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VIOLATION OF THE WEAK EQUIVALENCE PRINCIPLE IN BEKENSTEIN'S THEORY

L. KRAISELBURD*
Grupo de Gravitación, Astrofísica y Cosmología,
Facultad de Ciencias Astronómicas y Geofísicas,
Universidad Nacional de La Plata,
Paseo del Bosque S/N, CP 1900 La Plata, Argentina
lulikrai@gmail.com
lkrai@fcaglp.fcaglp.unlp.edu.ar

H. VUCETICH
Grupo de Gravitación, Astrofísica y Cosmología,
Facultad de Ciencias Astronómicas y Geofísicas,
Universidad Nacional de La Plata,
Paseo del Bosque S/N, CP 1900 La Plata, Argentina
vucetich@fcaglp.fcaglp.unlp.edu.ar

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Bekenstein has shown that violation of Weak Equivalence Principle is strongly suppressed in his model of charge variation. In this paper, it is shown that nuclear magnetic energy is large enough to produce observable effects in Eötvös experiments.

Keywords: Large number hypothesis; weak equivalence principle violation.

1. Introduction

The variation of fundamental constants has been an important subject of research since Dirac stated the Large Number Hypothesis (LNH).[1,2] In the latter times, the interest in that subject has been aroused again since such a variation is a common prediction of several "Theories of Everything" (TOEs), such as string theories.[3] One possible low-energy limit of these TOEs is Bekenstein's variable charge model,[4–6] since it has all desirable properties that such low-energy limit should exhibit.

Since Dirac's proposal, many attempts have been made to detect the proposed variations, most of them with null results (for reviews, see Refs. 3, 7 and 8). An interesting possibility is that a space variation of fundamental constants should produce a violation of the Weak Equivalence Principle,[9] a fact that can be proved easily using energy conservation.[10]

*Member of CONICET.

The Weak Equivalence Principle (WEP) states that the world line of a body immersed in a gravitational field is independent of its composition and structure,[9,11] a generalization of Galileo's law of Universality of Free Fall: the local acceleration **g** of a body is independent of its composition and structure. Since General Relativity has the Equivalence Principle as one of its consequences, testing for its validity is an important form of the search of "new physics".

The most sensitive forms of those tests are the *Eötvös experiments*: testing the equality of acceleration for bodies of different composition or structure.[9,12,13] Several accurate tests have been carried in the second half of the 20th century and up to now.[14–19] These tests impose strict bounds on parameters describing WEP violations.[11,20–24]

However, in his 2002 paper, Bekenstein[5] proved that a violation of WEP is highly unlikely in his model. We shall discuss briefly this issue later on, but the origin of this statement is a wonderful cancellation of electrostatic sources of the ψ field, leading to a null effect in the lowest order. No such cancellation happens for magnetostatic contribution, but a simple examination of the Solar System magnetic energy density suggests that a breakdown of WEP should be inobservable.

In this paper, we discuss the detection of a space variation of α in Bekenstein's model, considering the fluctuations of magnetic fields in quantum systems.

The rest of the paper is structured as follows: in Sec. 2 we make a short summary of the main results on Bekenstein's model related to our problem; Sec. 3 deals with the motion of a composite nonrelativistic body in external gravitational plus dilaton fields, to find an expression for its anomalous acceleration; in Sec. 4 we discuss the magnetostatic energy of matter in a quantum system and in Sec. 5 we state our results and conclusions. Appendix A is devoted to a simple proof of Eq. (19), and Appendix B to an explanation of how the electrical contribution to "Bekenstein's field" can be canceled in the small fields approximation.

2. A Survey of Bekenstein Model

Bekenstein's proposal[4,5] was to modify Maxwell's electromagnetic theory introducing a field ϵ to describe α variation. A unique form of the theory (up to a parameter) was found from the following hypotheses:

(i) The theory should reduce to Maxwell's for a constant α.
(ii) α variation must be dynamical (i.e., generated by a field $\psi = \ln \epsilon$).
(iii) The dynamics of the field is derived from a variational principle.
(iv) The theory must be causal, gauge and time-reversal invariant.
(v) The smallest length in the theory should be the Planck length ℓ_P.

