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Nonpositive curvature in p-Schatten class

Cristian Conde

Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutierrez 1150 (B1613GSX) Los Polvorines, Argentina

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ABSTRACT

We study the geometry of the set Δ_p, with $1 < p < \infty$, which consists of perturbations of the identity operator by p-Schatten class operators, which are positive and invertible as elements of $B(H)$. These manifolds have natural and invariant Finsler structures. In [C. Conde, Geometric interpolation in p-Schatten class, J. Math. Anal. Appl. 340 (2008) 920–931], we introduced the metric d_p and exposed several results about this metric space. The aim of this work is to prove that the space (Δ_p, d_p) behaves in many senses like a nonpositive curvature metric space.

1. Introduction

Let $B(H)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H, $Gl(H)$ the group of invertible elements of $B(H)$ and $Gl(H)^+$ the set of all positive elements of $Gl(H)$.

If $X \in B(H)$ is compact we denote by $\{s_j(X)\}$ the sequence of singular values of X (decreasingly ordered). For $1 \leqslant p < \infty$, let

$$\|X\|_p = \left(\sum s_j(X)^p\right)^{1/p} = \left(tr|X|^p\right)^{1/p},$$

where tr is the usual trace functional, this defines a norm on the set

$$B_p(H) = \left\{X \in B(H)\colon \ \|X\|_p < \infty\right\},$$

called the *p-Schatten class* of $B(H)$ (to simplify notation we use B_p). A reference for this subject is [8].

The Clarkson–McCarthy inequalities on B_p (see [11]) assert that for $2 \leqslant p < \infty$,

$$2\left(\|A\|_p^p + \|B\|_p^p\right) \leqslant \|A - B\|_p^p + \|A + B\|_p^p \leqslant 2^{p-1}\left(\|A\|_p^p + \|B\|_p^p\right), \tag{1}$$

and for $1 \leqslant p \leqslant 2$, the inequality is reversed.

In a fascinating paper [1], Ball, Carlen and Lieb defined that a Banach space X is q-uniformly convex with $q \geqslant 2$, if there exists a positive constant $K > 0$ such that

$$2\left(\frac{1}{K}\|v\|^q + \|w\|^q\right) \leqslant \|v - w\|^q + \|v + w\|^q,$$

for any $u, v \in X$ (it is easily to see that any Banach space q-uniformly convex is uniformly convex). In this paper Ball et al. proved that for $1 < p \leqslant 2$, B_p is 2-uniformly convex (see Proposition 3.3). For the cases $2 \leqslant p < \infty$, the Clarkson–McCarthy inequalities imply the p-uniform convexity of B_p.

E-mail addresses: cconde@ungs.edu.ar, conde@fceia.unr.edu.ar.

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On B_p we define the following norm associated with $a \in Gl(H)^+$:

$$\|X\|_{p,a} := \left\|a^{-1/2} X a^{-1/2}\right\|_p.$$

This metric is invariant under the action by $Gl(H, B_p) = \{x \in Gl(H):\ x - 1 \in B_p\}$,

$$l : Gl(H, B_p) \times \Delta_p \to \Delta_p, \qquad l_g(X) = gXg^*.$$

For each $a \in \Delta_p$, the tangent space $T_a(\Delta_p)$ can be identified with B_p^{sa}, the selfadjoint part of B_p and the geometry induced by the norm $\|.\|_{p,a}$ has a rich geometrical structure, for example uniqueness of the short curve (or geodesic) connecting two points.

In this work we expose several results about the geometrical structure of the manifold

$$\Delta_p = \{a = 1 + X:\ X \in B_p\} \subset Gl(H)^+,$$

for $1 < p < \infty$. This study relates to previous work on differential geometry of positive operators (or positive definite matrices). Mainly a series of papers [4–6] by Corach, Porta and Recht, where the geometry of the set of positive invertible elements of a C* algebra was studied. Also this study is related to classical work on the geometry of positive matrices [12]. In all these works, the authors have obtained among other properties that the distance function between two short curves is a convex function, that is if γ_1, γ_2 are geodesics then for all $t \in [0, 1]$,

$$d\big(\gamma_1(t), \gamma_2(t)\big) \leqslant (1 - t)d\big(\gamma_1(0), \gamma_2(0)\big) + td\big(\gamma_1(1), \gamma_2(1)\big).$$

The notion of convexity of the distance between short curves (with the same initial point) or the strict convexity of the power 2 of the distance are related with the notion of nonpositive curvate in metric spaces in sense of Busemann or Alexandrov respectively. For the development of the theory of nonpositively curved metric spaces, we shall consider works that have been carried out in two different directions: the works of H. Busemann and the works of A.D. Alexandrov and his collaborators. Both Busemann and Alexandrov started their works in the 1940s, and they showed that the notions of upper and lower curvature bounds make sense for a more general class of metric spaces than Riemannian manifolds, namely for "geodesic spaces." Let us briefly describe these spaces: a complete metric space (X, d) is called a *geodesic length space*, or simply a *geodesic space*, if for any two points $x, y \in X$, there exists a shortest geodesic joining them, i.e. a continuous curve such that $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$, and $d(x, y) = L^d(\gamma)$. Here, $L^d(\gamma)$ denotes the *length* of γ (respect to the metric d) and it is defined as

$$L^d(\gamma) := \sup\left\{\sum_{i=1}^{n} d\big(\gamma(t_{i-1}), \gamma(t_i)\big):\ 0 = t_0 < t_1 < \cdots < t_n = 1,\ n \in \mathbb{N}\right\}.$$

