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Strongly smooth paths of idempotents

Esteban Andruchow [a,b,*]

[a] Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina
[b] Instituto Argentino de Matemática, CONICET, Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina

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ABSTRACT

It is shown that a curve $q(t)$, $t \in I$ $(0 \in I)$ of idempotent operators on a Banach space $\mathcal{X}$, which verifies that for each $\xi \in \mathcal{X}$, the map $t \mapsto q(t)\xi \in \mathcal{X}$ is continuously differentiable, can be lifted by means of a regular curve G_t, of invertible operators in $\mathcal{X}$:

$$q(t) = G_t q(0) G_t^{-1}, \quad t \in I.$$

This is done by using the transport equation of the Grassmannian manifold, introduced by Corach, Porta and Recht. We apply this result to the case when the idempotents are conditional expectations of a C* algebra $\mathcal{A}$ onto a field of C*-subalgebras $\mathcal{B}_t \subset \mathcal{A}$. In this case the invertible operators, restricted to $\mathcal{B}_0$, induce C*-isomorphisms between $\mathcal{B}_0$ and $\mathcal{B}_t$. We examine the regularity condition imposed on the curve of expectations, in the case when these expectations are induced by discrete decompositions of a Hilbert space (also called systems of projectors in the literature).

1. Introduction: the transport equation

In the study of the Grassmann manifold of a Hilbert space $\mathcal{H}$, one may choose to identify closed subspaces with orthogonal projections,

$$\mathcal{S} \subset \mathcal{H} \leftrightarrow p_{\mathcal{S}} \in \mathcal{B}(\mathcal{H}),$$

where $p_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$ and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded and linear operators in $\mathcal{H}$. This identification allows one to compute all relevant geometric quantities in terms of operators. For instance, the parallel transport equation (of the Levi–Civita connection of the Grassmann manifold): if $p(t)$, $t \in [0,1]$, is a curve of projections, then the unique solution of the differential equation

$$\begin{cases} \dot{g} = [\dot{p}, p]g, \\ g(0) = 1 \end{cases}$$

([,] is the conmutator of operators) is a curve $g(t)$ of unitary operators in $\mathcal{H}$ which performs the parallel transport: if x is a tangent vector at $p(0)$ (it is a self-adjoint operator acting in $\mathcal{H}$ in this framework), then its parallel transport along $p(t)$ is $g(t)xg(t)^*$. This operator theoretic point of view was introduced in the papers [7,8,3] by G. Corach, H. Porta and L. Recht, for abstract Banach and C*-algebras (remarkably, no need to refer to any underlying vector space). The topology (and smooth structure) considered in their work is the one provided by the spectral norm of the algebra.

* Address for correspondence: Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina.
E-mail address: eandruch@ungs.edu.ar.

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The purpose of this paper is the study of the transport equation under weaker or more general conditions. Namely, given a Banach space $\mathcal{X}$ and a curve $q(t)$ of idempotent operators in $\mathcal{X}$ ($q^2(t) = q(t)$), $t \in I$ ($0 \in I$), such that for each $\xi \in \mathcal{X}$, the map $I \ni t \mapsto q(t)\xi \in \mathcal{X}$ is C^1, consider the following analogous to the transport equation:

$$\begin{cases} \dot{\gamma}(t) = [X(t), q(t)]\gamma(t), \\ \gamma(0) = \xi_0 \end{cases}$$

where γ takes values in $\mathcal{X}$, ξ_0 is fixed and $X(t)$ is the field of operators obtained by differentiating q, i.e.

$$X(t)\xi = \frac{d}{dt} q(t)\xi.$$

In order to apply the theory of linear differential equations in Banach spaces, one must first check that the commutators $[X(t), q(t)]$ are bounded, and that for each $\xi \in \mathcal{X}$, the map $t \mapsto [X(t), q(t)]\xi$ is continuous.

Once these facts are established, it is shown that the propagator of this equation, that is, the field G_t of invertible operators acting in $\mathcal{X}$, given by $G_t(\xi_0) = \gamma(t)$, where γ is the unique solution of the above equation with initial condition $\gamma(0) = \xi_0$, satisfies the following lifting property (as in the operator algebra context which motivated this study):

$$G_t q(0) G_t^{-1} = q(t), \quad t \in I.$$

Also, by construction, for each $\xi \in \mathcal{X}$, the map $I \ni t \mapsto G_t(\xi) \in \mathcal{X}$ is C^1.

Next this paper considers a special case, when $\mathcal{X} = \mathcal{A}$ is a C*-algebra and the idempotents are conditional expectations $E_t : \mathcal{A} \to \mathcal{B}_t \subset \mathcal{A}$ onto C*-subalgebras of $\mathcal{A}$. It is shown that in this particular case, the propagators G_t are unital, $*$-preserving maps, which restricted $\mathcal{B}_0$ give $*$-algebraic isomorphisms between $\mathcal{B}_0$ and $\mathcal{B}_t$.

In the last section a particular case of this situation is examined, when the conditional expectations arise from orthogonal decompositions, or systems of projections, of a Hilbert space. A system of projections is a finite or infinite collection

$$P = (p_1, p_2, \ldots),$$

such that $p_i p_j = \delta_{ij} p_i$ and $p_1 + p_2 + \cdots = 1$ (strongly). A system P gives rise to a conditional expectation E_P from $\mathcal{B}(\mathcal{H})$ onto the subalgebra of operators which commute with all the p_i:

$$E_P(x) = \sum_{i \geqslant 1} p_i x p_i.$$

This expectation preserves the ideal $\mathcal{K}(\mathcal{H})$ of compact operators, and we shall restrict it there. A curve $P(t)$ of systems of projectors gives rise then to a curve of conditional expectations in $\mathcal{K}(\mathcal{H})$. We characterize what regularity condition must be verified, in order that the differential equation above makes sense (i.e. the curve of idempotents is smooth in the sense discussed above). Namely, for each $\xi \in \mathcal{H}$, each map $I \ni t \mapsto p_i(t)\xi \in \mathcal{H}$ must be C^1, and for each closed and bounded interval $J \subset I$, there exists a constant $C_{\xi,J}$ such that

$$\sum_{i \geqslant 1} \|\dot{p}_i(t)\xi\|^2 < C_{\xi,J} < \infty.$$

The properties of the propagators G_t are studied. For example it is shown that they preserve the Schatten ideals $\mathcal{B}_p(\mathcal{H})$. For $p = 2$, G_t induces a unitary operator in $\mathcal{B}_2(\mathcal{H})$.

2. Strongly smooth paths of idempotents

Let $I \subset \mathbb{R}$ be an interval ($0 \in I$), $\mathcal{X}$ a Banach space and $q(t)$, $t \in I$, a path in $\mathcal{B}(\mathcal{X})$ whose values are idempotents. We shall suppose that $q(t)$ is strongly continuously differentiable, which we shall abreviate *strongly smooth*, and which means that for every $\xi \in \mathcal{X}$, the map

$$I \ni t \mapsto q(t)\xi \in \mathcal{X}$$

is continuously differentiable. Such a map defines a map of derivatives $t \mapsto X(t)$, given by $X(t)\xi = \dot{q}(t)\xi$. The operators $X(t)$ are linear and everywhere defined, and as a consequence of the Uniform Boundedness Principle (UBP), they are bounded. Let us state and prove this fact.

Lemma 2.1. *With the above hypothesis, for every $t \in I$, $X(t) \in \mathcal{B}(\mathcal{X})$.*

Proof. Fix $t \in I$. For an integer $n \geqslant 1$, consider $c_n \in \mathcal{B}(\mathcal{X})$ given by

$$c_n(\xi) = n\left\{q\left(t + \frac{1}{n}\right) - q(t)\right\}\xi.$$

Apparently, for each $\xi \in \mathcal{X}$, $c_n\xi \to X(t)\xi$ as $n \to \infty$. Then for each $\xi \in \mathcal{X}$, there exists C_ξ such that $\|c_n\xi\| \leqslant C_\xi$. Therefore, by the UBP there exists C such that $\|c_n\xi\| \leqslant C\|\xi\|$. In particular, this implies that $\|X(t)\xi\| \leqslant C\|\xi\|$. $\square$

Also it is apparent that the map $t \mapsto X(t)$ is strongly continuous. The main result in this section is that idempotents belonging to such curves are pairwise similar. More precisely, there exists a curve of invertible operators G_t, smooth in the above sense, such that $q(t) = G_t X(0) G_t^{-1}$.

Lemma 2.2. *Let $q(t)$, $t \in I$, be a strongly smooth path of idempotents, and let $X(t) = \dot{q}(t)$. Then for any $\xi \in \mathcal{X}$ and $t \in I$, one has*

$$X(t)\xi = q(t)X(t)\xi + X(t)q(t)\xi.$$

Proof. Since $q(t)q(t) = q(t)$, it follows that for $\xi \in \mathcal{X}$,

$$\begin{aligned} X(t)\xi &= \lim_{h\to 0}\frac{1}{h}\big(q(t+h)q(t+h)\xi - q(t)q(t)\xi\big) \\ &= \lim_{h\to 0}\frac{1}{h}q(t+h)\big\{q(t+h)\xi - q(t)\xi\big\} + \lim_{h\to 0}\frac{1}{h}\big\{q(t+h)q(t)\xi - q(t)q(t)\xi\big\}. \end{aligned}$$

The second summand converges to $X(t)q(t)\xi$. On the other hand, if $h \to 0$, then $q(t+h) \to q(t)$ strongly, and $\frac{1}{h}\{e(t+h)\xi - e(t)\xi\} \to X(t)\xi$ in $\mathcal{H}$. Moreover, the norms $\|q(t+h)\|$ are bounded (on bounded intervals), again by an elementary application of the UBP: by strong continuity of $q(t)$, for each $\xi \in \mathcal{X}$, and h in a bounded interval, $\|q(t+h)\xi\| \leqslant M_h < \infty$.

