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Maximal Subalgebras of $\mathrm{MV_n}$-algebras. A Proof of a Conjecture of A. Monteiro

Abstract. For each integer $n \geq 2$, $\mathbb{MV}_n$ denotes the variety of MV-algebras generated by the MV-chain with n elements. Algebras in $\mathbb{MV}_n$ are represented as continuous functions from a Boolean space into a n-element chain equipped with the discrete topology. Using these representations, maximal subalgebras of algebras in $\mathbb{MV}_n$ are characterized, and it is shown that proper subalgebras are intersection of maximal subalgebras. When $A \in \mathbb{MV}_3$, the mentioned characterization of maximal subalgebras of A can be given in terms of prime filters of the underlying lattice of A, in the form that was conjectured by A. Monteiro.

Keywords: MV-algebras, Moisil - Łukasiewicz algebras, Łukasiewicz many-valued logics, Boolean spaces, subalgebras.

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Introduction

A pioneering work on the algebraic treatment of many-valued logics was done by Moisil, who in his 1941 paper [15] introduced the *three-valued Łukasiewicz algebras*, as the algebraic counterpart of Łukasiewicz three-valued logic [14]. These algebras were deeply investigated by A. Monteiro in the early sixties, who related them with other algebras arising from logic, like monadic Boolean algebras and Nelson algebras [17, 18, 19, 20, 21].

Among A. Monteiro's personal files, the second author found a conjecture on the structure of maximal subalgebras of three-valued Łukasiewicz algebras in terms of prime (lattice) filters. The main aim of this paper is to prove that conjecture (see Corollary 3.5).

On the other hand, Iturrioz [12], motivated by results of Sachs [25] on maximal subalgebras of Boolean algebras, showed that the subalgebras that we call of type I in this paper, are maximal subalgebras that contain the Boolean elements of three-valued and four-valued Łukasiewicz algebras, and asked whether all maximal subalgebras of these algebras containing the boolean elements are of type I. We also give a positive answer for a class of

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algebras containing three-valued and four-valued Łukasiewicz algebras (see Theorem 3.2).

In the late fifties, Chang [4, 5] introduced MV-algebras as the algebraic counterpart of Łukasiewicz infinite valued logic [14]. Later, Grigolia [11] introduced MV_n-algebras, the MV-algebras corresponding to the n-valued Łukasiewicz logic, for $n \geq 2$. Moisil three-valued and four-valued Łukasiewicz algebras coincide with MV_3-algebras and MV_4 algebras, respectively. For each $n \geq 2$, MV_n-algebras can be represented by continuous functions on Boolean spaces taking values in finite chains equipped with the discrete topology. We use this representation to characterize maximal subalgebras of MV_n-algebras. When $n = 3$ we obtain that the maximal subalgebras are those conjectured by A. Monteiro. We also show that every proper subalgebra of an MV_n-algebra is an intersection of maximal subalgebras, generalizing a result of Sachs [25] for Boolean algebras (that coincide with MV_2-algebras).

Since MV_n-algebras form a variety generated by a semiprimal algebra, the mentioned representation by continuous functions can be derived from general results of universal algebra [22, 13]. As it plays a fundamental role in this paper, we give a direct elementary proof (Theorem 1.5).

For details on Łukasiewicz-Moisil algebras the reader can consult the monograph [3], and [9] for MV-algebras.

1. MV_n-algebras

In this section we collect some definitions, notations and results that we shall use in the paper.

A *De Morgan algebra* is an algebra $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\neg$ is a unary operation satisfying: $\neg\neg x = x$, and $\neg(x \wedge y) = \neg x \vee \neg y$. We assume the reader familiar with the basic properties of distributive lattices and De Morgan algebras that can be found in [1, 3, 23] (in this last reference, De Morgan algebras are called *quasi-Boolean algebras*).

An *MV-algebra*(also known as *Wajsberg algebra*) is a structure $\langle A, \oplus, \neg, 0\rangle$ where $\langle A, \oplus, 0\rangle$ is a commutative monoid with neutral element 0, satisfying the following equations: $\neg\neg x = x$, $\quad x \oplus \neg 0 = \neg 0$, and

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x\,. \tag{1.1}$$

The real unit interval $[0, 1]$ equipped with negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$ is an MV-algebra, which is called *the standard MV-algebra*. The defining equations of MV-algebras express simple properties of this concrete model. It was proved by Chang [5] (see also [8] or [9]) the variety $\mathbb{MV}$ of MV-algebras is generated by the standard MV-algebra.

