

Spherically symmetric static spacetimes in vacuum $f(T)$ gravity

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We show that Schwarzschild geometry remains as a vacuum solution for those four-dimensional $f(T)$ gravitational theories behaving as ultraviolet deformations of general relativity. In the gentler context of three-dimensional gravity, we also find that the infrared-deformed $f(T)$ gravities, like the ones used to describe the late cosmic speed up of the Universe, have as the circularly symmetric vacuum solution a Deser-de Sitter or a Bañados, Teitelboim and Zanelli-like spacetime with an effective cosmological constant depending on the infrared scale present in the function $f(T)$.

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I. INTRODUCTION

$f(T)$ theories [1–5] have been proposed in the last few years as an alternative to $f(R)$ modified gravity [6], and they constitute a very promising research area as is witnessed by the increasing interest in the field [7–26]. The dynamical object in $f(T)$ theories is the field of frames (vierbeins or tetrads), which involves the basis $\{e_a(x)\}$ of the tangent space and its respective co-basis $\{e^a(x)\}$:

$$e_\mu^a e_b^\mu = \delta_b^a, \quad e_\mu^a e_a^\nu = \delta_\mu^\nu. \quad (1)$$

The Lagrangian density

$$\mathcal{L} \propto e f(T) \quad (2)$$

is built with the product of the determinant e of the matrix e_μ^a (the components of the forms e^a in a coordinate basis) times a function of the scalar T , which is quadratic in the set of 2-forms $T^a \equiv de^a$. The components $T^c(e_a, e_b) = T^c_{\mu\nu} e_a^\mu e_b^\nu$ are directly related with the commutator of the e_a 's,

$$[e_a, e_b] = T^c(e_a, e_b) e_c. \quad (3)$$

Since coordinate bases commute, then T^a measures to what extent the basis $\{e_a(x)\}$ departs from a coordinate basis.

The theory makes contact with the metric tensor by declaring the basis orthonormal:

$$g^{\mu\nu} e_\mu^a e_\nu^b = \eta^{ab}. \quad (4)$$

Using the Eq. (1), the metric is solved as

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (5)$$

so $e = \sqrt{-\det(g_{\mu\nu})}$. Although the metric (5) is invariant under *local* Lorentz transformations of the tetrad, the

structure (3) and the $f(T)$ action are preserved only under *global* Lorentz transformations in the tangent space:

$$e_{a'} = \Lambda_{a'}^a e_a, \quad e^{a'} = \Lambda_a^{a'} e^a. \quad (6)$$

Therefore, $f(T)$ theories are dynamical theories not just for the metric (5) but for the entire tetrad. As will be mentioned later, only in the special case of the so-called teleparallel equivalent of general relativity (GR) the theory acquires invariance under local Lorentz transformations, so becoming a dynamical theory just for the metric. In general, it could be said that an $f(T)$ theory provides the spacetime not only with a metric but with an absolute parallelization.

As a consequence of the lack of local Lorentz invariance, the field of frames solving the dynamical equations can substantially differ from a naive “square root” of the metric tensor, as tends to be the guess when the metric is diagonal. In the case of Friedmann-Robertson-Walker (FRW) universes, this naive ansatz only works for spatially flat ($K = 0$) cosmologies, but in Ref. [27] it was shown that the frames parallelizing spatially curved FRW cosmologies are far from being the “squared root” of the corresponding (diagonal) metric. For instance, in the closed FRW model, whose topology is $R \times S^3$, the preferred frame inherits the structure $e^a = (dt, \hat{e}^i)$, where the fields \hat{e}^i , $i = 1, 2, 3$ are responsible for the parallelization of the three sphere S^3 . In this way, the spatial part of e^a turns out to be highly non diagonal in the usual spherical coordinates. Other less symmetric spacetimes, like the homogeneous and anisotropic (open) Bianchi type I cosmologies, are however susceptible of a more graceful treatment because their spatial topological structure R^3 is expressed also as a product of parallelizable submanifolds. In this case the tetrads with the form $e^a = (dt, a(t)dx, b(t)dy, c(t)dz)$, being $a(t)$, $b(t)$, $c(t)$ the (time dependent) scale factors, are certainly representative of the parallelization process and they have been also used recently [28].