A curve $\gamma : [0, 1] \to X$ is called a *geodesic* if there exists $\epsilon > 0$ such that

$$L^d(\gamma|_{[t,t']}) = d\big(\gamma(t), \gamma(t')\big) \quad \text{whenever } |t - t'| < \epsilon.$$

Finally, a geodesic $\gamma : [0, 1] \to X$ is called a *shortest geodesic* if $L^d(\gamma) = d(\gamma(0), \gamma(1))$.

For more details on such metric spaces we refer to [9] and [2]. The ramifications of these two theories continue to grow today, especially since the rekindling of interest that was given to nonpositive curvature by M. Gromov in the 1970s. Let us briefly describe the basic underlying ideas of these works.

A geodesic space (X, d) is said to be a *Busemann nonpositive curvature space* (BNPC) if for every $p \in X$ there exists $\delta_p > 0$ such that for any $x, y, z \in B(p, \delta_p)$ and any shortest geodesic $\gamma_1, \gamma_2 : [0, 1] \to X$ with $\gamma_1(0) = \gamma_2(0) = x \in B(p, \delta_p)$ and with endpoints $\gamma_1(1), \gamma_2(1) \in B(p, \delta_p)$, we have

$$d\left(\gamma_1\left(\frac{1}{2}\right), \gamma_2\left(\frac{1}{2}\right)\right) \leqslant \frac{1}{2} d\big(\gamma_1(1), \gamma_2(1)\big).$$

A geodesic space (X, d) is said to be an *Alexandrov nonpositive curvature space* (ANPC) if for every $p \in X$ there exists $\rho_p > 0$ such that for any $x, y, z \in B(p, \rho_p)$ and any shortest geodesic $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, $\gamma(1) = z$, we have for $0 \leqslant t \leqslant 1$,

$$d^2\big(y, \gamma(t)\big) \leqslant (1 - t)d^2\big(y, \gamma(0)\big) + td^2\big(y, \gamma(1)\big) - t(1 - t)L^d(\gamma)^2.$$

This paper is organized as follows: In Section 2 we introduce a metric in the tangent spaces of Δ_p, which are r-uniformly convex with $r = \max\{2, p\}$. This strong geometric condition implies the uniqueness of short curves in the metric space (Δ_p, d_p), where d_p is the metric given by the infima of the lengths of curves joining two given points in Δ_p, measured with the Finsler metric $\|.\|_{p,a}$. This space was previously studied in [3].

In Section 3 we introduce the notion of p-uniform convexity of a metric space (see Definition 3.2) and we prove that the space Δ_p is r-uniformly convex. As a consequence of this metric condition we obtain existence and uniqueness of best approximation to convex closed subsets of Δ_p (where by a convex set K we mean a set such that any geodesic joining two elements of K is entirely contained in K).

2. The geometry of Δ_p

We shall need the following facts concerning the geometric structure of the submanifold Δ_p, which we take from [3].

The first fact states that if we measure the length of a smooth curve $\gamma(t) \subset \Delta_p$, $t \in [a,b]$, using the Finsler norm, i.e.

$$L_p(\gamma) = \int_a^b \left\| \dot{\gamma}(t) \right\|_{p,\gamma(t)} dt,$$

then the curves of the form $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$, with $a, b \in \Delta_p$, have minimal length along their paths for $t \in [0,1]$. Note that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, and thus

$$d_p(a,b) = \inf\{L_p(\gamma)\colon\ \gamma \in \Omega_{a,b}\} = L_p(\gamma_{a,b}) = \left\| \log\left(a^{-1/2}ba^{-1/2}\right) \right\|_p,$$

where we denote by

$$\Omega_{a,b} = \{\alpha : [0,1] \to \Delta_p\colon\ \alpha \text{ is a } C^1 \text{ curve},\ \alpha(0) = a \text{ and } \alpha(b) = 1\}$$

the set of smooth curves in Δ_p joining a to b.

Lemma 2.1. *(See [3, Proposition 5.2].) For all $X, Y \in B_p^{sa}$,*

$$\|Y\|_p \leqslant \left\| e^{-X/2} d\exp_X(Y) e^{-X/2} \right\| p,$$

where $d\exp_X$ denotes the differential of the exponential map at X.

We give now a different proof of the existence of minimal curves. This proof involves the uniform convexity of the tangent spaces.

For a piecewise differentiable curve $\beta : [0,1] \to B_p^{sa}$, one computes the *length* of the curve β by

$$L(\beta) = \int_0^1 \left\| \dot{\beta}(t) \right\|_p dt.$$

Let δ be a piecewise smooth curve $\delta \subset \Delta_p$. Then $\delta = e^\beta$ for a uniquely determined piecewise smooth curve $\beta = \log(\delta)$ such that $\beta \subseteq B_p^{sa}$. By the previous lemma, we get

$$\dot{\delta} = d\exp_\beta(\dot{\beta}) = \int_0^1 e^{(1-t)\beta} \dot{\beta} e^{t\beta}\, dt.$$

We begin comparing the lengths of the curves δ and β.