If we add $x \oplus x = x$ to the equations of MV-algebras, then we obtain the variety of boolean algebras. Thus MV-algebras may be regarded as a non-idempotent equational generalization of boolean algebras. We shall use the following abbreviations, where x, y denote arbitrary elements of an MV-algebra:

$$1 = \neg\, 0, \quad x \odot y = \neg(\neg x \oplus \neg y), \quad x \ominus y = x \odot \neg y,$$

$$x \vee y = x \oplus \neg(x \oplus \neg y), \quad x \wedge y = x \odot \neg(x \odot \neg y).[1]$$

Note that equation (1.1) states that the join operation over $[0,1]$ is commutative. For every MV-algebra A, the reduct $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a De Morgan algebra. Notice that the underlying lattice order of the standard MV-algebra coincides with the usual order of real numbers.

Given an MV-algebra A, we set

$$B(A) := \{x \in A : x \oplus x = x\}.$$

It follows that $B(A)$ is a subalgebra of A, which is a Boolean algebra. Indeed, it is the Boolean algebra of the complemented elements of the lattice reduct of A. For each $x \in B(A)$, the Boolean complement of x is $\neg x$. The elements of $B(A)$ are called the *boolean elements of* A.

For each integer $n \geq 2$, we denote by L_n the subalgebra of the standard MV-algebra formed by the n fractions of denominator $n-1$: $0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1$. We shall denote by $\mathbb{MV}_n$ the subvariety of $\mathbb{MV}$ generated by L_n. The algebras in $\mathbb{MV}_n$ are called MV_n-algebras.

The following property, which is well known and easy to check, will play an important role in what follows:

LEMMA 1.1. *Given integers* $m, n \geq 2$*, we have:*

(i) *The only automorphism of L_n is the identity,*

(ii) *L_m is a subalgebra of L_n if and only if $m-1$ is a divisor of $n-1$.* □

Since the equation $x \oplus x = x$ holds in L_n if and only if $n = 2$, we have that $\mathbb{MV}_2$ is the variety of Boolean algebras, and that $B(L_n) = L_2$ for each $n \geq 2$.

[1] Unless otherwise specified, all MV-algebras in this paper shall be nontrivial, i.e., $0 \neq 1$.

The varieties $\mathbb{MV}_n$, for each $n \geq 3$ have been axiomatized by Grigolia [11] (see also [10, 9]).

For every $n \geq 2$, we can define one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language of MV-algebras such that evaluated on the algebras L_n give

$$\sigma_i^n\left(\frac{j}{n-1}\right) = \begin{cases} 1 & \text{if } i+j \geq n, \\ 0 & \text{if } i+j < n. \end{cases} \tag{1.2}$$

(see [6] or [24]). From this it follows that every MV_n-algebra admits a structure of an n-valued Łukasiewicz-Moisil algebra (see [16, 3]). For $n = 3$ and $n = 4$, the converse is also true: MV_3-algebras and MV_4-algebras are termwise equivalent to Moisil's three-valued and four-valued Łukasiewicz algebras, respectively [6, 3].

In the next lemma we collect, for further reference, some well known properties of the operations σ_i^n, $i = 1, \ldots, n-1$.

Lemma 1.2. *Let following properties hold in every $A \in \mathbb{MV}_n$ for each integer $n \geq 2$, where x, y denote arbitrary elements of A:*

(i) *$x \in B(A)$ if and only if $x = \sigma_i^n x$ for some $1 \leq i \leq n-1$ if and only if $x = \sigma_i^n x$ for all $1 \leq i \leq n-1$,*

(ii) *σ_i^n is a lattice homomorphism from A onto $B(A)$, for each $1 \leq i \leq n-1$,*

(iii) *$\sigma_i^n \sigma_j^n x = \sigma_j^n x$ for all $1 \leq i, j \leq n-1$,*

(iv) *$\sigma_1^n x \leq \sigma_2^n x \leq \cdots \leq \sigma_{n-1}^n$,*

(v) *if $\sigma_i^n x = \sigma_i^n y$ for $i = 1, \ldots, n-1$, then $x = y$.* □

By a *Boolean space* we understand a totally disconnected compact Hausdorff topological space. As usual, a set that is simultaneously open and closed is called clopen. The Boolean algebra of clopen sets of a Boolean space X will be denoted by $\mathrm{Clop}(X)$. The characteristic function of a set $S \subset X$ will be denoted by γ_S.

Given an integer $n \geq 1$, $\mathrm{Div}(n)$ will denote the set of divisors of n, and $\mathrm{Div}^*(n)$ the set of proper divisors of n, i. e., $\mathrm{Div}^*(n) = \{d \in \mathrm{Div}(n) : d < n\}$. Both sets become distributive lattices under the divisibility order.

Given an integer $n \geq 2$, an *n-valued Boolean space* is a pair $\langle X, \rho \rangle$, such that X is a Boolean space and ρ is a meet-homomorphism from the lattice of divisors of $n-1$ into the lattice of closed subsets of X, such that $\rho(n-1) = X$.