The parallelization involved in the dynamical fields e^a represents a serious trouble for characterizing a solution on the basis of the symmetry of the metric tensor. For

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instance, hypersurfaces of constant t in the Schwarzschild spacetime have topology $R \times S^2$. This submanifold is three-dimensional, so it is certainly parallelizable. However, the form of the parallel fields is not trivial because S^2 is not parallelizable. So, in spite of the simple structure of the metric tensor, the obtention of the *Schwarzschild frame*, i.e. the frame field which leads to the Schwarzschild solution in $f(T)$ theories, is a difficult task that requires a more systematic approach not yet developed. In particular, the recent study of spherically symmetric static spacetimes in $f(T)$ theories [29–34] did not get the correct parallelization. Actually, vacuum solutions with spherical symmetry are inconsistent with the frames considered in those works, so the existence of Birkhoff’s theorem in four-dimensional $f(T)$ theories remains as an open question.

After reviewing the essentials of $f(T)$ theories in Sec. II, we construct the *Schwarzschild frame* in Sec. III, which give rise to the Schwarzschild metric. This special frame turns out to be a boost of the isotropic (diagonal) frame, and represents the static, spherically symmetric vacuum solution of every ultraviolet smooth deformation of general relativity with $f(T)$ structure. In Sec. IV, and in the modest environment of the three-dimensional $f(T)$ gravity, we show that circularly symmetric vacuum solutions are given by Bañados, Teitelboim and Zanelli (BTZ)-like black holes or “Deser-de Sitter” spacetimes (depending on the sign of the effective cosmological constant that naturally arises by virtue of the new scales present in the function $f(T)$), provided the function $f(T)$ be of an infrared sort. In the case of high energy $f(T)$ modifications, the general circularly symmetric solution is given by the standard conical geometry characterizing the ($\Lambda = 0$) three-dimensional Einstein theory. Both results together constitute the Birkhoff’s theorem for three-dimensional $f(T)$ gravities. Finally, we display the conclusions in Sec. V.

II. THE *abc* OF $f(T)$ GRAVITY

Very much alike the $f(R)$ gravitational schemes, where the scalar curvature R characterizing the Hilbert-Einstein Lagrangian is replaced by an arbitrary function $f(R)$, the $f(T)$ gravities come into the scene as a result of replacing the Weitzenböck scalar T , which is the cornerstone of the *teleparallel equivalent* of GR [35], by an arbitrary function $f(T)$. The Weitzenböck scalar T is a quantity quadratic in the 2-forms T^a , which are usually introduced as the torsion associated to the Weitzenböck connection $\Gamma_{\mu\nu}^\lambda = e_a^\lambda \partial_\mu e_\nu^a$. In fact,

$$T^\lambda{}_{\mu\nu} \equiv e_a^\lambda T^a{}_{\mu\nu} = \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda = e_a^\lambda (\partial_\nu e_\mu^a - \partial_\mu e_\nu^a). \quad (7)$$

It is not difficult to verify that Weitzenböck connection is metric compatible and has null curvature.

In Refs. [35,36] the more general Lagrangian quadratic in T^a has been displayed as a linear combination of the

quadratic scalars associated with the three irreducible parts of T^a under the Lorentz group $SO(1, 3)$:

$$L_{||} = \frac{1}{16\pi G} [de^a \wedge *(a_1^{(1)}T^a + a_2^{(2)}T^a + a_3^{(3)}T^a)], \quad (8)$$

where the tensorial, vectorial and axial-vectorial parts $^{(1)}T^a$, $^{(2)}T^a$ and $^{(3)}T^a$ are

$$\begin{aligned} ^{(1)}T^a &= T^a - ^{(2)}T^a - ^{(3)}T^a, & ^{(2)}T^a &= \frac{1}{3}e^a \wedge (e_b]T^b), \\ ^{(3)}T^a &= -\frac{1}{3}*(e^a \wedge *(T^b \wedge e_b)). \end{aligned} \quad (9)$$

In the last two equations $]]$ is the interior product and $*$ refers to the Hodge star operator. In general, these Lagrangians are not invariant under local Lorentz transformation of the tetrad, since T^a acquires an additional term coming from the derivatives of the transformation matrix $\Lambda_a^{a'}$:

$$T^a \rightarrow T^{a'} = \Lambda_a^{a'} T^a - e^a \wedge d\Lambda_a^{a'}. \quad (10)$$

Therefore these theories describe global frames rather than just a metric tensor. Notice that the Weitzenböck covariant derivative of a vector field V^ν is

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + V^\lambda e_a^\nu \partial e_\lambda^a = e_a^\nu \partial_\mu (V^\lambda e_\lambda^a). \quad (11)$$

So, a global frame allows to call *constant* to those vector fields keeping constant its projections on the frame. Of course, such a notion of *constant* would not make sense in theories admitting local Lorentz transformations of the frames. The selection of global frames constitutes an intrinsic feature of these theories: the spacetime is endowed with an “absolute parallelism” determined by the grid of field lines of vectors e_a .

