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Convergence of distributed optimal control problems governed by elliptic variational inequalities

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Abstract First, let u_g be the unique solution of an elliptic variational inequality with source term g. We establish, in the general case, the error estimate between $u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2}$ and $u_4(\mu) = u_{\mu g_1 + (1-\mu) g_2}$ for $\mu \in [0, 1]$. Secondly, we consider a family of distributed optimal control problems governed by elliptic variational inequalities over the internal energy g for each positive heat transfer coefficient h given on a part of the boundary of the domain. For a given cost functional and using some monotony property between $u_3(\mu)$ and $u_4(\mu)$ given in Mignot (J. Funct. Anal. 22:130–185, 1976), we prove the strong convergence of the optimal controls and states associated to this family of distributed optimal control problems governed by elliptic variational inequalities to a limit Dirichlet distributed optimal control problem, governed also by an elliptic variational inequality, when the parameter h goes to infinity. We obtain this convergence without using the adjoint state problem (or the Mignot's conical differentiability) which is a great advantage with respect to the proof given in Gariboldi and Tarzia (Appl. Math. Optim. 47:213–230, 2003), for optimal control problems governed by elliptic variational equalities.

Keywords Elliptic variational inequalities · Convex combinations of the solutions · Distributed optimal control problems · Convergence of the optimal controls · Obstacle problem · Free boundary problems

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1 Introduction

Let V a Hilbert space, V' its topological dual, K be a closed, convex and non empty set in V, g in V' and a bilinear form $a : V \times V \to \mathbb{R}$, which is symmetric, continuous and coercive form on V, that to say, there exists a constant $m > 0$ such that $m\|v\|^2 \leq a(v, v)$ for all v in V. It is well known [23, 26, 34] that for each $g \in V'$ there exists a unique solution $u \in K$, such that

$$a(u, v - u) \geq \langle g, v - u\rangle, \quad \forall v \in K, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V and V'. So we can consider $g \mapsto u = u_g$ as a function from V' to K. Let $u_i = u_{g_i}$ be the corresponding solution of (1.1) with $g = g_i$ for $i = 1, 2$. We define for $\mu \in [0, 1]$

$$u_3(\mu) = \mu u_1 + (1 - \mu)u_2, \qquad g_3(\mu) = \mu g_1 + (1 - \mu)g_2, \quad \text{and} \quad u_4(\mu) = u_{g_3(\mu)}. \tag{1.2}$$

In [7], we established the necessary and sufficient condition to obtain that the convex combination $u_3(\mu)$ is the unique solution of the elliptic variational inequality (1.1) with source term $g_3(\mu)$, namely

$$u_4(\mu) = u_3(\mu) \quad \forall \mu \in [0, 1] \text{ if and only if } \alpha = \beta = 0, \tag{1.3}$$

with

$$\alpha = \alpha(g_1) := a(u_1, u_2 - u_1) - \langle g_1, u_2 - u_1\rangle, \tag{1.4}$$

$$\beta = \beta(g_2) := a(u_2, u_1 - u_2) - \langle g_2, u_1 - u_2\rangle. \tag{1.5}$$

In Sect. 2, we establish the error estimate between $u_3(\mu)$ and $u_4(\mu)$ in the case where α and β defined by (1.4) and (1.5) are not equal to zero. We obtain also some other information concerning $u_3(\mu)$ and $u_4(\mu)$ which will be used in Sect. 4. We can not obtain, for an arbitrary convex K, a needed monotony property of $u_3(\mu)$ and $u_4(\mu)$ that $u_4(\mu) \leq u_3(\mu)$ $\forall \mu \in [0, 1]$ [29] but we can obtain this inequality for the complementarity free boundary problems given in Sect. 3.

In Sect. 3, we consider a family of free boundary problems with mixed boundary conditions associated to particular cases of the elliptic variational inequality (1.1). We study some dependence properties of the solutions to this family of elliptic variational inequalities, on the internal energy g (see more details in the complementarity problem (3.1) or the variational inequalities (3.5) or (3.6)) and also on the heat transfer coefficient h which is characterized in the Newton law or the Robin boundary condition (3.3) (see also the variational inequality (3.6)). Note that mixed boundary conditions play an important role in various applications [12, 35].

In Sect. 4, first for a given constant $M > 0$ we consider g as a control variable for the cost functional (4.1), then we formulate the distributed optimal control problem associated to the variational inequality (3.5). We also formulate the family of distributed optimal control problems associated to the variational inequality of (3.6), which depend on a positive parameter h. With the above dependence properties obtained in Sect. 3, the inequality obtained in Sect. 2 and by using the monotony property [29] between $u_3(\mu)$ and $u_4(\mu)$, we obtain a new proof of the strict convexity of the cost functional which is not given in [29] and then the existence and the uniqueness of the optimal control g_{op} holds. We obtain similar results for the optimal control g_{op_h}. We remark here that the strict convexity of the cost functional is automatically true (then the uniqueness of the optimal control problems holds) when the equivalence (1.3) is verified.

Then, we prove that the optimal control g_{op_h} and its corresponding state $u_{g_{op_h}h}$ are strongly convergent to g_{op} and $u_{g_{op}}$ respectively, when $h \to +\infty$, in adequate functional spaces. This asymptotic behavior can be considered very important in the optimal control of heat transfer problems because the Dirichlet boundary condition, given in (3.2), is not a relevant physical condition to impose on the boundary; the true relevant physical condition is given by the Newton law or the Robin boundary condition (3.3) [9]. Therefore, the goal of this paper is to approximate a Dirichlet optimal control problem, governed by an elliptic variational inequality, by Neumann optimal control problems, governed by elliptic variational inequalities, for a large positive coefficient h. Moreover, from a numerical analysis point of view it may be preferable to consider approximating Neumann problems in all space V (see the variational inequality (3.6)), with parameter h, rather than a Dirichlet problem in a restriction of the space V (see the variational inequality (3.5)).

We note here that we do not need to consider the adjoint state for problems (3.5) and (3.6) as in [10, 27] in order to prove the convergence when $h \to +\infty$. This is a very important advantage of our proof with respect to the previous one given for variational equalities in [10]. This fact was possible because we do not need to use the Mignot's conical differentiability of the cost functional [29].