The latter statement should be dropped if the theory is considered a low-energy limit of some TOEs, since these introduce other fundamental length scales. Indeed, there is a second length scale in string theory, the "string tension parameter"; so it is possible to constraint any length with a "new" length scale which is smaller than ℓ_P.

The application of the above hypothesis leads to a unique form of the action

$$S = S_{\text{em}} + S_\psi + S_{\text{mat}} + S_G \,, \tag{1}$$

where

$$S_{\text{em}} = -\frac{1}{16\pi}\int e^{-2\psi} f^{\mu\nu} f_{\mu\nu}\sqrt{-g}\,d^4x \tag{2a}$$

is the modified Maxwell action $[f_{\mu\nu} \equiv a_{\nu,\mu} - a_{\mu,\nu} = (e^\psi A_\nu)_{,\mu} - (e^\psi A_\mu)_{,\nu} = e^\psi F_{\mu\nu}]$;

$$S_\psi = \frac{-\hbar c}{2\ell_B^2}\int (\partial_\mu \psi)^2 \sqrt{-g}\,d^4x \tag{2b}$$

is the ψ field action and S_{mat} and S_G are the matter and Einstein actions, and ℓ_B is Bekenstein's fundamental length scale. The local value of the electric charge is

$$e(x^\mu) = e_0 e^{\psi(x^\mu)} \qquad \alpha(x^\mu) = e^{2\psi(x^\mu)}\alpha_0 \,, \tag{3}$$

where e_0 and α_0 are reference values of these magnitudes.

The general equations of motions for these fields are

$$(e^{-\psi}F^{\mu\nu})_{,\nu} = 4\pi j^\mu \,, \tag{4a}$$

$$j^\mu \equiv e_0 c v^\mu \frac{\delta^3(\mathbf{x} - \mathbf{z}(\tau))}{\gamma\sqrt{-g}} \,, \tag{4b}$$

$$\Box\psi = \frac{\ell_B^2}{\hbar c}\left(\frac{\partial\sigma}{\partial\psi} - \frac{F^{\mu\nu}F_{\mu\nu}}{8\pi}\right), \tag{4c}$$

$$\sigma = \sum mc^2\gamma^{-1}(-g)^{-1/2}\delta^3[\mathbf{x} - \mathbf{z}(\tau)] \,, \tag{4d}$$

the latter quantity being the rest mass energy density.

A word of advice is due: in his papers[4,5] Bekenstein uses an ensemble of classical particles to represent matter. This is not a good model of matter wherever quantum phenomena are important, neither at high energy scales nor small distances scales, since fermions have a "natural length scale", namely the Compton wavelength of the particle $\lambda_C = \hbar/mc$. One must be wary of jumping to conclusions in these regimes (see also Ref. 25).

From the above equations of motion, Bekenstein[5] derives several affirmations which he used in other papers such as Ref. 26.

Cancellation statement

For an electrostatic field equation (4c) can be written in the form

$$\boldsymbol{\nabla}\cdot(e^{-2\psi}\mathbf{E}) = 4\pi\rho \,, \tag{5a}$$

$$\nabla^2\psi = 4\pi\kappa^2\left[\frac{\partial\sigma}{\partial\psi} + e^{-2\psi}\frac{E^2}{4\pi}\right] , \tag{5b}$$

$$\kappa^2 = \frac{\ell_B^2}{4\pi\hbar c}. \tag{5c}$$

In the source term for ψ the first term cancels almost exactly the second and the asymptotic value of ψ is almost exactly suppressed.

WEP for electric charges

The equation of motion of a system of charges in an electric field, in the limit of very small velocities, reduces to

$$M\ddot{\mathbf{Z}} = Q\mathbf{E}\,, \tag{6}$$

where M and Q are the total mass and charge of the system, respectively. Thus, there is no WEP violation.