If the set L_n is equipped with the discrete topology, and (X, ρ) is an n-valued Boolean space, then $\mathcal{C}_n(X, \rho)$ denotes the MV_n-algebra formed by the continuous functions f from X into L_n such that $f(\rho(d)) \subseteq L_{d+1}$ for each $d \in \mathrm{Div}^*(n-1)$, with the algebraic operations defined pointwise. Clearly, the correspondence $U \mapsto \gamma_U$ defines an isomorphism from $\mathrm{Clop}(X)$ onto $B(\mathcal{C}_n(X,\rho))$.

By a filter of an algebra $A \in \mathbb{MV}_n$ we understand a filter of the underlying lattice of A.[2]

LEMMA 1.3. *The prime filters of $A = \mathcal{C}_n(X,\rho)$ are of the form*

$$P_x^i = \{f \in A : f(x) \geq \frac{i}{n-1}\}$$

for each $x \in X$ and $i = 1, \ldots, n-1$.

PROOF. It is clear that P_x^i is a prime filter for each $x \in X$ and each $1 \leq i \leq n-1$, and that $P_x^i \subseteq P_x^j$ for $1 \leq j \leq i \leq n-1$. To prove that they are the only prime filters of A, we shall prove first the following:
Claim: If F is a proper filter of A, then there is $z \in X$ such that $f(z) > 0$ for all $f \in F$.

To prove the claim, we use a standard argument: suppose that for each $x \in X$ there is a function $f_x \in F$ such that $f_x(x) = 0$. By continuity, $U_x = f_x^{-1}(\{0\})$ is clopen, and by compactness, there are $x_1, \ldots, x_n$ in X such that $X = U_{x_1} \cup \cdots \cup U_{x_n}$. Hence $0 = f_{x_1} \wedge \cdots \wedge f_{x_n} \in F$, and F is not proper. This contradiction proves the claim.

Let F be a prime filter of A. By the claim, there is a nonempty set $S \subseteq X$ such that $f(s) > 0$ for all $s \in S$ and all $f \in F$. If there were two elements $u, v \in S$, $u \neq v$, then there would be a clopen U such that $u \in U$ and $v \in V = X \setminus U$. Then $\gamma_U \vee \gamma_V \in F$, $\gamma_U \notin F$ and $\gamma_V \notin F$, and F would not be prime. Hence there is $x \in X$ and $1 \leq i \leq n-1$ such that $F = P_x^i$. ■

REMARK 1.4. Note that the filters of the form P_x^{n-1} are the minimal prime filters of $\mathcal{C}_n(X,\rho)$, and those of the form P_x^1 are the maximal filters. Moreover, $P_x^1 = \varphi(P_x^{n-1})$, where φ is the Bialinycki-Birula and Rasiowa transformation defined on prime filters of a De Morgan algebra by the prescription $\varphi(P) = A \setminus \neg P$ (see [2] or [23]). Finally, observe that $\rho(1) = \{x \in X : P_x^1 = P_x^{n-1}\}$.

Since the variety $\mathbb{MV}_n$ is generated by the algebra L_n, which is a semiprimal algebra,[3] the next theorem follows at once from [13, Theorem 6.5] (cf [7, 22]). For the sake of completeness, we shall give a simple direct proof,

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[2]Notice that the lattice filters we are considering in this paper are not congruence filters of the corresponding MV-algebra.

[3]$\tau(x,y,z) = ((\sigma_1^n((x \to y) \wedge (y \to x)) \wedge z) \vee (\neg\sigma_1^n((x \to y) \wedge (z \to x)) \wedge x)$, where $\to$ denotes Łukasiewicz implication, is a discriminator term for L_n.

which gives an explicit description of the spaces $X(A)$ and the meet homomorphism ρ_A in terms of the elements of A, as well as an explicit construction of an isomorphism α_A from A onto $\mathcal{C}_n(X(A), \rho_A))$.

THEOREM 1.5. *For each $A \in \mathbb{MV}_n$, there is a Boolean space $X(A)$ and a meet homomorphism ρ_A from* $\mathrm{Div}(n-1)$ *into the lattice of closed subsets of $X(A)$, satisfying $\rho_A(n-1) = X(A)$, such that $A \cong \mathcal{C}_{\mathbf{n}}(X(A), \rho_A)$. Moreover, $X(A)$ is isomorphic to the Stone space of the Boolean algebra $B(A)$.*

PROOF. Given $A \in \mathbb{MV}_n$, let $X(A)$ be the set of all homomorphisms $\chi\colon A \to L_n$. Accordingly, $X(B(A))$ denotes the set of all homomorphisms $\chi\colon B(A) \to L_2$. It is well known that the map $\chi \mapsto \chi^{-1}(\{1\}) \cap B(A)$ is a one-one correspondence between $X(A)$ and the set of prime filters of $B(A)$. Therefore, if for each $\chi \in X(A)$, we let $\varphi_A(\chi)$ denote the restriction of χ to $B(A)$, then we obtain a bijective map φ_A from $X(A)$ onto $X(B(A))$.

