Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 302 (2006) 848–861
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Stone duality for Dedekind σ-complete ℓ-groups with order-unit ☆

Roberto Cignoli [a], Daniele Mundici [b,*]

[a] *Instituto Argentino de Matemática, CONICET, Saavedra 15, Piso 3, C1083ACA Buenos Aires, Argentina*
[b] *Department of Mathematics "Ulisse Dini", University of Florence, Viale Morgagni 67/A, 50134 Florence, Italy*

Received 4 March 2005
Available online 30 June 2006
Communicated by Michel Broué

Abstract

Building on the Goodearl–Handelman–Lawrence functional representation theorem, we provide a purely topological representation (specifically, a categorical duality) for a large class of Dedekind σ-complete ℓ-groups G with order-unit u, including all G where u has a finite index of nilpotence. Our duality is a far-reaching generalization of the well-known Stone duality between σ-complete boolean algebras and basically disconnected compact Hausdorff spaces.

Keywords: Lattice-ordered abelian group; Dedekind σ-complete group; Basically disconnected space; Stone duality; Regular biregular ring; Bounded index of nilpotence; σ-Complete MV-algebra

1. Introduction

In this paper by an ℓ-group we understand a lattice-ordered *abelian* group. An *order-unit*[1] of an ℓ-group G is an element $u \in G$ such that for every $a \in G$ there is an $n_a \in \mathbb{N}$ satisfying

$$-n_a u \leqslant a \leqslant n_a u. \tag{1.1}$$

☆ Partially supported by Project of Scientific and Technological Cooperation between Argentina and Italy, on "Many-valued Logic", Code 33I.
* Corresponding author.
E-mail addresses: cignoli@dm.uba.ar (R. Cignoli), mundici@math.unifi.it (D. Mundici).
[1] "Unité forte" in [1].

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.05.029

Following [4], we say that an ℓ-group G is *Dedekind σ-complete* (or a *$\sigma\ell$-group* for short), if every non-empty countable subset of G which is bounded above in G has a supremum in G.

As shown in [6, Section II, 13-14] the class of Dedekind σ-complete ℓ-groups with order-unit arises via the Grothendieck functor K_0 from an interesting class of $\aleph_0$-continuous regular rings.

Further, in [8, Theorem 3.9] a categorical equivalence Γ is established between lattice-ordered abelian groups with order-unit, and MV-algebras. In particular, by [8, Theorem 2.5] Dedekind σ-complete ℓ-groups with order-unit correspond via Γ to σ-complete MV-algebras: the latter are an interesting generalization of σ-complete boolean algebras (see, e.g., [9] and references therein).

Let $\mathbb{DED}$ denote the category whose objects are Dedekind σ-complete ℓ-groups with a distinguished order-unit, and whose morphisms are the unit preserving ℓ-group homomorphisms that also preserve all denumerable infima and suprema.

A fundamental representation theorem by Goodearl, Handelman and Lawrence (Theorem 2.3 below) shows that, up to isomorphism, every object $(G,u) \in \mathbb{DED}$ is an ℓ-group of continuous real-valued functions over the space $X = \partial_e S(G,u)$ of extremal states of G, the constant function 1 being the order-unit u. The space X can be equivalently characterized as the (Stone) maximal ideal space of the boolean algebra of characteristic elements of G (see [4, 8.14]).

Letting (G,u) be so represented, for each $x \in X$ let us denote by $G(x)$ the set of possible values of functions in G at x. In symbols,

$$G(x) = \{f(x)\colon\ f \in G\} = G/x.$$

It is well known [6, 1.4.7] that $G(x)$ either coincides with the set of all real numbers, or else for some integer $n_x \geqslant 1$ (called the rank of x), $G(x)$ coincides with the set of integer multiples of $1/n_x$. In this latter case x is said to be *discrete*. Assuming the finite ranks n_x to be bounded by some fixed n only depending on (G,u), we obtain the subcategory $\mathbb{BFR} \subseteq \mathbb{DED}$ of Dedekind σ-complete ℓ-groups with order unit of *bounded finite rank*.

The class of objects of $\mathbb{BFR}$ strictly includes all Dedekind σ-complete ℓ-groups (G,u) with order unit *of finite index*. By [5, Theorem 4.4] (also see the main result in [3]) any such (G,u) arises as $(K_0(R), [R])$ for some regular, biregular ring with bounded index (of nilpotence).[2]

In Theorem 4.2 below we establish an adjunction between $\mathbb{DED}^{\mathrm{op}}$ and a category $\mathbb{W}$ consisting of basically disconnected (always compact Hausdorff) spaces equipped with a system of closed sets satisfying very simple axioms (see Definitions 3.1 and 3.6 below). By restriction, this adjunction gives a duality between $\mathbb{BFR}$ and the full subcategory of $\mathbb{W}$ whose objects are basically disconnected compact Hausdorff spaces equipped with a *finite* system of closed sets satisfying the mentioned conditions.

For the very particular subcategory of $\mathbb{BFR}$ obtained when the rank of $G(x)$ is constantly equal to 1, our duality amounts to composition of the Γ functor with the Stone duality between σ-complete boolean algebras and basically disconnected spaces [10, Sections 8 and 22].

We refer to [1,4] for background on ℓ-groups, and to [7] for all the notions and results from category theory used in the paper.

Notation. Throughout this paper, $\mathbb{R}$ will denote the set of real numbers as well as the additive group of the real numbers with the usual order (and topology). $\mathbb{Z}$ will denote the set of integers

[2] Satisfying suitable additional continuity properties.

and $\mathbb{N}$ the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. We denote by $\frac{1}{n}\mathbb{Z}$ the subset of $\mathbb{R}$ given by the integer multiples of $\frac{1}{n}$. When the space X is clear from the context, $\overline{S}$ denotes the closure of S in X, and $\operatorname{int}(S)$ denotes its interior.

2. Dedekind σ-complete ℓ-groups with order-unit

Given a compact Hausdorff space X, we denote by $\mathrm{Cont}(X, \mathbb{R})$ the ℓ-group of real-valued continuous functions on X, with the pointwise operations. All the sub-ℓ-groups of $\mathrm{Cont}(X, \mathbb{R})$ considered in this paper have the constant function 1 as a distinguished order-unit.

Given an ℓ-group G with order-unit u, we denote by $\mathcal{X}(G, u)$ the set of all ℓ-group homomorphisms $\chi : G \to \mathbb{R}$ such that $\chi(u) = 1$. It is well known that the map $\chi \mapsto \mathrm{Ker}(\chi) = \{a \in G\colon\ \chi(a) = 0\}$ is a bijective correspondence between $\mathcal{X}(G, u)$ and the set of maximal ℓ-ideals of G. Thus in particular, $\mathcal{X}(G, u)$ is non-empty. From (1.1) it follows that $\mathcal{X}(G, u)$ is a closed subspace of the product $\prod_{a \in G}[-n_a, n_a]$, where the segments $[-n_a, n_a]$ are endowed with the usual topology inherited from $\mathbb{R}$. Consequently $\mathcal{X}(G, u)$ with the restriction topology is a compact Hausdorff space. Its topology is said to be the *natural topology* of $\mathcal{X}(G, u)$. For each $a \in G$ let the function $\hat{a} : \mathcal{X}(G, u) \to \mathbb{R}$ be defined by

$$\hat{a}(\chi) = \chi(a). \tag{2.2}$$

Then the map $a \mapsto \hat{a}$ is an ℓ-group isomorphism of G into $\mathrm{Cont}(\mathcal{X}(G, u), \mathbb{R})$ sending u to the constant function 1.