These facts imply that

$$\frac{1}{h}q(t+h)\big\{q(t+h)\xi_0 - q(t)\xi\big\} \to q(t)X(t)\xi. \qquad \square$$

Let $[\ ,\ ]$ denote the commutator of operators, $[a,b] = ab - ba$. Note that for each $\xi \in \mathcal{H}$, the map

$$t \mapsto \big[X(t), q(t)\big]\xi \in \mathcal{X}$$

is continuous. This is apparent, because the operators $q(t)$ are uniformly norm bounded, $\|q(t)\| \leqslant M$, on closed bounded sub-intervals of I. We shall consider the following linear differential equation in $\mathcal{X}$: for a given strongly smooth curve of idempotents $q(t)$, $t \in I$ ($0 \in I$), and for fixed $\xi \in \mathcal{X}$, $s \in I$,

$$\begin{cases} \dot{\gamma}(t) = \big[X(t), q(t)\big]\gamma(t), \\ \gamma(s) = \xi. \end{cases} \tag{1}$$

Note that this equation has a unique solution γ_s, defined in the interval I, taking values in $\mathcal{X}$. Indeed, this is a classical result in the theory of linear differential equations in Banach spaces [5]. There exists a two parameter family of invertible operators $G(s,t)$, $s,t \in I$, such that:

1. $G(r,s)G(s,t) = G(r,t)$.
2. $G(t,t) = 1$.
3. $G(s,t)$ is jointly strongly continuous in s, t.
4. The unique solution γ_s of (1) is given by

$$\gamma_s(t) = G(t,s)\xi.$$

The family $G(s,t)$ is called the *propagator* of the equation.

Denote $G_t := G(t,0)$.

Lemma 2.3. *Let γ be a solution of Eq. (1). Then $q(\gamma)$ is also a solution. In particular, if $\gamma(t_0) \in R(q(t_0))$ for some t_0, then $\gamma(t) \in R(q(t))$ for all t.*

Proof. By an argument similar to the one given in the above lemma, the map $t \mapsto q(t)(\gamma(t))$ is C^1, and the Leibniz rule holds:

$$\frac{d}{dt}q(t)\big(\gamma(t)\big) = X(t)\big(\gamma(t)\big) + q(t)\big(\dot{\gamma}(t)\big).$$

Then, since γ is a solution,

$$\frac{d}{dt}q(t)\big(\gamma(t)\big) = X(t)\big(\gamma(t)\big) + q(t)\big(X(t)q(t)\gamma(t) - q(t)X(t)\gamma(t)\big).$$

By Lemma 2.2, $q(t)X(t)q(t) = 0$ and $X(t) - q(t)X(t) = X(t)q(t)$. Then

$$\frac{d}{dt}q(t)\big(\gamma(t)\big) = X(t)q(t)\gamma(t).$$

On the other hand, by the same lemma

$$\big[X(t), q(t)\big]q(t)\gamma(t) = X(t)q(t)\gamma(t),$$

and $q(\gamma)$ is a solution. If $\gamma(t_0) \in R(q(t_0))$, then $q(t_0)(\gamma(t_0)) = \gamma(t_0)$, and thus $q(\gamma)$ and γ are two solutions satisfying the same initial condition. □

Theorem 2.4. *Let $q(t)$, $t \in I$, be a strongly smooth curve of idempotents. Then $q(t)$ are pairwise similar. More specifically, $q(t) = G_t q(0) G_t^{-1}$. The curve G_t is strongly C^1.*

Proof. Let us compare $q(t)G_t(\xi)$ and $G_t q(0)(\xi)$ for an arbitrary $\xi \in \mathcal{X}$. By the lemma above, $\alpha(t) = q(t)G_t(\xi)$ is a solution of (1), and then

$$\frac{d}{dt}\alpha = X(t)\alpha(t).$$

Note that $\alpha(0) = q(0)\xi$ On the other hand, $\beta(t) = G_t q(0)(\xi)$ is another solution, with initial condition $q(0)\xi$. Therefore $\alpha = \beta$. □

Note that the requirement that the curve $q(t)$ be strongly smooth is necessary. There are elementary examples of strongly continuous curves of projections linking non-similar projections. For instance, consider $q(t)$ the multiplication operator in $L^2(0,1)$ by the characteristic function $\chi_{[0,t]}$ of the interval $[0,t]$. Then $q(t)$ are projections, with $q(0) = 0$ and $q(1) = 1$. The curve $q(t)$ is strongly continuous: if $\xi \in L^2(0,1)$,

$$\big\|q(t+r)\xi - q(t)\xi\big\|_2^2 = \left|\int_t^{t+r} \big|\xi(s)\big|^2\, ds\right| \to 0 \quad (h \to 0).$$

In certain special cases more can be said. If $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $q(t) = e(t)$ are self-adjoint projections, then $G_t = U_t$ are unitary operators. This follows by noting that in this case $X(t)$ are self-adjoint, and therefore the commutators $[X(t), e(t)]$ (being the commutant of self-adjoint operators) are skew-hermitian. Thus the propagators are unitary operators [10]. Let us state this as a corollary.

Corollary 2.5. *Let $e(t)$, $t \in I$, be a strongly smooth curve of self-adjoint projections. Then $e(t)$ are pairwise unitarily equivalent. More specifically, $e(t) = U_t e(0) U_t^*$. The curve U_t is strongly C^1.*

Next we shall consider a special class of idempotents, namely conditional expectations in C*-algebras. See [1] for the basic facts on conditional expectations. Let $\mathcal{A}$ be a unital C*-algebra and suppose that for $t \in I$ one has subalgebras $1 \in \mathcal{B}_t \subset \mathcal{A}$ and conditional expectations $E_t : \mathcal{A} \to \mathcal{B}_t$. The smoothness assumption states that for each $a \in M$, the map $t \mapsto E_t(a) \in \mathcal{A}$ is continuously differentiable. Denote by $dE_t : \mathcal{A} \to \mathcal{A}$ the derivative of E_t: $dE_t(a) = \frac{d}{dt}E_t(a)$. For each fixed t, the operator $dE_t : \mathcal{A} \to \mathcal{A}$ is bounded. For each $t \in I$ and $a \in \mathcal{A}$, one has

$$dE_t\big(E_t(a)\big) + E_t\big(dE_t(a)\big) = dE_t(a).$$

Therefore we may consider the analogous differential equation, for $a \in \mathcal{A}$, $s \in I$

$$\begin{cases} \dot\alpha(t) = [dE_t, E_t]\big(\alpha(t)\big), \\ \alpha(s) = a. \end{cases} \tag{2}$$

In this case, the propagators G_t have the following properties:

Theorem 2.6. *The invertible operators $G_t : \mathcal{A} \to \mathcal{A}$ have the following properties*:

1. *For each $a \in \mathcal{A}$, the map $I \ni t \mapsto G_t(a) \in \mathcal{A}$ is C^1.*
2. $G_t(a^*) = G_t(a)^*$.
3. $G_t \circ E_0 \circ G_t^{-1} = E_t$.

4. *From the preceding point, it follows that G_t maps $\mathcal{B}_0$ onto $\mathcal{B}_t$. Moreover,*

$$G_t|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t$$

is a $*$-isomorphism. In particular, $G_t(1) = 1$.

5. *If $\mathcal{A}$ is a finite von Neumann algebra, $\mathcal{B}_t$ are von Neumann subalgebras, τ is a tracial normal and faithful state, and E_t are the unique τ-invariant expectations onto $\mathcal{B}_t$, then G_t is isometric for the $\| \, \|_2$ norm given by τ, therefore it extends to a unitary operator in $\mathcal{H} = L^2(\mathcal{A}, \tau)$.*

Proof. Given $a \in \mathcal{A}$, let $\alpha(t) = G_t(a)$. The first fact is clear. Since E_t and dE_t are $*$-preserving, it follows that $\alpha^*(t)$ is a solution, with initial condition $\alpha^*(0) = a^*$, and then $G_t(a^*) = G_t(a)^*$.

The fact that G_t intertwines E_0 and E_t follows from the general result in the previous section.