However a very peculiar choice of the coefficients in Eq. (8),

$$a_2 = -2a_1, \quad a_3 = -a_1/2, \quad (12)$$

makes the Lagrangian invariant under local Lorentz transformations in the tangent space. In such case the Lagrangian results (including a cosmological constant term) [35,37]

$$L_{\text{T}}[e^a] = \frac{1}{16\pi G} e(T - 2\Lambda), \quad (13)$$

where

$$T = S^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu}, \quad (14)$$

and

$$S^\rho{}_{\mu\nu} = \frac{1}{4}(T^\rho{}_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2}\delta_\mu^\rho T_{\sigma\nu}{}^\sigma - \frac{1}{2}\delta_\nu^\rho T_{\sigma\mu}{}^\sigma. \quad (15)$$

The reason why this particular Lagrangian is invariant under local Lorentz transformations is because the Weitzenböck scalar T can be rewritten as the sum of a term depending just on the metric (5)—which possesses

local Lorentz invariance—and a (nonlocally invariant) surface term:

$$T = -R + 2e^{-1}\partial_\nu(eT_\sigma^{\sigma\nu}). \quad (16)$$

The term depending just on the metric is the Levi-Civita scalar curvature R . This means that the theory (13) reveals itself as a theory completely equivalent to GR. This teleparallel form of the Hilbert-Einstein Lagrangian is known as the teleparallel equivalent of general relativity [35–38]. $f(T)$ theories of gravity are deformations of the Lagrangian (13), so their dynamics are governed by the action

$$I = \frac{1}{16\pi G} \int d^4x e f(T) + \int d^4x e L_{\mathcal{M}}, \quad (17)$$

where $L_{\mathcal{M}}$ refers to the scalar Lagrangian of matter. Excepting for linear f s, the action (17) does not possess local Lorentz invariance due to the behavior of the second term in the expression (16). This implies that $f(T)$ theories involve more degrees of freedom than General Relativity [39,40]. This situation can be compared with $f(R)$ theories in the metric formalism: although their dynamical variable is just the metric tensor, it appears one additional degree of freedom due to the fact that the dynamical equations are fourth-order in $f(R)$ theories. Instead, $f(T)$ theories have second order dynamical equations—since no second order derivatives appear in the Lagrangian—, but their dynamical variable—the tetrad or vierbein—has more components than the metric tensor.

By varying the action (17) with respect to the vierbein components $e_a^\mu(x)$ we obtain the dynamical equations

$$e^{-1}\partial_\mu(eS_a^{\mu\nu})f_T + e_a^\lambda S_\rho^{\nu\mu}T^\rho_{\mu\lambda}f_T + S_a^{\mu\nu}\partial_\mu T f_{TT} + \frac{1}{4}e_a^\nu f = 4\pi G \mathbb{T}_a^\nu, \quad (18)$$

where $\mathbb{T}_a^\nu = e_a^\mu \mathbb{T}_\mu^\nu$ refers to the matter energy-momentum tensor $\mathbb{T}_{\mu\nu}$, and f_T, f_{TT} are the first and second derivatives of f .

Equations (18) hide a crucial and very important property. Let us consider a vacuum solution $e_a^\nu(x)$ of Einstein equations. This means that the tetrad $e_a^\nu(x)$ solves Eqs. (18) with $f(T) = T - 2\Lambda$ and $\mathbb{T}_\mu^\nu = 0$, i.e., $e_a^\nu(x)$ satisfies

$$e^{-1}\partial_\mu(eS_a^{\mu\nu}) + e_a^\lambda S_\rho^{\nu\mu}T^\rho_{\mu\lambda} + \frac{1}{4}e_a^\nu(T - 2\Lambda) = 0. \quad (19)$$

It is quite important to note that, in spite of the fact that $e_a^\nu(x)$ is a *vacuum* solution, it does not generally lead to a null or constant Weitzenböck scalar T . This point can be easily seen by contracting the Eq. (19) with the inverse tetrad $e_\nu^a(x)$. In so doing, one obtains the scalar equation

$$e_\nu^a \partial_\mu(eS_a^{\mu\nu}) = 2e\Lambda, \quad (20)$$

which, in principle, does not compel the invariant T to be null or constant for vacuum solutions. However, by virtue

of the relation (16) we can see that the scalar T must reduce to a total derivative in that case.