A sub-ℓ-group G of $\mathrm{Cont}(X, \mathbb{R})$ is said to be *separating* if for any two distinct points $x, y \in X$, there is $c \in G$ such that $c(x) \neq c(y)$. Given a compact Hausdorff space X and a separating sub-ℓ-group G of $\mathrm{Cont}(X, \mathbb{R})$ containing the constant function 1, let the evaluation map $\upsilon : X \to \mathcal{X}(G, 1)$ be defined for each $x \in X$ and each $c \in G$ by the stipulation

$$\upsilon(x)(c) = c(x). \tag{2.3}$$

It is well known that the correspondence $x \mapsto \{c \in G\colon\ c(x) = 0\}$ is a bijection from X onto $\mathcal{X}(G, 1)$ (see, for instance, [2, pp. 351–352]). From this it follows easily that υ is a homeomorphism of X onto $\mathcal{X}(G, 1)$.

In what follows we will only consider $\sigma\ell$-groups with order-unit, and (G, u) will always denote a $\sigma\ell$-group G with a distinguished order-unit u. By a *sub-$\sigma\ell$-group* S of a $\sigma\ell$-group G we understand a sub-ℓ-group S of G having the following property: whenever a denumerable subset of S has a supremum s in G then s belongs to S.

Let X be a compact Hausdorff space. From [4, Corollary 9.3] it follows that the ℓ-group $\mathrm{Cont}(X, \mathbb{R})$ is Dedekind σ-complete if and only if X is *basically disconnected*, i.e., the closure of every open F_σ-set in X is open.[3]

Notation. When dealing with sequences $\{c_i\}$ in $\mathrm{Cont}(X, \mathbb{R})$, for X a basically disconnected compact Hausdorff space, we shall denote by $\bigvee_{i \in \mathbb{N}} c_i$ their supremum in $\mathrm{Cont}(X, \mathbb{R})$, and by $\bigwedge_{i \in \mathbb{N}} c_i$ their infimum in $\mathrm{Cont}(X, \mathbb{R})$. As is well known, $\bigvee_{i \in \mathbb{N}} c_i$ need not coincide with the pointwise supremum of the c_i's, and $\bigwedge_{i \in \mathbb{N}} c_i$ need no coincide with their pointwise infimum.

[3] As is well known, basically disconnected compact Hausdorff spaces are precisely the dual (Stone) spaces of σ-complete boolean algebras.

By [6, 1.4.3–1.4.6], $\mathcal{X}(G,u)$ coincides with the set $\partial_{\mathrm{e}}S(G,u)$ of extremal states of (G,u),

$$\mathcal{X}(G,u) = \partial_{\mathrm{e}}S(G,u). \tag{2.4}$$

To maintain the notation of [4], we shall write $\partial_{\mathrm{e}}S(G,u)$ instead of $\mathcal{X}(G,u)$ whenever G is a $\sigma\ell$-group. By [4, 8.11, 9.9, 9.10] *the space $\partial_{\mathrm{e}}S(G,u)$ with its natural topology is a basically disconnected compact Hausdorff space*.

Given a natural number n and $\chi \in \partial_{\mathrm{e}}S(G,u)$ we say that χ *is of rank* n, and we write $\mathrm{rank}(\chi) = n$, if $\chi(G) = \frac{1}{n}\mathbb{Z}$. If there is at least one χ of rank n we denote by $F_n(G,u)$ (abbreviated $F_n(G)$ if there is no danger of confusion) the set of points $\chi \in \partial_{\mathrm{e}}S(G,u)$ such that $\mathrm{rank}(\chi)$ is a divisor of n. If no such point exists, then by definition $F_n(G,u) = \emptyset$.

Let $\mathrm{Disc}(G,u) = \bigcup_{n\in\mathbb{N}} F_n(G,u)$. Then the elements of $\mathrm{Disc}(G,u)$ are said to be *discrete*. If $\chi \in \partial_{\mathrm{e}}S(G,u) \setminus \mathrm{Disc}(G,u)$, then χ is said to have *infinite rank*, and we write $\mathrm{rank}(\chi) = \infty$. The *numerical spectrum* $\mathrm{NumSpec}(G,u)$ is defined by $\mathrm{NumSpec}(G,u) = \{n \in \mathbb{N}\colon\ F_n(G,u) \neq \emptyset\}$, ordered by divisibility. If $\mathrm{NumSpec}(G,u)$ is finite we say that (G,u) has *bounded finite ranks*; we do not exclude the case $\mathrm{NumSpec}(G,u) = \emptyset$, i.e., as proved in [6, 1.4.7], $\chi(G) = \mathbb{R}$ for all $\chi \in \partial_{\mathrm{e}}S(G,u)$.

Remark 2.1. The set $\mathrm{NumSpec}(G,u)$ inherits the divisibility order of the natural numbers, but not necessarily the divisibility lattice structure: as a matter of fact, it may happen that two elements of $\mathrm{NumSpec}(G,u)$ have neither meet, nor join in $\mathrm{NumSpec}(G,u)$, or that they have a meet in $\mathrm{NumSpec}(G,u)$ different for their greatest common divisor, or a join different from their least common multiple. Trivial examples G of these situations are easily given by direct products $G = \frac{1}{n_1}\mathbb{Z} \times \frac{1}{n_2}\mathbb{Z} \times \frac{1}{n_3}\mathbb{Z}$ for suitably chosen natural numbers n_1, n_2 and n_3.

Definition 2.2. For S a subset of $\partial_{\mathrm{e}}S(G,u)$, a function $f \in \mathrm{Cont}(\partial_{\mathrm{e}}S(G,u), \mathbb{R})$ is said to be *compatible with S* (for short, *S-compatible*), if for each $\chi \in S$, $f(\chi) \in \chi(G)$.

The following theorem of Goodearl, Handelman and Lawrence [6, Proposition 1.4.7, Theorem 1.9.4], [4, 9.12–9.15] plays a fundamental role in the theory of $\sigma\ell$-groups with order-unit:

Theorem 2.3. *Let G be a $\sigma\ell$-group with order-unit u. Then the map $a \mapsto \hat{a}$ given by* (2.2) *defines an embedding $\eta_{(G,u)} : G \to \mathrm{Cont}(\partial_{\mathrm{e}}S(G,u), \mathbb{R})$ that preserves denumerable infima and suprema.*

The image $\eta_{(G,u)}(G)$ is the sub-$\sigma\ell$-group $\hat{G}$ of $\mathrm{Cont}(\partial_{\mathrm{e}}S(G,u), \mathbb{R})$ *given by all real-valued continuous functions on $\partial_{\mathrm{e}}S(G,u)$ that are simultaneously $F_n(G)$-compatible for all $n \in \mathbb{N}$.*

Remark 2.4. *For each $n \in \mathbb{N}$ the set $F_n(G)$ is closed.* Indeed, if $F_n(G) \neq \emptyset$, then $F_n(G) = \bigcap_{a\in G} \hat{a}^{-1}(\frac{1}{n}\mathbb{Z})$. One now recalls that each function $\hat{a}$ is continuous and that $\frac{1}{n}\mathbb{Z}$ is a closed subset of $\mathbb{R}$.

