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Cyclic homology of Hopf crossed products

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Abstract

We obtain a mixed complex, simpler than the canonical one, given the Hochschild, cyclic, negative and periodic homology of a crossed product $E = A\#_f H$, where H is an arbitrary Hopf algebra and f is a convolution invertible cocycle with values in A. We actually work in the more general context of relative cyclic homology. Specifically, we consider a subalgebra K of A which is stable under the action of H, and we find a mixed complex computing the Hochschild, cyclic, negative and periodic homology of E relative to K. As an application we obtain two spectral sequences converging to the cyclic homology of E relative to K. The first one works in the general setting and the second one (which generalizes those previously found by several authors) works when f takes its values in K.

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0. Introduction

Let G be a group acting on a differential or algebraic manifold M. Then G acts naturally on the ring A of regular functions of M, and the algebra ${}^G A$ of invariants of this action consists of the functions that are constants on each of the orbits of M. So, the naif idea is considering ${}^G A$ as a replacement for M/G in non-commutative geometry. Under suitable conditions the invariant algebra ${}^G A$ and the smash product $A\#k[G]$, associated with the action of G on A, are Morita equivalent. Since K-theory, Hochschild homology and cyclic homology are Morita invariant, there is no loss of information if ${}^G A$ is replaced by $A\#k[G]$. In the general case the experience has shown that smash products are better choices than invariants rings for algebras playing the role of non-commutative quotients. In fact, except in favorable cases (in which ones the smash product also works) the invariant ring of an action never is used in non-commutative geometry, because it is a very coarse invariant to measure properties of the action. For instance, consider the action $t \cdot (x, y) := (e^{it}x, e^{i\theta t}y)$ of $(\mathbb{R}, +)$ on the torus $S^1 \times S^1$, where θ is an irrational number. Then the ring of invariants of the induced action on the algebra A of regular (continuous, differentiable or analytical) functions is the ring of constant functions, but the associated smash product $A\#\mathbb{R}[(\mathbb{R}, +)]$ has a very rich structure (which depends on θ). This was a motivation for the interest to develop tools to compute the cyclic homology of smash products algebras. This problem was considered in [6,8,14]. In the first paper it was obtained a spectral sequence converging to the cyclic homology of the smash product algebra $A\#k[G]$. In [8], this result was derived from the theory of paracyclic modules and cylindrical modules developed by the authors. The main tool for this computation was a version for cylindrical modules of Eilenberg–Zilber theorem. In [1] this theory was used to obtain a Feigin–Tsygan type spectral sequence for smash products $A\#H$, of a Hopf algebra H with an H-module algebra A.

It is natural to try to extend this result to the general crossed products $A\#_f H$ introduced in [2] and [5]. Crossed Products, and more general algebras such as Hopf Galois extensions, have been homologically studied in several papers (see for instance [9,10,13,15]) but almost all of them deal with its Hochschild (co)homology. In [10] the relative to A cyclic homology of a Galois H extension C/A was studied, and the results obtained apply to the Hopf crossed products $A\#_f H$, giving the absolute cyclic homology when A is a separable algebra. As far as we know, [12] is the only work dealing with the absolute cyclic homology of a crossed product $A\#_f H$, with A non-separable and f non-trivial. In that paper the authors get a Feigin–Tsygan type spectral sequence for a crossed products $A\#_f H$, under the hypothesis that H is cocommutative and f takes values in k.

The goal of this article is to present a mixed complex $(\overline{X}, \overline{d}, \overline{D})$, simpler than the canonical one, giving the Hochschild, cyclic, negative and periodic homology groups of a crossed product $E = A\#_f H$. Under the assumptions of [12] our complex is isomorphic to the one obtained there. We actually work in the more general context of relative cyclic homology. Specifically, we consider a subalgebra K of A which is stable under the action of H (that is $\lambda^h \in K$ for all $\lambda \in K$ and $h \in H$), and we find a mixed complex computing the Hochschild, cyclic, negative and periodic homology groups of E relative to K (which we simply call the Hochschild, cyclic, negative and periodic homology groups of the K-algebra E). Our main result is Theorem 3.2, in which is proved that $(\overline{X}, \overline{d}, \overline{D})$ is homotopically equivalent to the canonical normalized mixed complex $(E \otimes \overline{E}^{\otimes^*} \otimes, b, B)$. As an application we obtain two spectral sequences converging to the cyclic homology of the K-algebra E. The first one works in the general setting and the second one (which generalizes those of [1] and [12]) works when f takes values in K.