Theorem 2.2. *Let $\delta = e^\beta \subseteq \Delta_p$ be a piecewise smooth curve. Then $L(\beta) \leqslant L_p(\delta)$.*

Proof. Let us compute the speed of δ using Lemma 2.1:

$$\|\dot{\beta}\|_p \leqslant \left\| e^{-\frac{\beta}{2}} d\exp_\beta(\dot{\beta}) e^{-\frac{\beta}{2}} \right\|_p = \|\dot{\delta}\|_{p,\delta}. \qquad \square$$

Corollary 2.3 *(EMI property). Let $X, Y \in B_p^{sa}$. Then*

$$\|X - Y\|_p \leqslant d_p\left(e^X, e^Y\right).$$

Proof. Let $\delta \in \Omega_{e^X, e^Y}$, put $\delta = e^\beta$ as before. Then

$$\|Y - X\|_p = \left\| \beta(1) - \beta(0) \right\|_p = \left\| \int_0^1 \dot{\beta}\, dt \right\|_p \leqslant \int_0^1 \|\dot{\beta}\|_p\, dt = L(\beta) \leqslant L_p(\delta).$$

Hence

$$\|Y - X\|_p \leqslant d_p\left(e^X, e^Y\right). \qquad \square \tag{2}$$

Theorem 2.4. *Assume that, for the geometry induced by the norm $\|\cdot\|_p$, the unique short curve joining 0 to v in B_p^{sa} is the straight segment $\gamma(t) = tX$. Then $\gamma_{a,b}$ is the unique short piecewise smooth curve joining a to b in Δ_p.*

Proof. Let $\delta = e^{\beta}$ be a short, piecewise smooth curve joining 1 and e^X in Δ_p. Now $L(\beta) = \|X\|_p$. Since $L(\beta) \leqslant L_p(\delta)$, then β is a piecewise smooth curve in B_p^{sa}, joining 0 to X, with length less or equal than $\|X\|_p$, which is the length of the straight segment tX ($t \in [0,1]$) in B_p^{sa}. Then $\beta(t) = tX$, and $\delta(t) = e^{tX}$. The general case follows by the homogeneity of the metric of Δ_p. □

Remark 2.5. The hypothesis of Theorem 2.4 holds for any $p \in (1,\infty)$, since it is a simple consequence of the uniform convexity of these spaces.

We summarize in the following proposition the basic properties of the metric space Δ_p.

Proposition 2.6. *Given $a, b \in \Delta_p$ and $g \in Gl(H, B_p)$, we get*

1. $$d_p(a,b) = d_p\big(a^{-1}, b^{-1}\big).$$
2. *For all $t \in \mathbb{R}$*
$$d_p\big(a, \gamma_{a,b}(t)\big) = |t| d_p(a,b).$$
3. *Invariance of the metric under the action by $Gl(H, B_p)$,*
$$d_p(a,b) = d_p(gag^*, gbg^*).$$
4. *If $X, Y \in B_p^{sa}$ commute, we have*
$$\|X - Y\|_p = d_p\big(e^X, e^Y\big).$$
In particular on each line $\mathbb{R}X \subseteq B_p^{sa}$ the exponential map preserves distances.
5. *If 1 lies on the geodesic $\gamma_{a,b}$, then a and b commute and*
$$\log(b) = -\frac{1-t}{t}\log(a),$$
where $t = d_p(a,1)/d_p(a,b)$.
6. *Let $s \in [1,\infty)$. Then the s-energy functional*
$$E_s : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_s(\beta) := \int_0^1 \big\|\dot\beta(t)\big\|_{p,\beta(t)}^s \, dt$$
has its global minimum $d_p^s(a,b)$ precisely at $\gamma_{a,b}$.
7. *Δ_p is a complete metric space with the geodesic distance d_p.*

A midpoint map for a metric space (X,d) is a map $m : X \times X \to X$ satisfying

$$d\big(m(x,y), x\big) = \frac{1}{2} d(x,y) = d\big(m(x,y), y\big) \quad \forall x, y \in X.$$

Given $a, b \in \Delta_p$ and $\gamma_{a,b}$ the shortest curve joining them, we can define the following midpoint map

$$m : \Delta_p \times \Delta_p \to \Delta_p, \qquad m(a,b) := \gamma_{a,b}\left(\frac{1}{2}\right).$$

Definition 2.7. Let $K \subseteq \Delta_p$. We say that K is convex if, for any $a, b \in K$, $\gamma_{a,b}(t) \in K$ for any $t \in [0,1]$.