Our next aim is to establish a topological characterization of the sets $F_n(G)$. To this purpose we give the following definition.

Definition 2.5. Let X be a basically disconnected compact Hausdorff space. We say that a set $F \subseteq X$ is *special* if for each denumerable sequence $\{C_i\}_{i\in\mathbb{N}}$ of clopen subsets of $X \setminus F$, $\overline{\bigcup_{i\in\mathbb{N}} C_i} \cap F = \emptyset$.[4]

Lemma 2.6. *Let G be a $\sigma\ell$-group with order-unit u. Fix $n \in \mathbb{N}$. Then $F_n(G)$ is a special closed set.*

Proof. Write for short $X = \partial_e S(G,u)$ and $F_n = F_n(G,u)$. We have already noted that F_n is closed. Trivially, $\emptyset$ is a special closed set. So suppose $F_n \neq \emptyset$, and let $\{C_j\}_{j\in\mathbb{N}}$ be a sequence of clopen sets in $X \setminus F_n$ such that there is $\zeta \in \overline{\bigcup_{j\in\mathbf{N}} C_j} \cap F_n$ (absurdum hypothesis). Without loss of generality we can suppose that the C_j are pairwise disjoint.

By hypothesis for each $j \in \mathbb{N}$ and $\chi \in C_j$, $\operatorname{rank}(\chi)$ is not a divisor of n. Whenever $r \in \mathbb{N}$ is not a divisor of n the rational number

$$d_r = \inf_{p\in\mathbb{Z}}\left\{\left|\frac{1}{r} - \frac{p}{n}\right|\right\} = \min_{p\in\mathbb{Z}}\left\{\left|\frac{1}{r} - \frac{p}{n}\right|\right\}$$

is > 0. Define $K \subseteq \mathbb{N}$ and d, e in $\mathbb{Q}$ by

$$K = \{r \in \mathbb{N}\colon\ r \leqslant 10n \text{ and } r \text{ does not divide } n\},$$
$$d = \min\{d_r\}_{r\in K}, \qquad e = \min\left\{\frac{d}{3}, \frac{1}{10n}\right\}.$$

Claim. *Fix $j \in \mathbb{N}$. Then there is a clopen neighborhood $U_\chi \subseteq C_j$ of χ and an element $a_\chi \in G$ such that $\widehat{a_\chi}(U_\chi) \cap [\frac{p}{n} - e, \frac{p}{n} + e] = \emptyset$ for all $p \in \mathbb{Z}$.*

To prove the claim, note that in case $\operatorname{rank}(\chi) = r \leqslant 10n$, by the definition of rank and Theorem 2.3, there is $a_\chi \in G$ such that $\widehat{a_\chi}(\chi) = \frac{1}{r}$. We choose the clopen $U_\chi \ni \chi$ such that $U_\chi \subseteq C_j$ and

$$\widehat{a_\chi}(U_\chi) \subseteq \left[\frac{1}{r} - \frac{d}{3}, \frac{1}{r} + \frac{d}{3}\right].$$

In case $\operatorname{rank}(\chi) = \infty$, we let $a_\chi \in G$ satisfy $\widehat{a_\chi} = \frac{1}{2n}$, and choose the clopen $U_\chi \ni \chi$ in such a way that $U_\chi \subseteq C_j$ and

$$\widehat{a_\chi}(U_\chi) \subseteq \left[\frac{1}{2n} - \frac{1}{10n}, \frac{1}{2n} + \frac{1}{10n}\right].$$

Finally, in case $\operatorname{rank}(\chi) = r > 10n$, by taking a suitable multiple of an element $c \in G$ with $\hat{c}(\chi) = \frac{1}{r}$, we can obtain an element $a_\chi \in G$ and a clopen $U_\chi \ni \chi$ such that $U_\chi \subseteq C_j$ and

$$\widehat{a_\chi}(U_\chi) \subseteq \left[\frac{1}{2n} - \frac{1}{10n}, \frac{1}{2n} + \frac{1}{10n}\right].$$

[4] Compare with [10, §21.6].

This completes the proof of the claim.

Since C_j is compact, there are finitely many elements $\chi(j1), \ldots, \chi(jz_j)$ in C_j together with a cover of C_j by clopens $U_{\chi(j1)}, \ldots, U_{\chi(jz_j)}$, and elements $a_{\chi(j1)}, \ldots, a_{\chi(jz_j)}$ in G such that for all $t = 1, \ldots, z_j$,

$$\widehat{a_{\chi(jt)}}(U_{\chi(jt)}) \cap \left[\frac{p}{n} - e, \frac{p}{n} + e\right] = \emptyset \quad \text{for all } p \in \mathbb{Z}. \tag{2.5}$$

We now modify the sets $U_{\chi(jt)}$ and the elements $a_{\chi(jt)}$ as follows: we first cut down $U_{\chi(jt)}$ to $U'_{\chi(jt)} \subseteq U_{\chi(jt)}$ so that $U'_{\chi(j1)}, \ldots, U'_{\chi(jz_j)}$ form a clopen partition of C_j. Since the C_j are pairwise disjoint then so are the sets $U'_{\chi(jt)}$ for $j \in \mathbb{N}$ and $1 \leqslant t \leqslant z_j$. Second, we replace each $a_{\chi(jt)}$ by the element $a'_{\chi(jt)}$ coinciding with $a_{\chi(jt)}$ over $U'_{\chi(jt)}$ and vanishing outside $U'_{\chi(jt)}$. We now define $a \in G$ by

$$a = \bigvee \left\{a'_{\chi(jt)}\colon\ j \in \mathbb{N},\ t = 1, \ldots, z_j\right\}.$$

By definition of F_n we have:

$$\text{There is } q \in \mathbb{Z} \quad \text{such that} \quad \hat{a}(\zeta) = \frac{q}{n}. \tag{2.6}$$

By definition of the $U'_{\chi(jt)}$ we can write

$$\overline{\bigcup_{j \in \mathbb{N}} \left(U'_{\chi(j1)} \cup \cdots \cup U'_{\chi(jz_j)}\right)} = \overline{\bigcup_{j \in \mathbb{N}} C_j} \ni \zeta,$$

whence for every clopen neighborhood W of ζ there is $j \in \mathbb{N}$ and $t \in \{1, \ldots, z_j\}$ such that $W \cap U'_{\chi(jt)} \neq \emptyset$. By (2.6), there is a clopen $V \ni \zeta$ such that

$$\hat{a}(V) \subseteq \left[\frac{q}{n} - \frac{e}{3}, \frac{q}{n} + \frac{e}{3}\right].$$

Let $\bar{j}, \bar{t}$ be such that $V \cap U'_{\chi(\bar{j}\bar{t})} \neq \emptyset$. By (2.5) we can now refine the inequality $a \geqslant a'_{\chi(\bar{j}\bar{t})}$ to

$$\hat{a} > \widehat{a'_{\chi(\bar{j}\bar{t})}} \quad \text{over } V \cap U'_{\chi(\bar{j}\bar{t})}. \tag{2.7}$$