Since the Einstein theory or its teleparallel equivalent allows for local Lorentz transformations of the tetrad, one can wonder whether this GR solution can be rephrased as a frame $\bar{e}^a(x) = \Lambda_b^a(x)e^b(x)$ making constant the Weitzenböck scalar T , let us say $T[\bar{e}^a] = 2\bar{\Lambda}$. Of course, the existence of such a frame is independent of the chosen coordinates, and has not effect at all in the spacetime metric because g is given by the tetrad through $g = \eta_{ab}e^a \otimes e^b$, which is Lorentz invariant. According to Eq. (16), we are asking for a Lorentz transformation $\Lambda_b^a(x)$ such that

$$\bar{\Lambda} = e^{-1}\partial_\nu(eT_\sigma^{\sigma\nu}[\bar{e}^a]), \quad (21)$$

because $R[\bar{e}^a] = R[e^a] = 0$ for vacuum solutions, and $\bar{e} = e$. Let us suppose that we have found such a frame and replace it in the deformed equations (18). If $f_T(2\bar{\Lambda}) \neq 0$, the result is

$$e^{-1}\partial_\mu(eS_a^{\mu\nu}) + e_a^\lambda S_\rho^{\nu\mu}T^\rho_{\mu\lambda} + \frac{1}{4}e_a^\nu \frac{f(2\bar{\Lambda})}{f_T(2\bar{\Lambda})} = 0. \quad (22)$$

Since $\bar{e}^a(x)$ is already a solution of the Einstein equations (19) with $T = 2\bar{\Lambda}$, then it will also solve the $f(T)$ vacuum equations provided that

$$2(\bar{\Lambda} - \Lambda) = \frac{f(2\bar{\Lambda})}{f_T(2\bar{\Lambda})}. \quad (23)$$

This argument can be extended to nonvacuum solutions, by adding the substitution $G \rightarrow G/f_T(2\bar{\Lambda})$. In this way, we have the remarkable result that, if we can find in Einstein's theory the (locally rotated) frame $\bar{e}^a(x)$ such that $T[\bar{e}^a] = 2\bar{\Lambda}$, then this special frame will also solve those $f(T)$ theories accomplishing the relation (23). For instance, a GR vacuum solution with $\bar{\Lambda} = 0 = \Lambda$ also solves any ultraviolet deformation of GR:

$$f(T) = T + \mathcal{O}(T^2), \quad \text{i.e.,} \quad f(0) = 0, \quad f'(0) = 1. \quad (24)$$

In next section, we will take advantage of this remarkable property to show that Schwarzschild geometry solves any $f(T)$ theory fulfilling the condition (24).

III. SPHERICAL SYMMETRY AND THE SCHWARZSCHILD FRAME

Despite some claims on the contrary [32], it is easy to verify that the frame

$$\begin{aligned} e^0 &= \left(1 - \frac{2M}{r}\right)^{1/2} dt, & e^1 &= \left(1 - \frac{2M}{r}\right)^{-1/2} dr, \\ e^2 &= r d\theta, & e^3 &= r \sin\theta d\varphi, \end{aligned} \quad (25)$$

which certainly leads to the Schwarzschild interval

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\Omega^2, \quad (26)$$

is not a consistent solution of the $f(T)$ dynamical equations. In fact the r - θ equation of motion (18) yields

$$f_{TT}(16M^3 - 8M^2r - 2Mr^2 + r^3) = 0, \quad (27)$$

which clearly cannot be satisfied except in the case $f(T) = T - 2\Lambda$. Worse yet, if $f_{TT} \neq 0$ the Eq. (26) is not fulfilled not even if $M = 0$; i.e., the tetrad (25) results unsatisfactory even for the description of Minkowski spacetime in arbitrary $f(T)$ theories. Of course, the parallelization of Minkowski spacetime is generated by the Cartesian basis $\{dx^a\}$. On the contrary, the frame (25) generates circles, which are certainly not autoparallel curves of flat spacetime.

Of course, this fact does not mean that the Schwarzschild solution is absent in $f(T)$ theories, but, in turn, that the frame (25) is not correct for the description of such a spacetime. Actually, in this section, we will show that the Schwarzschild spacetime appears as solution of every $f(T)$ theory satisfying (24), but in a somewhat tricky way.