Our method of proof is different from that used in [1,8,12], since they are based in the results obtained in [9] and the perturbation lemma instead of a generalization of the Eilenberg–Zilber theorem.

The paper is organized in the following way: in Section 1 we summarize the material on mixed complexes, perturbation lemma and Hochschild homology of Hopf crossed products necessary for our purpose. Moreover we set up notation and terminology. For proofs we refer to [4] and [9]. In Section 2 we obtain a mixed complex $(\widehat{X}, \widehat{d}, \widehat{D})$, simpler that the canonical one, giving the Hochschild, cyclic, periodic and negative homology of the K-algebra $E = A\#_f H$, which works without the usual assumption that f is convolution invertible. Finally in Section 3, we show that when f is convolution invertible, then $(\widehat{X}, \widehat{d}, \widehat{D})$ is isomorphic to a simpler mixed complex $(\overline{X}, \overline{d}, \overline{D})$. Finally, as an application we derive the above mentioned spectral sequences.

1. Preliminaries

Here we fix the general terminology and notation used in the following sections, and give a brief review of the background necessary for the understanding of this paper.

Let k be a commutative ring, A a k-algebra and H a Hopf k-algebra. We will use the Sweedler notation $\Delta(h) = h^{(1)} \otimes_k h^{(2)}$, with the summation implicitly understood and superindices instead of subindices. Recall from [2] and [5] that a *weak action* of H on A is a bilinear map $(h, a) \mapsto a^h$, from $H \times A$ to A, such that for $h \in H$, $a, b \in A$:

(1) $(ab)^h = a^{h^{(1)}} b^{h^{(2)}}$,
(2) $1^h = \epsilon(h)1$,
(3) $a^1 = a$.

Given a weak action of H on A and a k-linear map $f : H \otimes_k H \to A$, we let $A\#_f H$ denote the k-algebra (in general non-associative and without 1) with underlying k-module $A \otimes_k H$ and multiplication map

$$(a \otimes_k h)(b \otimes_k l) = ab^{h^{(1)}} f\big(h^{(2)}, l^{(1)}\big) \otimes_k h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes_k h$ of $A\#_f H$ will usually be written $a\#h$ to remind us H is weakly acting on A. The algebra $E := A\#_f H$ is called a *crossed product* if it is associative with $1\#1$ as identity element. It is easy to check that this happens if and only if f and the weak action satisfy the following conditions:

(i) (Normality of f) for all $h \in H$, we have $f(h, 1) = f(1, h) = \epsilon(h)1_A$.
(ii) (Cocycle condition) for all $h, l, m \in H$, we have

$$f\big(l^{(1)}, m^{(1)}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)} m^{(2)}\big) = f\big(h^{(1)}, l^{(1)}\big) f\big(h^{(2)} l^{(2)}, m\big).$$

(iii) (Twisted module condition) for all $h, l \in H$, $a \in A$ we have

$$\big(a^{l^{(1)}}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)}\big) = f\big(h^{(1)}, l^{(1)}\big) a^{h^{(2)} l^{(2)}}.$$

Next we establish some notations that we will use throughout the paper.

Notations 1.1. Let K be a subalgebra of A and let $C = A$ or $C = E$.