Different problems with distributed optimal control governed by partial differential equations can be found in the following books [3, 25, 31, 38]. Moreover, we describe briefly some works on optimal control governed by elliptic variational inequalities, see for example: [1, 30] on optimality conditions for the penalized problem, [4] on augmented Lagrangian algorithms, [5, 6, 17, 22] on Lagrange multipliers, [39] on quasilinear elliptic variational inequalities, [15] on estimation of a parameter involved in a variational inequality model, [8] on optimal control problems of variational inequalities for Signorini problem, [32] on optimal control for variational inequalities governed by a pseudomonotone operator, [13] when optimal control problem for a variational inequality is approximated by a family of finite-dimensional problems, [14] on the identification of a distributed parameter, and [28] on regularization techniques with state constraints. In conclusion, many practical applications ranging from physical and engineering sciences to mathematical finance are modeled properly by elliptic and parabolic variational inequalities (see [15, 16, 18] and their references within them).

2 Some general results

In [7] we proved the equivalence (1.3). In order to study optimal control problems in Sect. 4 it is useful for us, to obtain the error estimate between $u_3(\mu)$ and $u_4(\mu)$ when the equivalence (1.3) is not satisfied.

Theorem 2.1 *Let u_1 and u_2 be the two solutions of the variational inequality* (1.1) *with respectively as source term g_1 and g_2, then we have the following estimate*

$$m\|u_4(\mu)-u_3(\mu)\|_V^2+\mu I_{14}(\mu)+(1-\mu)I_{24}(\mu)\leq\mu(1-\mu)(\alpha+\beta),\quad\forall\mu\in[0,1]$$

where α and β are defined by (1.4) *and* (1.5) *respectively and*

$$I_{14}(\mu)=a(u_1,u_4(\mu)-u_1)-\langle g_1,u_4(\mu)-u_1\rangle\geq 0,$$
$$I_{24}(\mu)=a(u_2,u_4(\mu)-u_2)-\langle g_2,u_4(\mu)-u_2\rangle\geq 0.$$

Proof As $u_4(\mu)$ is the unique solution of the variational inequality

$$a(u_4(\mu),v-u_4(\mu))-\langle g_3(\mu),v-u_4(\mu)\rangle\geq 0,\quad\forall v\in K$$

and $u_3(\mu)\in K$ so taking $v=u_3(\mu)$ in this variational inequality, we have

$$m\|u_4(\mu)-u_3(\mu)\|_V^2\leq a(u_3(\mu),u_3(\mu)-u_4(\mu))-\langle g_3(\mu),u_3(\mu)-u_4(\mu)\rangle.$$

Using that $u_3(\mu)=\mu(u_1-u_2)+u_2$ and $g_3(\mu)=\mu(g_1-g_2)+g_2$ we obtain

$$\begin{aligned}m\|u_4(\mu)-u_3(\mu)\|_V^2&\leq\left[a(u_2,u_2-u_4(\mu))-\langle g_2,u_2-u_4(\mu)\rangle\right]\\&\quad+\mu\left[a(u_2,u_1-u_2)-\langle g_2,u_1-u_2\rangle\right]\\&\quad+\mu^2\left[a(u_1-u_2,u_1-u_2)-\langle g_1-g_2,u_1-u_2\rangle\right]\\&\quad+\mu\left[a(u_1-u_2,u_2-u_4(\mu))-\langle g_1-g_2,u_2-u_4(\mu)\rangle\right]\\&\leq -I_{24}(\mu)+\mu\beta-\mu^2\beta-\mu^2\alpha+\mu I_{24}(\mu)\\&\quad+\mu\left[a(u_1,u_2-u_4(\mu))-\langle g_1,u_2-u_4(\mu)\rangle\right],\end{aligned}$$

so

$$m\|u_4(\mu)-u_3(\mu)\|_V^2\leq\mu(1-\mu)(\alpha+\beta)-[\mu I_{14}(\mu)+(1-\mu)I_{24}(\mu)],$$

which is the required result. □

The result of Theorem 2.1 will be used in Sect. 4 (see Lemma 4.1). Moreover, from Theorem 2.1 we deduce the result obtained in [7] and more information concerning $u_3(\mu)$ and $u_4(\mu)$ in the following corollary.

Corollary 2.1

$$\alpha(g_1)=\beta(g_2)=0\quad\Longrightarrow\quad\begin{cases}\text{(i) } u_3(\mu)=u_4(\mu) & \forall\mu\in[0,1],\\ \text{(ii) } I_{14}(\mu)=I_{24}(\mu)=0 & \forall\mu\in[0,1].\end{cases}$$

Remark 2.1 We can not obtain a monotony property between $u_3(\mu)$ and $u_4(\mu)$ for a general variational inequality (1.1), precisely for any convex set K. But we can obtain it when we consider the particular obstacle problems (see Sect. 3).

3 Dependence properties of solution of obstacle problem

Let Ω an open bounded set in $\mathbb{R}^n$ with its boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We suppose that $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $meas(\Gamma_1) > 0$. We consider the following complementarity problem:

$$u \geq 0, \quad u(-\Delta u - g) = 0, \quad -\Delta u - g \geq 0 \quad \text{a.e. in } \Omega, \tag{3.1}$$

$$u = b \quad \text{on } \Gamma_1, \qquad -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2 \tag{3.2}$$

and for a parameter $h > 0$, we consider the complementarity problem (3.1) with the mixed boundary conditions:

$$-\frac{\partial u}{\partial n} = h(u - b) \quad \text{on } \Gamma_1, \qquad -\frac{\partial u}{\partial n} = q \quad \text{on } \Gamma_2 \tag{3.3}$$

where h is the heat transfer coefficient on Γ_1, g is the internal energy, b is the temperature on Γ_1, q is the heat flux on Γ_2.

It is well known that the regularity of the mixed problem is problematic in the neighborhood of some part of the boundary, see for example the book [11]. A regularity for elliptic problems with mixed boundary conditions is given in [2, 24]. Moreover, sufficient hypothesis on the data in order to have the H^2 regularity for elliptic variational inequalities are [33, p. 139]:

$$\partial\Omega \in C^{1,1}, \qquad g \in H = L^2(\Omega), \qquad q \in H^{3/2}(\Gamma_2) \tag{3.4}$$

which are assumed from now on.