The above results use the classical point charges model of matter. On the other hand, the equation of motion for ψ for a static system of magnetic dipoles is

$$\nabla^2\psi = -4\pi\kappa^2 e^{-2\psi}\frac{B^2}{4\pi}\,, \tag{7}$$

and there is no cancellation of sources. From an estimate of the field intensities in the Solar System, Bekenstein states that no observable WEP violation can be detected in laboratory experiments. This latter result is also based on the classical point charges model of matter.

3. Motion of a Composite Body in the ψ Field

Let us now find the Lagrangian of a body composed of point-like charges, such as an atom or an atomic nucleus. We shall work in the nonrelativistic limit for the charges, but we shall keep for the moment the full expression for the electromagnetic field. We shall treat the system as classical and later on quantize it in a simple way. The techniques we use are a lightweight version of those used in the $TH\epsilon\mu$ formalism.[11,27] We assume that there are external dilaton ψ and Newtonian gravitational ϕ_N fields acting over the body, but we shall neglect the self-fields generated. With these approximations and using Bekenstein's notation ($c^{-1}\mathbf{E} \equiv \{f^{01}, f^{02}, f^{03}\}$, $\mathbf{B} \equiv \{f^{23}, f^{31}, f^{12}\}$), the Lagrangian of the system takes the form

$$\begin{aligned} L = &-M_{\text{tot}}[\psi]c^2 \\ &+ \sum_l \left[\frac{1}{2}m_l v_l^2 - m_l\phi_N(\mathbf{x}_l) - \frac{1}{c^2}a_\mu(\mathbf{x}_l)j^\mu(\mathbf{x}_l)\right] \\ &- \int \frac{e^{-2\psi}(E^2 - B^2)dV}{16\pi}\,. \end{aligned} \tag{8}$$

To eliminate the electromagnetic fields we use the equations of motion (4a) together with the Lorentz gauge condition to obtain

$$j^\mu_{\text{ef}} = e^\psi j^\mu + \frac{c}{2\pi}\psi_{,\nu}f^{\mu\nu}\,. \tag{9}$$

These equations can be solved using retarded potentials

$$a_0(\mathbf{x},t) = \int \frac{\rho_{\mathrm{ef}}(t_{\mathrm{ret}})}{R} dV', \quad a_i(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{j}_{\mathrm{ef}}(t_{\mathrm{ret}})}{R} dV', \quad R = |\mathbf{x}-\mathbf{x}'|\,, \tag{10}$$

whose slow-motion approximations are

$$a_0 = \int \frac{\rho_{\mathrm{ef}}}{R} dV', \quad a_i = \frac{1}{c}\int \frac{\mathbf{j}_{\mathrm{ef}}}{R} dV', \quad \mathbf{B} = \frac{1}{c}\int \frac{\mathbf{j}_{\mathrm{ef}}\times\mathbf{R}}{R^3} dV\,. \tag{11}$$

In these equations we shall neglect the contribution of $\psi_{,\nu}$ since they are much smaller than the usual current contribution. After some transformations the Lagrangian can be written as

$$\begin{aligned} L = &-M_{\mathrm{tot}}[\psi]c^2 \\ &+ \sum_l \left[\frac{1}{2}m_l v_l^2 - m_l\phi_N(\mathbf{x}_l)\right] - \frac{1}{2}\int e^{2\psi}\frac{\rho_c(\mathbf{x})\rho_c(\mathbf{x}')}{R} dV dV' \\ &+ \frac{1}{2c^2}\int e^{2\psi}\frac{\mathbf{j}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x}')}{R} dV dV' \end{aligned} \tag{12}$$

where we have replaced sums over pair of particles or currents with integrals.