(1) We set $\overline{C} = C/K$ and $\overline{H} = H/k$. Moreover for $c \in C$ we also let c denote its class in $\overline{C}$, and similarly for $h \in H$.
(2) We use the unadorned tensor symbol $\otimes$ to denote the tensor product $\otimes_K$.
(3) We write $\overline{H}^{\otimes_k^l} = \overline{H} \otimes_k \cdots \otimes_k \overline{H}$ (l times) and $\overline{C}^{\otimes^l} = \overline{C} \otimes \cdots \otimes \overline{C}$ (l times).
(4) Given $c_0 \otimes \cdots \otimes c_r \in C^{\otimes^{r+1}}$ and $0 \leqslant i < j \leqslant r$, we write $\mathbf{c}_{ij} = c_i \otimes \cdots \otimes c_j$.
(5) Given $h_1 \otimes_k \cdots \otimes_k h_s \in H^{\otimes_k^s}$ and $1 \leqslant i < j \leqslant s$, we write $\mathbf{h}_{ij} = h_i \otimes_k \cdots \otimes_k h_j$.
(6) Given $\mathbf{h}_{ij} \in H^{\otimes_k^{j-i+1}}$, we set

$$\mathbf{h}_{ij}^{(1)} \otimes_k \mathbf{h}_{ij}^{(2)} = h_i^{(1)} \otimes_k \cdots \otimes_k h_j^{(1)} \otimes_k h_i^{(2)} \otimes_k \cdots \otimes_k h_j^{(2)}.$$

(7) Given $a \in A$ and $\mathbf{h}_{ij} \in H^{\otimes_k^{j-i+1}}$, we write $a^{\mathbf{h}_{ij}} = (\cdots (a^{h_j})^{h_{j-1}} \cdots)^{h_i}$.
(8) Given $\mathbf{a}_{ij} \in A^{\otimes^{j-i+1}}$ and $h \in H$, we write $\mathbf{a}_{ij}^h = a_i^{h^{(1)}} \otimes \cdots \otimes a_j^{h^{(j-i+1)}}$.
(9) The symbol $\gamma(h)$ stands for $1\#h \in E$. Moreover we also use the same symbol to denote its class in E/A.
(10) Given $\mathbf{h}_{ij} \in H^{\otimes_k^{j-i+1}}$, we set

$$\gamma(\mathbf{h}_{ij}) = \gamma(h_i) \otimes \cdots \otimes \gamma(h_j) \quad \text{and} \quad \overline{\gamma(\mathbf{h}_{ij})} = \gamma(h_i) \otimes_A \cdots \otimes_A \gamma(h_j).$$

(11) We will denote by $\mathcal{H}$ the image of the canonical inclusion of H into $A\#H$.
(12) Given $h_1, \dots, h_i \in H$, we will denote by $\langle h_1, \dots, h_i \rangle$ the Hopf subalgebra of H generated by $h_1, \dots, h_i$.

1.1. A simple resolution

Let Υ be the family of all epimorphisms of E-bimodules which split as left E-module maps. In this subsection we review the construction of the Υ-projective resolution (X_*, d_*), of E as an E-bimodule, given in Section 1 of [9]. We are going to modify the sign of some maps in order to obtain expressions for the boundary maps d_* and the comparison maps between (X_*, d_*) and the normalized bar resolution of E, simpler than those of the above mentioned paper. Let K be a subalgebra of A, closed under the weak action of H on A. Since we want to consider the cyclic homology of the K-algebra E, in the sequel Υ will be the family of all epimorphisms of E-bimodules which split as (E, K)-bimodule maps.

For all $r, s \geqslant 0$, let

$$Y_s = E \otimes_A (E/A)^{\otimes_A^s} \otimes_A E$$

and

$$X_{rs} = E \otimes_A (E/A)^{\otimes_A^s} \otimes \overline{A}^{\otimes^r} \otimes E.$$

Consider the diagram of E-bimodules and E-bimodule maps

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\Big\downarrow{\scriptstyle -\partial_2} & & & & & & \\
Y_2 & \xleftarrow{\mu_2} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & \cdots \\
\Big\downarrow{\scriptstyle -\partial_2} & & & & & & \\
Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \cdots \\
\Big\downarrow{\scriptstyle -\partial_1} & & & & & & \\
Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \cdots,
\end{array}$$

where (Y_*, ∂_*) is the normalized bar resolution of the A-algebra E, introduced in [7]; for each $s \geqslant 0$, the complex (X_{*s}, d^0_{*s}) is $(-1)^s$ times the normalized bar resolution of the K-algebra A, tensored on the left over A with $E \otimes_A (E/A)^{\otimes_A^s}$, and on the right over A with E; and for each $s \geqslant 0$, the map μ_s is the canonical projection.