$X(A)$ becomes a Boolean space with the topology inherited from the product space $(L_n)^A$, where L_n is equipped with the discrete topology. The sets $W_{a,j} = \{\chi \in X(A) : \chi(a) = \frac{j}{n-1}\}$, for $a \in A$ and $0 \leq j \leq n-1$ form a subbasis for this topology. Notice that $X(B(A))$ coincides with the Stone space of the Boolean algebra $B(A)$. We have that $\varphi_A\colon X(A) \to X(B(A))$ is a homeomorphism. Indeed, since $\varphi_A(\chi)(b) = \chi(b)$ for all $\chi \in X(A)$ and all $b \in B(A)$, the inverse image of a clopen subset $X(B(A))$ is clopen in $X(A)$. Therefore φ_A is continuous, and a continuous bijection between compact Hausdorff spaces is a homeomorphism.

To each $a \in A$, associate the function $\hat{a}\colon X(A) \to L_n$ defined by $\hat{a}(\chi) = \chi(a)$ for all $\chi \in X(A)$. By the definition of the topologies in $X(A)$ and in L_n, $\hat{a}$ is continuous. For each $d \in \mathrm{Div}(n-1)$, let $\rho_A(d) = \{\chi \in X(A) : \chi(A) \subseteq L_{d+1}\}$. Since $\rho_A(d) = \bigcap_{a\in A} \hat{a}^{-1}(L_{d+1})$, and L_{d+1} is a clopen subset of L_n, the continuity of the functions $\hat{a}$ for $a \in A$ implies that $\rho_A(d)$ is a closed subset of $X(A)$. Clearly $\rho_A(n-1) = X(A)$, and it easy to check that ρ_A is a meet homomorphism from $\mathrm{Div}(n-1)$ into the lattice of closed subsets of $X(A)$. Taking into account that the operations in $\mathcal{C}_n(X(A), \rho_A)$ are defined pointwise, we have that the correspondence $a \mapsto \hat{a}$ defines an injective homomorphism $\alpha_A\colon A \to \mathcal{C}_n(X(A), \rho_A)$.

To complete the proof we have to show that α_A is surjective. We start by showing that for each $U \in \mathrm{Clop}(X(A))$, there is $b \in B(A)$ such that $\hat{b} = \gamma_U$. Indeed, since $X(B(A))$ is the Stone space of the Boolean algebra $B(A)$, there is $b \in B(A)$ such that for each $\chi \in X(B(A))$, $\gamma_{\varphi_A^{-1}(U)}(\chi) = \chi(b)$. Hence $\gamma_U(\varphi_A(\chi)) = \hat{b}(\varphi(\chi))$ for all $\chi \in X(B(A))$, and since φ_A is surjective, we have that $\gamma_U = \hat{b}$.

Let $f \in \mathcal{C}_n(X(A), \rho_A)$, and suppose that the clopen set $U_j = f^{-1}(\{\frac{j}{n-1}\}) \neq \emptyset$. From the definition of the closed sets $\rho_A(d)$, $d \in \mathrm{Div}(n-1)$, we have that for each $\chi \in X(A)$, $f(\chi) \in \chi(A) \subseteq L_n$. Hence for each $\xi \in U_j$ there is $a_\xi \in A$ such that $\widehat{a_\xi}(\xi) = f(\xi) = \frac{j}{n-1}$. By continuity, there is a clopen V_ξ such that $\chi \in V_\xi \subseteq U_j$ and $\widehat{a_\xi}(\chi) = f(\chi)$ for all $\chi \in V_\xi$. Let $b_\xi \in B(A)$ be such that $\widehat{b_\xi} = \gamma_{V_\xi}$ and let $c_\xi = a_\xi \wedge b_\xi$. Hence $\widehat{c_\xi}(\chi) = f(\chi)$ for $\chi \in V_\xi$ and $\widehat{c_\xi}(\chi) = 0$ for $\chi \notin V_\xi$. By compactness there are $\xi_1, \ldots, \xi_n$ in U_j such that $U_j = V_{\xi_1} \cup \cdots \cup V_{\xi_n}$. Then $c_j = c_{\xi_1} \vee \cdots \vee c_{\xi_n} \in A$ and $\widehat{c_j} = f$ on U_j and $\widehat{c_j}$ is 0 outside U_j. Taking as c the join of the c_j such that $U_j \neq \emptyset$, we have that $\hat{c}(\chi) = f(\chi)$ for all $\chi \in X(A)$. Therefore α_A is surjective. ■