For a macroscopic solid body we shall be interested in the motion of the center of mass. The separation of this motion is easily achieved with the usual substitutions and developing the slowly varying external fields ϕ_N and ψ:

$$\mathbf{R}_{\mathrm{CM}} = \frac{\sum_l m_l \mathbf{x}_l}{M_{\mathrm{tot}}}, \quad \mathbf{V}_{\mathrm{CM}} = \sum_l \mathbf{v}_l\,, \tag{13}$$

$$\mathbf{x}_l = \mathbf{R}_{\mathrm{CM}} + \mathbf{x}'_l, \quad \mathbf{v}_l = \mathbf{V}_{\mathrm{CM}} + \mathbf{v}'_l\,, \tag{14}$$

$$\phi_N(\mathbf{x}) \simeq \phi_N(\mathbf{R}_{\mathrm{CM}}) + O({x'}^2), \quad \psi(\mathbf{x}) \simeq \psi(\mathbf{R}_{\mathrm{CM}}) + O({x'}^2)\,; \tag{15}$$

and besides

$$M_{\mathrm{tot}}[\psi]c^2 \simeq M_{\mathrm{tot}}[0] + \frac{\partial M c^2}{\partial\psi}\psi(\mathbf{R}_{\mathrm{CM}}) + O(\psi^2)\,. \tag{16}$$

Substitution of the above Lagrangian leads to

$$\begin{aligned} L = -M_{\mathrm{tot}}&\left[c^2 - \frac{V_{\mathrm{CM}}^2}{2} - \phi_N(\mathbf{R}_{\mathrm{CM}})\right] \\ &+ 2\psi(\mathbf{R}_{\mathrm{CM}})E_m + \cdots\,. \end{aligned} \tag{17}$$

The electrostatic contribution cancels with the mass dependence on ψ, according to Bekenstein statement (see Appendix B.1), and the neglected terms are of either tidal order, negligible in laboratory tests of WEP, or of higher order in ψ.

The above Lagrangian shows that a body immersed in external gravitational and Bekenstein fields will suffer an acceleration

$$\ddot{\mathbf{R}}_{\mathrm{CM}} = \mathbf{a} = \mathbf{g} + 2\frac{E_m}{M}\boldsymbol{\nabla}\psi|_{\mathrm{CM}}\,. \tag{18}$$

The latter term is the anomalous acceleration generated by the Bekenstein field. The acceleration difference (Eq. (18)) is tested in Eötvös experiments.

4. Magnetic Energy of Matter

In a quantum model of matter, magnetic fields originate in not only in the stationary electric currents that charged particle originate and their static magnetic moments but also in quantum fluctuations of the number density. These contributions to the magnetic energy have been computed in Refs. 11, 28 from a minimal nuclear shell model. The matrix elements of the current operator can be related to the strength of the dipole resonance, with the result (for more details see Appendix A.1)

$$E_m = \int d^3x \frac{B^2}{8\pi} \simeq \frac{1}{2c^2} \int d^3x d^3x' \frac{\mathbf{j}(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \simeq \frac{3}{20\pi} \frac{\hat{E}}{R(A)\hbar c} \int \sigma dE \,, \tag{19}$$

where $R(A)$ is the nuclear radius, $\hat{E}$ is the giant dipole mean absorption energy, and $\int \sigma dE$ its integrated strength function. These quantities have the following approximate representation

$$R(A) = 1.2 \text{ A}^{\frac{1}{3}} \text{ fm}, \quad \hat{E} \sim 25 \text{ MeV}, \quad \int \sigma dE \simeq 1.6A \text{ MeV fm}^2 \,. \tag{20}$$

Since the magnetic energy density is concentrated near atomic nuclei, it can be represented in the form

$$e_m(\mathbf{x}) = \sum_a E_m^a \delta(\mathbf{x} - \mathbf{x}_a) \simeq \sum_b E_m^b n_b(\mathbf{x}) \,, \tag{21}$$

where index b runs over different nuclear species. Define

$$\zeta_m^b = \frac{E_m^b}{M_b c^2} \tag{22}$$

as the fractional contribution of the magnetic energy to rest mass. Then

$$e_m(\mathbf{x}) = \bar{\zeta}_m(\mathbf{x})\rho(\mathbf{x})c^2 \,, \tag{23}$$

where $\rho(\mathbf{x})$ is the local mass density and

$$\bar{\zeta}_m(\mathbf{x}) = \frac{\sum_b \zeta_b \rho_b(\mathbf{x})}{\rho(\mathbf{x})} \tag{24}$$

is the local mass-weighted average of ζ_m.