Note that $X_{rs} \simeq E \otimes_k \overline{H}^{\otimes_k^s} \otimes \overline{A}^{\otimes^r} \otimes E$, where the right action of K on $E \otimes_k \overline{H}^{\otimes_k^s}$ is the one obtained by translation of structure through the canonical bijection from $E \otimes_k \overline{H}^{\otimes_k^s}$ to $E \otimes_A (E/A)^{\otimes_A^s}$. Moreover, each one of the rows of this diagram is contractible as an (E, K)-bimodule complex. A contracting homotopy

$$\sigma^0_{0s} : Y_s \to X_{0s} \quad \text{and} \quad \sigma^0_{r+1,s} : X_{rs} \to X_{r+1,s},$$

of the s-th row, is given by

$$\sigma^0_{0s}\big(\overline{\gamma(\mathbf{h}_{0,s+1})}\big) = \overline{\gamma(\mathbf{h}_{0s})} \otimes \gamma(h_{s+1})$$

and

$$\sigma^0_{r+1,s}\big(\overline{\gamma(\mathbf{h}_{0s})} \otimes \mathbf{a}_{1r} \otimes a_{r+1}\gamma(h)\big) = (-1)^{r+s+1}\overline{\gamma(\mathbf{h}_{0s})} \otimes \mathbf{a}_{1,r+1} \otimes \gamma(h).$$

(To see that the maps σ^0 are right K-linear it is necessary to use that K is stable under the action of H.)

Let $\widetilde{\mu} : Y_0 \to E$ be the multiplication map. The complex of E-bimodules

$$E \xleftarrow{-\widetilde{\mu}} Y_0 \xleftarrow{-\partial_1} Y_1 \xleftarrow{-\partial_2} Y_2 \xleftarrow{-\partial_3} Y_3 \xleftarrow{-\partial_4} Y_4 \xleftarrow{-\partial_5} Y_5 \xleftarrow{-\partial_6} \cdots$$

is also contractible as a complex of (E, K)-bimodules. A chain contracting homotopy

$$\sigma^{-1}_0 : E \to Y_0 \quad \text{and} \quad \sigma^{-1}_{s+1} : Y_s \to Y_{s+1} \quad (s \geqslant 0),$$

is given by $\sigma^{-1}_{s+1}(\mathbf{x}) = (-1)^s \mathbf{x} \otimes_A 1_E$.

For $r \geqslant 0$ and $1 \leqslant l \leqslant s$, we define E-bimodule maps $d^l_{rs}: X_{rs} \to X_{r+l-1,s-l}$ recursively on l and r, by:

$$d^l(\mathbf{x}) = \begin{cases} \sigma^0 \circ \partial \circ \mu(\mathbf{x}) & \text{if } l = 1 \text{ and } r = 0, \\ -\sigma^0 \circ d^1 \circ d^0(\mathbf{x}) & \text{if } l = 1 \text{ and } r > 0, \\ -\sum_{j=1}^{l-1} \sigma^0 \circ d^{l-j} \circ d^j(\mathbf{x}) & \text{if } 1 < l \text{ and } r = 0, \\ -\sum_{j=0}^{l-1} \sigma^0 \circ d^{l-j} \circ d^j(\mathbf{x}) & \text{if } 1 < l \text{ and } r > 0, \end{cases}$$

for $\mathbf{x} \in E \otimes_A (E/A)^{\otimes_A^s} \otimes \overline{A}^{\otimes^r} \otimes K$.