Let us show that $G_t|_{\mathcal{B}_0}$ is an isomorphism, i.e. that it is multiplicative. To that purpose note that if the initial data $a = \alpha(0)$ in Eq. (2) belongs to $\mathcal{B}_0$, then, by Lemma 2.3, $\alpha(t) = G_t(a) \in \mathcal{B}_t$. Therefore, under this assumption, this equation is equivalent to the condition

$$E_t\big(\dot\alpha(t)\big) = 0.$$

Indeed, differentiating $E_t(\alpha(t)) = \alpha(t)$, one obtains

$$dE_t\big(\alpha(t)\big) + E_t\big(\dot\alpha(t)\big) = \dot\alpha(t) = dE_t\big(E_t\big(\alpha(t)\big)\big) - E_t\big(dE_t\big(\alpha(t)\big)\big).$$

In the right hand term, $dE_t(E_t(\alpha(t))) = dE_t(\alpha(t))$ and $E_t(dE_t(\alpha(t))) = 0$ (as remarked above, $E_t \circ dE_t \circ E_t = 0$). Thus

$$E_t\big(\dot\alpha(t)\big) = 0,$$

and viceversa. Therefore if $a_1, a_2 \in \mathcal{B}_0$ and α_1 and α_2 are the solutions of (2) with these initial conditions, then, using that E_t are $\mathcal{B}_t$-valued conditional expectations,

$$E_t\left(\frac{d}{dt}\big\{\alpha_1(t)\alpha_2(t)\big\}\right) = E_t\big(\dot\alpha_1(t)\alpha_2(t)\big) + E_t\big(\alpha_1(t)\dot\alpha_2(t)\big),$$
$$E_t\big(\dot\alpha_1(t)\big)\alpha_2(t) + \alpha_1(t)E_t\big(\dot\alpha_2(t)\big) = 0.$$

That is, $\alpha_1(t)\alpha_2(t)$ is a solution of (2), with initial condition a_1a_2, hence

$$G_t(a_1a_2) = \alpha_1(t)\alpha_2(t) = G_t(a_1)G_t(a_2).$$

Suppose that $\mathcal{A}$ is a finite von Neumann algebra with trace τ, and that E_t are τ-invariant. Then $\alpha(t)$ can be regarded as a curve in the completion $\mathcal{H}$ of $\mathcal{A}$, which is differentiable in $\mathcal{H}$, because it is C^1 with the structure given by the norm of $\mathcal{A}$. The conditional expectations extend to self-adjoint projections in $\mathcal{H}$, and their derivatives dE_t define symmetric operators, whose domains include $\mathcal{A} \subset \mathcal{H}$. Since $E_t(\mathcal{A}) \subset \mathcal{A}$ and $dE_t(\mathcal{A}) \subset \mathcal{A}$, the commutators $[dE_t, E_t]$ are defined in $\mathcal{A}$, and are skew-symmetric operators. Then

$$\frac{d}{dt}\big\langle\alpha(t), \alpha(t)\big\rangle = \big\langle\dot\alpha(t), \alpha(t)\big\rangle + \big\langle\alpha(t), \dot\alpha(t)\big\rangle = \big\langle[dE_t, E_t]\alpha(t), \alpha(t)\big\rangle + \big\langle\alpha(t), [dE_t, E_t]\alpha(t)\big\rangle = 0,$$

and therefore $\langle\alpha(t), \alpha(t)\rangle = \langle a, a\rangle$, i.e. $\|G_t(a)\|_2 = \|a\|_2$. Thus G_t extends to an isometry of $\mathcal{H}$, whose image contains the dense subspace $\mathcal{A} \subset \mathcal{H}$, and therefore is a unitary operator. □

Remark 2.7. As seen above, if the initial value belongs to the range of E_0, then at time t the solution remains inside the range of E_t. The same is true for the kernels. Pick $z_0 \in \ker E_0$. Then $E_t(G_t(z_0)) = G_t(E_0(z_0)) = 0$. Also Eq. (2) has a simpler form in this case,

$$[dE_t, E_t]\big(G_t(z_0)\big) = -E_t\big(dE_t\big(G_t(z_0)\big)\big).$$

Using the identity $dE = dE(E) + E(dE)$, the above term equals $-dE_t(G_t(z_0))$. Thus if the initial value z_0 belongs to the kernel of E_0, Eq. (2) transforms into

$$\begin{cases} \dot z(t) = -dE_t\big(z(t)\big), \\ z(0) = z_0. \end{cases}$$

Remark 2.8. In the above theorem, when $\mathcal{A}$ is a finite von Neumann algebra with a finite faithful and normal trace τ, acting by left multiplication in $\mathcal{H} = L^2(\mathcal{A}, \tau)$, the $*$-isomorphism

$$G_t|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t,$$

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can be extended to a surjective isometry

$$V_t : L^2(\mathcal{B}_0, \tau) \to L^2(\mathcal{B}_t, \tau),$$

which implements the isomorphism: $G_t(b) = V_t b V_t^*$, or more precisely, $L_{G_t(b)} = V_t L_b V_t^*$, for $b \in \mathcal{B}_0$ (and $L_b =$ left multiplication by b in $L^2(\mathcal{B}_o, \tau)$). Indeed, for $x \in \mathcal{B}_t$ dense in $L^2(\mathcal{B}_t, \tau)$,

$$V_t L_b V_t^*(x) = G_t\big(bG_t^{-1}(x)\big) = G_t\big(G_t^{-1}\big(G_t(b)x\big)\big) = G_t(b)x = L_{G_t(b)}(x).$$

In [11], C. Skau established the one to one correspondence between subalgebras $\mathcal{B} \subset \mathcal{A}$ and what he called finite projections in $\mathcal{H}$, associated to the algebra $\mathcal{A}$ and a cyclic and separating vector $\xi_0 \in \mathcal{H}$. If we fix here ξ_0 equal to the unit element $1 \in \mathcal{A} \subset \mathcal{H}$ (regarded as a vector in $\mathcal{H}$). The correspondence is given by

$$\mathcal{B} \longleftrightarrow p_{\mathcal{B}} \text{ the orthogonal projection onto } L^2(\mathcal{B}, \tau),$$

i.e. this projection $p_{\mathcal{B}}$ is the completion of the trace invariant conditional expectation $E_{\mathcal{B}}$ (also called the Jones projection of the inclusion $\mathcal{B} \subset \mathcal{A}$).

In the notation of the above remark, if one extends trivially V_t as 0 on $L^2(\mathcal{B}_0, \tau)^\perp \subset L^2(\mathcal{A}, \tau)$, then it becomes a partial isometry which verifies

$$V_t V_t^* = p_{\mathcal{B}_t} \quad \text{and} \quad V_t^* V_t = p_{\mathcal{B}_0}.$$

Moreover, the linear isomorphisms $G_t : \mathcal{A} \to \mathcal{A}$ extend to unitary operators $U_t : \mathcal{H} \to \mathcal{H}$. These verify

$$U_t p_{\mathcal{B}_0} U_t^* = p_{\mathcal{B}_t}.$$

Indeed, if one evaluates this identity in elements $a \in \mathcal{A} \subset \mathcal{H}$, it is the intertwining property of G_t.

3. Decompositions of a Hilbert space

3.1. Expectation onto the commutant of a decomposition

Let $I \subset \mathbb{R}$ be an interval containing the origin, and for each $t \in I$, a system of orthogonal projections

$$P(t) = \big(p_1(t), p_2(t), \ldots\big)$$

in $\mathcal{H}$ is defined. Recall that a system of projections $P = (p_1, p_2, \ldots)$ is a collection of self-adjoint projections in $\mathcal{H}$ such that $p_i p_j = \delta_{ij} p_i$ and $\sum_{i \geqslant 1} p_i = 1$. See [2] and [4] for related results on systems of projections. We make the assumption that $p_i(t)$ are strongly continuously differentiable, i.e for each fixed $\xi \in \mathcal{H}$ and $i \geqslant 1$, the map

$$I \ni t \mapsto p_i(t)\xi \in \mathcal{H}$$

is continuously differentiable. Furthermore, we shall make the following boundedness assumption: For each $\xi \in \mathcal{H}$, and each closed bounded sub-interval $J \subset I$, there exists a constant $C_{\xi,J} < \infty$ such that

$$\sum_{i \geqslant 1} \big\| \dot{p}_i(t)\xi \big\|^2 \leqslant C_{\xi,J} < \infty, \tag{3}$$

for all $t \in J$.

Remark 3.1. For a fixed $t \in I$, consider the Hilbert space $\mathcal{H}^+ = \bigoplus_{i \geqslant 1} \mathcal{H}_i$, where $\mathcal{H}_i = p_i(\mathcal{H})$, and the linear map

$$\pi_t : \mathcal{H} \to \mathcal{H}^+, \quad \pi_t(\xi) = \big(\dot{p}_1(t)\xi, \dot{p}_2(t)\xi, \ldots\big).$$

Hypothesis (3) above implies that π_t is well defined. It has an everywhere defined adjoint, namely

$$\sigma_t : \mathcal{H}^+ \to \mathcal{H}, \quad \sigma_t(\eta_1, \eta_2, \ldots) = \sum_{i \geqslant 1} \dot{p}_i(t)\eta_i,$$

where the series above is weakly convergent in $\mathcal{H}$. Therefore π_t is bounded [9]. Moreover, hypothesis (3) means that for any closed bounded sub-interval $J \subset I$, there exists $C_{\xi,J}$ such that

$$\big\|\pi_t(\xi)\big\| \leqslant C_{\xi,J}^{1/2}$$

for $t \in J$. Therefore, by the uniform boundedness principle, there exists a constant $C_J > 0$ such that $\|\pi_t\| \leqslant C_J$ for all $t \in J$. Therefore, hypothesis (3) is equivalent to

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$$\sum_{i\geqslant 1}\|\dot p_i\xi\|^2\leqslant C_J\|\xi\|^2, \tag{4}$$

with C_J independent of ξ and $t\in J$.

A system of projections in $\mathcal{H}$ gives rise to a conditional expectation in $\mathcal{B}(\mathcal{H})$, namely

$$E(X)=\sum_{i\geqslant 1}p_iXp_i.$$

Moreover, if X is compact, then also $E(X)$ is compact. Indeed, denote by $\xi\otimes\eta$ the rank one operator given by $\xi\otimes\eta(v)=\langle v,\eta\rangle\xi$. Then

$$E(\xi\otimes\eta)=\sum_{i\geqslant 1}p_i(\xi\otimes\eta)p_i=\sum_{i\geqslant 1}p_i\xi\otimes p_i\eta.$$

Clearly this series is absolutely convergent in $\mathcal{B}(\mathcal{H})$, because $\|p_i\xi\otimes p_i\eta\|=\|p_i\xi\|\|p_i\eta\|$, and the sequences $\|p_i\xi\|$ and $\|p_i\eta\|$ are square summable, by Parseval's identity. It follows that the operator $E(\xi\otimes\eta)$ is compact. Therefore E maps finite rank operators into compact operators, and thus compacts into compacts.