3. Clarkson's inequalities and uniform convexity

3.1. Weak semi parallelogram law

Let $(V, \langle .,. \rangle)$ be an euclidean space, i.e. V is a real vector space (finite or infinite dimensional) and $\langle .,. \rangle$ is a positive definite symmetric bilinear form on V. Then $\|v\| = \sqrt{\langle v, v \rangle}$ defines a norm on V and the parallelogram law states that for $u, v \in V$ we have

$$\|v - w\|^2 + \|v + w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big).$$

If we consider a parallelogram with vertices x, x_1, x_2 and $x_3 = x_1 + x_2 - x$, then the parallelogram law reads

$$d(x_1, x_2)^2 + d(x, x_3)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2$$

where $d(a, b) := \|a - b\|$. If $z = \frac{x_1+x_2}{2}$ is the midpoint of x_1 and x_2, then we get

$$d(x_1, x_2)^2 + 4d(x, z)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2.$$

Definition 3.1. Let (X, d) be a metric space. We say that (X, d) satisfies the *semi parallelogram law* (SPL) if for $x_1, x_2 \in X$ there exists a point $z \in X$ such that for each $x \in X$ we have

$$d(x_1, x_2)^2 + 4d(x, z)^2 \leqslant 2d(x, x_1)^2 + 2d(x, x_2)^2.$$

Note that the point z occuring in the preceding definition plays the role of a midpoint between x_1 and x_2.

The above condition can be rephrased as follows in a geodesic length space (see [9]):

For any $x \in X$ and any minimal curve $\eta : [0, 1] \to X$ with $\eta(0) = x_1$, $\eta(1) = x_2$, we have

$$d\left(x, \eta\left(\frac{1}{2}\right)\right)^2 \leqslant \frac{1}{2}d\big(x, \eta(0)\big)^2 + \frac{1}{2}d\big(x, \eta(1)\big)^2 - \frac{1}{4}d\big(\eta(0), \eta(1)\big)^2. \tag{3}$$

One natural generalization of the p-uniform convexity to a metric space is the following:

Definition 3.2. Let (X, d) be a metric space and $p \geqslant 2$. We say that (X, d) is p-uniformly convex if for any $x \in X$ and any minimal curve $\eta : [0, 1] \to X$ with $\eta(0) = x_1$, $\eta(1) = x_2$, there exists a constant $c_p > 0$ such that

$$d\left(x, \eta\left(\frac{1}{2}\right)\right)^p \leqslant \frac{1}{2}d\big(x, \eta(0)\big)^p + \frac{1}{2}d\big(x, \eta(1)\big)^p - \frac{1}{4}c_p d\big(\eta(0), \eta(1)\big)^p. \tag{4}$$

If $p = 2$ and $c_2 = 1$, then the inequality (4) corresponds to the SPL.

At this point we can easily prove the r-uniform convexity "at the origin" of Δ_p.

The proof of this fact requires some preliminaries. We begin with the following inequalities, proved in [1] by Ball, Carlen and Lieb.

Proposition 3.3. *For $A, B \in B_p(H)$ and $1 \leqslant p \leqslant 2$ it holds that*

$$\|A\|_p^2 + (p - 1)\|B\|_p^2 \leqslant \frac{1}{2}\big(\|A + B\|_p^2 + \|A - B\|_p^2\big). \tag{5}$$

Lemma 3.4. *Let $X, B \in B_p^{sa}$ and $\gamma : [0, 1] \to \Delta_p$ be the geodesic joining e^B with e^{-B}. Then for $1 < p < \infty$,*

$$d_p\left(e^X, \gamma\left(\frac{1}{2}\right)\right)^r \leqslant \frac{1}{2}\big(d_p\big(e^X, \gamma(0)\big)^r + d_p\big(e^X, \gamma(1)\big)^r\big) - \frac{1}{4}c_r d_p\big(\gamma(0), \gamma(1)\big)^r, \tag{6}$$

with $r = \max\{p, 2\}$ and $c_r = p - 1$ if $r = 2$ or $c_r = \frac{1}{2^{p-2}}$ if $r \neq 2$.

Proof. By (1), if $2 < p < \infty$, then

$$\begin{aligned} 2\big(\|X\|_p^p + \|B\|_p^p\big) &= 2\big(d_p\big(e^X, 1\big)^p + d_p\big(e^B, 1\big)^p\big) \\ &\leqslant \|X - B\|_p^p + \|X + B\|_p^p \\ &\leqslant d_p\big(e^X, e^B\big)^p + d_p\big(e^X, e^{-B}\big)^p. \end{aligned}$$

Since $d_p(e^B,1)=\frac{1}{2}d_p(e^B,e^{-B})=\frac{1}{2}L_p(\gamma)$ we have

$$\frac{1}{2^p}L_p(\gamma)^p \leqslant \frac{1}{2}\big(d_p\big(e^X,\gamma(0)\big)^p+d_p\big(e^X,\gamma(1)\big)^p\big)-d_p\left(e^X,\gamma\left(\frac{1}{2}\right)\right)^p.$$

Now, we consider the case $1<p\leqslant 2$. Applying the exponential map and using the EMI property in the inequality (5) we obtain

$$\begin{aligned}\|X\|_p^2+(p-1)\|B\|_p^2 &= d_p\big(e^X,1\big)^2+(p-1)d_p\big(e^B,1\big)^2\\ &\leqslant \frac{1}{2}\big(\|X+B\|_p^2+\|X-B\|_p^2\big)\\ &\leqslant \frac{1}{2}\big(d_p\big(e^X,e^B\big)^2+d_p\big(e^X,e^{-B}\big)^2\big).\end{aligned}$$

Since $d_p(e^B,1)=\frac{1}{2}d_p(e^B,e^{-B})=\frac{1}{2}L_p(\gamma)$, we have

$$(p-1)\frac{1}{2^2}L_p(\gamma)^2 \leqslant \frac{1}{2}\big(d_p\big(e^X,\gamma(0)\big)^2+d_p\big(e^X,\gamma(1)\big)^2\big)-d_p\left(e^X,\gamma\left(\frac{1}{2}\right)\right)^2. \qquad \square$$

The following proposition establishes the r-uniform convexity for Δ_p.