COROLLARY 1.6. *For each $A \in \mathbb{MV}_n$, we have that A is a Boolean algebra if and only if $\rho_A(1) = X(A)$, and that A is a Post algebra of order n if and only if $\rho_A(d) = \emptyset$ for each $d \in \mathrm{Div}^*(n-1)$.*

2. Subalgebras of $\mathcal{C}_n(X, \rho)$

Unless otherwise specified, through this section, A will denote the MV_n-algebra $\mathcal{C}_n(X, \rho)$, where X is a Boolean space, and ρ is a meet-homomorphism from $\mathrm{Div}(n-1)$ into the lattice of closed subsets of X, such that $\rho(n-1) = X$.

For each subalgebra S of A let $\equiv_S$ be the equivalence relation defined on X by the prescription $x \equiv_S y$ if and only if $f(x) = f(y)$ for all $f \in S$.

Let $\Pi(S)$ be the partition of X determined by the equivalence classes of $\equiv_S$, and for each $d \in \mathrm{Div}(n-1)$, let $\Gamma_d(S) = \{\alpha \in \Pi(S) : f(\alpha) \subseteq L_{d+1}$ for all $f \in S\}$. It follows from the definition of $\equiv_S$ that f is constant on each block $\alpha \in \Pi(S)$, for every $f \in S$. Hence if $\alpha \in \Pi(S)$ and $\alpha \cap \rho(d) \neq \emptyset$, then $\alpha \in \Gamma_d(S)$.

We say that a subset $Z \subseteq X$ is *S-saturated* provided that whenever $x \in Z$ and $y \equiv_S x$ imply $y \in Z$. In other words, Z is S-saturated if and only if Z is a union of equivalence classes of $\equiv_S$.

For each subalgebra S of A, we have that $B(S) = S \cap B(A)$. Moreover the set of all $U \in \mathrm{Clop}(X)$ such that $\gamma_U \in S$ form a subalgebra $\mathrm{Clop}_S(X)$ of $\mathrm{Clop}(X)$, which is isomorphic to $B(S)$.

LEMMA 2.1. *Let S be a subalgebra of A. A clopen U belongs to* $\mathrm{Clop}_S(X)$ *if and only if U is S-saturated.*

PROOF. It is obvious that $\gamma_U \in S$ implies that U is S-saturated. To prove the converse, suppose that $\gamma_U \notin \mathrm{Clop}_S(X)$, and let $F = \{V \in \mathrm{Clop}_S(X) :$

$U \subset V\}$. If $V_1, \ldots, V_k$ are in F, then $V_1 \cap \cdots \cap V_k \cap X \setminus U \neq \emptyset$, because otherwise we should have $U \subseteq V_1 \cup \cdots \cup V_k \subseteq U$, contradicting the hypothesis that $U \notin \mathrm{Clop}_S(X)$. Hence $F \cup \{X \setminus U\}$, being a family of clopen sets with the finite intersection property, has a nonempty intersection. Let $v \in (X \setminus U) \cap \bigcap_{V \in F} V$. Let $J = \{W \in \mathrm{Clop}_S(X) : v \notin W\}$. We have that for $W_1, \ldots, W_n \in F$, $U \cap X \setminus (W_1 \cup \ldots \cup W_n) \neq \emptyset$, because otherwise $W_1 \cup \ldots \cup W_n \in F \cap J$, which is impossible because $F \cap J = \emptyset$. Therefore the family formed by the complements of the sets in J together with U has the finite intersection property. Let $u \in U \cap \bigcap_{W \in J}(X \setminus W)$. Since for each $V \in \mathrm{Clop}_S$, $u \in V$ if and only if $v \in V$, it follows that $f(u) = f(v)$ for all $f \in B(S)$. Since for each $f \in S$, $\sigma_i^n f \in B(S)$, we have that $\sigma_i^n f(u) = \sigma_i^n f(v)$ for $i = 1, \ldots, n-1$. Hence by (v) in Lemma 1.2, $f(u) = f(v)$. Therefore $u \equiv_S v$, $u \in U$ and $v \notin U$, which shows that U is not S-saturated. ■

THEOREM 2.2. *Let S be a subalgebra of A. If a function $f \in A$ satisfies the conditions*

(i) *f is constant on each $\alpha \in \Pi(S)$, and*

(ii) *$f(\alpha) \subseteq L_{d+1}$ for each $\alpha \in \Gamma_d(S)$,*

then $f \in S$.