Therefore the mapping $t\mapsto P(t)$, induces a curve of conditional expectations $t\mapsto E_t$. We may regard each E_t acting in $\mathcal{B}(\mathcal{H})$, or in the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. The range of E_t consists of all operators which commute with the projections $p_i(t)$ in the system $P(t)$. In the compact case, they are compact.

Our main result regarding this example, is that hypothesis (3) above is precisely what is required in order that, for each operator $X\in\mathcal{K}(\mathcal{H})$, the map $t\mapsto E_t(X)\in\mathcal{K}(\mathcal{H})$ is continuously differentiable.

Let us show the following:

Theorem 3.2. *Hypothesis* (3) *holds if and only if there exists a curve $t\to\Omega_t$ of unitary operators, which is strongly continuously differentiable, and such that*

$$\Omega_0=1\quad\text{and}\quad\Omega_tp_i(0)\Omega_t^*=p_i(t),\quad\text{for all } t\in I \text{ and } i\geqslant 1.$$

Proof. Suppose first that there exists a strongly $\mathcal{C}^1$ curve Ω_t of unitaries in $\mathcal{H}$ such that $\Omega_0=1$ and $\Omega_tp_i(0)\Omega_t^*=p_i(t)$ for all $t\in I$, $i\geqslant 1$. Then

$$\|\dot p_i(t)\xi\|=\|\dot\Omega_tp_i(0)\Omega_t^*\xi+\Omega_tp_i(0)\dot\Omega_t^*\xi\|\leqslant\|\dot\Omega_tp_i(0)\Omega_t^*\xi\|+\|p_i(0)\dot\Omega_t^*\xi\|.$$

Thus it suffices to show that both sequences

$$\|\dot\Omega_tp_i(0)\Omega_t^*\xi\|\quad\text{and}\quad\|p_i(0)\dot\Omega_t^*\xi\|$$

are square summable. First note that since $t\mapsto\Omega_t\xi$ is $\mathcal{C}^1$, if $J\subset I$ is a closed bounded interval, then set of operators $\{\dot\Omega_t:\ t\in J\}$ is bounded at each $\xi\in\mathcal{H}$: $\|\dot\Omega_t\xi\|\leqslant k_\xi<\infty$ for all t in J. Thus, by the uniform boundedness principle, it follows that

$$\sup_{t\in I}\|\dot\Omega_t\|\leqslant k<\infty.$$

Consider the first sequence:

$$\sum_{i\geqslant 1}\|\dot\Omega_tp_i(0)\Omega_t^*\xi\|^2\leqslant k^2\sum_{i\geqslant 1}\|p_i(0)\Omega_t^*\xi\|^2,$$

which by Bessel's inequality is bounded by

$$k^2\|\Omega_t^*\xi\|^2=k^2\|\xi\|^2.$$

The second sequence, again using Bessel's inequality, is bounded by the same constant:

$$\sum_{i\geqslant 1}\|p_i(0)\dot\Omega_t^*\xi\|^2\leqslant\|\dot\Omega_t^*\xi\|^2\leqslant k^2\|\xi\|^2.$$

Conversely, assume that hypothesis (3) holds. Fix $t\in I$. Then for each $\xi\in\mathcal{H}$, the sum $\sum_{i\geqslant 1}p_i(t)\dot p_i(t)\xi$ is convergent in $\mathcal{H}$. Indeed, the terms of this sum are orthogonal vectors, and $\|p_i(t)\dot p_i(t)\xi\|^2\leqslant\|\dot p_i(t)\xi\|^2$, i.e. their norms are square summable. Thus the series is convergent. Let us denote by Δ_t the operator

$$\Delta_t:\mathcal{H}\to\mathcal{H},\quad \Delta_t\xi=\sum_{i\geqslant 1}p_i(t)\dot p_i(t)\xi. \tag{5}$$

Note that this operator is bounded. Its adjoint is defined by the weakly convergent series

$$\Delta_t^*\eta = \sum_{i\geqslant 1} \dot p_i(t)p_i(t)\eta.$$

Indeed, the equality

$$\langle \Delta_t\xi, \eta\rangle = \sum_{i\geqslant 1}\bigl\langle p_i(t)\dot p_i(t)\xi, \eta\bigr\rangle = \sum_{i\geqslant 1}\bigl\langle \xi, \dot p_i(t)p_i(t)\eta\bigr\rangle = \bigl\langle \xi, \Delta_t^*\eta\bigr\rangle,$$

proves both assumptions. Therefore Δ_t is bounded. Next note that it is anti-hermitian. For each $i \geqslant 1$ and each pair of vectors $\xi, \eta \in \mathcal{H}$, differentiating the identity

$$p_i(t)\xi = p_i(t)p_i(t)\xi$$

one obtains

$$\bigl\langle p_i(t)\dot p_i(t)\xi, \eta\bigr\rangle + \bigl\langle \dot p_i(t)p_i(t)\xi, \eta\bigr\rangle = \bigl\langle \dot p_i(t)\xi, \eta\bigr\rangle.$$

In particular, this implies that the series $\sum_{i\geqslant 1}\langle \dot p_i(t)\xi, \eta\rangle$ is convergent. Since $\sum_{i\geqslant 1} p_i(t) = 1$ (strongly) for all $t \in I$, then $\sum_{i\geqslant 1}\langle \dot p_i(t)\xi, \eta\rangle = 0$. Therefore adding the equalities above one has

$$0 = \sum_{i\geqslant 1}\bigl\langle p_i(t)\dot p_i(t)\xi, \eta\bigr\rangle + \bigl\langle \dot p_i(t)p_i(t)\xi, \eta\bigr\rangle = \langle \Delta_t\xi, \eta\rangle + \bigl\langle \Delta_t^*\xi, \eta\bigr\rangle.$$

Finally, let us show that for each fixed $\xi \in \mathcal{H}$, the curve $I \ni t \mapsto \Delta_t\xi \in \mathcal{H}$ is continuous. As remarked above, if $s, t \in I$ both series $\sum_{i\geqslant 1} p_i(s)\dot p_i(s)\xi$ and $\sum_{i\geqslant 1} p_i(t)\dot p_i(t)\xi$ are convergent in $\mathcal{H}$, and moreover, by hypothesis (3), they are uniformly convergent if s, t lie on a closed sub-interval J. Thus their tails tend, uniformly with respect to s, t, to zero. Therefore it suffices to check continuity of the finite sums $t \mapsto \sum_{i=1}^N p_i(t)\dot p_i(t)\xi$. This follows from the strongly continuous differentiability of the maps $p_i(t)$.

We may consider, for each $\xi_0 \in \mathcal{H}$ the linear differentiable equation

$$\begin{cases} \dot\omega(t) = -\Delta_t\omega(t), \\ \omega(0) = \xi_0. \end{cases} \tag{6}$$

Since $t \mapsto \Delta_t$ is a strongly continuous map of skew-hermitic operators, the general theory of linear differential equations in Hilbert spaces [10] implies the existence of the unitary propagator

$$I \ni t \mapsto \Omega_t,$$

which is strongly continuously differentiable, verifying that the unique solution of (6) is $\omega_{\xi_0}(t) = \Omega_t\xi_0$. Therefore

$$\begin{aligned}
\frac{d}{dt}\bigl\langle \Omega_t^* p_j(t)\Omega_t\xi, \eta\bigr\rangle &= \frac{d}{dt}\bigl\langle p_j(t)\Omega_t\xi, \Omega_t\eta\bigr\rangle \\
&= \bigl\langle \dot p_j(t)\Omega_t\xi, \Omega_t\eta\bigr\rangle + \bigl\langle p_j(t)\dot\Omega_t\xi, \Omega_t\eta\bigr\rangle + \bigl\langle p_j(t)\Omega_t\xi, \dot\Omega_t\eta\bigr\rangle \\
&= \bigl\langle \dot p_j(t)\Omega_t\xi, \Omega_t\eta\bigr\rangle - \bigl\langle p_j(t)\Delta_t\Omega_t\xi, \Omega_t\eta\bigr\rangle - \bigl\langle \Omega_t\xi, p_j(t)\Delta_t\Omega_t\eta\bigr\rangle \\
&= \bigl\langle \dot p_j(t)\Omega_t\xi, \Omega_t\eta\bigr\rangle - \bigl\langle p_j(t)\dot p_j(t)\xi, \Omega_t\eta\bigr\rangle - \bigl\langle \Omega_t\xi, p_j(t)\dot p_j(t)\eta\bigr\rangle \\
&= \bigl\langle \bigl(\dot p_j(t) - p_j(t)\dot p_j(t) - \dot p_j(t)p_j(t)\bigr)\Omega_t\xi, \Omega_t\eta\bigr\rangle = 0,
\end{aligned}$$

by identity (5).

It follows that, for all $t \in I$,

$$\Omega_t^* p_j(t)\Omega_t = \Omega_0^* p_j(0)\Omega_0 = p_j(0). \qquad \square$$

Using this result we may characterize when a curve E_t of conditional expectations in $\mathcal{K}(\mathcal{H})$, arising from a system of projectors, is smooth in the sense of the previous section. First, the following elementary lemma will be useful.