Theorem 3.5. *Let $X\in B_p^{sa}$, and $\gamma:[0,1]\to\Delta_p$ be a geodesic. Then for $1<p<\infty$,*

$$d_p\left(e^X,\gamma\left(\frac{1}{2}\right)\right)^r \leqslant \frac{1}{2}\big(d_p\big(e^X,\gamma(0)\big)^r+d_p\big(e^X,\gamma(1)\big)^r\big)-\frac{1}{4}c_r d_p\big(\gamma(0),\gamma(1)\big)^r, \tag{7}$$

with r and c_r as above.

Proof. Given $a=\gamma(0)$, $b=\gamma(1)\in\Delta_p$, let $m=m(a,b)$ be the midpoint of a and b. We claim that there exist $g\in GL(H,B_p)$ and $X\in B_p^{sa}$ with

$$l_g(a)=e^X, \qquad l_g(b)=e^{-X}.$$

First, observe that $h_1=b^{-\frac{1}{2}}$ satisfies $l_{h_1}(b)=1$. Let $x:=l_{h_1}(a)$ and we define $h_2:x^{-\frac{1}{4}}$ and $g:=h_2h_1$. Then

$$\begin{aligned}&l_g(a)=h_2h_1ah_1h_2=l_{h_2}(x)=x^{\frac{1}{2}},\\ &l_g(b)=h_2h_1bh_1h_2=l_{h_2}(1)=x^{-\frac{1}{2}}.\end{aligned}$$

Now the claim above follows with $x=e^{2X}$.

By the invariance of the distance under the action of $GL(H,B_p)$ and the claim, it suffices to verify the inequality for pairs a,b with $b=a^{-1}$. This case follows from the previous lemma. $\square$

Following [7], one can define two different notions of convexity for metric spaces.

Definition 3.6. Let (X,d) be a metric space admitting a midpoint map. (X,d) is called

1. *ball convex* if for all $x,y,z\in X$,

$$d\big(m(x,y),z\big)\leqslant \max\big\{d(x,z),d(y,z)\big\} \tag{8}$$

for any midpoint m. It is called strictly ball convex if the inequality is strict whenever $x\neq y$;
2. *distance convex* if for all $x,y,z\in X$,

$$d\big(m(x,y),z\big)\leqslant \frac{1}{2}\big[d(x,z)+d(y,z)\big], \tag{9}$$

for any midpoint map m.

Note that the condition (9) implies condition (8), and also that strict ball convexity implies the uniqueness of a midpoint map.

Now, we give the definition of uniform ball convexity of metric spaces, see [7].

Definition 3.7. Let (X,d) be a metric space admitting a midpoint map. (X,d) is called uniformly ball convex if for all $\epsilon > 0$ there exists a $\rho(\epsilon) > 0$ such that for all $x, y, z \in X$ satisfying $d(x,y) > \epsilon \max\{d(x,z), d(y,z)\}$, it holds that

$$d\big(m(x,y),z\big) \leqslant \big(1-\rho(\epsilon)\big)\max\big\{d(x,z),d(y,z)\big\}$$

for the (unique) midpoint map m.

Corollary 3.8. *The metric space (Δ_p, d_p) is strictly ball convex for $1 < p < \infty$.*

Proof. Let $a, b, c \in \Delta_p$ with $a \neq b$, then

$$d_p\left(c, \gamma_{a,b}\left(\frac{1}{2}\right)\right)^r \leqslant \frac{1}{2}\big(d_p(c,a)^r + d_p(c,b)^r\big) - \frac{1}{4}c_r d_p(a,b)^r < \big(\max\big\{d_p(c,a), d_p(c,b)\big\}\big)^r. \qquad \square$$

Corollary 3.9. *If $a, b, c \in \Delta_p$ $(1 < p < \infty)$ are three arbitrary points, then we have*

$$2^r(L_A)^r \leqslant 2^{r-1}B^r + 2^{r-1}C^r - c_r A^r$$

with $A = d_p(b,c)$, $B = d_p(c,a)$, $C = d_p(a,b)$ and L_A the length of the geodesic joining a to $m(b,c)$.

Another interesting consequence of Theorem 3.5 is the uniform convexity of (Δ_p, d_p).

Corollary 3.10. *For $1 < p < \infty$, the metric space (Δ_p, d_p) is uniformly ball convex.*

One can to compute the modulus of uniform convexity explicitly in these cases.