To begin with the construction, let us take the spherically symmetric Schwarzschild metric in isotropic coordinates, given by

$$ds^2 = A(\rho)^2 dt^2 - B(\rho)^2 (dx^2 + dy^2 + dz^2), \quad (28)$$

where the functions A and B depend on the radial coordinate $\rho = \sqrt{x^2 + y^2 + z^2}$, and they are

$$A(\rho) = \frac{2\rho - M}{2\rho + M}, \quad B(\rho) = \left(1 + \frac{M}{2\rho}\right)^2. \quad (29)$$

The isotropic chart covers just the exterior region of the Schwarzschild spacetime, as results clear from the relation between the coordinate ρ and the radial coordinate r of the Schwarzschild gauge, which is

$$\sqrt{r^2 - 2Mr} + r - M = 2\rho. \quad (30)$$

We introduce now the *asymptotic* frame, which is supposed to be a good approximation to the actual (physical) frame just at spatial infinity, when the spacetime has a Minkowskian structure. This frame comes by taking the squared root of the metric (28), so it reads

$$\begin{aligned} e^0 &= A(\rho)dt, & e^1 &= B(\rho)dx, \\ e^2 &= B(\rho)dy, & e^3 &= B(\rho)dz. \end{aligned} \quad (31)$$

The frame (31), unlike (25), is particularly useful as starting point because it captures the asymptotic geometrical meaning of the parallelization process, reflected in the fact that, at spatial infinity, we have the *Minkowskian* frame $e_\mu^a(\infty) = \delta_\mu^a$ which gives a null torsion tensor. However, the *asymptotic* frame is also incapable to globally describe the Schwarzschild spacetime, as can be checked by replacing it in the motion equations.

According to the results of the previous section, we have to look for a Lorentz transformation in such a way that, after its action on the frame (31), we be able to achieve a null Weitzenböck scalar. Bearing in mind this, and the fact that the Weitzenböck scalar coming from the tetrad (31) involves the functions $A(\rho)$ and $B(\rho)$ as well as its first derivatives, we focus the quest in a radial boost depending solely on the radial coordinate ρ . With the usual definitions

$$\gamma(\rho) = (1 - \beta^2(\rho))^{-1/2}, \quad \beta(\rho) = v(\rho)/c, \quad (32)$$

we found that the most general boosted (radial) frame coming from (31) yields

$$\begin{aligned} \bar{e}^0 &= A(\rho)\gamma(\rho)dt - \frac{B(\rho)}{\rho}\sqrt{\gamma^2(\rho) - 1}[x dx + y dy + z dz], \\ \bar{e}^1 &= -\frac{A(\rho)}{\rho}\sqrt{\gamma^2(\rho) - 1}x dt + B(\rho)\left[\left(1 + \frac{\gamma(\rho) - 1}{\rho^2}x^2\right)dx \right. \\ &\quad \left. + \frac{\gamma(\rho) - 1}{\rho^2}xy dy + \frac{\gamma(\rho) - 1}{\rho^2}xz dz\right], \\ \bar{e}^2 &= -\frac{A(\rho)}{\rho}\sqrt{\gamma^2(\rho) - 1}y dt + B(\rho)\left[\frac{\gamma(\rho) - 1}{\rho^2}xy dx \right. \\ &\quad \left. + \left(1 + \frac{\gamma(\rho) - 1}{\rho^2}y^2\right)dy + \frac{\gamma(\rho) - 1}{\rho^2}yz dz\right], \\ \bar{e}^3 &= -\frac{A(\rho)}{\rho}\sqrt{\gamma^2(\rho) - 1}z dt + B(\rho)\left[\frac{\gamma(\rho) - 1}{\rho^2}xz dx \right. \\ &\quad \left. + \frac{\gamma(\rho) - 1}{\rho^2}yz dy + \left(1 + \frac{\gamma(\rho) - 1}{\rho^2}z^2\right)dz\right]. \end{aligned} \quad (33)$$

The expression for the Weitzenböck scalar corresponding to the boosted frame (33) is rather complicated and it is not worth of be showed here. However, it can be easily checked that with the form (29) for the functions $A(\rho)$ and $B(\rho)$, the Weitzenböck invariant becomes zero if and only if the Lorentz factor $\gamma(\rho)$ is chosen according to

$$\gamma(\rho) = \frac{4\rho^2 + M^2}{4\rho^2 - M^2}. \quad (34)$$

In fact, in this case the non-null components of $T_{\lambda\mu\nu}$ and $S_{\lambda\mu\nu}$ are

$$\begin{aligned} T_{00\alpha} &= S_{00\alpha} = -\frac{M}{\rho^3}\frac{A(\rho)}{B(\rho)}x_\alpha \\ T_{\alpha 0\beta} &= 2S_{\alpha 0\beta} - \frac{M}{\rho^2}\delta_{\alpha\beta} = \frac{2M}{\rho^4}x_\alpha x_\beta - \frac{M}{\rho^2}\delta_{\alpha\beta} \\ T_{\alpha\alpha\beta} &= \frac{M}{\rho^3}\frac{B(\rho)}{A(\rho)}x_\beta \quad \alpha \neq \beta, \end{aligned} \quad (35)$$