Lemma 3.3. *Suppose that $f(t), g(t) \in \mathcal{H}$ are vector valued continuously differentiable functions for $t \in I$. Then the map $t \mapsto f(t) \otimes g(t)$ is continuously differentiable, with values in $\mathcal{K}(\mathcal{H})$. Its derivative is*

$$t \mapsto f'(t) \otimes g(t) + f(t) \otimes g'(t).$$

Proof.

$$\left\| \frac{1}{h}\big(f(t+h)\otimes g(t+h) - f(t)\otimes g(t)\big) - f'(t)\otimes g(t) - f(t)\otimes g'(t) \right\|$$
$$\leqslant \left\| \frac{1}{h}\big(f(t+h)\otimes g(t+h) - f(t)\otimes g(t+h)\big) - f'(t)\otimes g(t+h) \right\|$$
$$+ \left\| \frac{1}{h}\big(f(t)\otimes g(t+h) - f(t)\otimes g(t)\big) - f(t)\otimes g'(t) \right\| + \left\| f'(t)\otimes g(t+h) - f'(t)\otimes g(t) \right\|.$$

The first term equals

$$\left\| \left\{ \frac{1}{h}\big(f(t+h) - f(t)\big) - f'(t) \right\} \otimes g(t+h) \right\| = \left\| \frac{1}{h}\big(f(t+h) - f(t)\big) - f'(t) \right\| \left\| g(t+h) \right\|,$$

and tends to 0 as $h \to 0$. The second and third terms are dealt similarly. It is apparent that the derivative

$$I \ni t \mapsto f'(t)\otimes g(t) + f(t)\otimes g'(t) \in \mathcal{K}(\mathcal{H})$$

is continuous. □

Theorem 3.4. *The map*

$$I \ni t \mapsto E_t(X) \in \mathcal{K}(\mathcal{H})$$

is continuously differentiable for every compact operator X, if and only if hypothesis (3) *holds.*

Proof. Suppose first that hypothesis (3) holds. There exist orthogonal systems of vectors $\{\xi_k\}$ and $\{\eta_k\}$, such that $X = \sum_{k\geqslant 1} \xi_k \otimes \eta_k$. Denote by $X_N = \sum_{k=1}^N \xi_k \otimes \eta_k$. Clearly $X_N \to X$ in norm, in $\mathcal{K}(\mathcal{H})$. We claim that the map

$$I \ni t \mapsto E_t(X_N) \in \mathcal{K}(\mathcal{H})$$

is $\mathcal{C}^1$. To prove this it suffices to prove that $t \mapsto E_t(\xi \otimes \eta)$ is $\mathcal{C}^1$ for any pair of vectors $\xi, \eta \in \mathcal{H}$. As shown above

$$E_t(\xi \otimes \eta) = \sum_{i\geqslant 1} p_i(t)\xi \otimes p_i(t)\eta.$$

First note that, by the above lemma, each term $p_i(t)\xi \otimes p_i(t)\eta$ is a $\mathcal{K}(\mathcal{H})$-valued $\mathcal{C}^1$ function. Indeed, by hypothesis, the map $t \mapsto p_i(t)v$ is $\mathcal{C}^1$ for each fixed $v \in \mathcal{H}$. Note that $\sum_{i\geqslant 1} p_i(t)\xi \otimes p_i(t)\eta$ is absolutely and uniformly summable, on closed and bounded sub-intervals, by the inequalities of Bessel and Hölder,

$$\sum_{i\geqslant 1} \left\| p_i(t)\xi \otimes p_i(t)\eta \right\| = \sum_{i\geqslant 1} \left\| p_i(t)\xi \right\| \left\| p_i(t)\eta \right\| \leqslant \|\xi\| \|\eta\|.$$

Therefore the sum $\sum_{i\geqslant 1} p_i(t)\xi \otimes p_i(t)\eta$ is continuous in t. By a similar computation, using the hypothesis that $\sum_{i\geqslant 1} \|\dot{p}_i(t)\xi\|^2 \leqslant C_\xi < \infty$, the series of derivatives converges absolutely and uniformly:

$$\begin{aligned}\sum_{i\geqslant 1} \left\| \dot{p}_i(t)\xi \otimes p_i(t)\eta + p_i(t)\xi \otimes \dot{p}_i(t)\eta \right\| &\leqslant \sum_{i\geqslant 1} \left\| \dot{p}_i(t)\xi \otimes p_i(t)\eta \right\| + \left\| p_i(t)\xi \otimes \dot{p}_i(t)\eta \right\| \\ &= \sum_{i\geqslant 1} \left\| \dot{p}_i(t)\xi \right\| \left\| p_i(t)\eta \right\| + \left\| p_i(t)\xi \right\| \left\| \dot{p}_i(t)\eta \right\| \leqslant 2C\|\eta\|\|\xi\|,\end{aligned}$$

where the constant C comes from the equivalent form (4) of hypothesis (3). Therefore the series of derivatives $\sum_{i\geqslant 1} \dot{p}_i(t)\xi \otimes p_i(t)\eta + p_i(t)\xi \otimes \dot{p}_i(t)\eta$ defines a continuous function, which is the derivative of the former series (see for instance Theorem 7.2 in [6]). Therefore $t \mapsto E_t(X_N)$ is $\mathcal{C}^1$. Let us prove that $t \mapsto E_t(X)$ is $\mathcal{C}^1$. First note that $E_t(X)$ is continuous:

$$\left\| E_t(X) - E_t(X_N) \right\| = \left\| E_t(X - X_N) \right\| \leqslant \|X - X_N\|.$$

That is, $E_t(X)$ is the uniform limit of the functions $E_t(X_N)$, which are $\mathcal{C}^1$. To prove that it is differentiable, we exhibit first its derivative. Put

$$F_t(X) = \sum_{i\geqslant 1} \dot{p}_i(t)Xp_i(t) + p_i(t)X\dot{p}_i(t), \quad X \in \mathcal{K}(\mathcal{H}).$$

Note that since the ranges of p_i are pairwise orthogonal, for any $\xi \in \mathcal{H}$,

$$\left\| \sum_{i \geqslant 1} p_i(t) X \dot{p}_i(t)\xi \right\|^2 = \sum_{i \geqslant 1} \left\| p_i(t) X \dot{p}_i(t)\xi \right\|^2 \leqslant \|X\|^2 \sum_{i \geqslant 1} \left\| \dot{p}_i(t)\xi \right\|^2 \leqslant \|X\|^2 C \|\xi\|^2.$$

The operators

$$X \mapsto \sum_{i \geqslant 1} p_i(t) X \dot{p}_i(t)$$

are everywhere defined, and bounded by $C^{1/2}$, for all $t \in I$,

$$\left\| \sum_{i \geqslant 1} p_i(t) X \dot{p}_i(t) \right\| \leqslant C^{1/2} \|X\|.$$

Then also the adjoints are bounded by the same constant: $\| \sum_{i \geqslant 1} \dot{p}_i(t) X p_i(t) \| \leqslant C^{1/2} \|X\|$. This proves that $F_t(X)$ is convergent, and that it is the uniform limit of $F_t(X_N)$:

$$\left\| F_t(X) - F_t(X_N) \right\| \leqslant \left\| \sum_{i \geqslant 1} p_i(t)(X - X_N)\dot{p}_i(t) \right\| + \left\| \sum_{i \geqslant 1} \dot{p}_i(t)(X - X_N)p_i(t) \right\| \leqslant 2C^{1/2} \|X - X_N\|.$$

By the same argument as above (involving Theorem 7.2 of [6]), it follows that $E_t(X)$ is differentiable and that $\frac{d}{dt} E_t(X) = F_t(X)$, which is continuous.

Conversely, suppose that the curve $I \ni t \mapsto E_t(X) \in \mathcal{K}(\mathcal{H})$ is continuously differentiable. Then Theorem 2.6 applies: there exist bounded linear isomorphisms

$$G_t : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}), \quad t \in I,$$

with the following properties:

1. For each $X \in \mathcal{K}(\mathcal{H})$, $I \ni t \mapsto G_t(X) \in \mathcal{K}(\mathcal{H})$ is continuously differentiable.
2. G_t preserves adjoints.
3. $G_t \circ E_0 \circ G_t^{-1} = E_t$.
4. G_t maps the commutant of $P(0)$, $\mathcal{B}_0 = \mathcal{K}(\mathcal{H}) \cap \{p_i(0)\colon i \geqslant 1\}'$, onto $\mathcal{B}_t$, the commutant of $P(t)$, and it is a $*$-isomorphism between these C*-algebras.