With expression (23) we can write Eq. (7) in the form

$$\nabla^2 \psi = -8\pi\kappa^2 c^2 e^{-2\psi} \bar{\zeta}_m \rho \,. \tag{25}$$

For small ψ we can find a solution for an arbitrary distribution of sources

$$\psi = 8\pi\kappa^2 c^2 \frac{1}{r} \int_0^r x^2 \bar{\zeta}_m(x)\rho(x) dx \,, \tag{26}$$

whose asymptotic behavior can be expressed in terms of the Newtonian gravitational potential

$$\psi \asymp \frac{8\pi\kappa^2}{GM}\phi_N(r)\int_0^\infty x^2\bar{\zeta}_m(x)\rho(x)dx = 2\left(\frac{\ell_B}{\ell_P}\right)^2\tilde{\zeta}_m\frac{\phi_N(r)}{c^2}, \tag{27}$$

where $\tilde{\zeta}_m$ is the mass-averaged value of ζ_m and we have introduced the Planck length ℓ_P.

5. Results and Conclusion

From Eq. (27) we obtain for the differential acceleration of a pair A, B of different bodies

$$\eta(A,B) = \frac{a_A - a_B}{g} = 4\left(\frac{\ell_B}{\ell_P}\right)^2\zeta_S(\zeta_A - \zeta_B) = C_f\left(\frac{\ell_B}{\ell_P}\right)^2, \tag{28}$$

where ζ_S, ζ_A, and ζ_B are the magnetic energy fractions of the source, body A, and body B, respectively.

$$\zeta_I = \frac{E_I}{M_I c^2}\,. \tag{29}$$

Table 1 shows the results of the most accurate versions of the Eötvös experiment. A simple least squares fit with the statistical model $y = C_f x^2$ yields

$$\left(\frac{\ell_B}{\ell_P}\right)^2 = 0.0003 \pm 0.0006 \tag{30}$$

from which we get the "3σ" upper bound

$$\left(\frac{\ell_B}{\ell_P}\right)^2 < 0.002, \qquad \frac{\ell_B}{\ell_P} < 0.05\,. \tag{31}$$

This last equation encodes the main result of this paper: strict upper bounds can be set from Eötvös experiments on the Bekenstein parameter ℓ_B/ℓ_P even if the electrostatic field does not generate ψ field. These bounds are much larger than the ones that would result if electrostatic energy density would generate ψ field intensity. This calculation was carried in the 1982 paper of Bekenstein[4] and has been repeated several times (e.g., Refs. 20, 29 and 30) with the result

$$\left(\frac{\ell_B}{\ell_P}\right)_{\rm el} < 8.7\times 10^{-3}\,, \tag{32}$$

one order of magnitude smaller than Eq. (31).

It is interesting to compare our result Eq. (31) with the results obtained from an analysis of all evidence from time variation of the fine-structure constant α.[30] In that paper, an effective value of $\zeta = 10^{-4}$ was used, following the suggestion of Ref. 6 and a 1σ bound on $(\ell_B/\ell_P)^2 < 0.003$ was found. From the estimate of $\zeta_{\rm H}$ in

Table 1. Results of Eötvös experiments. The columns show the composition of the bodies, the source, the coefficient of $(\ell B/\ell P)^2$ in Eq. (28), the measured value of η, and its 1σ error.

A	B	Source	$10^{11}C_f$	$10^{11}\eta(A,B)$	Ref.
Al	Au	Sun	17.5	1.0 ± 1.5	14
Al	Pt	Sun	17.5	0.03 ± 0.045	15
Cu	W	Sun	8.8	0.0 ± 2.0	16
Be	Al	Earth	6.8	-0.02 ± 0.23	17
Be	Cu	Earth	10.4	-0.19 ± 0.25	17
Be	Al	Sun	16.1	0.40 ± 0.98	17
Be	Cu	Sun	24.6	-0.51 ± 0.61	17
Si/Al	Cu	Sun	8.8	0.51 ± 0.67	17
EC	MM	Sun	-7.6	0.001 ± 0.032	18
Be	Ti	Earth	6.9	0.004 ± 0.018	19

Ref. 5 we compute an effective value of $\zeta_U = 2.7 \times 10^{-5}\Omega_B \simeq 1.4 \times 10^{-6}$ and so we find a 3σ upper bound

$$\frac{\ell_B}{\ell_P} < 0.8 \tag{33}$$

one order of magnitude larger than in Eq. (31).