and those coming from the antisymmetric behavior (x_α stands for x, y, z); then it can be easily verified that $T = S^{\lambda\mu\nu}T_{\lambda\mu\nu}$ cancels out. It is evident that the boosted frame (33) asymptotically reduces to the one given in Eq. (31), which, by virtue of the definition (32), means that the boost velocity $v(\rho)$ asymptotically goes to zero. Moreover, as ρ

approaches the black hole horizon located at $\rho_h = M/2$, the boost velocity goes to the speed of light. It should be remarked that the expression (33) is not longer valid in the interior region of the black hole, since the isotropic chart does not cover that part of the spacetime. However, one can change coordinates in order to cover the interior piece of the black hole without affecting the fact that $T[\bar{e}]$ is null. For instance, we can obtain the maximum analytical extension of the Schwarzschild spacetime after performing a change to Kruskal coordinates by means of the change law

$$\bar{e}^a_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \bar{e}^a_\mu, \quad (36)$$

where $x^{\mu'}$ refers to the Kruskal chart and x^μ to the isotropic one. The $\bar{e}^a_{\mu'}$ would be, then, the *same* frame in a different coordinate system.

The existence of the *Schwarzschild frame* (33) automatically proves that the Schwarzschild spacetime is a solution of every $f(T)$ theory satisfying the condition (24). It is quite important to emphasize that, despite that both frames (31) and (33) lead to the same metric (28), which is a trivial fact in the context of GR, the only consistent frame (up to *global* Lorentz transformations) that reproduces the Schwarzschild spacetime in the context of $f(T)$ -like gravities is the *Schwarzschild frame* (33).

IV. CIRCULAR SYMMETRY IN $D = 2 + 1$ DIMENSIONS

In three-dimensional spacetime, the most general metric compatible with circular symmetry can be cast in the form

$$ds^2 = N^2(t, r) dt^2 - \frac{Y^2(t, r)}{N^2(t, r)} dr^2 - r^2 (N^\theta(t, r) dt + d\theta)^2. \quad (37)$$

A suitable dreibein field for the metric (37) is given by

$$\begin{aligned} \bar{e}^0 &= N(t, r) dt, & \bar{e}^1 &= \frac{Y(t, r)}{N(t, r)} dr, \\ \bar{e}^2 &= r N^\theta(t, r) dt + r d\theta. \end{aligned} \quad (38)$$

The Weitzenböck invariant for this frame reads

$$T = \frac{2(N^2)' + r^3(N^{\theta'})^2}{2rY^2}, \quad (39)$$

where the primes indicate differentiation with respect to the radial coordinate. Note that no time derivatives are involved in the expression (39), so all circularly symmetric spacetimes are also stationary [41]. Because of this reason, we can write the *lapse* N and the *shift* N^θ just as functions of r . The presence of the shift function in the off-diagonal term of this ‘‘spherically’’ symmetric line element is one of the typical subtleties in three-dimensional gravity.

The Einstein (teleparallel equivalent) vacuum solution with the form (38) is

$$N^\theta(r) = -\frac{J}{2r^2}, \quad N^2(r) = -M - \Lambda r^2 + \frac{J^2}{4r^2}, \quad Y = 1, \quad (40)$$

M and J being two integration constants associated with the mass and angular momentum, respectively. When the cosmological constant Λ is negative, we have the spinning BTZ black hole [42]. In the case of null and positive Λ , in turn, we have the conical geometry first studied by Deser, Jackiw and 't Hooft [43] and by Deser and Jackiw in Ref. [44]. The key point here is that the Weitzenböck invariant (39) is constant for the particular GR solution (40); Its value is $T = -2\Lambda$. This fact simplifies the analysis enormously, because the triad (38) plays the role of the ‘‘transformed’’ frame mentioned in the construction made at the end of Sec. II, which means that the solution (40) will be also a solution of $f(T)$ theories satisfying Eq. (23) with $\bar{\Lambda} = -\Lambda$. We will show this point by solving the motion equations (18) in an explicit way. For this, we replace the ansatz (38) in (18), and find three independent equations

$$f - 2f_T T = 0, \quad f_T = c_1 Y, \quad f_T N^{\theta'} = c_2 \frac{Y}{r^3}, \quad (41)$$

where $c_{1,2}$ are two integration constants and T is given in Eq. (39). If f_T is non null, we divide the last two equations to obtain (up to an additive constant which can be eliminated by a coordinate change)

$$N^\theta = -\frac{J}{2r^2}, \quad J \equiv \frac{c_2}{c_1}, \quad (42)$$

which gives the shift function of Eq. (40). It is important to note that this result is independent of the specific choice for the function $f(T)$.