Consider $\epsilon > 0$, set $d_p(a,b), d_p(a,c) \leqslant s$ and $d_p(b,c) \geqslant \epsilon s$. Then,

$$d_p\big(a, m(b,c)\big)^r \leqslant s^r - \frac{1}{4}c_r d_p(b,c)^r \leqslant s^r - \frac{1}{4}c_r(\epsilon s)^r = \left[1 - c_r\left(\frac{\epsilon}{2^{2/r}}\right)^r\right]s^r.$$

Then an admissible value for the modulus of uniform convexity is $\rho_{\Delta_p}(\epsilon) = 1 - [1 - c_r(\frac{\epsilon}{2^{2/r}})^r]^{1/r}$.

Note that the formula of the modulus is similar with the one obtained by Clarkson for the space L_p.

Proposition 3.11. *Let $X \in B_p^{sa}$, $\gamma : [0,1] \to \Delta_p$ a geodesic and $1 < p \leqslant 2$. Then for all $t \in [0,1]$,*

$$d_p\big(e^X, \gamma(t)\big)^r \leqslant (1-t)d_p\big(e^X, \gamma(0)\big)^r + t d_p\big(e^X, \gamma(1)\big)^r - t(1-t)c_r d_p\big(\gamma(0), \gamma(1)\big)^r. \tag{10}$$

Proof. Let us denote $W_2(t) = t(1-t)$. Given any geodesic $\gamma : [0,1] \to \Delta_p$, it suffices to prove the previous inequality for all dyadic $t \in [0,1]$. It obviously holds for $t = 0$ and $t = 1$. Assume that it holds for all $t = k2^{-n}$ with $k = 0, 1, \dots, 2^n$. We want to prove that (10) also holds for all $t = k2^{-(n+1)}$ with $k = 0, 1, \dots, 2^{n+1}$. For k even this is clear. Fix $t = k2^{-(n+1)}$ with k odd; and put $\Delta t = 2^{-(n+1)}$. Then by (7)

$$d_p\big(e^X, \gamma(t)\big)^r \leqslant \frac{1}{2}\big(d_p\big(e^X, \gamma(t-\Delta t)\big)^r + d_p\big(e^X, \gamma(t+\Delta t)\big)^r\big) - \frac{1}{4}c_r d_p\big(\gamma(t-\Delta t), \gamma(t+\Delta t)\big)^r. \tag{11}$$

By the assumption for multiples of 2^{-n},

$$d_p\big(e^X, \gamma(t \pm \Delta t)\big)^r \leqslant (1 - t \mp \Delta t)d_p\big(e^X, \gamma(0)\big)^r + (t \pm \Delta t)d_p\big(e^X, \gamma(1)\big)^r - W_2(t \pm \Delta t)c_r d_p\big(\gamma(0), \gamma(1)\big)^r.$$

Thus, by (11)

$$d_p\big(e^X, \gamma(t)\big)^r \leqslant (1-t)d_p\big(e^X, \gamma(0)\big)^r + t d_p\big(e^X, \gamma(1)\big)^r - \big[g(t, \Delta t)\big]c_r d_p\big(\gamma(0), \gamma(1)\big)^r,$$

where $g(t, \Delta t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t)$.

Since

$$W_2(t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t) = g(t, \Delta t), \tag{12}$$

then

$$d_p\big(e^X, \gamma(t)\big)^r \leqslant (1-t)d_p\big(e^X, \gamma(0)\big)^r + t d_p\big(e^X, \gamma(1)\big)^r - W_2(t)c_r d_p\big(\gamma(0), \gamma(1)\big)^r. \qquad \square$$

Corollary 3.12. *Let $\gamma, \eta : [0,1] \to \Delta_p$, $1 < p \leqslant 2$ and $t \in [0,1]$, then*

$$d_p\big(\eta(t),\gamma(t)\big)^r \leqslant (1-t)^2 d_p\big(\eta(0),\gamma(0)\big)^r + t^2 d_p\big(\eta(1),\gamma(1)\big)^r - t(1-t)c_r\big(L_p(\eta)^r + L_p(\gamma)^r\big) + t(1-t)\big[d_p\big(\eta(0),\gamma(1)\big)^r + d_p\big(\eta(1),\gamma(0)\big)^r\big].$$

Proof. Applying (10) twice, we obtain that

$$\begin{aligned} d_p\big(\eta(t),\gamma(t)\big)^r &\leqslant (1-t)d_p\big(\eta(0),\gamma(t)\big)^r + t d_p\big(\eta(1),\gamma(t)\big)^r - t(1-t)c_r L_p(\eta)^r \\ &\leqslant (1-t)\big[(1-t)d_p\big(\eta(0),\gamma(0)\big)^r + t d_p\big(\eta(0),\gamma(1)\big)^r - t(1-t)c_r L_p(\gamma)^r\big] \\ &\quad + t\big[(1-t)d_p\big(\eta(1),\gamma(0)\big)^r + t d_p\big(\eta(1),\gamma(1)\big)^r - t(1-t)c_r L_p(\gamma)^r\big] - t(1-t)c_r L_p(\eta)^r \\ &= (1-t)^2 d_p\big(\eta(0),\gamma(0)\big)^r + t^2 d_p\big(\eta(1),\gamma(1)\big)^r - t(1-t)c_r\big(L_p(\eta)^r + L_p(\gamma)^r\big) \\ &\quad + t(1-t)\big[d_p\big(\eta(0),\gamma(1)\big)^r + d_p\big(\eta(1),\gamma(0)\big)^r\big]. \qquad \square \end{aligned}$$

In particular if $p = r = 2$, the metric space (Δ_2, d_2) is an Alexandrov nonpositive curvature space and we get

$$d_2\big(\eta(t),\gamma(t)\big) < (1-t)d_2\big(\eta(0),\gamma(0)\big) + t d_2\big(\eta(1),\gamma(1)\big),$$

or equivalently, d_2 is strictly convex on geodesics.