Let f be the real-valued continuous function over X that coincides with $\hat{a}$ outside $V \cap U'_{\chi(\bar{j}\bar{t})}$ and coincides with $\widehat{a'_{\chi(\bar{j}\bar{t})}}$ over $V \cap U'_{\chi(\bar{j}\bar{t})}$. By (2.7), $f < \hat{a}$. By construction, f is an upper bound of the sequence $\{\widehat{a'_{\chi(jt)}}\colon\ j \in \mathbb{N},\ t \in \{1, \ldots, z_j\}\}$. By Theorem 2.3 there is $b \in G$ such that $\hat{b} = f$. Now b is an upper bound in G of the sequence $\{a'_{\chi(jt)}\colon\ j \in \mathbb{N},\ t \in \{1, \ldots, z_j\}\}$, and is strictly smaller than a, which is impossible. $\square$

For a converse of the foregoing proposition, fix $n \in \mathbb{N}$. Let X be an arbitrary basically disconnected compact Hausdorff space. For any closed subset F of X define $\Theta(F)$ by

$$\Theta(F) = \Theta_{X,n}(F) = \left\{ f \in \mathrm{Cont}(X, \mathbb{R}) \colon \forall x \in F,\ f(x) \in \frac{1}{n}\mathbb{Z} \right\}.$$

Direct inspection shows that $\Theta(F)$ is a sub-ℓ-group of the $\sigma\ell$-group $\mathrm{Cont}(X, \mathbb{R})$. The next lemma shows that when F is special, $\Theta(F)$ is a sub-$\sigma\ell$-group of $\mathrm{Cont}(X, \mathbb{R})$.

Lemma 2.7. *Let X be a basically disconnected compact Hausdorff space. Let n be a fixed but otherwise arbitrary natural number. Let F be a closed subset of X. If F is special, then for every sequence $\{f_i\}_{i\in\mathbb{N}}$ of functions in $\Theta_{X,n}(F)$ such that $f = \bigvee_{i\in\mathbb{N}} f_i$ exists, we have $f \in \Theta_{X,n}(F)$.*

Proof. Write $\Theta(F)$ instead of $\Theta_{X,n}(F)$. Suppose that F is a closed subset of X and that there is a sequence $\{f_i\}_{i\in\mathbb{N}}$ of functions in $\Theta(F)$ with $f = \bigvee_{i\in\mathbb{N}} f_i$ and $f \notin \Theta(F)$. We will show that F is not special. By hypothesis there is $z \in F$ such that $f(z)$ is not an integer multiple of $\frac{1}{n}$. Since $\frac{1}{n}\mathbb{Z}$ is a closed subset of $\mathbb{R}$, there is $\varepsilon > 0$ such that the interval $[f(z) - \varepsilon, f(z) + \varepsilon]$ is disjoint from $\frac{1}{n}\mathbb{Z}$. For each integer $k \geqslant 3$, defining

$$I_k = \bigcup_{p\in\mathbb{Z}} \left[\frac{p}{n} + \frac{1}{kn}, \frac{p+1}{n} - \frac{1}{kn} \right],$$

we get an increasing sequence of closed subsets of $\mathbb{R}$ whose union coincides with the complement of $\frac{1}{n}\mathbb{Z}$ in $\mathbb{R}$. For each $i \in \mathbb{Z}$ and each integer $k \geqslant 3$, let

$$C_{ik} = \overline{f_i^{-1}\big(\mathrm{int}(I_k)\big)}.$$

Since $\mathrm{int}(I_k)$ is an open F_σ in $\mathbb{R}$, from the compactness of X and the continuity of f_i it follows that $f_i^{-1}(\mathrm{int}(I_k))$ is an open F_σ in X, and C_{ik} is clopen in X. Moreover, $C_{ik} \subseteq f_i^{-1}(I_k) \subseteq X \setminus F$. Let

$$U = \overline{\bigcup_{i\in\mathbb{N}, k\geqslant 3} C_{ik}}.$$

To complete the proof it suffices to settle the following

Claim. $F \cap U \neq \emptyset$.

By way of contradiction suppose $F \cap U = \emptyset$. Since U is the closure of an F_σ and X is basically disconnected, then U is an open subset of X containing all points $x \in X$ such that $f_i(x) \notin \frac{1}{n}\mathbb{Z}$ for some $i \in \mathbb{N}$. Thus there is a clopen subset $C \subseteq X$ such that $z \in C \subseteq X \setminus U$ and $f(C) \subseteq [f(z) - \varepsilon, f(z) + \varepsilon]$. From $f_i(x) \leqslant f(x)$ and $[f(z) - \varepsilon, f(z) + \varepsilon] \cap \frac{1}{n}\mathbb{Z} = \emptyset$, we obtain that for all $x \in C$ and $i \in \mathbb{N}$, $f_i(x) < f(z) - \varepsilon$. The function $g : X \to \mathbb{R}$ taking constantly the value $f(z) - \varepsilon$ over C, and the value 1 over $X \setminus C$ is continuous. Therefore, the function $g \wedge f$ is a continuous upper bound of the sequence $\{f_i\}_{i\in\mathbb{N}}$ and is strictly smaller than f, which contradicts the hypothesis $f = \bigvee_{i\in\mathbb{N}} f_i$. $\square$

3. The category $\mathbb{W}$

As the reader will recall, $\mathbb{DED}$ denotes the category whose objects are Dedekind σ-complete ℓ-groups with a distinguished order-unit, and whose morphisms are the unit preserving ℓ-group homomorphisms that also preserve all denumerable infima and suprema. The full subcategory of $\mathbb{DED}$ whose objects are the $\sigma\ell$-groups of bounded finite rank will be denoted by $\mathbb{BFR}$.

Our duality is in terms of a category $\mathbb{W}$ whose objects are given by the following definition (morphisms will be defined in 3.6 below):

Definition 3.1. *Objects* of $\mathbb{W}$ are triples $\langle X, S, \varphi\rangle$ such that X is a basically disconnected compact Hausdorff space, S is a (possibly empty) set of natural numbers and φ is a one–one map from S into the set of subsets of X satisfying the following three conditions:

(A1) For each $n \in S$, $\varphi(n)$ is a non-empty special closed subset of X;
(A2) For each $n \in S$, $\varphi(n) \supsetneqq \bigcup\{\varphi(j)\colon\ j \in S,\ j < n,\ j \text{ divides } n\}$;
(A3) For all $m, n \in S$,

$$\varphi(m) \cap \varphi(n) = \bigcup\bigl\{\varphi(j)\colon\ j \in S,\ j \text{ is a common divisor of } m \text{ and } n\bigr\}.$$

Notation. For each object $\langle X, S, \varphi\rangle$ in $\mathbb{W}$, and $n \in S$, let $\varphi(n)'$ be defined by

$$\varphi(n)' = \varphi(n) \setminus \bigcup\bigl\{\varphi(i)\colon\ n > i \in S \text{ and } i \text{ divides } n\bigr\}.$$

With this notation, (A2) can be equivalently restated as

(A2′) $\varphi(n)' \neq \emptyset$ for all $n \in S$.