We define the spaces $V = H^1(\Omega)$, $V_0 = \{v \in V : v_{|\Gamma_1} = 0\}$ and the convex sets given by

$$K = \{v \in V : v|_{\Gamma_1} = b, \ v \geq 0 \text{ in } \Omega\},$$
$$K_+ = \{v \in V : v \geq 0 \text{ in } \Omega\}.$$

It is classical that, for a given positive $b \in H^{\frac{1}{2}}(\Gamma_1)$, $q \in L^2(\Gamma_2)$, and $g \in H$, the two free boundary problems (3.1)–(3.2) and (3.1), (3.3) lead respectively to the following elliptic variational problems: Find $u \in K$ such that

$$a(u, v-u) \geq (g, v-u) - \int_{\Gamma_2} q(v-u)ds, \quad \forall v \in K \tag{3.5}$$

and find $u \in K_+$ such that

$$a_h(u, v-u) \geq (g, v-u) - \int_{\Gamma_2} q(v-u)ds + h\int_{\Gamma_1} b(v-u)ds, \quad \forall v \in K_+ \tag{3.6}$$

respectively, where

$$a(u, v) = \int_\Omega \nabla u \nabla v dx, \qquad (g, v) = \int_\Omega g v dx,$$

$$a_h(u,v) = a(u,v) + h\int_{\Gamma_1} uvds.$$

It is evident that [23]

$$\exists\lambda > 0 \quad \text{such that} \quad \lambda\|v\|_V^2 \le a(v,v), \quad \forall v \in V_0.$$

Moreover [35, 36]

$$\exists\lambda_1 > 0 \quad \text{such that} \quad \lambda_h\|v\|_V^2 \le a_h(v,v), \quad \forall v \in V, \text{ with } \lambda_h = \lambda_1 \min\{1,h\}$$

that is a_h is a bilinear continuous, symmetric and coercive form on V, as a.

Remark 3.1 Note that we can easily obtain the same results of this paper for more general problem than (3.1)–(3.2) and (3.1), (3.3) governed by elliptic variational inequalities under the assumption that the form a must be bilinear, continuous and coercive.

Remark 3.2 The variational inequalities (3.5) and (3.6) are the particular cases of (1.1) for the particular convex sets K and K_+ and

$$\langle g,v\rangle = (g,v) - \int_{\Gamma_2} qvds, \tag{3.7}$$

$$\langle g,v\rangle = (g,v) - \int_{\Gamma_2} qvds + h\int_{\Gamma_1} bvds \tag{3.8}$$

respectively. Moreover for $g \ge 0$ in Ω, $q \le 0$ on Γ_2 and $b \ge 0$ on Γ_1, then by the weak maximum principle, the unique solution of (3.5) is in K and the unique solution of (3.6) is in K_+ for each $h > 0$.

For all $h > 0$ and all $g \in H$, we associate $u = u_{g_h}$ the unique solution of (3.6) and $u = u_g$ the unique solution of (3.5).

Lemma 3.1

(a) *Let u_{g_n}, u_g two solutions of (3.5) with g_n and g in H then we have*

$$g_n \rightharpoonup g \quad \text{in } H \text{ (weak) as } n \to +\infty \quad \text{then} \quad u_{g_n} \to u_g \quad \text{in } V \text{ (strong)}. \tag{3.9}$$

Moreover, we have

$$g_1 \ge g_2 \quad \text{in } \Omega \quad \text{then} \quad u_{g_1} \ge u_{g_2} \quad \text{in } \Omega, \tag{3.10}$$

$$u_{min(g_1,g_2)} \le u_4(\mu) \le u_{max(g_1,g_2)}, \quad \forall\mu \in [0,1]. \tag{3.11}$$

(b) *Let $u_{g_n h}$, u_{gh} two solutions of (3.6) with g_n and g in H and $h > 0$ then we have*

$$g_n \rightharpoonup g \quad \text{in } H \text{ (weak) as } n \to +\infty \quad \text{then} \quad u_{g_n h} \to u_{gh} \quad \text{in } V \text{ (strong)}. \tag{3.12}$$

Proof (a) Let $g_n \rightharpoonup g$ in H as $n \to +\infty$, u_{g_n} and u_g in K such that

$$a(u_{g_n}, v - u_{g_n}) \geq (g_n, v - u_{g_n}) - \int_{\Gamma_2} q(v - u_{g_n})ds, \quad \forall v \in K. \tag{3.13}$$

Set $z_n = u_{g_n} - B$ where $B \in K$ such that $B|_{\Gamma_1} = b$, and taking $v = B$ in (3.13) we obtain the following inequalities

$$\lambda \|z_n\|_V^2 \leq a(z_n, z_n) \leq -a(z_n, B) + (g_n, z_n) - \int_{\Gamma_2} q z_n ds. \tag{3.14}$$

As $g_n \rightharpoonup g$ in H then $\|g_n\|_H$ is bounded, then from (3.14) there exists a positive constant C which do not depend on n such that $\|u_{g_n}\|_V \leq C$. Thus

$$\exists \eta \in V \quad \text{such that} \quad u_{g_n} \rightharpoonup \eta \text{ weakly in } V \text{ (strongly in } H), \tag{3.15}$$

taking $n \to +\infty$ in (3.13), we get

$$a(\eta, v - \eta) \geq (g, v - \eta) - \int_{\Gamma_2} q(v - \eta)ds, \quad \forall v \in K. \tag{3.16}$$

By the uniqueness of the solution of (3.5) we obtain that $\eta = u_g$. Taking now $v = u_g$ in (3.13), and taking $v = u_{g_n}$ in (3.5) with $u = u_g$, then by addition we get

$$a(u_{g_n} - u_g, u_{g_n} - u_g) \leq (g_n - g, u_{g_n} - u_g),$$

that is (3.9).

Taking in (3.5) $v = u_1 + (u_1 - u_2)^-$ (which is in K) where $u = u_1$ and $g = g_1$. Then taking in (3.5) $v = u_2 - (u_1 - u_2)^-$ (which also is in K) where $u = u_2$ and $g = g_2$. By addition we get

$$a((u_1 - u_2)^-, (u_1 - u_2)^-) \leq (g_2 - g_1, (u_1 - u_2)^-)$$

so if $g_2 - g_1 \leq 0$ in Ω then $\|(u_1 - u_2)^-\|_V = 0$, and as $(u_1 - u_2)^- = 0$ on Γ_1 we have $u_1 - u_2 \geq 0$ in Ω. This gives (3.10). Finally (3.11) follows from (3.10) because

$$\min\{g_1, g_2\} \leq \mu g_1 + (1 - \mu) g_2 \leq \max\{g_1, g_2\}, \quad \forall \mu \in [0, 1].$$

(b) It is similar to (a) for all $h > 0$. □

Let now g_1, g_2 in H, and $u_{g_1 h}$, $u_{g_2 h}$ two solutions of the variational inequality (3.6) with $g = g_1$ and $g = g_2$ respectively, and the same q and h. We define also

$$u_{3h}(\mu) = \mu u_{g_1 h} + (1 - \mu) u_{g_2 h} \quad \text{and} \quad u_{4h}(\mu) = u_{(\mu g_1 + (1-\mu) g_2) h}.$$

So we obtain as in (3.11) that

$$u_{min(g_1, g_2)h} \leq u_{4h}(\mu) \leq u_{max(g_1, g_2)h}, \quad \forall \mu \in [0, 1]. \tag{3.17}$$

Remark 3.3 Taking $v = u^+$ in (3.6) we deduce that

$$a_h(u^-, u^-) \leq -(g, u^-) + \int_{\Gamma 2} q u^- ds - h \int_{\Gamma 1} b u^- ds$$

so for $h > 0$ sufficiently large we can have $u_{gh} \geq 0$ in Ω with $g \leq 0$ in Ω, for given $q \geq 0$ on Γ_2 and $b \geq 0$ on Γ_1.

