Surfaces refracting and reflecting collimated beams

Cristian E. Gutiérrez*

Department of Mathematics, Temple University, Philadelphia, PA 19122

Federico Tournier

Instituto Argentino de Matemática, CONICET, Saavedra 15, Buenos Aires (1083), Argentina

**Corresponding author: gutierre@temple.edu*

OCIS codes: 080.2720, 080.2740, 080.4225, 080.4228, 230.4040.

1. Introduction

In this note we show how to construct surfaces that refract collimated radiation into a given set of positions or directions. To deduce the equations governing the problem it is convenient to use the Snell law of refraction in vector form. This leads to a system of first order partial differential equations that using elementary calculus can be solved explicitly. We give a necessary and sufficient condition on the image transformation for the problem to be solvable, condition (2.5). Likewise, we also study the corresponding problem for mirrors or reflectors that was considered by R. A. Hicks and collaborators in [HB01] and [HP05] for certain important applications such us the design of a driver mirror without a blind spot. For mirrors, we also give a necessary and sufficient condition on the image transformation for the problem to be solvable, condition (3.9). In particular, when the image transformation is the scaling magnification this condition is not satisfied, and we obtain as in [HP05] that there is no solution in this case. In contrast with this, the scaling magnification does verify (2.5) and therefore there is a refracting surface in this case. These refracting surfaces yield an optical lenses as the region cut above by the surface and below by a horizontal plane. In the plane we show that mirrors and refracting curves always exist with no conditions on the image transformation. All solutions are given by explicit formulas.

In case conditions (2.5) or (3.9) are not satisfied, it is possible to find a solution that is closest in L^2 norm to the prescribed image transformation, see Section 4.

For applications to laser beam shaping see [She00].

2. The lens problem

We look for a surface in the 3d-space with equation $y = u(x, y)$ that separates glass and air (or in general two materials with different refractive indices) and such that a ray emanating from the point $(x, y, 0)$ is refracted by the surface into the point $(t(x, y), m)$. We assume the incident ray emanating from $(x, y, 0)$ has direction $\mathbf{k} = (0, 0, 1)$, that is, all emanating rays are parallel. The unit outer normal to the curve is $\mathbf{N} = \dfrac{(-Du(x, y), 1)}{\sqrt{1 + |Du(x, y)|^2}}$. Suppose all points $(x, y, 0)$ are surrounded by glass and all points $(t(x, y), m)$ surrounded by air, see Figure 1. Glass has index of refraction n_1 and air has index of refraction $n_2 (\approx 1)$. Let $\kappa = n_2/n_1$, so in this case we have $\kappa < 1$. The geometry of the refracting surfaces depend on weather $\kappa < 1$ or $\kappa > 1$, see [GH09]. From the Snell law of refraction in vector form, see also [GH09, Section 2.1] for a discussion, if the incident ray has unit direction $\mathbf{k}$, then the unit direction of the refracted ray is $\mathbf{v}$ with

$$\mathbf{k} - \kappa\,\mathbf{v} = \lambda\,\mathbf{N}, \tag{2.1}$$

and from the physical constraint for refraction we need $\mathbf{k} \cdot \mathbf{v} \geq \kappa$. Making the dot product with $\mathbf{N}$ in (2.1) yields

$$\lambda = \mathbf{k} \cdot \mathbf{N} - \kappa\, \mathbf{v} \cdot \mathbf{N}$$

If θ_1 is the angle of incidence and θ_2 is the angle of refraction we have $\sin\theta_2 = \dfrac{1}{\kappa}\sin\theta_1$. So $\mathbf{v} \cdot \mathbf{N} = \cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \sqrt{1 - \kappa^{-2}\sin^2\theta_1}$, and $\sin^2\theta_1 = 1 - \cos^2\theta_1 = 1 - (\mathbf{k} \cdot \mathbf{N})^2$. So

$$\mathbf{v} \cdot \mathbf{N} = \sqrt{1 - \kappa^{-2}\left(1 - (\mathbf{k} \cdot \mathbf{N})^2\right)}$$

and therefore

$$\lambda = \mathbf{k} \cdot \mathbf{N} - \kappa \sqrt{1 - \kappa^{-2}\left(1 - (\mathbf{k} \cdot \mathbf{N})^2\right)}.$$