Now, we try to extend this result for $p > 1$ with $p \neq 2$. Recently, Larotonda [10] using the theory of dissipative operators and the theory of entire functions, derived several operator inequalities for unitarily invariant norms. Among them, if $X, Y \in B_p^{sa}$,

$$\big\|\log\big(e^{\frac{-t}{2}X} e^{tY} e^{\frac{-t}{2}X}\big)\big\|_p \leqslant t\big\|\log\big(e^{\frac{-X}{2}} e^{Y} e^{\frac{-X}{2}}\big)\big\|_p. \tag{13}$$

This inequality establishes the convexity of the geodesic distance d_p in the Finsler manifold Δ_p, that is:

Proposition 3.13. *Let $a, b, c, d \in \Delta_p$, then*

$$d_p\big(\gamma_{a,b}(t),\gamma_{c,d}(t)\big) \leqslant t d_p(a,c) + (1-t)d_p(b,d). \tag{14}$$

Proof. Consider the geodesic rectangle with vertices a, b, c, d. Let $\gamma_{c,b}$ be the short curve joining c to b in Σ_Φ, and consider the triangle with sides c, b, d, and the geodesic triangle with sides b, a, c. Note that $\gamma_{c,b}(t) = \gamma_{b,c}(1-t)$ and the same holds for $\gamma_{a,b}$. Then, by the triangle inequality

$$d_p\big(\gamma_{a,b}(t),\gamma_{c,d}(t)\big) \leqslant d_p\big(\gamma_{a,b}(t),\gamma_{c,b}(t)\big) + d_p\big(\gamma_{c,b}(t),\gamma_{c,d}(t)\big),$$

and by (13)

$$d_p\big(\gamma_{c,b}(t),\gamma_{c,d}(t)\big) \leqslant t\, d_p(b,d).$$

Also

$$d_p\big(\gamma_{b,c}(1-t),\gamma_{b,a}(1-t)\big) \leqslant (1-t)\, d_p(a,b).$$

Adding these two inequalities yields the convexity of d_p. $\square$

3.2. Best approximation

Given a subset $K \subseteq \Delta_p$ and an element $a \in \Delta_p$, put

$$d_p(a,K) = \inf\big\{d_p(a,k):\ k \in K\big\}.$$

In Theorem 3.14, we shall prove that, as in a Hilbert space, one can define a metric projection onto convex closed subsets of Δ_p. In other words given K a convex closed subset of Δ_p and $a \in \Delta_p$, there is a unique $k_0 \in K$ such that the length of the geodesic joining a with k_0 is the distance between a and K. That is, there is a unique solution to the minimization problem

$$\begin{cases} k_0 \in K, \\ d_p(a,k_0) \leqslant d_p(a,k) \quad \forall k \in K. \end{cases} \tag{15}$$

Theorem 3.14 *(Best approximation). Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and $a \in \Delta_p$. Then the problem* (15) *has a unique solution. In other words, there is a unique $\pi_K(a) = k_0 \in K$ such that $d_p(a, q_0) = d_p(a, K)$. In addition, if $\tilde{a}$ belongs to the geodesic segment $[a, \pi_K(a)]$, then $\pi_K(\tilde{a}) = \pi_K(a)$.*

Proof. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence in K, such that $d_p(a, k_n) \to d_p(a, K)$. By Theorem 3.5 we obtain that

$$\begin{aligned}\frac{1}{4}c_r d_p(k_n, k_m)^r &\leqslant \frac{1}{2}\big(d_p(k_n, a)^r + d_p(a, k_m)^r\big) - d_p(a, k_{n,m})^r \\ &\leqslant \frac{1}{2}\big(d_p(k_n, a)^r + d_p(a, k_m)^r\big) - d_p(a, K)^r,\end{aligned} \tag{16}$$

where $k_{n,m} = \gamma(\frac{1}{2}) \in K$, with γ the geodesic joining k_n and k_m.