Lemma 3.2. *Let $\langle X, S, \varphi\rangle$ be an object in $\mathbb{W}$. For all m, n in S, $\varphi(m) \subseteq \varphi(n)$ if and only if m is a divisor of n.*

Proof. The "if" part follows at once from (A2). To prove the "only if" part, suppose m is not a divisor of n. Then by (A3) $\varphi(m) \cap \varphi(n)$ is the union of $\varphi(j)$ for $j \in S$ and j a common divisor of m and n. By hypothesis $j \neq m$. Thus $j < m$, and $\varphi(m) \cap \varphi(n) \subseteq \bigcup\{\varphi(j)\colon\ j \in S,\ j \text{ divides } m,\ j < m\}$. By (A2), this union is properly contained in $\varphi(m)$. Hence $\varphi(m) \nsubseteq \varphi(n)$, as required. □

Lemma 3.3. *Let $\langle X, S, \varphi\rangle$ be an object in $\mathbb{W}$. For each $x \in \bigcup_{n\in S}\varphi(n)$ there is $n_x \in S$ with $x \in \varphi(n_x)$ having the additional property that for every $m \in S$, $x \in \varphi(m)$ if and only if n_x divides m. In other words, n_x is the minimum $n \in S$ such that $x \in \varphi(n)$.*

Proof. Let n_x be minimal (with respect to the divisibility order) in the set $\{n \in S\colon\ x \in \varphi(n)\}$. If $x \in \varphi(m)$ then $x \in \varphi(n_x) \cap \varphi(m)$, and by (A3), there is a common divisor d of n_x and of m such that $x \in \varphi(d)$. By definition of n_x we have $n_x = d$, whence n_x divides m. □

Definition 3.4. Let $\langle X, S, \varphi\rangle$ be an object in $\mathbb{W}$ and let $x \in X$. In the light of Lemma 3.3, we define the *virtual rank*

$$\mathrm{vrank}(x)$$

of x as the minimum $n \in S$ such that $x \in \varphi(n)$ in case $x \in \bigcup_{n \in S} \varphi(n)$, and we set $\operatorname{vrank}(x) = \infty$ otherwise.

Remark 3.5. For each $n \in S$, $\varphi(n)' = \{x \in X\colon \operatorname{vrank}(x) = n\}$. Hence (A2) asserts that *for each $n \in S$ there is at least one $x \in X$ such that* $\operatorname{vrank}(x) = n$.

Given basically disconnected spaces X, Y, we say that a function $f : X \to Y$ is *σ-continuous* if it is continuous, and for each sequence $(U_n\colon n \in \mathbb{N})$ of clopen subsets of Y, we have the identity

$$\operatorname{int}\Bigl(\bigcap_{n \in \mathbb{N}} f^{-1}(U_n)\Bigr) = f^{-1}\Bigl(\operatorname{int}\Bigl(\bigcap_{n \in \mathbb{N}} U_n\Bigr)\Bigr). \tag{3.8}$$

As a matter of fact, $f : X \to Y$ is σ-continuous if and only if it induces a σ-homomorphism from the dual σ-boolean algebra of Y into the dual σ-boolean algebra of X (cf. [10, §22]). Notice that homeomorphisms between basically disconnected compact Hausdorff spaces are σ-continuous functions.

We can now complete the definition of the category $\mathbb{W}$:

Definition 3.6. Whenever $\langle X, S, \varphi\rangle$ and $\langle Y, T, \psi\rangle$ are objects in $\mathbb{W}$, by a *morphism* $\langle X, S, \varphi\rangle \to \langle Y, T, \psi\rangle$ we understand a σ-continuous function $f : X \to Y$ such that for every $x \in X$, if $\operatorname{vrank}(x) < \infty$, then $\operatorname{vrank}(f(x))$ divides $\operatorname{vrank}(x)$.

Lemma 3.7. *Objects $\langle X, S, \varphi\rangle$ and $\langle Y, T, \psi\rangle$ are isomorphic in the category $\mathbb{W}$ if and only if $S = T$ and there is a homeomorphism f from X onto Y such that* $\operatorname{vrank}(f(x)) = \operatorname{vrank}(x)$ *for all $x \in X$.*

Proof. For the non-trivial direction, suppose that $f : \langle X, S, \varphi\rangle \to \langle Y, T, \psi\rangle$ is an isomorphism in $\mathbb{W}$. Then the inverse function $f^{-1} : Y \to X$ is also a morphism in $\mathbb{W}$. Hence f is a homeomorphism from X onto Y. Let $n \in S$. By Remark 3.5 there is $x \in X$ such that $\operatorname{vrank}(x) = n$. If $m = \operatorname{rank}(f(x))$ then $m \in T$ and m divides n. Since f^{-1} is a morphism, $n = \operatorname{vrank}(f^{-1}(f(x)))$ is a divisor of $\operatorname{vrank}(f(x)) = m$. It follows that $m = n$. Hence $S \subseteq T$ and $\operatorname{vrank}(f(x)) = \operatorname{vrank}(x)$ for all $x \in \bigcup_{n \in S} \varphi(n)$. The proof is completed by interchanging the roles of f and f^{-1}. $\square$

Our duality for the category $\mathbb{BFR}$ is given in terms of the following category:

Definition 3.8. We let $\mathbb{W}_{\text{fin}}$ be the full subcategory of $\mathbb{W}$ whose objects are all triples $\langle X, S, \varphi\rangle$ with *S finite*.

Below in Theorem 5.3 we will prove that $\mathbb{BFR}^{\text{op}}$ is equivalent to $\mathbb{W}_{\text{fin}}$. But first we shall construct an adjunction between $\mathbb{DED}^{\text{op}}$ and $\mathbb{W}$.

4. An adjunction between $\mathbb{DED}^{\text{op}}$ and $\mathbb{W}$

Given an object (G, u) of $\mathbb{DED}$, let

$$\mathsf{F}(G, u) = \bigl\langle \partial_{\mathrm{e}} S(G, u), \operatorname{NumSpec}(G, u), \varphi\bigr\rangle,$$

where $\varphi(n) = F_n(G)$ for all $n \in \mathrm{NumSpec}(G,u)$. From Lemma 2.6 and the definition of φ it follows that $\mathsf{F}(G,u)$ is an object of $\mathbb{W}$.

For any morphism $(G,u) \xrightarrow{h} (H,v)$ in $\mathbb{DED}$ let

$$h^*: \partial_{\mathrm{e}} S(H,v) \to \partial_{\mathrm{e}} S(G,u)$$

be defined by

$$h^*(\chi) = \chi \circ h \tag{4.9}$$

for all $\chi \in \partial_{\mathrm{e}} S(H,v)$. Then h^* is continuous and $\chi(h(G)) \subseteq \chi(H)$. Thus, in order to prove that h^* is a morphism in $\mathbb{W}$ we have only to check that it is σ-continuous, i.e., that condition (3.8) is satisfied by h^*. In what follows, for any subset E of a set L, we shall denote by $\mathrm{char}(E): L \to \{0,1\}$ the characteristic function of E in L. Let $(U_n\colon n \in \mathbb{N})$ be a sequence of clopen subsets of $\partial_{\mathrm{e}} S(G,u)$, and let $U = \mathrm{int}(\bigcap_{n\in\mathbb{N}} U_n)$. We have the identity $\bigwedge_{n\in\mathbb{N}} \mathrm{char}(U_n) = \mathrm{char}(U)$ in $\mathrm{Cont}(\partial_{\mathrm{e}} S(G,u),\mathbb{R})$. Trivially, all functions $\mathrm{char}(U_n)$, as well as $\mathrm{char}(U)$, are $F_n(G)$-compatible for each $n \in \mathrm{NumSpec}(G,u)$. Thus by Theorem 2.3 there are elements b_n and b in G such that $\widehat{b_n} = \mathrm{char}(U_n)$, for all $n \in \mathbb{N}$, $\hat{b} = \mathrm{char}(U)$, and $\bigwedge b_n = b$. It follows that $h(b) = \bigwedge_{n\in\mathbb{N}} h(b_n)$ in H. Recalling (2.2) and (4.9), for each $n \in \mathbb{N}$ we have the identities $\widehat{h(b_n)} = \widehat{b_n} \circ h^* = \mathrm{char}(h^{*-1}(U_n))$. Since by Theorem 2.3 the map $a \mapsto \hat{a}$ preserves all denumerable infima we can write