PROOF. For each $z \in X$ let α_z be the only block of $\Pi(S)$ such that $z \in \alpha_z$, and let d be the smallest divisor of $n-1$ such that $f(z) \in L_{d+1}$. Condition (ii) implies that $\alpha_z \in \Gamma_{jd}(S)$ for some $1 \leq j \leq \frac{n-1}{d}$. Then there is $f_z \in S$ such that $f_z(z) = f(z)$.

Suppose $f(z) = f_z(z) = r \in L_{jd}$, and let $U_z = f_z^{-1}(\{r\}) \cap f^{-1}(\{r\})$. Clearly U_z is clopen. We claim that it is saturated. By the definition of $\Pi(S)$ and condition (i), we have that $\alpha_z \subseteq U_z$ and $f(x) = f_z(x)$ for all $x \in \alpha_z$. Let $s \in U_z \setminus \alpha_z$. If $t \in \alpha_s$, then $f_z(t) = f_z(s) = r = f(s)$, and by (i), $f(t) = f(s)$. This proves our claim, and we have that $f_z(x) = f(x)$ for all $x \in U_z$.

Since $\{U_z\}_{z \in X}$ is an open covering of X, by compactness there are points $z_1, \ldots, z_n$ such that $X = U_{z_1} \cup \cdots \cup U_{z_n}$. By Lemma 2.1, $g_i = f_{z_i} \wedge \gamma_{U_{z_i}} \in S$, therefore $f = g_1 \vee \cdots \vee g_n \in S$. ■

Given $d_1, \ldots, d_k \in \mathrm{Div}^*(n-1)$ and subsets $W_1, \ldots, W_k$ of X such that $W_i \cap \rho(d_i) = \emptyset$, let $S(W_1, d_1, \ldots, W_k, d_k) = \{f \in A : f(W_i) \subseteq L_{d_i+1}, i = 1, \ldots, k\}$. It is clear that $S(W_1, d_1, \ldots, W_k, d_k)$ is a proper subalgebra of A that contains $B(A)$. When $k = 1$ and $W_1 = \{w\}$, we write $S(w, d)$ instead of $S(\{w\}, d)$.

COROLLARY 2.3. *The following are equivalent conditions for each subalgebra S of A:*

(i) $B(A) \subseteq S$,

(ii) *S is separating, i. e., given different elements x, y in X there is $f \in S$ such that $f(x) \neq f(y)$,*

(iii) *all the blocks in $\Pi(S)$ are singletons,*

(iv) *$S = S(W_1, d_1, \ldots, W_k, d_k)$ for some $d_1, \ldots, d_k \in \mathrm{Div}^*(n-1)$ and some subsets $W_1, \ldots, W_k$ of X, $k \geq 1$.*

PROOF. Given $x \neq y$ in X, there is an $U \in \mathrm{Clop}(X)$ such that $\gamma_U(x) \neq \gamma_U(y)$. Hence (i) implies (ii). It is obvious that (ii) implies (iii). Suppose that (iii) holds, and let $d_1, \ldots, d_k$ be the proper divisors of $n-1$ such that $\Gamma_{d_i}(S) \neq \emptyset$, and let W_i be the union of all singletons $\{w\} \in \Gamma_{d_i}(S)$ such that $w \notin \rho(d_i)$. It follows from Theorem 2.2 that $S = S(W_1, d_1, \ldots, W_k, d_k)$. Therefore (iii) implies (iv), and we have already observed that (iv) implies (i). ■

If S, T are subalgebras of A such that $S \subseteq T$, then $x \equiv_T y$ implies $x \equiv_S y$ for all $x, y \in X$, and consequently $\Pi(T)$ is a refinement of $\Pi(S)$, that is, each element of $\Pi(S)$ is a union of elements of $\Pi(T)$. Moreover, if $\alpha \in \Gamma_d(T)$ and $\alpha \subseteq \beta \in \Pi(S)$, then $\beta \in \Gamma_d(S)$. The next corollary is an easy consequence of these remarks and Corollary 2.3.

COROLLARY 2.4. *The maximal subalgebras of A containing $B(A)$ are the subalgebras $S(w, d)$, for d a coatom of $\mathrm{Div}(n-1)$ and $z \notin \rho(d)$.* □

Given a set $W \subseteq X$ with $\mathrm{card}(W) \geq 2$, where $\mathrm{card}(W)$ denotes the cardinal of W, define

$$S^W = \{f \in A : f(s) = f(t) \text{ for all } (s, t) \in W \times W\}.$$

Clearly, S^W is a non-separating subalgebra of A, and if x, y are in W, $x \neq y$, then $S^W \subseteq S^{\{x,y\}}$.