In order to proceed, let us suppose first that $T = 0$. In such case, the first equation in (41) is accomplished by any ultraviolet deformation (24). Moreover, Eqs. (39) and (42) completely determine the lapse function:

$$N^2(r) = M + \frac{J^2}{4r^2}, \quad (43)$$

which, essentially, is the second equation in (40) with $\Lambda = 0$. Besides, the relation

$$f_T(0) = c_1 Y \quad (44)$$

in Eq. (41) says that we can choose $Y = 1$ with no loss of generality. In this way, we see that the conical geometry characterizing the elementary solution of Einstein's theory in three-dimensional spacetime also emerges out as the solution of *any* ultraviolet $f(T)$ theory.

Although the last conclusion resembles the one obtained in $3 + 1$ dimensions for the Schwarzschild frame, some differences distinguishing GR in $D = 3 + 1$ and $D = 2 + 1$ should be here mentioned. While M in Schwarzschild solution characterizes the local geometry, M and J in the metric (37) with $Y = 1$,

$$ds^2 = [d(Mt + J\theta/(2M))]^2 - \frac{dr^2}{J^2/(4r^2) + M^2} - \frac{r^2}{M^2}(J^2/(4r^2) + M^2)d\theta^2, \quad (45)$$

could be regarded as aspects of the global geometry. In fact, they can be absorbed by performing the coordinate change

$$\rho = M^{-2}(J^2/4 + M^2r^2)^{1/2}, \quad \tau = Mt + J\theta/(2M), \quad \varphi = M\theta, \quad (46)$$

given so the metric

$$ds^2 = d\tau^2 - d\rho^2 - \rho^2 d\varphi^2, \quad (47)$$

where now we have $0 \leq \varphi \leq 2\pi M$ and the new time coordinate τ is forced to a jump of $\Delta\tau = J/\pi M$ as a consequence of the discontinuity of φ . This means that the geometry is locally flat whatever M and J are. As a consequence, the pair M, J in Eqs. (38) and (40) with $\Lambda = 0$, characterizes different frames for the *same* local geometry. Moreover, since T is zero for all the M, J family of frames, the theory is unable to pick a frame and remove this ambiguity. For instance, the frames (38) and

$$e^0 = d\tau, \quad e^1 = d\rho, \quad e^2 = \rho d\varphi, \quad (48)$$

which are connected by the boost

$$\bar{e}^0 = \frac{e^0 - J/(2M^2\rho)e^2}{\sqrt{1 - J^2/(4M^4\rho^2)}}, \quad \bar{e}^1 = e^1, \quad \bar{e}^2 = \frac{e^2 - J/(2M^2\rho)e^0}{\sqrt{1 - J^2/(4M^4\rho^2)}}, \quad (49)$$

both lead to a null Weitzenböck scalar T , in contrast with what happens in the four-dimensional analogue (remember that, in general, T is not invariant under local Lorentz transformations of e^a , see Eq. (16)). This freedom in the choice of the triad could be removed in a more general theory, as the one proposed in Ref. [27] where M and J do characterize the local geometry. From another point of view, the use of the metric (45) in the (2 + 1)-Einstein equations implies the energy-momentum tensor $T^{ii} \propto \mu \delta^2(\mathbf{r})$, $T^{ij} \propto J \varepsilon^{ij} \partial_j \delta^2(\mathbf{r})$, where $\mu = (1 - M)/4$. This means that a spinning massive particle is located at the origin $r = 0$. Although the spinning particle at the origin does not locally curve the spacetime, it does confer global properties to the manifold; while the presence of mass generates a wedge in Minkowski space, the effect of spin is to give the spacetime a kind of helical structure, for a complete rotation about the source is accompanied by a shift in time [43]. This could be regarded as a physical reason to have different triads for the same local geometry.