Notice that since $\kappa < 1$, then for refraction to occur the incidence angle θ_1 we must satisfy $\theta_1 \leq \theta_c$, with θ_c the critical angle, that is, $\sin\theta_c = \kappa$. Therefore $\sin\theta_1 \leq \kappa$ and so $1 - (\mathbf{k} \cdot \mathbf{N})^2 \leq \kappa^2$ and consequently $(\mathbf{k} \cdot \mathbf{N})^2 \geq 1 - \kappa^2$. Since

$$\mathbf{k} \cdot \mathbf{N} = \frac{1}{\sqrt{1 + |Du(x,y)|^2}},$$

we obtain the following bound for the gradient of the refracting surface

$$|Du(x,y)| \leq \frac{\kappa}{\sqrt{1 - \kappa^2}}. \tag{2.2}$$

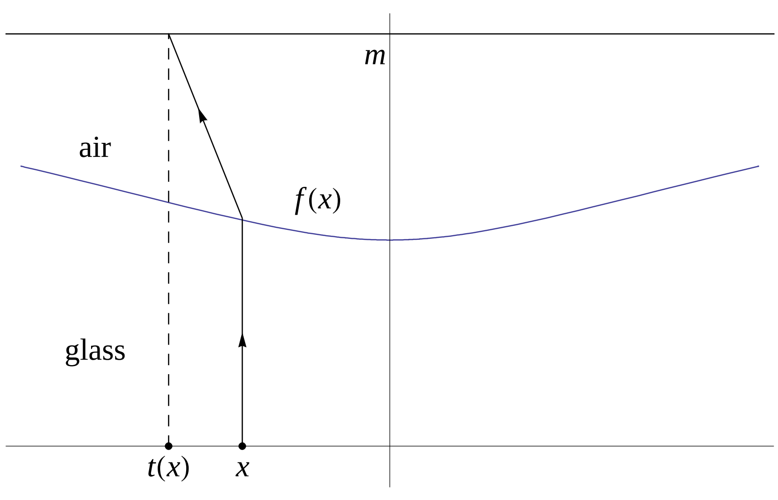

Fig. 1. Lens problem

We show now that the gradient $Du(x,y)$ is determined by the problem. Indeed, the line with direction $\mathbf{v}$ emanating from $(x,y,u(x,y))$ has equation $(x,y,u(x,y))+\alpha\,\mathbf{v}$, and it reaches the point $(t(x,y),m)$ when

$$\alpha = d := \sqrt{|t(x,y)-(x,y)|^2+|m-u(x,y)|^2},$$

that is,

$$(t(x,y)-(x,y), m-u(x,y)) = d\,\mathbf{v}$$

and from (2.1) we obtain

$$\begin{aligned} t(x,y)-(x,y) &= \frac{d\,\lambda}{\kappa}\frac{Du(x,y)}{\sqrt{1+|Du(x,y)|^2}} \\ m-u(x,y) &= \frac{d}{\kappa}\left(1-\frac{\lambda}{\sqrt{1+|Du(x,y)|^2}}\right). \end{aligned}$$

Therefore

$$Du(x,y) = \frac{t(x,y)-(x,y)}{\dfrac{d}{\kappa}-m+u(x,y)}. \tag{2.3}$$

2.A. Far field case

If $g(x,y)$ is a fixed function, we called it the image transformation, and we let $t(x,y) = m\,g(x,y)$, then (2.3) becomes

$$Du(x,y) = \frac{(x,y)-m\,g(x,y)}{m-u(x,y)-\dfrac{1}{\kappa}\sqrt{|m\,g(x,y)-(x,y)|^2+|m-u(x,y)|^2}}$$

and letting $m\to\infty$ we obtain that

$$Du(x,y) = \frac{g(x,y)}{\dfrac{1}{\kappa}\sqrt{|g(x,y)|^2+1}-1} \tag{2.4}$$

We notice that $Du(x,y)$ verifies (2.2) for any $g(x,y)$ because the maximum value of the function $f(z) = \dfrac{z}{\dfrac{1}{\kappa}\sqrt{1+z^2}-1}$ on $(0,+\infty)$ equals to $\kappa/\sqrt{1-\kappa^2}$.