Lemma 3.2 *Let g_1, g_2 in H and u_{g_1h}, u_{g_2h} two solutions of the variational inequality (3.6) with the same q and h. Suppose that b is a positive constant and $q \geq 0$, then we have*

$$g \leq 0 \quad \text{in } \Omega \implies u_{g_h} \leq b \quad \text{in } \Omega, \quad \text{and} \quad u_{g_h} \leq b \quad \text{on } \Gamma_1, \tag{3.18}$$

$$g_2 \leq g_1 \leq 0 \quad \text{in } \Omega, \quad \text{and} \quad h_2 \leq h_1 \implies u_{g_2h_2} \leq u_{g_1h_1} \quad \text{in } \Omega, \tag{3.19}$$

$$g \leq 0 \quad \text{in } \Omega \implies u_{g_h} \leq u_g \quad \text{in } \Omega, \ \forall h > 0. \tag{3.20}$$

Moreover $\forall g \in H$, $\forall q \in L^2(\Gamma_2)$ and $\forall b \in H^{\frac{1}{2}}(\Gamma_1)$, we have

$$h_2 \leq h_1 \implies \|u_{g_{h_2}} - u_{g_{h_1}}\|_V \leq \frac{\|\gamma_0\|}{\lambda_1 \min(1, h_2)} \|b - u_{g_{h_1}}\|_{L^2(\Gamma_1)}(h_1 - h_2) \tag{3.21}$$

where γ_0 is the trace embedding from V to $L^2(\Gamma_1)$ and $\|\gamma_0\|$ is its norm.

Proof Taking in (3.6) $u = u_{g_h}$ and $v = u_{g_h} - (u_{g_h} - b)^+$ (which in K_+), we get

$$-a_h(u_{g_h}, (u_{g_h} - b)^+) \geq -(g, (u_{g_h} - b)^+) + \int_{\Gamma_2} q(u_{g_h} - b)^+ ds - h \int_{\Gamma_1} b(u_{g_h} - b)^+ ds,$$

then

$$a_h((u_{g_h} - b)^+, (u_{g_h} - b)^+) \leq (g, (u_{g_h} - b)^+) - \int_{\Gamma_2} q(u_{g_h} - b)^+ ds \leq 0,$$

so (3.18) holds.

To check (3.19) we take first in (3.6) $v = u_{g_1h_1} + (u_{g_2h_2} - u_{g_1h_1})^+$, which is in K_+, where $u = u_{g_1h_1}$ is in K_+ with $g = g_1$ and $h = h_1$, and taking in (3.6) $v = u_{g_2h_2} - (u_{g_2h_2} - u_{g_1h_1})^+$, which is also in K_+, where $u = u_{g_2h_2}$ is in K_+ with $g = g_2$ and $h = h_2$, then adding the two obtained inequalities we get

$$a_{h_2}((u_{g_2h_2} - u_{g_1h_1})^+, (u_{g_2h_2} - u_{g_1h_1})^+) \leq (g_2 - g_1, (u_{g_2h_2} - u_{g_1h_1})^+) ds - (h_2 - h_1) \int_{\Gamma_1} (u_{g_1h_1} - b)(u_{g_2h_2} - u_{g_1h_1})^+ ds$$

and from (3.18) we get (3.19).

To check (3.20), let $W = u_{g_h} - u_g$ and choose in (3.6) $v = u_{g_h} - W^+$ which is in K_+, so

$$a(u_{g_h}, W^+) \le (g, W^+) - \int_{\Gamma_2} q W^+ ds. \tag{3.22}$$

We choose, in (3.5), $v = u_g + W^+$, which is in K because from (3.18), then we have $W^+ = 0$ on Γ_1, so

$$a(u_g, W^+) \ge (g, W^+) - \int_{\Gamma_2} q W^+ ds. \tag{3.23}$$

So from (3.22) and (3.23) we deduce that $a(W^+, W^+) \le 0$. Then (3.20) holds.

To finish the proof it remains to check (3.21). We choose $v = u_{g_{h_2}}$ in (3.6) where $u = u_{g_{h_1}}$, and $v = u_{g_{h_1}}$ in (3.6) where $u = u_{g_{h_2}}$, adding the two inequalities we get

$$\begin{aligned}\lambda_1 \min\{1, h_2\} \|u_{g_{h_1}} - u_{g_{h_2}}\|_V^2 &\le (h_1 - h_2)\|b - u_{g_{h_1}}\|_{L^2(\Gamma_1)} \|u_{g_{h_1}} - u_{g_{h_2}}\|_{L^2(\Gamma_1)} \\ &\le \|\gamma_0\|(h_1 - h_2)\|b - u_{g_{h_1}}\|_{L^2(\Gamma_1)} \|u_{g_{h_1}} - u_{g_{h_2}}\|_V.\end{aligned}$$

Thus (3.21) holds. □

Remark 3.4 The Lemma 3.2 gives as a first additional information that, for all $g \le 0$ in Ω and all $h > 0$, the sequence (u_{g_h}) is increasing and bounded, so it is convergent in some space. We study, in the next sections, the optimal control problems associated to the variational inequalities (3.5) and (3.6) and the convergence when $h \to +\infty$ in Lemma 4.2 and Theorem 4.1 for all g, without restriction to $g \le 0$ in Ω.

4 Optimal control problems and convergence for $h \to +\infty$

We will first study in this section two kinds of distributed optimal control problems, their existence, uniqueness results and the relation between them. In fact the existence and uniqueness, of the solution to the two variational inequalities (3.5) and (3.6) allow us to consider $g \mapsto u_g$ and $g \mapsto u_{gh}$ as a functions from H to V, for any $h > 0$.