The structure of the commutant algebras $\mathcal{B}_t$ is apparent:

$$\mathcal{B}_t = \bigoplus_{i \geqslant 1} \mathcal{B}_{i,t},$$

where the factors $\mathcal{B}_{i,t}$ are

$$\mathcal{B}_{i,t} = \mathcal{K}\big(R\big(p_i(t)\big)\big),$$

i.e. the space of compact operators acting on the range of $p_i(t)$. Let $\hat{\mathcal{B}}_{i,t}$ be the unitization of $\mathcal{B}_{i,t}$:

$$\hat{\mathcal{B}}_{i,t} = \mathbb{C} p_i(t) + \mathcal{B}_{i,t}$$

(note that $p_i(t)$ is the unit element in the algebra of operators acting in $R(p_i(t))$). The isomorphisms G_t extend canonically to the unitizations, and the curve $t \mapsto G_t(X)$ is continuously differentiable for each $X \in \hat{\mathcal{B}}_{i,0}$. We claim that $G_t(p_i(0)) = p_i(t)$. Indeed, G_t maps each factor $\mathcal{B}_{k,0}$ onto a factor of the decomposition of $\mathcal{B}_t$, say $\mathcal{B}_{k(t),t}$, with $k(t) \in \mathbb{N}_0$. Since $p_k(t)$ is the unit element of the algebra $\hat{\mathcal{B}}_{k,t}$, it follows that $G_t(p_k(0)) = p_{k(t)}(t)$. A continuity argument shows that $k(t) = k$. To prove this assertion, let ξ_0 be a unit vector in the range of $p_k(0)$, and put $\xi_t = p_k(t)\xi_0$. Then if $k(t) \neq k$, since the ranges of $p_k(t)$ and $p_{k(t)}(t)$ are orthogonal, $G_t(p_k(0))\xi_t = p_{k(t)}(\xi_0) = 0$. On the other hand the map $t \mapsto G_t(p_k(0))\xi_t$ is continuous in the parameter t, and at $t = 0$ is equal to ξ_0. Thus there exists $r > 0$ such that $G_t(p_k(0))\xi_t \neq 0$ if $t \in [0, r]$. It follows that $k(t) = k$ for $t \in [0, r]$. A similar argument shows that $k(t)$ is locally constant at every $t \in I$, and therefore it is constant, and our claim is proved. Then

$$\dot{p}_k(t) = \frac{d}{dt} G_t\big(p_k(0)\big).$$

The curve of operators G_t is continuously differentiable at every $X \in \hat{\mathcal{B}}_{i,0}$, therefore the operators $\dot{G}_t$ are well defined, as linear maps from $\hat{\mathcal{B}}_{i,0}$ to $\mathcal{B}(\mathcal{H})$. By a standard argument involving the uniform boundedness principle (as in the first section), it follows that on closed bounded sub-intervals $J \subset I$, the norms $\|\dot{G}_t\|$ are uniformly bounded by a constant C_J. Therefore, for any $\xi \in \mathcal{H}$,

$$\sum_{k\geqslant 1} \left\| \dot{p}_k(t)\xi \right\|^2 \leqslant \sum_{k\geqslant 1} \|\dot{G}_t\|^2 \left\| p_k(0)\xi \right\|^2 \leqslant C_J^2 \sum_{k\geqslant 1} \left\| p_k(0)\xi \right\|^2 = C_J^2 \|\xi\|^2,$$

i.e. hypothesis (3) is verified. □

The curve of unitaries Ω_t obtained in Theorem 3.2 (under hypothesis (3)), which intertwine the systems of projections $P(t)$ and $P(0)$ can be used to obtain a curve of inner automorphisms intertwinning the expectations E_t and E_0. Namely,

$$Ad(\Omega_t) : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}), \qquad Ad(\Omega_t)(X) = \Omega_t X \Omega_t^*$$

clearly verifies that

$$Ad(\Omega_t) \circ E_0 \circ Ad(\Omega_t)^{-1} = Ad(\Omega_t) \circ E_0 \circ Ad\big(\Omega_t^*\big) = E_t.$$

Also the fact that $t \mapsto \Omega_t$ is strongly continuously differentiable implies that for each $X \in \mathcal{K}(\mathcal{H})$, the curve $t \mapsto Ad(\Omega_t)(X) \in \mathcal{K}(\mathcal{H})$ is continuously differentiable (this fact will be proved below in Proposition 3.10).

It is a natural question if $Ad(\Omega_t)$ and G_t coincide, and if not, what is the relation between these two maps. We emphasize the fact that, a priori, G_t is not multiplicative in $\mathcal{K}(\mathcal{H})$. First we show that $Ad(\Omega_t)$ and G_t coincide on the range of E_0,

$$\mathcal{B}_0 = \big\{p_i(0)\colon\ i \geqslant 1\big\}' \cap \mathcal{K}(\mathcal{H}).$$

Proposition 3.5. *The maps G_t and $Ad(\Omega_t)$ coincide in the range of E_0.*

Proof. Pick A in the range of E_0. We must show that $\omega(t) = Ad(\Omega_t)(A)$ verifies

$$\begin{cases} \dot{\omega}(t) = [dE_t, E_t]\omega(t), \\ \omega(0) = A. \end{cases}$$

Clearly $\omega(0) = A$ because $\Omega_0 = 1$. Also it is apparent that since $Ad(\Omega_t)$ intertwines E_0 and E_t, then $\omega(t)$ takes values in the range of E_t. As it was shown in Theorem 2.6, a curve $\omega(t)$ taking values in the ranges of E_t is a solution of the above equation if and only if

$$E_t\big(\dot{\omega}(t)\big) = 0.$$

In this case $E_t(\dot{\omega}(t)) = E_t(\dot{\Omega}_t A \Omega_t^* + \Omega_t A \dot{\Omega}_t^*)$. Recall that Ω_t satisfies the equation

$$\dot{\Omega}_t = -\sum_{i\geqslant 1} p_i(t)\dot{p}_i(t)\Omega_t,$$

where the series converges strongly. Therefore for each $j \geqslant 1$, $p_j(t)\dot{\Omega}_t = -p_j(t)\dot{p}_j(t)\Omega_t$, and taking adjoints, $\dot{\Omega}_t^* p_j(t) = -\Omega_t^* \dot{p}_j(t)p_j(t)$. Then

$$E_t\big(-\dot{\omega}(t)\big) = \sum_{j\geqslant 1} p_j(t)\big(\dot{\Omega}_t A \Omega_t^* + \Omega_t A \dot{\Omega}_t^*\big)p_j(t) = -\sum_{j\geqslant 1} p_j(t)\dot{p}_j(t)\Omega_t A \Omega_t^* p_j(t) + p_j(t)\Omega_a A \Omega_t^* \dot{p}_j(t)p_j(t).$$

The fact that $\omega(t) = \Omega_t A \Omega_t^*$ lies in the range of E_t means that it commutes with $p_j(t)$ for all $j \geqslant 1$. Thus this sum equals

$$-\sum_{j\geqslant 1} p_j(t)\dot{p}_j(t)p_j(t)\Omega_t A \Omega_t^* - \Omega_t A \Omega_t^* \sum_{j\geqslant 1} p_j(t)\dot{p}_j(t)p_j(t).$$

Recall from Section 1 that a strongly continuously differentiable curve of idempotents $p_j(t)$ verifies $p_j(t)\dot{p}_j(t)p_j(t) = 0$ for all t, and therefore the proof follows. □

Our next result shows that if the system $P(t)$ consists of two projections, i.e. $P(t) = (p(t), 1 - p(t))$, then $Ad(\Omega_t)$ and G_t coincide.

Proposition 3.6. *If $P(t) = (p(t), 1 - p(t))$, then $G_t(A) = \Omega_t A \Omega_t^*$ for all $A \in \mathcal{K}(\mathcal{H})$ and all $t \in I$.*

Proof. The proof is simpler if we refer to symmetries instead of projections: put $\epsilon(t) = 2p(t) - 1$. A straightforward computation shows that

$$[dE_t, E_t](X) = \big[\dot p(t), p(t)\big]X + X\big[p(t), \dot p(t)\big] = \big[\big[\dot p(t), p(t)\big], X\big] = \frac{1}{4}\big[\big[\dot\epsilon(t), \epsilon(t)\big], X\big].$$

On the other hand the equation satisfied by Ω is also simplified:

$$\Delta_t = p(t)\dot p(t) + \big(1 - p(t)\big)\big(-\dot p(t)\big) = \big(2p(t) - 1\big)\dot p(t) = \frac{1}{2}\epsilon(t)\dot\epsilon(t),$$

and thus

$$\dot\Omega_t = -\frac{1}{2}\epsilon(t)\dot\epsilon(t)\Omega_t.$$

As above, fix $A \in \mathcal{K}(\mathcal{H})$ and put $\omega(t) = \Omega_t A \Omega_t^*$. Note that since $\epsilon(t)^2 = 1$ and $\epsilon(t)^* = \epsilon(t)$, then $\dot\epsilon(t)\epsilon(t) + \epsilon(t)\dot\epsilon(t) = 0$. Then

$$\begin{aligned}\big[\big[\dot\epsilon(t), \epsilon(t)\big], \omega(t)\big] &= \dot\epsilon(t)\epsilon(t)\omega(t) - \epsilon(t)\dot\epsilon(t)\omega(t) - \omega(t)\dot\epsilon(t)\epsilon(t) + \omega(t)\epsilon(t)\dot\epsilon(t)\\ &= -2\epsilon(t)\dot\epsilon(t)\Omega_t A \Omega_t^* - 2\Omega_t A \Omega_t^* \dot\epsilon(t)\epsilon(t).\end{aligned}$$

Since $\epsilon(t)\dot\epsilon(t)\Omega_t = -2\dot\Omega_t$ (and therefore, taking adjoints, $\Omega_t^*\dot\epsilon(t)\epsilon(t) = -2\dot\Omega_t^*$), this expression above equals

$$4\dot\Omega_t A \Omega_t^* + 4\Omega_t A \dot\Omega_t^* = 4\dot\omega(t),$$

i.e. $\dot\omega(t) = \frac{1}{4}[[\dot\epsilon(t), \epsilon(t)], \omega(t)]$. $\square$

For systems with more than two projections, G_t and $Ad(\Omega_t)$ may differ, as the following example shows.