If $g(x,y) = (g_1(x,y), g_2(x,y))$, then from an elementary calculus theorem about exact differentials there is $u\in C^2$ solving (2.4) if and only if

$$\left(\frac{g_1(x,y)}{\dfrac{1}{\kappa}\sqrt{|g(x,y)|^2+1}-1}\right)_y = \left(\frac{g_2(x,y)}{\dfrac{1}{\kappa}\sqrt{|g(x,y)|^2+1}-1}\right)_x. \tag{2.5}$$

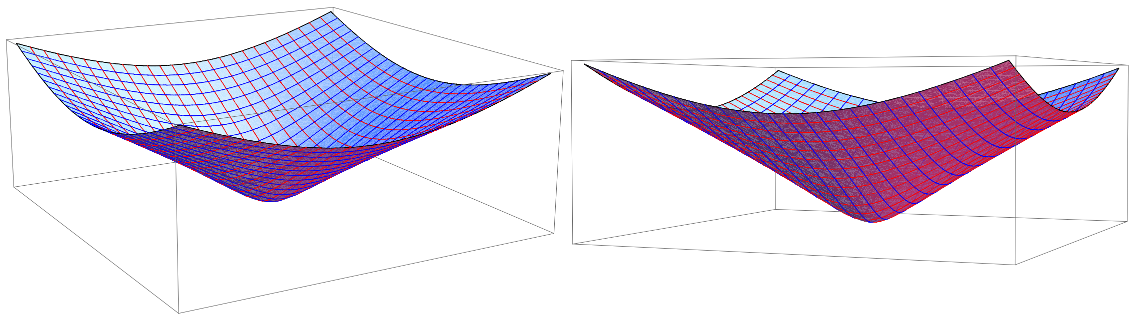

Fig. 2. Two views refracting surface with $\kappa = 2/3$ and $\beta = 2$

In particular, when we seek for a surface magnifying the image, that is, if $g_1(x,y) = \beta x$ and $g_2(x,y) = \beta y$ with $\beta > 0$ it is easy to see that this condition is satisfied and so there exists a refracting surface solving (2.4). Therefore, in this case one can solve explicitly the equation (2.4) obtaining

$$u(x,y) = \int_a^x \frac{\beta s}{\frac{1}{\kappa}\sqrt{|g(s,y)|^2+1}-1}\,ds + \int_b^y \frac{\beta t}{\frac{1}{\kappa}\sqrt{|g(a,t)|^2+1}-1}\,dt. \tag{2.6}$$

An elementary integration shows that

$$u(x,y) = \psi\left(\beta^2(x^2+y^2)\right) - \psi\left(\beta^2(a^2+b^2)\right),$$

where

$$\psi(r) = \frac{\kappa}{\beta}\sqrt{r+1} + \frac{\kappa^2}{\beta}\log\left(\sqrt{r+1}-\kappa\right).$$

If we look at the problem in the plane, then equation (2.4) becomes

$$u'(x) = \frac{g(x)}{\frac{1}{\kappa}\sqrt{|g(x)|^2+1}-1} \tag{2.7}$$

and therefore

$$u(x) = \int_a^x \frac{g(t)}{\frac{1}{\kappa}\sqrt{|g(t)|^2+1}-1}\,dt$$

that can be integrated for many functions g for example when the magnification g is a continuous function. This provides by rotational symmetry surfaces that refract radiation with magnification g. Another approach to find these curves using the Legendre transformation in the plane is in the paper [Gut11].