$$\widehat{h(b)} = \bigwedge_{n\in\mathbb{N}} \mathrm{char}\big(h^{*-1}(U_n)\big) = \mathrm{char}\Bigg(\mathrm{int}\bigg(\bigcap_{n\in\mathbb{N}} h^{*-1}(U_n)\bigg)\Bigg).$$

Similarly, $\widehat{h(b)} = \hat{b} \circ h^* = \mathrm{char}(h^{*-1}(U))$, whence

$$h^{*-1}\Bigg(\mathrm{int}\bigg(\bigcap_{n\in\mathbb{N}} U_n\bigg)\Bigg) = \mathrm{int}\bigg(\bigcap_{n\in\mathbb{N}} h^{*-1}(U_n)\bigg).$$

We conclude that h^* is σ-continuous, and consequently, h^* is a morphism in $\mathbb{W}$.

Setting $\mathsf{F}(h) = h^*$ we have a contravariant functor F from $\mathbb{DED}$ into $\mathbb{W}$.

Conversely, given an object $\langle X,S,\varphi\rangle$ in $\mathbb{W}$ let

$$G\big(\langle X,S,\varphi\rangle\big) = \Bigg\{c \in \mathrm{Cont}(X,\mathbb{R})\colon \forall x \in \bigcup_{n\in S}\varphi(n),\ c(x) \in \frac{1}{\mathrm{vrank}(x)}\mathbb{Z}\Bigg\}.$$

Let $n \in S$. Lemma 3.3 implies that $c(x) \in \frac{1}{n}\mathbb{Z}$ for each $x \in \varphi(n)$, and by (A1), $\varphi(n)$ is a special closed set. Hence from Lemma 2.7 it follows that $G(\langle X,S,\varphi\rangle)$ is a sub-$\sigma\ell$-group of $\mathrm{Cont}(X,\mathbb{R})$ containing $\mathrm{Cont}(X,\mathbb{Z})$. Defining now $\mathsf{G}(\langle X,S,\varphi\rangle) = (G(\langle X,S,\varphi\rangle),u)$, with u the constant function 1 over X, we obtain an object in $\mathbb{DED}$.

Let $\langle Y,T,\psi\rangle \xrightarrow{f} \langle X,S,\varphi\rangle$ be a morphism in $\mathbb{W}$. For each $c \in G(\langle X,S,\varphi\rangle)$ the composite map $f^*(c) = c \circ f$ belongs to $G(\langle Y,T,\psi\rangle)$. Indeed, if $n \in T$ and $y \in \psi(n)$, then $m = \mathrm{vrank}(f(y))$ is a divisor of $\mathrm{vrank}(y)$, which is a divisor of n. Hence $c(f(y)) \in \frac{1}{m}\mathbb{Z} \subseteq \frac{1}{n}\mathbb{Z}$. Consequently, $f^*: \mathsf{G}(\langle X,S,\varphi\rangle) \to \mathsf{G}(\langle Y,T,\psi\rangle)$ is an ℓ-group homomorphism preserving the distinguished order-units. We shall show that f^* also preserves infima and suprema of denumerable families of elements in $G(\langle X, S, \varphi\rangle)$. To this purpose it is sufficient to show that for every sequence of functions $(c_n\colon n \in \mathbb{N})$ in $G(\langle X, S, \varphi\rangle)$, if $\bigwedge_{n\in\mathbb{N}} c_n = 0$ then $\bigwedge_{n\in\mathbb{N}} f^*(c_n) = 0$. By way of contradiction, let the sequence $(c_n\colon n \in \mathbb{N})$ in $G(\langle X, S, \varphi\rangle)$ be a counterexample: thus $\bigwedge c_n = 0$ but the element $c = \bigwedge_{n\in\mathbb{N}}(c_n \circ f)$ is non-zero. Then there is some $z \in Y$ such that $c(z) > 0$. By continuity there is $\varepsilon > 0$ and a non-empty clopen $U \subseteq Y$ such that $c(y) > \varepsilon$ for each $y \in U$. For each $n \in \mathbb{N}$, $f(U)$ is contained in the open set $H_n = \{x \in X\colon c_n(x) > \varepsilon\}$. Hence for each $x \in f(U)$ there is a clopen V_x in X such that $x \in V_x \subseteq H_n$. Since $f(U)$ is compact in X, we can find a clopen V_n such that $f(U) \subseteq V_n \subseteq H_n$. It follows that

$$U \subseteq f^{-1}\big(f(U)\big) \subseteq \bigcap f^{-1}(V_n) \quad \text{and} \quad \operatorname{int}\Big(\bigcap_{n\in\mathbb{N}} f^{-1}(V_n)\Big) \neq \emptyset.$$

On the other hand, $\bigcap V_n \subseteq \bigcap_{n\in\mathbb{N}} H_n \subseteq \{x \in X\colon \inf_{n\in\mathbb{N}} c_n(x) \geqslant \varepsilon\}$, where $\inf_{n\in\mathbb{N}} c_n$ denotes the pointwise infimum of the sequence (c_n). Since $G(\langle X, S, \varphi\rangle)$ is a sub-$\sigma\ell$-group of $\mathrm{Cont}(X, \mathbb{R})$, from [4, Theorem 9.2 and Lemma 9.1] it follows that, if $d = \bigwedge_{n\in\mathbb{N}} c_n$, then

$$0 = d(x) = \sup_{W\in\mathcal{N}(x)} \Big(\inf_{w\in W} \Big(\inf_{n\in\mathbb{N}} c_n(w)\Big)\Big),$$

where $\mathcal{N}(x)$ is the set of open neighborhoods of $x \in X$. Therefore, $\operatorname{int}(\{x \in X\colon \inf_{n\in\mathbb{N}} c_n(x) \geqslant \varepsilon\}) = \emptyset$, whence $\operatorname{int}(\bigcap_{n\in\mathbb{N}} V_n) = \emptyset \neq \operatorname{int}(\bigcap_{n\in\mathbb{N}} f^{-1}(V_n))$, thus contradicting (3.8). This shows that $f^*\colon G(\langle X, S, \varphi\rangle) \to G(\langle Y, T, \psi\rangle)$ is a morphism in $\mathbb{DED}$.

Setting $\mathsf{G}(f) = f^*$ *we have a contravariant functor* G *from* $\mathbb{W}$ *into* $\mathbb{DED}$.