Example 3.7. Consider the system of projections in $M_3(\mathbb{C})$:

$$p_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad p_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad p_3 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and the curve of unitaries

$$U_t = \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) & -\sin(t)\\ -\sin(t) & \cos(t) & 0\\ \cos(t)\sin(t) & \sin^2(t) & \cos(t)\end{pmatrix},$$

defined on any interval I with $0 \in I$. Note that $U_0 = 1$. Consider the system of projections $p_i(t) = U_t p_1 U_t^*$, $i = 1, 2, 3$. A straightforward computation shows that

$$U_t^*\dot U_t = \begin{pmatrix} 0 & 1 & -\cos(t)\\ -1 & 0 & -\sin(t)\\ \cos(t) & \sin(t) & 0\end{pmatrix}.$$

The fact that $U_t^*\dot U_t$ has zeros on the diagonal implies that $p_i U_t^*\dot U_t p_i = 0$ and thus

$$\Delta_t = p_1(t)\dot p_1(t) + p_2(t)\dot p_2(t) + p_3(t)\dot p_3(t) = U_t p_1 \dot U_t^* + U_t p_2 \dot U_t^* + U_t p_3 \dot U_t^* = U_t \dot U^*.$$

Taking adjoints, $-\Delta_t = \dot U_t U_t^*$, or equivalently $\dot U_t = -\Delta_t U_t$. That is, in the notation above, $U_t = \Omega_t$, corresponding to this system of projections. On the other hand, as in the previous proposition,

$$[dE_t, E_t](X) = \sum_{i=1}^{3}\big[\dot p_i(t), p_i(t)\big]X p_i(t) - p_i(t)X\big[\dot p_i(t), p_i(t)\big].$$

Note that $p_i(t)\dot p_i(t) = \Omega_t p_i \dot\Omega_t^*$ and $\dot p_i(t) p_i(t) = \dot\Omega_t p_i \Omega_t^*$. Then a straightforward computation shows that

$$[dE_t, E_t]\big(\Omega_t A \Omega_t^*\big) = \frac{d}{dt}\big\{\Omega_t E_0(A)\Omega_t^*\big\} + \Omega_t E_0\big(\big[\Omega_t^*\dot\Omega_t, A\big]\big)\Omega_t^*.$$

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Take for instance

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that if $\omega(t) = \Omega_t A \Omega_t^*$, then

$$\dot{\omega}(t) \neq [dE_t, E_t]\big(\omega(t)\big),$$

i.e. $G_t(A) \neq \Omega_t A \Omega_t^*$.

Let us show that under the equivalent conditions of the previous theorem (hypothesis (3)), the invertible maps G_t preserve also de p-Schatten ideals.

Remark 3.8. It is also well known that this type of conditional expectation, obtained by diagonal compression with a system of projections, also preserves the p-Schatten classes $\mathcal{B}_p(\mathcal{H}) = \{X \in \mathcal{K}(\mathcal{H})\colon\ tr(|X|^p) < \infty\}$. For $1 \leqslant p < \infty$, the set $\mathcal{B}_p(\mathcal{H})$ is a Banach space with the p-norm $\|X\|_p = tr(|X|^p)^{1/p}$. If E_t is as above, then

$$E_t\big(\mathcal{B}_p(\mathcal{H})\big) \subset \mathcal{B}_p(\mathcal{H}) \quad \text{and} \quad \big\|E_t(X)\big\|_p \leqslant \|X\|_p.$$

We shall need the following lemma.

Lemma 3.9. *Let $U_t, V_t, t \in I$, be strongly continuously differentiable curves of unitaries in $\mathcal{H}$. Fix $X \in \mathcal{B}_p(\mathcal{H})$, $1 \leqslant p \leqslant \infty$. Then the map*

$$I \ni t \mapsto U_t X V_t^* \in \mathcal{B}_p(\mathcal{H})$$

is continuously differentiable.

Proof. By the spectral theorem of compact self-adjoint operators and the polar decomposition, X can be written

$$X = \sum_{i \geqslant 1} \xi_i \otimes \eta_i,$$

where $\{\xi_i\}$ and $\{\eta_i\}$ are orthogonal sequences with $\|\eta_i\| = 1$ and $\|\xi_i\|$ p-summable (or tending to 0 if $p = \infty$). Let $X_N = \sum_{i=1}^N \xi_i \otimes \eta_i$, so that $\|X - X_N\|_p \to 0$ as N goes to infinity. Clearly the map

$$t \mapsto U_t X_N V_t^* = \sum_{i=1}^N U_t \xi_i \otimes V_t \eta_i \in \mathcal{B}_p(\mathcal{H})$$

is continuously differentiable, and its derivative equals

$$t \mapsto \dot{U}_t X_N V_t^* + U_t X_N \dot{V}_t^* = \sum_{i=1}^N \dot{U}_t \xi_i \otimes V_t \eta_i + U_t \xi_i \otimes \dot{V}_t \eta_i.$$

Note that

$$\big\|U_t X V_t^* - U_t X_N V_t^*\big\|_p = \|X - X_N\|_p,$$

which implies that $t \mapsto U_t X V_t$ is continuous in the p-norm, and that

$$\big\|\dot{U}_t X V_t^* + U_t X \dot{V}_t^* - \dot{U}_t X_N V_t^* - U_t X_N \dot{V}_t^*\big\|_p \leqslant \big\|\dot{U}_t (X - X_N)\big\|_p + \big\|(X - X_N)\dot{V}_t^*\big\|_p.$$

The first term is bounded by $\|\dot{U}_t\| \|X - X_N\|_p$. By yet another standard application of the uniform boundedness principle, the norms $\|\dot{U}_t\|$ are uniformly bounded on closed bounded sub-intervals of I. The other term is dealt similarly. Thus the series of derivatives is also uniform convergent in the p-norm. Therefore [6, Theorem 7.2] the map $t \mapsto U_t X V_t$ is differentiable, and its derivative is $t \mapsto \dot{U}_t X V_t + U_t X \dot{V}_t$, which is continuous. □

Proposition 3.10. *Let $1 \leqslant p < \infty$. Under hypothesis* (3)*, the maps G_t preserve the p-Schatten ideals, $G_t(\mathcal{B}_p(\mathcal{H})) = \mathcal{B}_p(\mathcal{H})$, and moreover, for any $X \in \mathcal{B}_p(\mathcal{H})$, $t \mapsto G_t(X)$ is continuously differentiable as a $\mathcal{B}_p(\mathcal{H})$-valued map.*

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Proof. Let us show that the linear operators $[dE_t, E_t]$ preserve the Schatten ideals, and that for any $X \in \mathcal{B}_p(\mathcal{H})$, the map

$$t \mapsto [dE_t, E_t](X) \in \mathcal{B}_p(\mathcal{H})$$

is continuous in the p-norm. Let Ω_t be as above, the strongly continuously differentiable curve of unitary operators obtained as solutions of

$$\dot{\Omega}_t = -\Delta_t \Omega_t, \qquad \Omega_0 = 1.$$

As shown before, these unitaries intertwine $p_i(0)$ and $p_i(t)$, and therefore

$$E_t(X) = \Omega_t E_0\big(\Omega_t^* X \Omega_t\big)\Omega_t^*.$$

For any fixed $X \in \mathcal{B}_p(\mathcal{H})$, by the above lemma the map $I \ni t \mapsto E_t(X) \in \mathcal{B}_p(\mathcal{H})$ is continuously differentiable. Therefore the commutators $t \mapsto [dE_t, E_t](X)$ are continuous in the p-norm, for any fixed $X \in \mathcal{B}_p(\mathcal{H})$. Indeed, fix t and X and let h tend to zero, then

$$\begin{aligned}\big\|E_{t+h}\, dE_{t+h}(X) - E_t\, dE_t(X)\big\|_p &\leqslant \big\|E_{t+h}\big(dE_{t+h}(X) - dE_t(X)\big)\big\|_p + \big\|E_{t+h}\, dE_t(X) - dE_t E_t(X)\big\|_p \\ &\leqslant \big\|dE_{t+h}(X) - dE_t(X)\big\|_p + \big\|E_{t+h}\, dE_t(X) - dE_t E_t(X)\big\|_p,\end{aligned}$$

which tend to zero. To deal analogously with $dE_t E_t$, note that (again) by the uniform boundedness principle, the operators dE_t have uniformly bounded norms, as operators on $\mathcal{B}_p(\mathcal{H})$:

$$\|dE_t\| \leqslant C$$

on closed bounded sub-intervals of I. Then

$$\begin{aligned}\big\|dE_{t+h}E_{t+h}(X) - dE_t E_t(X)\big\|_p &\leqslant \big\|dE_{t+h}\big(E_{t+h}(X) - E_t(X)\big)\big\|_p + \big\|dE_{t+h}E_t(X) - dE_t E_t(X)\big\|_p \\ &\leqslant C\big\|E_{t+h}(X) - E_t(X)\big\|_p + \big\|dE_{t+h}E_t(X) - dE_t E_t(X)\big\|_p.\end{aligned}$$

It follows that, for any $X \in \mathcal{B}_p(\mathcal{H})$, the differential equation

$$\begin{cases}\dot{\alpha}(t) = [dE_t, E_t]\alpha(t), \\ \alpha(0) = X\end{cases}$$

has a unique solution in $\mathcal{B}_p(\mathcal{H})$, and defines continuously differentiable propagators, which by the uniqueness of the solution in $\mathcal{K}(\mathcal{H})$, are precisely G_t. $\square$

For the special case $p = 2$ one has the following result:

Proposition 3.11. *Under hypothesis* (3), *for any $t \in I$,*

$$G_t|_{\mathcal{B}_2(\mathcal{H})} : \mathcal{B}_2(\mathcal{H}) \to \mathcal{B}_2(\mathcal{H})$$

is a unitary operator, which verifies $G_t(X^*) = X^*$ and $G_t(1) = 1$.