3. Reflector problem

We seek for a surface with equation $z = u(x, y)$ reflecting all collimated rays emanating with direction $\mathbf{k} = (0, 0, 1)$ from (x, y) into the points $(g_1(x, y), m, g_2(x, y))$. From the Snell law the reflected ray has unit direction

$$\mathbf{v} = \mathbf{k} - 2(\mathbf{k} \cdot \mathbf{N})\mathbf{N} = \left(\frac{2}{1 + |Du(x, y)|^2}\, u_x(x, y), \frac{2}{1 + |Du(x, y)|^2}\, u_y(x, y), 1 - \frac{2}{1 + |Du(x, y)|^2}\right),$$

with unit normal $\mathbf{N} = \dfrac{(Du(x, y), -1)}{\sqrt{1 + |Du(x, y)|^2}}$. The ray with direction $\mathbf{v}$ emanating from $(x, y, u(x, y))$ hits $(g_1(x, y), m, g_2(x, y))$ when

$$(x, y, u(x, y)) + d\,\mathbf{v} = (g_1(x, y), m, g_2(x, y))$$

with

$$d = \sqrt{(x - g_1(x, y))^2 + (y - m)^2 + (u(x, y) - g_2(x, y))^2}.$$

So $(g_1(x, y) - x, m - y) = \dfrac{2d}{1 + |Du(x, y)|^2}\, Du(x, y)$ and $g_2(x, y) - u(x, y) = d\left(1 - \dfrac{2}{1 + |Du(x, y)|^2}\right)$, and therefore

$$Du(x, y) = \frac{(g_1(x, y) - x, m - y)}{d - g_2(x, y) + u(x, y)}.$$

For the far field problem we assume $g_1(x, y) = m\,h_1(x, y)$ and $g_2(x, y) = m\,h_2(x, y)$, with m that will tend to ∞. We then have

$$Du(x, y) = \frac{m\,(h_1(x, y) - (x/m), 1 - (y/m))}{m\left(-h_2(x, y) + \dfrac{u(x, y)}{m} + \sqrt{\left(\dfrac{x}{m} - h_1(x, y)\right)^2 + \left(\dfrac{y}{m} - 1\right)^2 + \left(\dfrac{u(x, y)}{m} - h_2(x, y)\right)^2}\right)}$$

and letting $m \to \infty$ yields the equation

$$Du(x, y) = \frac{(h_1(x, y), 1)}{\sqrt{1 + h_1(x, y)^2 + h_2(x, y)^2} - h_2(x, y)}. \tag{3.8}$$

Therefore there exists a C^2 solution u to (3.8) if and only if

$$\left(\frac{h_1}{\sqrt{1+h_1^2+h_2^2}-h_2}\right)_y = \left(\frac{1}{\sqrt{1+h_1^2+h_2^2}-h_2}\right)_x . \tag{3.9}$$

Similarly as in (2.6), the solution u has the form

$$u(x,y) = \int_a^x \frac{h_1(s,y)}{\sqrt{1+h_1(s,y)^2+h_2(s,y)^2}-h_2(s,y)}\,ds + \int_b^y \frac{1}{\sqrt{1+h_1(a,t)^2+h_2(a,t)^2}-h_2(a,t)}\,dt. \tag{3.10}$$

When $h_1(x,y) = \lambda\, x$ and $h_2(x,y) = \lambda\, y$, it is easy to verify that (3.9) does not hold, which shows as in [HP05], that the reflector problem does not have a solution for constant scaling.