This implies that $\{k_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in K, hence convergent to some $k_0 \in K$. Since K is closed, $k_0 \in K$. By the continuity of the distance we have

$$d_p(k_0, a) = \lim_{n\to\infty} d_p(k_n, a) = d_p(a, K).$$

For the uniqueness part, let $k_1, k_2 \in K$ such that

$$d_p(k_1, a) = d_p(a, K) = d_p(k_2, a).$$

Replacing k_n and k_m by k_1 and k_2 respectively in (16), we obtain

$$d_p(a, k_{1,2})^r \leqslant \frac{1}{2}\big(d_p(k_1, a)^r + d_p(a, k_2)^r\big) - \frac{1}{4}c_r d_p(k_1, k_2)^r = d_p(a, K)^r - \frac{1}{4}c_r d_p(k_1, k_2)^r,$$

since $k_{1,2} \in K$, the above inequality proves that $d_p(k_1, k_2) = 0$. □

Definition 3.15. Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and $a \in \Delta_p$. By Theorem 3.14 there is exactly one point $\pi_K(a) \in K$ such that

$$d_p\big(a, \pi_K(a)\big) = d_p(a, K).$$

Then $\pi_K(a)$ is called the projection of a to K. The map $\pi_K : \Delta_p \to K$ is called the projection map to K.

Remark 3.16.

1. We remark that we have proved the existence of a unique projection without assuming any kind of compactness for K.
2. Clearly $\pi_K^2 = \pi_K$.

Theorem 3.17. *Let $K \subseteq \Delta_p$ be a convex closed set, $1 < p < \infty$ and π_K the projection map onto K. Then π_K is continuous.*

Proof. Let the sequence $\{c_n\}$ converge to c in Δ_p. For simplicity, denote $\pi_K(c_n)$ by u_n. Now $\{u_n\}$ is a Cauchy sequence in K, otherwise there are positive numbers ϵ and subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ such that $n_k < m_k$ and $d_p(u_{n_k}, u_{m_k}) \geqslant \epsilon$ for all k. Put $a_k = u_{n_k}$, $b_k = u_{m_k}$ and $M_k = \max\{d_p(c, a_k), d_p(c, b_k)\}$.

Note that $M_k \to d_p(c, K)$ as $k \to \infty$. Now $d_p(c, a_k) \leqslant M_k, d_p(c, b_k) \leqslant M_k$ and $d_p(a_k, b_k) \geqslant (\frac{\epsilon}{M_k})M_k$. This implies

$$d_p\big(c, m(a_k, b_k)\big) \leqslant M_k\left(1 - \rho_{\Delta_p}\left(\frac{\epsilon}{M_k}\right)\right) \leqslant M_k\left(1 - \rho_{\Delta_p}\left(\frac{d(a_k, b_k)}{M_k}\right)\right).$$

Also $\rho_{\Delta_p}(\frac{\epsilon}{M_k}) \leqslant 1 - \frac{d_p(c,K)}{M_k}$, letting $k \to \infty$, one has $\delta_{\Delta_p}(\frac{\epsilon}{M_k}) \to 0$ and ϵ cannot be positive. Thus $\{\pi_K(c_n)\}$ is a Cauchy sequence in K and therefore converges to a point z in K, as $d_p(c, z) = d_p(c, K)$, then $z = \pi_K(c)$. □

Another useful property of the Δ_p spaces with $1 < p \leqslant 2$ is the following Pythagoras type inequality.

Corollary 3.18. *Under the same conditions stated above, we have that for all $k \in K$ with K a closed and convex set and $t \in (0, 1]$,*

$$d_p\big(a, \pi_K(a)\big)^2 + (1 - t)(p - 1)d_p\big(\pi_K(a), k\big)^2 \leqslant d_p(a, k)^2, \tag{17}$$

in particular

$$d_p\big(a, \pi_K(a)\big)^2 + (p - 1)d_p\big(\pi_K(a), k\big)^2 \leqslant d_p(a, k)^2. \tag{18}$$

Proof. Let $\gamma : [0,1] \to \Delta_p$ be the geodesic joining $\gamma(0) = \pi_K(a)$ and $\gamma(1) = k$, then $\gamma(t) \in K$ by the convexity of K. Hence, by Proposition 3.11

$$d_p\big(a, \pi_K(a)\big)^2 \leqslant d_p\big(a, \gamma(t)\big)^2 \leqslant (1-t)d_p\big(a, \pi_K(a)\big)^2 + td_p(a,k)^2 - t(1-t)(p-1)d_p\big(\pi_K(a), k\big)^2,$$

and therefore

$$td_p\big(a, \pi_K(a)\big)^2 \leqslant td_p(a,k)^2 - t(1-t)(p-1)d_p\big(\pi_K(a), k\big)^2.$$

Now if $t \in (0,1]$, this is the desired inequality. □

We shall prove now that the inequality (18) characterizes solutions of the minimization problem.

Theorem 3.19. *Let $K \subseteq \Delta_p$ be a convex closed subset and $q \in \Delta_p$ with $1 < p \leqslant 2$. Suppose that $q_0 \in K$ verifies* (18)*, then q_0 is the unique solution of* (15).

Proof. For all $k \in K$ and $t \in (0,1]$ we have

$$d(q,q_0)^2 + (p-1)d(q_0,k)^2 \leqslant d(q,k)^2.$$

Then $d(q,q_0) \leqslant d(q,k)$. For the uniqueness part, let $q_0, q_1 \in K$ satisfying (18), then

$$(p-1)d(q_1,q_0)^2 \leqslant d(q,q_1)^2 - d(q,q_0)^2 = d(q,K)^2 - d(q,K)^2. \qquad \square$$

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