Proof. It suffices to show that the commutators $[dE_t, E_t]$ are anti-hermitic. We omit the parameter t for brevity,

$$\begin{aligned}\big\langle [dE, E]X, Y\big\rangle = tr\big(Y^*[dE, E]X\big) &= \sum_{i \geqslant 1} tr\big(Y^*(\dot{p}_i p_i X p_i + p_i X p_i \dot{p}_i - p_i \dot{p}_i X p_i - p_i X \dot{p}_i p_i)\big) \\ &= \sum_{i \geqslant 1} tr\big(\big(p_i Y^* \dot{p}_i p_i + p_i \dot{p}_i Y^* p_i - p_i Y^* p_i \dot{p}_i - \dot{p}_i p_i Y_i^p\big)X\big) \\ &= \sum_{i \geqslant 1} tr\big((p_i \dot{p}_i Y p_i + p_i Y \dot{p}_i p_i - \dot{p}_i p_i Y p_i - p_i Y p_i \dot{p}_i)^* X\big) = -\big\langle X, [dE, E]Y\big\rangle.\end{aligned}$$

The properties that G_t is unital and $*$-preserving hold in general (Theorem 2.6). $\square$

Finally, note that $G_t : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ can be extended to linear $*$-preserving isomorphisms of $\mathcal{B}(\mathcal{H})$, and such that the curve $t \mapsto G_t(X)$ is w^*-$\mathcal{C}^1$.

Remark 3.12. It was shown in Proposition 3.10 that the curve $t \mapsto G_t(Y)$ is a $\mathcal{C}^1$ $\mathcal{B}_1(\mathcal{H})$-valued curve for each $Y \in \mathcal{B}_1(\mathcal{H})$. Pick $X_0 \in \mathcal{B}(\mathcal{H})$, it induces a curve of bounded linear functionals in $\mathcal{B}_1(\mathcal{H})$, namely,

$$t \mapsto \varphi_{t,X_0}, \quad \varphi_{t,X_0}(Y) = tr\big(X_0 G_t(Y)\big), \quad \text{for } Y \in \mathcal{B}_1(\mathcal{H}).$$

Thus, by duality, there exist $X_{0,t} \in \mathcal{B}(\mathcal{H})$ such that

$$tr(X_{0,t}Y) = \varphi_{t,X_0}(Y) = tr\big(X_0 G_t(Y)\big).$$

Put $G_t(X_0) = X_{0,t}$. The properties of this extension are apparent.

3.2. Expectations onto algebras generated by the system

A system of projections is related to another type of C*-algebra which is the range of a conditional expectation, that is, the C*-algebra generated by the system. This is a commutative algebra which consists of as many copies of $\mathbb{C}$ as there are projections in the system, namely, if $P = (p_1, p_2, \ldots)$,

$$\mathcal{B} = \bigoplus_{i \geqslant 1} p_i \mathbb{C}.$$

A conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \mathcal{B}$ is of the form

$$E(X) = \sum_{i \geqslant 1} p_i \Phi_i(p_i X p_i),$$

where Φ_i is a state in $p_i \mathcal{B}(\mathcal{H}) p_i = \mathcal{B}(p_i(\mathcal{H}))$. Let us suppose that we have a curve $P(t) = (p_1(t), p_2(t), \ldots)$ of systems of projections as in the preceding section, and a curve of expectations $E_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}_t$, where $\mathcal{B}_t$ is the C*-algebra generated by $P(t)$. We first need to establish the following elementary fact.

Lemma 3.13. *Suppose that $I \ni t \mapsto A_t \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ is a curve of linear operators acting on $\mathcal{B}(\mathcal{H})$, such that for each $X \in \mathcal{B}(\mathcal{H})$ the map $t \mapsto A_t(X) \in \mathcal{B}(\mathcal{H})$ is $\mathcal{C}^1$. If $t \mapsto X(t) \in \mathcal{B}(\mathcal{H})$ is a $\mathcal{C}^1$ map, then*

$$I \ni t \mapsto A_t\big(X(t)\big) \in \mathcal{B}(\mathcal{H})$$

is $\mathcal{C}^1$.

Proof. Fix $t \in I$, then $\frac{1}{h}\{A_{t+h}(X(t+h)) - A_t(X(t))\}$ equals

$$\frac{1}{h}\big\{A_{t+h}\big(X(t+h)\big) - A_{t+h}\big(X(t)\big)\big\} + \frac{1}{h}\big\{A_{t+h}\big(X(t)\big) - A_t\big(X(t)\big)\big\}.$$

The right hand term tends to $\dot{A}_t(X(t))$ as $h \to 0$ by hypothesis. The left hand term equals

$$A_{t+h}\left\{\frac{1}{h}\big(X(t+h) - X(t)\big)\right\}.$$

Clearly the arguments $Y_h = \frac{1}{h}\{X(t+h) - X(t)\} \to \dot{X}(t) = Y_0$ as $h \to 0$. Thus we need to show that if $Y_h \to Y_0$, then $A_{t+h}(Y_h) \to A_t(Y_0)$ as $h \to 0$. To prove this, it suffices to show that the norms $\|A_t\|$ are uniformly bounded on closed bounded sub-intervals of I. This follows from the UBP. $\square$

Proposition 3.14. *Suppose that the curve of expectations is $\mathcal{C}^1$ in the sense given before (for each $X \in \mathcal{B}(\mathcal{H})$, the map $t \mapsto E_t(X) \in \mathcal{B}(\mathcal{H})$ is $\mathcal{C}^1$). Then for each $i \geqslant 1$ and each $X \in \mathcal{B}(p_i\mathcal{H})$, the maps*

$$t \mapsto p_i(t) \in \mathcal{B}(\mathcal{H})$$

and

$$t \mapsto \Phi_{i,t}\big(p_i(t) X p_i(t)\big) \in \mathbb{C}$$

are $\mathcal{C}^1$ (where $E_t(X) = \sum_{i \geqslant 1} p_i(t) \Phi_{i,t}(p_i(t) X p_i(t))$).

Proof. If the curve of expectations is $\mathcal{C}^1$, then Theorem 2.6 applies, and there exists a $\mathcal{C}^1$ curve $G_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $G_t \circ E_0 \circ G_t^{-1}$ and G_t restricted to $\mathcal{B}_0$ is a $*$-isomorphism onto $\mathcal{B}_t$. In particular, G_t maps each factor $p_i(0)\mathbb{C}$ of $\mathcal{B}_0$ onto a factor $p_{i(t)}\mathbb{C}$. By the same continuity argument as in the previous section, $i(t) = i$. That is, $G_t(p_i(0)) = p_i(t)$. This implies that the map $t \mapsto p_i(t) \in \mathcal{B}(\mathcal{H})$ is $\mathcal{C}^1$ in the norm topology. Therefore, for each $j \geqslant 1$ and each $X \in \mathcal{B}(\mathcal{H})$ the map

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$$t \mapsto p_j(t)E_t(X) = p_j(t)\Phi_{j,t}\big(p_j(t)Xp_j(t)\big) \in \mathcal{B}(\mathcal{H})$$

is $\mathcal{C}^1$. Then, by the above lemma, also the map

$$t \mapsto G_t^{-1}\big(p_j(t)E_t(X)\big) = p_j(0)\Phi_{j,t}\big(p_j(t)Xp_j(t)\big)$$

is $\mathcal{C}^1$, which implies that $t \mapsto \Phi_{j,t}(p_j(t)Xp_j(t)) \in \mathbb{C}$ is $\mathcal{C}^1$. $\square$

4. Curves of states

A state φ in a unital C*-algebra $\mathcal{A}$ is a special case of conditional expectation, where the range algebra is the subalgebra $\mathbb{C}\cdot 1$, and $E(a) = \varphi(a)1$. Suppose that one has a curve of states in $\mathcal{A}$, φ_t, $t \in I$, which is smooth in the sense above: for each $a \in \mathcal{A}$, $I \ni t \mapsto \varphi_t(a) \in \mathbb{C}$ is $\mathcal{C}^1$. For instance, if ξ_t are unit vectors in a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ acts, they induce pure states in $\mathcal{A}$: $\varphi_t(a) = \langle a\xi_t, \xi_t\rangle$. The smoothness condition is fulfilled if the curve $t \mapsto \xi_t \in \mathcal{H}$ is C^1.

Theorem 2.6 states the existence of linear unital $*$-preserving linear isomorphisms $G_t : \mathcal{A} \to \mathcal{A}$ which intertwine φ_0 and φ_t. In this case one can compute them explicitly. First note the following:

$$dE_t\big(E_t(a)\big) = \dot\varphi_t\big(\varphi_t(a).1\big).1 = \varphi_t(a)\dot\varphi_t(1).1 = 0,$$

because $\varphi_t(1) = 1$ for all $t \in I$. On the other hand

$$E_t\big(dE_t(a)\big) = \varphi_t\big(\dot\varphi_t(a).1\big) = \dot\varphi_t(a).\varphi_t(1) = \dot\varphi_t(a).$$

Thus the differential equation defining G_t is

$$\begin{cases} \dot\alpha = -\dot\varphi_t(\alpha).1, \\ \alpha(0) = a. \end{cases}$$

Note that $\dot\alpha$ takes scalar values. This implies that $\alpha(t) = a + \beta(t).1$. A straightforward computation shows then that $\alpha(t) = a + (\varphi_t(a) - \varphi_0(a)).1$. That is

$$G_t(a) = a + \big(\varphi_t(a) - \varphi_0(a)\big).1.$$

Note that G_t is not multiplicative.

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