In the case of the plane if we seek for a curve $y = u(x)$ reflecting off collimated rays emanating from each point $(x,0)$ with direction $\mathbf{j} = (0,1)$ to a point $(m, g(x)$, then proceeding in the same way we obtain that u satisfies the equation

$$u'(x) = \frac{m-x}{d-g(x)+u(x)},$$

where $d = \sqrt{(m-x)^2+(g(x)-u(x))^2}$. For the far field case letting $g(x) = m\,h(x)$ and letting $m \to \infty$, we obtain the equation

$$u'(x) = \frac{1}{\sqrt{1+h(x)^2}-h(x)},$$

giving

$$u(x) = \int_0^x \frac{1}{\sqrt{1+h(t)^2}-h(t)}\,dt.$$

4. Energy minimizing solution

Given a domain $\Omega \subset \mathbb{R}^n$ with sufficiently regular boundary and f a continuous function on the boundary $\partial\Omega$, and suppose we have a $F(x) = (F_1(x), \cdots, F_n(x))$ a field that is $C^1(\bar{\Omega})$ we want to find u defined in $\bar{\Omega}$ such that $u \in C^2(\Omega) \cap C(\Omega)$ such that u minimizes the integral

$$I(u) = \int_\Omega |Du(x) - F(x)|^2\,dx \tag{4.11}$$

among all functions $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $u = f$ on $\partial\Omega$. For our application, $n = 2$ and $F(x,y)$ is the right hand side of equation (3.8) in case of the reflector or $F(x,y)$

equals the right hand side of (2.4) in case of the refractor. Obviously, if (3.9) holds in case of the reflector or (2.5) holds in case of the refractor, then (4.11) is zero and the minumum is attained at the solutions of the reflector and refractor problems previously found, respectively. In general, the minimum of the integral (4.11), if it exists, is not zero. We show that this minimum is attained when u is the solution of the Poisson equation

$$\Delta u = D \cdot F, \qquad \text{in } \Omega \tag{4.12}$$

with boundary data $u = f$ on $\partial\Omega$; $D \cdot F$ denotes the divergence of the field F. Indeed, let $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $v = f$ on $\partial\Omega$. Let $\phi = v - u$. We have

$$\begin{aligned} I(v) &= \int_\Omega \left(D\phi + Du - F\right) \cdot \left(D\phi + Du - F\right)\, dx \\ &= 2\int_\Omega (D\phi) \cdot (Du - F)\, dx + \int_\Omega D\phi \cdot D\phi\, dx + \int_\Omega (Du - F) \cdot (Du - F)\, dx. \end{aligned}$$

We have from the first Green identity that

$$\int_\Omega D\phi \cdot Du\, dx = -\int_\Omega (\Delta u)\, \phi\, dx$$

since $\phi = 0$ on $\partial\Omega$. Also from the divergence theorem we have that

$$\int_\Omega D\phi \cdot F\, dx = -\int_\Omega \phi\, (D \cdot F)\, dx$$

since $\phi = 0$ on $\partial\Omega$. Therefore,

$$I(v) = \int_\Omega D\phi \cdot D\phi\, dx + I(u) \geq I(u),$$

and we are done.

Therefore, the solution u of the reflector or the refractor problems that minimizes the energy (4.11) for a given image transformation is the solution of the Poisson equation (4.12).

Acknowledgements

The first author was supported by NSF grant DMS–0901430.

References

[HB01] R. A. Hicks and R. Bajcsy, *Reflective surfaces as computational sensors*, Image and Vision Computing **19** (2001), 773–777.

[HP05] R. A. Hicks and R. K. Perline, *Blind-spot problem for motor vehicles*, Applied Optics **44** (2005), no. 19, 3893–3897.

[GH09] C. E. Gutiérrez and Qingbo Huang, *The refractor problem in reshaping light beams*, Arch. Rational Mech. Anal. **193** (2009), no. 2, 423–443.

[Gut11] C. E. Gutiérrez, *Reflection, refraction and the Legendre transform*, JOSA A **28** (2011), no. 2, 284–289.

[She00] D. L. Shealy, *Theory of geometrical methods for design of laser beam shaping systems*, Laser Beam Shaping Theory and Techniques (New York) (F. M. Dickey and S. C. Holswade, eds.), vol. 4095, Proceedings SPIE, Marcel Dekker, 2000, pp. 163–213.