Let a constant $M > 0$. We define the two cost functionals $J : H \to \mathbb{R}$ and $J_h : H \to \mathbb{R}$ such that [25] (see also [19–21])

$$J(g) = \frac{1}{2}\|u_g\|_H^2 + \frac{M}{2}\|g\|_H^2, \tag{4.1}$$

$$J_h(g) = \frac{1}{2}\|u_{gh}\|_H^2 + \frac{M}{2}\|g\|_H^2, \tag{4.2}$$

and we consider the family of distributed optimal control problems

$$\text{Find } g_{op} \in H \quad \text{such that} \quad J(g_{op}) = \min_{g \in H} J(g), \tag{4.3}$$

$$\text{Find } g_{op_h} \in H \quad \text{such that} \quad J(g_{op_h}) = \min_{g \in H} J_h(g). \tag{4.4}$$

Lemma 4.1 *Let g, g_1, g_2 in H and u_g, u_{g_1}, u_{g_2} are the associated solutions of* (3.5). *We have*

$$\|u_3(\mu) - u_4(\mu)\|_V^2 + \mu(1-\mu)\|u_{g_1} - u_{g_2}\|_V^2 + \frac{\mu}{\lambda} I_{14} + \frac{(1-\mu)}{\lambda} I_{24} \leq \frac{\mu(1-\mu)}{\lambda^2}\|g_1 - g_2\|_H^2. \tag{4.5}$$

For $u_{g_h}, u_{g_{1h}}, u_{g_{2h}}$ the associated solutions of (3.6), *we also have*

$$\|u_{4h}(\mu) - u_{3h}(\mu)\|_V^2 + \mu(1-\mu)\|u_{g_{2h}} - u_{g_{1h}}\|_V^2 + \frac{\mu}{\lambda_h} I_{14h} + \frac{(1-\mu)}{\lambda_h} I_{24h} \leq \frac{\mu(1-\mu)}{\lambda_h^2}\|g_1 - g_2\|_H, \tag{4.6}$$

Proof For $i = 1, 2$ we have

$$I_{i4}(\mu) = a(u_i, u_4(\mu) - u_i) - (g_i, u_4(\mu) - u_i) + \int_{\Gamma_2} q(u_4(\mu) - u_i)ds \geq 0$$

and therefore by using Theorem 2.1 and (3.7) we obtain

$$\lambda\|u_3(\mu) - u_4(\mu)\|_V^2 + \mu I_{14} + (1-\mu)I_{24} \leq \mu(1-\mu)(\alpha + \beta), \quad \forall \mu \in [0, 1].$$

As

$$\begin{aligned}
\alpha + \beta &= a(u_1, u_2 - u_1) - (g_1, u_2 - u_1) + \int_{\Gamma_2} q(u_2 - u_1)ds \\
&\quad + a(u_2, u_1 - u_2) - (g_2, u_1 - u_2) + \int_{\Gamma_2} q(u_1 - u_2)ds \\
&\leq -a(u_2 - u_1, u_2 - u_1) + (g_2 - g_1, u_2 - u_1) \\
&\leq -\lambda\|u_2 - u_1\|_V^2 + \|g_2 - g_1\|_H\|u_2 - u_1\|_H \\
&\leq -\lambda\|u_2 - u_1\|_V^2 + \frac{1}{\lambda}\|g_2 - g_1\|_H^2
\end{aligned}$$

thus (4.5) follows. (4.6) follows also from Theorem 2.1 and (3.8) as above. □

By using Lemma 4.1 and the references [3, 25], we can obtain firstly the existence (not the uniqueness) of optimal controls g_{op} and g_{op_h} solution of Problem (4.3) and Problem (4.4) respectively. Then, the corresponding uniqueness of the optimal control problems can be obtained by using [29, pp. 166 and 177]. Secondly, in order to avoid the use of the conical differentiability (see [29]) and by completeness of the proof of the result we can do another proof of the uniqueness of the optimal control problems which is not given in [29]. For that, we can prove two important equalities (4.7) and (4.8) which allow us to get that J and J_h are strictly convex applications on H, so there exist the unique solutions g_{op} and g_{op_h} in H to the Problem (4.3) and Problem (4.4) respectively. This fact is also very important for us because it permits us to obtain the convergence in Theorem 4.1, our mean result, without using the adjoint state problem.

Proposition 4.1 *Let given g in H and $h > 0$, there exist unique solutions g_{op} and g_{op_h} in H respectively for the Problems* (4.3) *and* (4.4).

Proof We remark first that using Lemma 4.1 and [3, 10, 25, 29] we can obtain the following classical results

$$\lim_{\|g\|_H \to +\infty} J(g) = +\infty, \quad \text{and} \quad \lim_{\|g\|_H \to +\infty} J_h(g) = +\infty,$$
$$J \text{ and } J_h \quad \forall h > 0, \text{ are lower semi-continuous on } H \text{ weak},$$

so we can deduce the existence, of at least, an optimal control g_{op} solution of Problem (4.3) and respectively an optimal control g_{op_h} solution of Problem (4.4).

The uniqueness of the solutions of Problems (4.3) and (4.4) can be also obtained by using [29, pp. 166 and 177]. For completeness we will prove that the cost functional J and J_h are strictly convex applications on H which are not given in [29]. Let $u = u_{g_i}$ and $u_{g_i h}$ be respectively the solution of the variational inequalities (3.5) and (3.6) with $g = g_i$ for $i = 1, 2$. We have

$$\|u_3(\mu)\|_H^2 = \mu^2 \|u_{g_1}\|_H^2 + (1-\mu)^2 \|u_{g_2}\|_H^2 + 2\mu(1-\mu)(u_{g_1}, u_{g_2})$$

then the following equalities hold

$$\|u_3(\mu)\|_H^2 = \mu \|u_{g_1}\|_H^2 + (1-\mu)\|u_{g_2}\|_H^2 - \mu(1-\mu)\|u_{g_2} - u_{g_1}\|_H^2, \tag{4.7}$$

$$\|u_{3h}(\mu)\|_H^2 = \mu \|u_{g_1 h}\|_H^2 + (1-\mu)\|u_{g_2 h}\|_H^2 - \mu(1-\mu)\|u_{g_2 h} - u_{g_1 h}\|_H^2. \tag{4.8}$$

Let now $\mu \in [0, 1]$ and $g_1, g_2 \in H$ so we have

$$\begin{aligned}
&\mu J(g_1) + (1-\mu) J(g_2) - J(g_3(\mu)) \\
&\quad = \frac{\mu}{2}\|u_{g_1}\|_H^2 + \frac{(1-\mu)}{2}\|u_{g_2}\|_H^2 \\
&\qquad - \frac{1}{2}\|u_4(\mu)\|_H^2 + \frac{M}{2}\left\{\mu\|g_1\|_H^2 + (1-\mu)\|g_2\|_H^2 - \|g_3(\mu)\|_H^2\right\}
\end{aligned}$$

and by using (4.7) for $g_3(\mu) = \mu g_1 + (1-\mu) g_2$ we obtain

$$\begin{aligned}
\mu J(g_1) + (1-\mu) J(g_2) - J(g_3(\mu)) &= \frac{1}{2}\{\mu\|u_{g_1}\|_H^2 + (1-\mu)\|u_{g_2}\|_H^2 - \|u_4(\mu)\|_H^2\} \\
&\quad + \frac{M}{2}\mu(1-\mu)\|g_1 - g_2\|_H^2.
\end{aligned} \tag{4.9}$$