Let us return to the problem posed in Eq. (41), for considering the cases where T is non-null. The first equation in (41) is an algebraic equation for T . At this point we have to prescribe the function $f(T)$ in order to carry on with

the analysis. Let us consider the case of the infrared deformations originally proposed to describe the late cosmic speed up [4]:

$$f(T) = T + \frac{\alpha}{(-T)^\beta}. \quad (50)$$

Thus, multiplying the first equation in (41) by the non-null quantity $(-T)^\beta$ we obtain

$$T = -[-\alpha(1 + 2\beta)]^{(1/1+\beta)}. \quad (51)$$

Therefore the second equation in (41) just says

$$\frac{1 + \beta}{1 + 2\beta} = c_1 Y, \quad (52)$$

which means that we can make $Y = 1$ by choosing the constant c_1 . As mentioned before, the function N^θ does not depend on the form of $f(T)$, so its expression is again (42). Finally, the Eq. (39) says that the lapse function is

$$N^2(r) = -M - \frac{1}{2}[-\alpha(1 + 2\beta)]^{(1/1+\beta)}r^2 + \frac{J^2}{4r^2}, \quad (53)$$

Thus, the infrared deformations (50) lead to an asymptotically de Sitter or anti de Sitter spacetime, according to the sign of the effective cosmological constant,

$$\Lambda = \frac{1}{2}[-\alpha(1 + 2\beta)]^{(1/1+\beta)}. \quad (54)$$

We see that a small effective cosmological constant can be obtained not only by considering $\alpha \approx 0$, but also setting $\beta = -1/2 + \epsilon$, with $|\epsilon| \ll 1$. In this case, the effective cosmological constant is positive: $\Lambda \approx 2\alpha^2\epsilon^2$. Although this choice for the parameters α and β is not expected to be meaningful for the four-dimensional analogue, nonetheless it points out a very interesting line of research.

Again we remark that the infrared-deformed solution (42)–(53) can be straightforwardly obtained from its GR partner (40). Since the GR solution (40) has a constant Weitzenböck invariant, $T = -2\Lambda$, then we can invoke the argument displayed at the end of Sec. II. Using that $\bar{\Lambda} = -\Lambda$, then the Eq. (23) for the function (50) leads to the Eq. (54) for the effective cosmological constant.

V. CONCLUDING REMARKS

We have shown in Sec. III that the Schwarzschild spacetime remains as solution of *ultraviolet* deformations of GR with $f(T)$ structure. The Schwarzschild frame (33), which emerges from a radial boost of the isotropic frame of Eq. (31), is the solution for any $f(T)$ subjected to the conditions of Eq. (24). This result rules out the possibility of obtaining a deformed (hopefully regular) black hole solution as a vacuum state of $f(T)$ schemes. However, getting the Schwarzschild frame constitutes the starting point in the search for potential deformations of *infrared* character, which are important nowadays in connection with the accelerated expansion experienced by the

Universe, and also to explore new physics in the Solar System. Additionally, it makes possible to formulate more realistic models for stars, since the matching between the exterior (Schwarzschild) frame and the interior one will bring additional constraints in the physical quantities describing the star, such as the energy density and the pressure of the fluid. Finally, we should emphasize that the results developed in the Sec. III of this article just applies in the static case; the existence of monopolar radiation coming from the additional degrees of freedom that certainly are present in $f(T)$ theories, and so, the very validity of Birkhoff's theorem in this context, remains as an open problem to be worked in the future.

With the aim of clarifying some of these issues, in Sec. IV we have worked in the gentler setting of three-dimensional gravity. We saw, by virtue of the time independence of the Weitzenböck invariant [Eq. (39)], that all circularly symmetric vacuum solutions of $f(T)$ theories are also stationary. Moreover, the elementary circular symmetric solution in vacuum for all high energy deformations is given by the conical spacetime of Deser *et al.* This seems to be a very stringent property of $f(T)$ theories, not shared by other ultraviolet GR deformations with absolute parallelism structure (see the regular spacetime obtained in the context of Born-Infeld gravity [45]). We also found that the “exterior” region of this spacetime can

be described by a family of triad fields connected by local Lorentz boosts in the $(t-\theta)$ plane, suggesting so, at least in three spacetime dimensions, the existence of a local subgroup of the full Lorentz group that officiates also as a symmetry group of the theory. More work is needed in order to determine this subgroup as well as its physical significance.

Concerning the infrared deformations in $D = 2 + 1$, we saw that the effect of having a $f(T)$ function other than $T - 2\Lambda$, translates into the presence of an effective cosmological constant coming from the new scales present in the function $f(T)$. For instance, in the extensively considered infrared model $f(T) = T + \alpha(-T)^{-\beta}$, the solution acquires an effective cosmological constant given in Eq. (54). If this cosmological constant turns out to be negative, the solution represent a BTZ-like black hole, while a positive $\bar{\Lambda}$ give rise to a “Deser-de Sitter” spacetime with shift and lapse functions given in Eqs. (42) and (53) respectively. We expect to achieve analogous results in four dimensions, though the formal proof constitutes a matter of current research.

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