From Theorem 2.3 it follows that the correspondence $a \mapsto \hat{a}$ defines an isomorphism $(G, u) \xrightarrow{\eta_{(G,u)}} \mathsf{G}(\mathsf{F}(G, u))$. Further, for any morphism $(G, u) \xrightarrow{h} (H, v)$ in $\mathbb{DED}$ the following diagram is promptly seen to commute:

$$\begin{tikzcd}
(G, u) \arrow[r, "h"] \arrow[d, "\eta_{(G,u)}"'] & (H, v) \arrow[d, "\eta_{(H,v)}"] \\
\mathsf{G}\big(\mathsf{F}(G, u)\big) \arrow[r, "\mathsf{G}(\mathsf{F}(h))"] & \mathsf{G}\big(\mathsf{F}(H, v)\big).
\end{tikzcd} \tag{4.10}$$

Let us now reverse arrows and compositions in this diagram. Then for each morphism $(H, v) \xrightarrow{h} (G, u)$ in the opposite category $\mathbb{DED}^{\mathrm{op}}$ of $\mathbb{DED}$, we obtain

$$h \circ \eta^{\mathrm{op}}_{(H,v)} = \eta^{\mathrm{op}}_{(G,u)} \circ \mathsf{G}\big(\mathsf{F}(h)\big). \tag{4.11}$$

We have proved

Proposition 4.1. *The isomorphisms* η^{op} *define a natural equivalence from* $\mathsf{G} \circ \mathsf{F}$ *to the identity functor of* $\mathbb{DED}^{\mathrm{op}}$.

For each object $\langle X, S, \varphi\rangle$ in $\mathbb{W}$ we have the inclusions

$$\mathrm{Cont}(X, \mathbb{Z}) \subseteq G\big(\langle X, S, \varphi\rangle\big) \subseteq \mathrm{Cont}(X, \mathbb{R}).$$

From the first inclusion it follows that $G(\langle X,S,\varphi\rangle)$ is a separating sub-$\sigma\ell$-group of $\mathrm{Cont}(X,\mathbb{R})$. Thus the evaluation map $\upsilon_X : X \to \partial_{\mathrm{e}} S(\mathsf{G}(\langle X,S,\varphi\rangle))$ given by (2.3)–(2.4) is a homeomorphism. Moreover, if $x \in X$ and $\mathrm{vrank}(x) = n < \infty$, then for each $c \in G(\langle X,S,\varphi\rangle)$ we have $\upsilon_X(x)(c) = c(x) \in \frac{1}{n}\mathbb{Z}$. Therefore, the integer $\mathrm{vrank}(\upsilon_X(x)) = \mathrm{rank}(\upsilon_X(x))$ divides $\mathrm{vrank}(x)$, and υ_X is a morphism in $\mathbb{W}$ from $\langle X,S,\varphi\rangle$ into $\mathsf{F}(\mathsf{G}(\langle X,S,\varphi,\rangle))$. In addition, for every morphism $f : \langle X,S,\varphi\rangle \to \langle Y,T,\psi\rangle$ in $\mathbb{W}$ we have a commutative diagram

$$\begin{tikzcd}
\langle X,S,\varphi\rangle \arrow[r, "f"] \arrow[d, "\upsilon_X"'] & \langle Y,T,\psi\rangle \arrow[d, "\upsilon_Y"] \\
\mathsf{F}\big(\mathsf{G}\big(\langle X,S,\varphi\rangle\big)\big) \arrow[r, "\mathsf{F}(\mathsf{G}(f))"] & \mathsf{F}\big(\mathsf{G}\big(\langle Y,T,\psi\rangle\big)\big).
\end{tikzcd} \tag{4.12}$$

Summing up, the morphisms $\upsilon_{(X,S,\varphi)} = \upsilon_X$ determine a natural transformation from the identity functor of $\mathbb{W}$ to the composite functor $\mathsf{F} \circ \mathsf{G}$.

Interpreting F as a (covariant) functor from $\mathbb{DED}^{\mathrm{op}}$ to $\mathbb{W}$, and G as a (covariant) functor from $\mathbb{W}$ to $\mathbb{DED}^{\mathrm{op}}$, for each object (G,u) of $\mathbb{DED}^{\mathrm{op}}$, we have

$$\Big[\mathsf{F}(G,u) \xrightarrow{\upsilon_{\mathsf{F}(G,u)}} \mathsf{F}\big(\mathsf{G}\big(\mathsf{F}(G,u)\big)\big) \xrightarrow{\mathsf{F}(\eta^{\mathrm{op}}_{(G,u)})} \mathsf{F}(G,u)\Big] = \Big[\mathsf{F}(G,u) \xrightarrow{\mathrm{id}_{\mathsf{F}(G,u)}} \mathsf{F}(G,u)\Big], \tag{4.13}$$

and for each object $\langle X,S,\varphi\rangle$ in $\mathbb{W}$,

$$\begin{aligned}
&\Big[\mathsf{G}\big(\langle X,S,\varphi\rangle\big) \xrightarrow{\mathsf{G}(\upsilon_X)^{\mathrm{op}}} \mathsf{G}\big(\mathsf{F}\big(\mathsf{G}\big(\langle X,S,\varphi\rangle\big)\big)\big) \xrightarrow{\eta_{\mathsf{G}(\langle X,S,\varphi\rangle)^{\mathrm{op}}}} \mathsf{G}\big(\langle X,S,\varphi\rangle\big)\Big] \\
&\quad = \Big[\mathsf{G}\big(\langle X,S,\varphi\rangle\big) \xrightarrow{id_{\mathsf{G}(\langle X,S,\varphi\rangle)}} \mathsf{G}\big(\langle X,S,\varphi\rangle\big)\Big].
\end{aligned} \tag{4.14}$$

Recalling now [7, §IV.1 Theorem 2(v)], from (4.11)–(4.14) we obtain

Theorem 4.2. *The functor* $\mathsf{G} : \mathbb{W} \to \mathbb{DED}^{\mathrm{op}}$ *is right adjoint to* $\mathsf{F} : \mathbb{DED}^{\mathrm{op}} \to \mathbb{W}$. *The natural transformation* υ *is the unit of the adjunction, and the counit is the natural equivalence* η^{op}.

In the next section we shall show that by restriction, this adjunction gives an equivalence between the categories $\mathbb{BFR}^{\mathrm{op}}$ and $\mathbb{W}_{\mathrm{fin}}$.

5. The equivalence between $\mathbb{BFR}^{\mathrm{op}}$ and $\mathbb{W}_{\mathrm{fin}}$

It is easy to see that the restriction of the functor F to $\mathbb{BFR}^{\mathrm{op}}$ is a functor $\mathsf{F}_{\mathrm{b}} : \mathbb{BFR}^{\mathrm{op}} \to \mathbb{W}_{\mathrm{fin}}$. We shall show that the restriction of G to $\mathbb{W}_{\mathrm{fin}}$ is a functor G_{b} from $\mathbb{W}_{\mathrm{fin}}$ to $\mathbb{BFR}^{\mathrm{op}}$.

Proposition 5.1. *Let* $\langle X,S,\varphi\rangle$ *be an object in* $\mathbb{W}_{\mathrm{fin}}$. *For each* $x \in X$, $\mathrm{vrank}(x) = \mathrm{rank}(\upsilon_X(x))$.