THEOREM 2.5. *The maximal subalgebras of A not containing $B(A)$ are the subalgebras $S^{\{s,t\}}$, with $\{s, t\} \subseteq \rho(d)$ or $\{s, t\} \cap \rho(d) = \emptyset$, for each $d \in \mathrm{Div}^*(n-1)$.*

PROOF. Let S be a maximal subalgebra of $A = \mathcal{C}_n(X, \rho)$ such that $B(A) \nsubseteq S$. Since $B(A) \nsubseteq S$, by Corollary 2.3 S is not separating, hence there is $W \subseteq X$ with at least two elements x, y such that $S \subseteq S^W$. The maximality of S implies that $S = S^W$ and that $W = \{x, y\}$. Suppose (absurdum hypothesis) that $x \in \rho(d)$ and $y \notin \rho(d)$. Then we would have $S = S(y, d) = \{f \in A : f(y) \in L_{d+1}\}$, and $B(A) \subseteq S$, a contradiction. Hence x and y are both in $\rho(d)$ or are both in $X \setminus \rho(d)$. To prove the converse, let $S = S^{\{s,t\}}$, with s, t both in $\rho(d)$ or both in $X \setminus \rho(d)$, for each $d \in \mathrm{Div}^*(n-1)$. We have that the only non-singleton block of $\Pi(A)$ is $\{s, t\}$. Suppose that T is a subalgebra of A such that $S \subsetneq T$. Since $\Pi(T)$ is a refinement of $\Pi(S)$, all the blocks of $\Pi(T)$ are singletons. If $f(x) \in L_{d+1}$ for all $f \in T$, then we also have that $f(x) \in L_{d+1}$ for all $f \in S$, and by the hypothesis on s, t, this implies that $x \in \rho(d)$. Hence $\Gamma_d(T)$ is the set of singletons $\{y\}$, for $y \in \rho(d)$ for all $d \in \mathrm{Div}^*(n-1)$. Therefore, taking into account Theorem 2.2, we conclude that $T = A$. Consequently, $S = S^{\{s,t\}}$ is maximal. ■

3. Subalgebras of MV$_n$-algebras

The next theorem generalizes a result of Sachs [25] for Boolean algebras:

THEOREM 3.1. *Every proper subalgebra of an MV$_n$-algebra A is an intersection of maximal subalgebras.*

PROOF. By Theorem 1.5 we can assume that $A = \mathcal{C}_n(X(A), \rho_A)$. Let S be a proper subalgebra of A, and suppose that $f \in A \setminus S$. Then by Theorem 2.2 we have two possible cases:
Case 1: f is non-constant on some $\alpha \in \Pi(S)$, or
Case 2: f is constant on all $\alpha \in \Pi(S)$, but there is $d \in \mathrm{Div}^*(n-1)$ and $\alpha \in \Gamma(S)$ so that $f(\alpha) \nsubseteq L_{d+1}$.

In Case 1, there are $x, y \in \alpha$ such that $f(x) \neq f(y)$. Hence $f \notin S^{\{x,y\}}$, and $S \subseteq S^{\{x,y\}}$.

In Case 2, there is $z \in \alpha$ such that $f(z) \notin L_{d+1}$. Since $f \in A$, this implies that $z \notin \rho(d)$. Therefore $f \notin S(z, d)$, and $S \subseteq S(z, d)$.

Since in all possible cases we have found a maximal subalgebra M such that $f \notin M$ and $S \subseteq M$, we conclude that S is an intersection of maximal subalgebras of A. ■

For $n = 3$ and $n = 4$, the "if" part of the next theorem was proved in [12] by quite different methods, and in that paper was also left open the question of the validity of the "only if" part for these values of n.

With the notations of Remark 1.4, we have:

THEOREM 3.2. *Let $A \in \mathbb{MV}_n$, with $n-1$ a prime. Then $S \subseteq A$ is the universe of a maximal subalgebra of A containing $B(A)$ if and only if there is a minimal prime filter P of A such that $P \subsetneq \varphi(P)$ and $S = P \cup \neg P$.*

PROOF. Since $n-1$ is a prime number, then the only proper subalgebra of L_n is $B(L_n) = L_2$. By Theorem 1.5, $A \cong \mathcal{C}_n(X,\rho)$, with $X = X(A)$ and $\rho = \rho_A$. With the notation of Lemma 1.3, we have that for each $x \in X \setminus \rho(1)$, $S_x = P_x^{n-1} \cup \neg P_x^{n-1}$, where $P_x^{n-1} \subsetneqq P_x^1 = \varphi(P_x^{n-1})$ (see Remark 1.4). Hence the result follows from Corollary 2.4. ∎

From Theorems 3.1 and 3.2 we obtain:

COROLLARY 3.3. *Let $A \in \mathbb{MV}_n$, with $n-1$ a prime. Then $S \subseteq A$ is the universe of a proper subalgebra of A containing $B(A)$ if and only if S is an intersection of maximal subalgebras of the form $P \cup \neg P$, for P a minimal prime filter of A such that $P \subsetneqq \varphi(P)$.* □

THEOREM 3.4. *Let $A \in \mathbb{MV}_3$. Then $S \subseteq A$ is the universe of a maximal subalgebra of A such that $B(A) \nsubseteq S$, if and only if there are two prime filters P_1, P_2 of A such that $P_1 \neq P_2$, $P_i \subseteq \varphi(P_i)$ for $i = 1, 2$, and*

$$S = (P_1 \cap P_2) \cup ((\varphi(P_1) \setminus P_1) \cap (\varphi(P_2) \setminus P_2)) \cup (\neg P_1 \cap \neg P_2). \tag{3.3}$$

Moreover $P_1 = \varphi(P_1)$ if and only if $P_2 = \varphi(P_2)$.

PROOF. By Theorem 1.5, $A \cong \mathcal{C}_3(X,\rho)$, with $X = X(A)$ and $\rho = \rho_A$. By Theorem 2.5, S is the universe of a maximal subalgebra of A not containing $B(A)$ if and only if there are two points, x_1, x_2 in X such that $x_1 \neq x_2$ and $S = S^{\{x_1,x_2\}}$. By Lemma 1.3, $P^2_{x_1}$ and $P^2_{x_2}$ are different minimal prime filters of A. By Remark 1.4, $\varphi(P^2_{x_i}) = P^1_{x_i}$. Note that $P^2_{x_1} = \varphi(P^2_{x_1})$ if and only if $x_1 \in \rho_A(1)$. But in the light of Theorem 2.5, this happens if and only if $x_2 \in \rho_A(1)$, i. e., if and only if $P^2_{x_2} = \varphi(P^2_{x_2})$. Since $f \in P^2_{x_1} \cap P^2_{x_2}$ if and only if $f(x_1) = f(x_2) = 1$, $f \in (\varphi(P^2_{x_1}) \setminus P^2_{x_1}) \cap (\varphi(P^2_{x_2}) \setminus P^2_{x_2})$ if and only if $f(x_1) = f(x_2) = \frac{1}{2}$, $f \in \neg P^2_{x_1} \cap \neg P^2_{x_2}$ if and only if $f(x_1) = f(x_2) = 0$, and $S = S^{\{x_1,x_2\}}$, we have the equality (3.3). ∎

Let $A \in \mathbb{MV}_3$. Maximal subalgebras of A of the form $P \cup \neg P$, with P as in Theorem 3.2, are called of *type I.* Those of the form given by (3.3) in Theorem 3.4 will be called of *type II* when $P_i \neq \varphi(P_i)$, and of *type III*, when $P_i = \varphi(P_i)$, $i = 1, 2$.

The next corollary, which proves the conjecture of A. Monteiro mentioned in the Introduction, is an immediate consequence of Theorems 3.2 and 3.4.

COROLLARY 3.5. *The maximal subalgebras of a MV_3-algebra are of type I, II or III.* □

Since for each prime filter P of a Boolean algebra $\varphi(P) = P$, we obtain the following result of Sachs [25] for Boolean algebras:

COROLLARY 3.6. *The maximal subalgebras of a Boolean algebra are of type III.* □

Taking into account Corollary 1.6 and the fact that all prime filters of a Post algebra of order three satisfy that $P \subsetneqq \varphi(P)$ [1, 3], we have:

COROLLARY 3.7. *The maximal subalgebras of a Post algebra of order three are of type I or III.* □

If A is a finite algebra in $\mathbb{MV}_3$, then $X(A)$ is a finite set endowed with the discrete topology, and $\rho_A(1)$ is a subset of $X(A)$. Hence $A = L_2^m \times L_3^{n-m}$, where $n = \mathrm{card}(X(A))$ and $m = \mathrm{card}(\rho_A(1))$. This shows that isomorphism classes of MV_3-algebras are in one to one correspondence with the pairs of integers (n, m) such that $n \geq 1$ and $0 \leq m \leq n$. The pairs $(n, 0)$ correspond to Post algebras, and the pairs (n, n) corresponds to Boolean algebras.

If p, q are nonnegative integers, we put $\binom{p}{q} = \frac{p\,!}{q!(p-q)!}$ if $p \geq q$, and $\binom{p}{q} = 0$ if $p < q$. With these notations we have:

COROLLARY 3.8. *Let A be a finite MV_3-algebra characterized, up to isomorphisms, by the pair (n, m). Then A has $(n-m)$ maximal subalgebras of type I, $\binom{n-m}{2}$ of type II, and $\binom{m}{2}$ of type III.* □

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