In conclusion, we have shown that very strict bound can be put on the Bekenstein model parameter ℓ_B/ℓ_P from the quantum fluctuations of the magnetic fields of matter. From Eq. (31) one should discard the Bekenstein model, but since it can be obtained as a low-energy limit of string models, the latter conclusion should be taken with a grain of salt.

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Appendix A

A.1. *Proof of Eq. (19)*

Reference 28 does not give a proof of Eq. (19). The following proof is based on their methods.

Let us write the total magnetic energy of the nucleus in the form

$$E_m \frac{1}{2c^2} \sum_\alpha \int d\mathbf{x} d\mathbf{x}' \frac{\langle 0|\mathbf{j}(\mathbf{x})|\alpha\rangle \cdot \langle\alpha|\mathbf{j}(\mathbf{x}')|0\rangle}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{A.1}$$

where α runs over a complete set of eigenstates of the nuclear Hamiltonian H. The current operator is defined as

$$\mathbf{j}(\mathbf{x}) = \sum_a \delta(\mathbf{x}-\mathbf{x}_a) e_a \frac{\mathbf{p}_a}{m_a}\,, \tag{A.2}$$

where the sum runs over all particles in the system. Neglecting the momentum dependence of the nuclear potential, we can write

$$\frac{\mathbf{p}_a}{m_a} = \frac{i}{\hbar}[x_a, H]\,.$$

Substitution of the above in Eq. (A.2) yields the result

$$\begin{aligned}\langle 0|\mathbf{j}(\mathbf{x})|\alpha\rangle &= \frac{i}{\hbar}\sum_a \delta(\mathbf{x}-\mathbf{x}_a)(E_0-E_\alpha)\langle 0|e_a\mathbf{x}_a|\alpha\rangle \\ &= \frac{i}{\hbar}\sum_a \delta\left(\mathbf{x}-\mathbf{x}_a\right)\left(E_0-E_\alpha\right)\mathbf{d}_{0\alpha}\,,\end{aligned} \tag{A.3}$$

with $\mathbf{d}(\mathbf{x})$ the polarization (dipole density) operator.

If we assume a constant density within the nucleus, the dipole density can be represented as

$$\mathbf{d}_{0\alpha} = \frac{d_{0\alpha}}{V_N}\hat{\mathbf{x}}\,,$$

where $V_N = \frac{4\pi}{3}R_N^3$ is the nuclear volume, and so

$$\langle 0|\mathbf{j}(\mathbf{x})|\alpha\rangle \cdot \langle \alpha|\mathbf{j}(\mathbf{x})|0\rangle \simeq \frac{|d_{0\alpha}|^2}{\hbar^2}\frac{E_{0\alpha}^2}{V_N^2}\cos\theta\,, \tag{A.4}$$

where θ is the angle between $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$. Thus, the magnetic energy can be expressed approximately as

$$E_m \simeq \frac{\sum_a E_{0\alpha}^2|d_{0\alpha}|^2}{2\hbar^2c^2}\frac{\int d\mathbf{x}d\mathbf{x}'\frac{\cos\theta}{|\mathbf{x}-\mathbf{x}'|}}{V_N^2}\,. \tag{A.5}$$

The last factor is equal to $\frac{3}{5R_N}$. The first one can be computed from the connection between the strength function and the photoabsorption cross section

$$\sigma_{0\alpha} = \frac{4\pi}{\hbar c}E_{\alpha 0}|d_{\alpha 0}|^2\,. \tag{A.6}$$

From this, we easily get

$$\sum_a E_{\alpha 0}^2|d_{\alpha 0}|^2 = \frac{\hbar c}{4\pi}\frac{\int E\sigma(E)dE}{\int \sigma(E)dE}\cdot\int\sigma(E)dE = \bar{E}\int\sigma(E)dE\,, \tag{A.7}$$

where $\bar{E} \sim 25$ MeV is the mean absorption energy, roughly independent of A.