Proof. Suppose first that $\mathrm{vrank}(x) = n < \infty$. Let J_n be the set of non-multiples of n in S,

$$J_n = \{k \in S\colon\ n \text{ does not divide } k\}.$$

Let further $Z_n = X \setminus \bigcup_{k\in J_n} \varphi(k)$. Since Z_n is the complement of a finite union of closed sets, it is open. Hence by (A2′) and Remark 3.5, $x \in \varphi(n)' \subseteq Z_n$. Therefore there is a clopen subset U of X such that $x \in U \subseteq Z_n$, and the function taking constantly the value $\frac{1}{n}$ on U and 0 outside U belongs to $G(\langle X, S, \varphi\rangle)$. This shows that $\{c(x)\colon c \in G(\langle X, S, \varphi\rangle)\} = \frac{1}{n}\mathbb{Z}$, i.e., $\operatorname{rank}(\upsilon_X(x)) = n$. If $\operatorname{vrank}(x) = \infty$, then x is in the open set $X \setminus \bigcup_{n\in S} \varphi(n)$. Let U be a clopen neighborhood of x contained in $X \setminus \bigcup_{n\in S} \varphi(n)$. The function taking constantly an irrational value on U and 0 outside U belongs to $G(\langle X, S, \varphi\rangle)$, and this shows that $\operatorname{rank}(\upsilon_X(x)) = \infty$. □

The next corollary shows that if $\langle X, S, \varphi\rangle$ is an object in $\mathbb{W}_{\mathrm{fin}}$, then $\mathsf{G}(\langle X, S, \varphi\rangle)$ is an object in $\mathbb{BFR}$.

Corollary 5.2. *For each object $\langle X, S, \varphi\rangle$ in $\mathbb{W}_{\mathrm{fin}}$ we have*:

(i) $\operatorname{NumSpec}(\mathsf{G}(\langle X, S, \varphi\rangle)) = S$;
(ii) $F_n(\mathsf{G}(\langle X, S, \varphi\rangle)) = \upsilon_X(\varphi(n))$ *for all* $n \in S$.

It follows that the restriction G_{b} of the functor G to the full subcategory $\mathbb{W}_{\mathrm{fin}}$ of $\mathbb{W}$ is a functor from $\mathbb{W}_{\mathrm{fin}}$ to $\mathbb{BFR}^{\mathrm{op}}$. Moreover, by Lemma 3.7 and Proposition 5.1, for any object $\langle X, S, \varphi\rangle$ in $\mathbb{W}_{\mathrm{fin}}$, υ_X is an isomorphism from $\langle X, S, \varphi\rangle$ to $\mathsf{F}_{\mathrm{b}}(\mathsf{G}_{\mathrm{b}}(\langle X, S, \varphi\rangle))$ in the category $\mathbb{W}$.

Given morphisms $(G, u) \xrightarrow{h} (H, v)$ in $\mathbb{BFR}$, and $\langle X, S, \varphi\rangle \xrightarrow{f} \langle Y, T, \psi\rangle$ in $\mathbb{W}$, if we replace in (4.10) and (4.12) F and G by F_{b} and G_{b}, respectively, we still have commutative diagrams. Since in this case both η and υ are isomorphisms, we obtain the main result of this paper:

Theorem 5.3. *The quadruple* $(\mathsf{G}_{\mathrm{b}}, \mathsf{F}_{\mathrm{b}}, \upsilon, \eta^{\mathrm{op}})$ *defines an equivalence between the categories* $\mathbb{BFR}^{\mathrm{op}}$ *and* $\mathbb{W}_{\mathrm{fin}}$.

As a final result, let us show that *every object in $\mathbb{DED}$ is a limit of objects in $\mathbb{BFR}$.*

To this purpose, for any object $\langle X, S, \varphi\rangle$ in $\mathbb{W}$ and integer $p \geqslant 0$ we let $S_p = \{n \in S\colon n \leqslant p\}$. Then $\{S_p\}_{p\geqslant 0}$ is a non-decreasing sequence of finite sets of natural numbers such that $S_0 = \emptyset$ and $\bigcup_{p\geqslant 0} S_p = S$. For each $p \geqslant 0$, letting φ_p denote the restriction of φ to S_p, it follows that $\langle X, S_p, \varphi_p\rangle$ is an object of $\mathbb{W}_{\mathrm{fin}}$. The identity over X is both a morphism $\rho_p : \langle X, S_p, \varphi_p\rangle \to \langle X, S_{p+1}, \varphi_{p+1}\rangle$ and a morphism $\lambda_p : \langle X, S_p, \varphi_p\rangle \to \langle X, S, \varphi\rangle$. More precisely, in the category $\mathbb{W}$ the system $(\langle X, S, \varphi\rangle, \{\lambda_p\}_{p\geqslant 0})$ is the colimit of $(\{\langle X, S_p, \varphi_p\rangle\}_{p\geqslant 0}, \{\rho_p\}_{p\geqslant 0})$. Note that ρ_p and λ_p are simultaneously monomorphisms and epimorphisms for every $p \geqslant 0$, but in general they are not isomorphisms in $\mathbb{W}$. Since by Theorem 4.2, G is a left adjoint of F, it transforms colimits in $\mathbb{W}$ into colimits in $\mathbb{DED}^{\mathrm{op}}$, i.e., limits in $\mathbb{DED}$. In conclusion, $\mathsf{G}(\langle X, S, \varphi\rangle) = \lim \mathsf{G}(\langle X, S_p, \varphi_p\rangle) = \lim \mathsf{G}_{\mathrm{b}}(\langle X, S_p, \varphi_p\rangle)$, as desired.

Acknowledgment

We are very grateful to the referee for her/his careful and competent reading of our paper, and for suggesting a simplification of Lemma 2.6 and a reorganization of the final part.

References

[1] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés, Lecture Notes in Math., vol. 608, Springer, Berlin, 1971.

[2] G. Birkhoff, Lattice Theory, third ed., Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, RI, 1967.
[3] W.D. Burgess, J.-M. Goursaud, K_0 d'un anneau dont les localisés centraux sont simples artiniens, Canad. Math. Bull. 25 (1982) 344–347.
[4] K.R. Goodearl, Partially Ordered Abelian Groups with Interpolation, Amer. Math. Soc., Providence, RI, 1986.
[5] K.R. Goodearl, K_0 of regular rings with bounded index of nilpotence, in: Contemp. Math., vol. 171, Amer. Math. Soc., 1994, pp. 173–199.
[6] K.R. Goodearl, D.E. Handelman, J.W. Lawrence, Affine representations of Grothendieck groups and applications to Rickart C^*-algebras and $\aleph_0$-continuous regular rings, Mem. Amer. Math. Soc. 234 (1980).
[7] S. Mac Lane, Categories for the Working Mathematician, second ed., Grad. Texts in Math., Springer, Berlin, 1998.
[8] D. Mundici, Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus, J. Funct. Anal. 65 (1986) 15–63.
[9] B. Riečan, D. Mundici, Probability on MV-algebras, in: E. Pap (Ed.), Handbook of Measure Theory, vol. II, North-Holland, Amsterdam, 2001, pp. 869–909.
[10] R. Sikorski, Boolean Algebras, Ergeb. Math. Grenzgeb., Springer, Berlin, 1960.