Following [29] we obtain the cornerstone monotony property

$$u_4(\mu) \leq u_3(\mu) \quad \text{in } \Omega, \ \forall \mu \in [0, 1], \tag{4.10}$$

and as $u_4(\mu) \in K$ so $u_4(\mu) \geq 0$ in Ω for all $\mu \in [0,1]$, we deduce

$$\|u_4(\mu)\|_H^2 \leq \|u_3(\mu)\|_H^2, \quad \forall \mu \in [0,1].$$

By using (4.7) we have

$$\begin{aligned}&\mu\|u_{g_1}\|_H^2 + (1-\mu)\|u_{g_2}\|_H^2 - \|u_4(\mu)\|_H^2\\&\quad = \|u_3(\mu)\|_H^2 - \|u_4(\mu)\|_H^2 + \mu(1-\mu)\|u_{g_1} - u_{g_2}\|_H^2\end{aligned}$$

which is positive for all $\mu \in [0,1]$. Finally we deduce from (4.9) that

$$\mu J(g_1) + (1-\mu)J(g_2) - J(g_3) \geq \frac{\mu(1-\mu)}{2}\big\{\|u_{g_1} - u_{g_2}\|_V^2 + M\|g_1 - g_2\|_H^2\big\} > 0 \tag{4.11}$$

for all $\mu \in]0,1[$ and for all g_1, g_2 in H. So J is a strictly convex functional, thus the uniqueness of the optimal control for the Problem (4.3) holds.

The uniqueness of the optimal control of the Problem (4.4) follows using the analogous inequalities (4.9)–(4.11) for any $h > 0$, that is

$$\begin{aligned}&\mu J_h(g_1) + (1-\mu)J_h(g_2) - J_h(g_3(\mu))\\&\quad = \frac{1}{2}\{\mu\|u_{g_1h}\|_H^2 + (1-\mu)\|u_{g_2h}\|_H^2 - \|u_{4h}(\mu)\|_H^2\}\\&\qquad + \frac{M}{2}\mu(1-\mu)\|g_1 - g_2\|_H^2\end{aligned} \tag{4.12}$$

from

$$u_{4h}(\mu) \leq u_{3h}(\mu) \quad \text{in } \Omega, \tag{4.13}$$

so we get

$$\|u_{4h}(\mu)\|_H^2 \leq \|u_{3h}(\mu)\|_H^2, \tag{4.14}$$

and obtain

$$\begin{aligned}&\mu J_h(g_1) + (1-\mu)J_h(g_2) - J_h(g_3)\\&\quad \geq \frac{\mu(1-\mu)}{2}\big\{\|u_{g_{1h}} - u_{g_{2h}}\|_V^2 + M\|g_1 - g_2\|_H^2\big\} > 0\end{aligned}$$

for all $\mu \in]0,1[$, for all $h > 0$ and for all g_1, g_2 in H. So J_h is also a strictly convex functional, thus the uniqueness of the optimal control for the Problem (4.4) holds. □

Remark 4.1 The Proposition 4.1 is automatically true (and then it is not necessary in order to study the convergence given in Theorem 4.1) when the equivalence (1.3) is verified for all g_1, g_2 in H.

Now we study the convergence of the state $u_{g_{op_h}h}$, and the optimal control g_{op_h}, when the heat transfer coefficient h on Γ_1, goes to infinity. For a given fixed $g \in H$, we have the following property which generalizes the one obtained for variational equality in [35, 36]. After that, we can study the limit $h \to +\infty$ for the general optimal control problems.

Lemma 4.2 *Let u_{g_h} the unique solution of the variational inequality* (3.6) *and u_g the unique solution of the variational inequality* (3.5), *then*

$$u_{g_h} \to u_g \quad \text{in } V \text{ strongly as } h \to +\infty,\ \forall g \in H.$$

Proof We take $v = u_g$ in (3.6) where $u = u_{g_h}$, recalling that $u_g = b$ on Γ_1 and $h > 1$, we obtain

$$\begin{aligned}
&a_1(u_{g_h} - u_g, u_{g_h} - u_g) + (h-1)\int_{\Gamma_1} (u_{g_h} - u_g)^2 ds \\
&\quad \le (g, u_{g_h} - u_g) - \int_{\Gamma_2} q(u_{g_h} - u_g)ds + \int_{\Gamma_1} b(u_{g_h} - u_g)ds - a_1(u_g, u_{g_h} - u_g) \\
&\quad \le (g, u_{g_h} - u_g) - \int_{\Gamma_2} q(u_{g_h} - u_g)ds - a(u_g, u_{g_h} - u_g). \qquad (4.15)
\end{aligned}$$

From what we deduce that $\|u_{g_h} - u_g\|_V$ and $(h-1)\|u_{g_h} - u_g\|_{L^2(\Gamma_1)}$ are bounded for all $h > 1$. So there exists $\eta \in V$ such that $u_{g_h} \rightharpoonup \eta$ weakly in V and $\eta \in K$. From (3.6) we have also

$$\begin{aligned}
&a(u_{g_h}, v - u_{g_h}) + h\int_{\Gamma_1} (u_{g_h} - b)(v - u_{g_h})ds \\
&\quad \ge (g, v - u_{g_h}) - \int_{\Gamma_2} q(v - u_{g_h})ds, \quad \forall v \in K_+,
\end{aligned}$$

taking $v \in K$ so $v = b$ on Γ_1, thus

$$a(u_{g_h}, u_{g_h}) \le a(u_{g_h}, v) - (g, v - u_{g_h}) + \int_{\Gamma_2} q(v - u_{g_h})ds, \quad \forall v \in K. \qquad (4.16)$$

Thus we can pass to the limit in (4.16), for $h \to +\infty$, to obtain

$$a(\eta, v - \eta) \ge (g, v - \eta) - \int_{\Gamma_2} q(v - \eta)ds, \quad \forall v \in K.$$

Using the uniqueness of the solution of (3.5) we get that $\eta = u_g$.

To prove the strong convergence of u_{g_h} to u_g, when $h \to +\infty$, it is sufficient to use the inequality (4.15) and the weak convergence of u_{g_h} to $\eta = u_g$ for all $g \in H$. This ends the proof. □

We give now the main result of the paper which generalizes, for optimal control problems governed by elliptic variational inequalities, the convergence result obtained in [10]. Moreover, this convergence is obtained without need of the adjoint states. We remark here the double dependence on the parameter h in the expression of state of the system $u_{g_{op_h}h}$ corresponding to the optimal control g_{op_h}.