The cross section satisfies the Thomas–Reiche–Kuhn sum rule

$$\int\sigma(E)dE = (1+x)\frac{2\pi^2e^2\hbar}{mc}\frac{NZ}{A} \simeq (1+x)\ 15\ \text{MeV mbarn}\ A\,, \tag{A.8}$$

where $x \sim 0.2$ takes into account exchange and velocity dependence of nuclear interactions. Combining Eqs. (A.5), (A.7), and (A.8) we obtain Eqs. (19) and (20).

Appendix B

B.1. *The Uniqueness of the Solution for Electrostatic Systems*

Using the approximations $\kappa|\Phi| \ll 1$, $\mathbf{E} \approx -\boldsymbol{\nabla}\Phi$, $e^{2\psi} \approx 1 + \kappa^2\Phi^2$ in Eq. (5b) (as Bekenstein did in his paper[5]),

$$\frac{\partial\sigma}{\partial\psi} = -\frac{\rho_0}{\kappa}\tan\kappa\Phi \approx \frac{\rho_0}{\kappa}\kappa\Phi\,, \tag{B.1}$$

and

$$\frac{1}{4\pi}e^{-2\psi}\mathbf{E}^2 \rightarrow \frac{1}{4\pi\nu}\int_\nu d^3x[(\nabla\Phi)^2 - \kappa^2\Phi^2(\boldsymbol{\nabla}\Phi)^2] \tag{B.2}$$

$$= \rho_0\Phi - \frac{1}{4\pi\nu}\left[\oint_{\partial\nu}\Phi(\boldsymbol{\nabla}\Phi)\cdot ds + \kappa^2\int_\nu d^3x\Phi^2(\boldsymbol{\nabla}\Phi)^2\right], \tag{B.3}$$

where ν is the volume of the distribution. So,

$$\nabla^2\psi = -4\pi\kappa^2\left[\frac{1}{4\pi\nu}\oint_{\partial\nu}\Phi(\boldsymbol{\nabla}\Phi)\cdot ds + \frac{\kappa^2}{4\pi\nu}\int_\nu d^3x\Phi^2(\boldsymbol{\nabla}\Phi)^2\right]. \tag{B.4}$$

The first term on the right side of the above equation can be neglected because it is a boundary term. Then the ψ generated is,

$$\psi(r) \approx 4\pi\nu\kappa^2\int\frac{\rho_b n_p(r')}{|r-r'|}d^3r'\,, \tag{B.5}$$

$$\rho_b \approx -\frac{1}{4\pi\nu}\kappa^2\int_\nu\Phi^2(\boldsymbol{\nabla}\Phi)^2 d^3x\,, \tag{B.6}$$

where n_p is the density of number of particles. Since this field should be dimensionless, it is multiplied and divided by m^3 (m is the mass of the distribution). Being $E_G = \frac{Gm^2}{R}$ the gravitational energy of the quantum system and writing κ as a function of the lengths ℓ_B and $\ell_P = \sqrt{\frac{\hbar G}{c^3}}$,

$$\psi(r) \approx -\frac{\widetilde{C}\zeta_C^2\ell_B^4}{c^2\ell_P^4}\left(\frac{E_G}{mc^2}\right)\int_\nu\frac{\rho_m(r')G}{|r-r'|}d^3r'\,, \tag{B.7}$$

where $\int_\nu \frac{\rho_m(r')G}{|r-r'|}d^3r'$ is the Newtonian potential of the system and $\widetilde{C} \approx 10^{-1}$ is an adimensional constant. Then, $(\frac{E_G}{mc^2}) \sim 10^{-39}A^{2/3}$ and $\zeta_c = (\frac{E_c}{mc^2}) \sim 10^{-3}$ (E_C is the Coulombian energy of the quantum system). The most important result is that ψ satisfies the Principle of Maximum for elliptic equations, so the solution is unique. Besides, it is very small to produce observable effects. This issue will be explained in detail in our next paper coming soon.

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