 0$.

Then, define a new constant $\lambda := \lambda_+ - i$. Note that the imaginary part of λ should be bigger than zero. In terms of this new constant equation (32) has the following form:

$$e^{-\frac{1}{\alpha}(\frac{ip^3}{3} - i\lambda p)} e^{-\frac{1}{\alpha}(\frac{iq^3}{3} - i\lambda q + q)}, \quad p > 0, \quad q > 0, \quad \text{Im } \lambda > 0. \quad (33)$$

With the change of variables given by $p \mapsto -q$ and $q \mapsto -p$, we reach the sector $p < 0, q < 0$. Writing $\mu_- = \lambda$, we obtain

$$e^{\frac{1}{\alpha}(\frac{iq^3}{3} - i\mu_- q)} e^{\frac{1}{\alpha}(\frac{ip^3}{3} - i\mu_- p + p)}, \quad p < 0, \quad q < 0, \quad \text{Im } \mu_- > 0. \quad (34)$$

The function in (34) is just $R_-(p)S_-(q)$. A simple calculation shows that $P_+(p)Q_+(q)$ and $R_-(p)S_-(q)$ have the same norm¹. Therefore the mapping

$$P_+(p)Q_+(q) \mapsto R_-(p)S_-(q) \quad (35)$$

leaves the Hilbert space norm invariant. The same property can be proven for the mapping

$$R_+(p)S_+(q) \mapsto P_-(p)Q_-(q). \quad (36)$$

Combining (35) and (36), we have a mapping U that maps functions in K_+ into functions in K_- that preserves the Hilbert space norm. This map can be extended by linearity and it results a continuous map from a dense set in K_+ into K_- . Since it preserves the norm, U is continuous and can be again extended to a continuous norm preserving linear map from K_+ into K_- . It is easy to prove that U is onto and therefore U is a unitary mapping from K_+ onto K_- .

Once we have constructed such a unitary operator, we have automatically a self adjoint extension of the Liouvillian \mathcal{L} and therefore a possible quantization of our dissipative system.

4 Concluding remarks.

We have shown, using an example that dissipative systems do not admit in general canonical quantization since this procedure will provide a non self adjoint Hamiltonian.

However, this canonical quantization is possible in the Liouville space. This is however not unique and worse of all, there is an infinite number of nonequivalent possibilities for this quantization, each one related with a self adjoint extension of the Liouvillian. The task of obtaining these self adjoint extensions can be expected to be difficult, although some particular cases

¹This norm is given by

$$\int_0^\infty e^{-\frac{2}{\alpha}(b_+ - 1)p} e^{-\frac{2}{\alpha}b_+ q} dp dq.$$

are expected to be readily obtained. As an example, we have obtained a self adjoint extension of the Liouvillian in a given example.

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5 Appendix.

Here, we present the proof of the Proposition in Section 2. This is a consequence of the following result proven in [11]: Let H be a maximal symmetric operator on the Hilbert space \mathcal{H} . Then, \mathcal{H} can be decomposed into an orthogonal direct sum as follows:

$$\mathcal{H} = \bigoplus_{n=0}^N \mathcal{H}_n, \quad (37)$$

where N is either finite or infinite. The subspaces \mathcal{H}_n are all invariant under the action of both H and H^\dagger and verify the following properties:

- i.) The restriction H_0 of H to \mathcal{H}_0 is self adjoint.
- ii.) For each $n \neq 0$, there exists a unitary operator U_n from \mathcal{H}_n onto $L^2(\mathbb{R}^+)$ ($\mathbb{R}^+ := [0, \infty)$), such that if H_n denotes the restriction of T into \mathcal{H}_n , the operator $U_n H_n U_n^{-1}$ is the operator $\pm iD$ with domain:

$$\begin{aligned} \mathcal{D}(D) = \{f \in L^2(\mathbb{R}^+) ; f(x) \text{ is absolutely continuous,} \\ f'(x) \in L^2(\mathbb{R}^+) ; f(0) = 0\}, \end{aligned} \quad (38)$$

and

$$Df(x) = f'(x). \quad (39)$$

We have $+iD$ for all n if the positive (n_+) deficiency index of H is zero and $-iD$ for all n if the negative (n_-) deficiency index of H is zero².

Let us consider now the decomposition (37) and take $n \neq 0$. Consider the orthogonal projection $P_n : \mathcal{H} \mapsto \mathcal{H}_n$ and $\mathcal{P}_n := P_n \otimes P_n$. Now, define

²This is our case.

$$\begin{aligned}
\mathcal{L}_n &:= \mathcal{P}_n \mathcal{L} \mathcal{P}_n = P_n \otimes P_n (H \otimes I - I \otimes H) P_n \otimes P_n \\
&= (P_n H P_n) \otimes P_n - P_n \otimes (P_n H P_n). \tag{40}
\end{aligned}$$

Since H leaves \mathcal{H}_n invariant, $P_n H P_n$ is the restriction of H to \mathcal{H}_n . This restriction is unitarily equivalent to either $+iD$ or $-iD$ on $L^2(\mathbb{R}^+)$. Thus, \mathcal{L}_n is unitarily equivalent to $K_n := \pm i(D \otimes I - I \otimes D)$. Since [12] $K_n^\dagger > D^\dagger \otimes I - I \otimes D^\dagger$, the equations $(K^\dagger \pm iI)\psi = 0$ have, at least, the following solutions:

$$\psi_\alpha(x, y) = C e^{\alpha x} e^{\alpha y} e^{\mp i y}, \tag{41}$$

with $\alpha < 0$. These solutions are in $L^2(\mathbb{R}^+ \times \mathbb{R}^\times)$. Since solutions with different values of α are linearly independent, we conclude that the deficiency indices of K_n are both infinite. Therefore, we conclude that the deficiency indices of \mathcal{L}_n are both infinite. Since \mathcal{L} is the closure of the orthogonal sum of the \mathcal{L}_n , this implies that \mathcal{L} has both deficiency indices equal to infinity.

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