Theorem 4.1 *Let $u_{g_{op_h}h}$, g_{op_h} and $u_{g_{op}}$, g_{op} are the states and the optimal controls defined in the problems* (4.4) *and* (4.3) *respectively. Then, we obtain the following asymptotic behavior*:

$$\lim_{h\to+\infty} \|u_{g_{op_h}h} - u_{g_{op}}\|_V = 0, \tag{4.17}$$

$$\lim_{h\to+\infty} \|g_{op_h} - g_{op}\|_H = 0. \tag{4.18}$$

Proof We have first

$$J_h(g_{op_h}) = \frac{1}{2}\|u_{g_{op_h}h}\|_H^2 + \frac{M}{2}\|g_{op_h}\|_H^2 \le \frac{1}{2}\|u_{g_h}\|_H^2 + \frac{M}{2}\|g\|_H^2, \quad \forall g \in H$$

then for $g = 0 \in H$ we obtain that

$$J_h(g_{op_h}) = \frac{1}{2}\|u_{g_{op_h}h}\|_H^2 + \frac{M}{2}\|g_{op_h}\|_H^2 \le \frac{1}{2}\|u_{0_h}\|_H^2 \tag{4.19}$$

where $u_{0_h} \in K_+$ is solution of the following elliptic variational inequality

$$a_h(u_{0_h}, v - u_{0_h}) \ge -\int_{\Gamma_2} q(v - u_{0_h})ds + h\int_{\Gamma_1} b(v - u_{0_h})ds, \quad \forall v \in K_+.$$

Taking $v = B$ with $B \in K_+$ such that $B = b$ on Γ_1, we get

$$a_1(u_{0_h}, u_{0_h}) + (h-1)\int_{\Gamma_1} (u_{0_h} - b)^2 ds \le a_1(u_{0_h}, B) + \int_{\Gamma_2} q(B - u_{0_h})ds + \int_{\Gamma_1} b(u_{0_h} - b)ds$$

thus $\|u_{0_h}\|_V$ is bounded independently of h, then from $\|u_{0_h}\|_H \le \|u_{0_h}\|_V$, we deduce that $\|u_{0_h}\|_H$ is bounded independently of h. So we deduce with (4.19) that $\|u_{g_{op_h}h}\|_H$ and $\|g_{op_h}\|_H$ are also bounded independently of h. So there exists f and ξ in H such that

$$g_{op_h} \rightharpoonup f \quad \text{in } H \text{ (weak)} \quad \text{and} \quad u_{g_{op_h}h} \rightharpoonup \xi \quad \text{in } H \text{ (weak)}. \tag{4.20}$$

Taking now $v = u_{g_{op}} \in K \subset K_+$ in (3.6) with $u = u_{g_{op_h}h}$ and $g = g_{op_h}$, we obtain

$$\begin{aligned} a_1(u_{g_{op_h}h}, u_{g_{op}} - u_{g_{op_h}h}) &+ (h-1)\int_{\Gamma_1} u_{g_{op_h}h}(u_{g_{op}} - u_{g_{op_h}h})ds \\ &\ge (g_{op_h}, u_{g_{op}} - u_{g_{op_h}h}) \\ &\quad - \int_{\Gamma_2} q(u_{g_{op}} - u_{g_{op_h}h})ds + h\int_{\Gamma_1} b(u_{g_{op}} - u_{g_{op_h}h})ds \end{aligned}$$

as $u_{g_{op}} = b$ on Γ_1 we obtain

$$a_1(u_{gop_h}h - u_{gop}, u_{gop} - u_{gop_h}h) - (h-1)\int_{\Gamma_1}(u_{gop_h}h - b)^2 ds$$
$$\geq (g_{op_h}, u_{gop} - u_{gop_h}h) - \int_{\Gamma_2} q(u_{gop} - u_{gop_h}h)ds$$
$$+ \int_{\Gamma_1} b(b - u_{gop_h}h)ds - a_1(u_{gop}, u_{gop} - u_{gop_h}h)$$

so

$$a_1(u_{gop_h}h - u_{gop}, u_{gop_h}h - u_{gop}) + (h-1)\int_{\Gamma_1}(u_{gop_h}h - b)^2 ds$$
$$\leq (g_{op_h}, u_{gop_h}h - u_{gop}) - \int_{\Gamma_2} q(u_{gop_h}h - u_{gop})ds - a(u_{gop}, u_{gop_h}h - u_{gop})$$

thus there exists a constant $C > 0$ which does not depend on h such that (as $h \to +\infty$ we can take $h > 1$):

$$\|u_{gop_h}h - u_{gop}\|_V \leq C \quad \text{and} \quad (h-1)\int_{\Gamma_1}|u_{gop_h}h - b|^2 ds \leq C,$$

then

$$u_{gop_h}h \rightharpoonup \xi \quad \text{in } V \text{ weak (in } H \text{ strong)}, \tag{4.21}$$
$$u_{gop_h}h \to b \quad \text{in } L^2(\Gamma_1) \text{ strong}, \tag{4.22}$$

and then $\xi \in K$.

Now taking $v \in K$ in (3.6) where $u = u_{gop_h}h$ and $g = g_{op_h}$ so

$$a_h(u_{gop_h}h, v - u_{gop_h}h) \geq (g_{op_h}, v - u_{gop_h}h) - \int_{\Gamma_2} q(v - u_{gop_h}h)ds$$
$$+ h\int_{\Gamma_1} b(v - u_{gop_h}h)ds$$

as $v \in K$ so $v = b$ on Γ_1, thus we obtain

$$a(u_{gop_h}h, u_{gop_h}h) + h\int_{\Gamma_1}(u_{gop_h}h - b)^2 ds \leq a(u_{gop_h}h, v) - (g_{op_h}, v - u_{gop_h}h)$$
$$+ \int_{\Gamma_2} q(v - u_{gop_h}h)ds.$$

Thus

$$a(u_{gop_h}h, u_{gop_h}h) \leq a(u_{gop_h}h, v) - (g_{op_h}, v - u_{gop_h}h) + \int_{\Gamma_2} q(v - u_{gop_h}h)ds,$$

using (4.20) and (4.21) we deduce that

$$a(\xi, v-\xi) \geq (f, v-\xi) - \int_{\Gamma_2} q(v-\xi)ds, \quad \forall v \in K,$$

so by the uniqueness of the solution of the variational inequality (3.5) we obtain that

$$u_f = \xi. \tag{4.23}$$

Now we prove that $f = g_{op}$. Indeed we have

$$\begin{aligned} J(f) &= \frac{1}{2}\|\xi\|_H^2 + \frac{M}{2}\|f\|_H^2 \\ &\leq \liminf_{h\to+\infty} \left\{ \frac{1}{2}\|u_{g_{op_h}h}\|_H^2 + \frac{M}{2}\|g_{op_h}\|_H^2 \right\} = \liminf_{h\to+\infty} J_h(g_{op_h}) \\ &\leq \liminf_{h\to+\infty} J_h(g) = \liminf_{h\to+\infty} \left\{ \frac{1}{2}\|u_{g_h}\|_H^2 + \frac{M}{2}\|g\|_H^2 \right\} \end{aligned}$$

using now the strong convergence $u_{g_h} \to u_g$ as $h \to +\infty$, $\forall\, g \in H$ (see Lemma 4.2), we obtain that

$$J(f) \leq \liminf_{h\to+\infty} J_h(g_{op_h}) \leq \frac{1}{2}\|u_g\|_H^2 + \frac{M}{2}\|g\|_H^2 = J(g), \quad \forall g \in H \tag{4.24}$$

then by the uniqueness of the optimal control problem (4.3) we get

$$f = g_{op}. \tag{4.25}$$

Now we prove the strong convergence of $u_{g_{op_h}h}$ to ξ in V, indeed taking $v = \xi$ in (3.6) where $u = u_{g_{op_h}h}$ and $g = g_{op_h}$ we get

$$\begin{aligned} a_h(u_{g_{op_h}h}, \xi - u_{g_{op_h}h}) &\geq (g_{op_h}, \xi - u_{g_{op_h}h}) - \int_{\Gamma_2} q(\xi - u_{g_{op_h}h})ds \\ &\quad + h\int_{\Gamma_1} b(\xi - u_{g_{op_h}h})ds, \end{aligned}$$

as $\xi \in K$ so $\xi = b$ on Γ_1, we obtain

$$\begin{aligned} & a_1(u_{g_{op_h}h} - \xi, u_{g_{op_h}h} - \xi) + (h-1)\int_{\Gamma_1} (u_{g_{op_h}h} - \xi)^2 ds \\ &\quad \leq (g_{op_h}, u_{g_{op_h}h} - \xi) + \int_{\Gamma_2} q(\xi - u_{g_{op_h}h})ds + a(\xi, \xi - u_{g_{op_h}h}) \end{aligned}$$

thus

$$\lambda_1 \|u_{g_{op_h}h} - \xi\|_V^2 \leq (g_{op_h}, u_{g_{op_h}h} - \xi) + \int_{\Gamma_2} q(\xi - u_{g_{op_h}h})ds + a(\xi, \xi - u_{g_{op_h}h}).$$

Using (4.21) we deduce that

$$\lim_{h\to+\infty} \|u_{g_{op_h}h} - \xi\|_V = 0,$$

and with (4.23) we deduce (4.17). Moreover, as $f \in H$, then from (4.24) with $g = f$ and (4.25) we can write

$$\begin{aligned} J(f) = J(g_{op}) &= \frac{1}{2}\|u_{g_{op}}\|_H^2 + \frac{M}{2}\|g_{op}\|_H^2 \\ &= \lim_{h\to+\infty} J_h(g_{op_h}) = \lim_{h\to+\infty}\left\{\frac{1}{2}\|u_{g_{op_h}h}\|_H^2 + \frac{M}{2}\|g_{op_h}\|_H^2\right\} \end{aligned} \tag{4.26}$$

and using (4.17) the strong convergence $u_{g_{op_h}h} \to \xi = u_f = u_{g_{op}}$ in V, we get

$$\lim_{h\to+\infty} \|u_{g_{op_h}h}\|_H = \|u_{g_{op}}\|_H, \tag{4.27}$$

thus from (4.26) and (4.27) we get

$$\lim_{h\to+\infty} \|g_{op_h}\|_H = \|g_{op}\|_H. \tag{4.28}$$

Finally

$$\lim_{h\to+\infty} \|g_{op_h} - g_{op}\|_H^2 = \lim_{h\to+\infty}\Big(\|g_{op_h}\|_H^2 + \|g_{op}\|_H^2 - 2(g_{op_h}, g_{op})\Big). \tag{4.29}$$

By the first part of (4.20) we obtain that

$$\lim_{h\to+\infty}\big(g_{op_h}, g_{op}\big) = \|g_{op}\|_H^2,$$

so from (4.28) and (4.29) we get (4.18). This ends the proof. □

Remark 4.2 Much of the recent literature on optimal control problems governed by variational inequalities (often called mathematical programs with equilibrium constraints (MPEC)) is focused on the numerical realization of stationary points to these problems. See for example recent works as e.g. [16] and their references within it. The numerical analysis of the convergence of optimal control problems governed by elliptic variational equalities [10] is given in [37] but the numerical analysis of the corresponding convergence of optimal control problems governed by elliptic variational inequalities given by Theorem 4.1 is an open problem.

Conclusions In this paper we have first established the error estimate between the convex combination $u_3(\mu) = \mu u_{g_1} + (1-\mu)u_{g_2}$ of two solutions u_{g_1} and u_{g_2} for elliptic variational inequality corresponding to the data g_1 and g_2 respectively, and the solution $u_4(\mu) = u_{g_3(\mu)}$ of the same elliptic variational inequality corresponding to the convex combination $g_3(\mu) = \mu g_1 + (1-\mu)g_2$ of the two data. This result complements and generalizes the previous one given in [7].

Using the existence and uniqueness of the solution to particular elliptic variational inequality, we consider a family of distributed optimal control problems on the internal energy g associated to the heat transfer coefficient h defined on a portion of the boundary of the domain. Using the monotony property [29] (see (4.10) and (4.13)) we can obtain the strict convexity of the cost functional (4.1) and (4.2), and the existence and uniqueness of the distributed optimal control problems (4.3) and (4.4) for any $h > 0$ holds by a different way used in [29] avoiding the conical differentiability of the cost functional. Then we prove that the optimal control g_{op_h} and its corresponding state of the system $u_{g_{op_h}h}$ are strongly convergent, when $h \to +\infty$, to g_{op} and $u_{g_{op}}$ which are respectively the optimal control and its corresponding state of the system, for a limit Dirichlet distributed optimal control problems. We obtain our results without using the notion of adjoint state (i.e. the Mignot's conical differentiability) of the optimal control problems which is a very important advantage with respect to the previous result given in [10] for elliptic variational equalities.

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