Theorem 1.2. *(See [9].) There is an Υ-projective resolution of E*

$$E \xleftarrow{-\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} \cdots, \tag{1.1}$$

where $\mu : X_{00} \to E$ is the multiplication map,

$$X_n = \bigoplus_{r+s=n} X_{rs} \quad \textit{and} \quad d_n = \sum_{l=1}^{n} d^l_{0n} + \sum_{r=1}^{n} \sum_{l=0}^{n-r} d^l_{r,n-r}.$$

In order to carry out our computations we also need to give an explicit contracting homotopy of the resolution (1.1). For this we define maps

$$\sigma^l_{l,s-l} : Y_s \to X_{l,s-l} \quad \text{and} \quad \sigma^l_{r+l+1,s-l} : X_{rs} \to X_{r+l+1,s-l}$$

recursively on l, by:

$$\sigma^l_{r+l+1,s-l} = -\sum_{i=0}^{l-1} \sigma^0 \circ d^{l-i} \circ \sigma^i \quad (0 < l \leqslant s \text{ and } r \geqslant -1).$$

Proposition 1.3. *(See [9].) The family*

$$\overline{\sigma}_0 : E \to X_0, \qquad \overline{\sigma}_{n+1} : X_n \to X_{n+1} \quad (n \geqslant 0),$$

defined by $\overline{\sigma}_0 = \sigma^0_{00} \circ \sigma_0^{-1}$ and

$$\overline{\sigma}_{n+1} = -\sum_{l=0}^{n+1} \sigma^l_{l,n-l+1} \circ \sigma^{-1}_{n+1} \circ \mu_n + \sum_{r=0}^{n} \sum_{l=0}^{n-r} \sigma^l_{r+l+1,n-r-l} \quad (n \geqslant 0),$$

is a contracting homotopy of (1.1).

Let $f\langle h_1, \dots, h_i\rangle := f(\langle h_1, \dots, h_i\rangle \otimes \langle h_1, \dots, h_i\rangle)$ and let $\widetilde{f}\langle h_1, \dots, h_i\rangle$ be the minimal K-subbimodule of A including $f\langle h_1, \dots, h_i\rangle$ and closed under the weak action of $\langle h_1, \dots, h_i\rangle$ on A.

Theorem 1.4. *(See [9].) Let* $\mathbf{x} = \overline{\gamma(\mathbf{h}_{0s})} \otimes \mathbf{a}_{1r} \otimes 1$. *The following assertions hold*:

$$d^1(\mathbf{x}) = \sum_{i=0}^{s-1} (-1)^i \overline{\gamma(\mathbf{h}_{0,i-1})} \otimes_A \gamma(h_i)\gamma(h_{i+1}) \otimes_A \overline{\gamma(\mathbf{h}_{i+1,s})} \otimes \mathbf{a}_{1r} \otimes 1$$
$$+ (-1)^s \overline{\gamma(\mathbf{h}_{0,s-1})} \otimes \mathbf{a}_{1r}^{h_s^{(1)}} \otimes \gamma\bigl(h_s^{(2)}\bigr)$$

and

$$d^2(\mathbf{x}) = (-1)^{s-1} \overline{\gamma(\mathbf{h}_{0,s-2})} \otimes f\bigl(h_{s-1}^{(1)}, h_s^{(1)}\bigr) \overline{*} \mathbf{a}_{1r} \otimes \gamma\bigl(h_{s-1}^{(2)} h_s^{(2)}\bigr),$$

where $f(h,l)\overline{*}\mathbf{a}_{1r} = \sum_{i=0}^{r} (-1)^i (\mathbf{a}_{1i}^{l^{(1)}})^{h^{(1)}} \otimes f(h^{(2)}, l^{(2)}) \otimes \mathbf{a}_{i+1,r}^{h^{(3)} l^{(3)}}$. *Moreover, for each* $l \geqslant 2$, *the map* d_{rs}^l *takes* $\mathbf{x}$ *into the* E*-subbimodule of* $X_{r+l-1,s-l}$ *generated by all the simple tensors*

$$1 \otimes x_1 \otimes_A \cdots \otimes_A x_{s-l} \otimes a_1 \otimes \cdots \otimes a_{r+l-1} \otimes 1$$

with one a_j *in* $f\langle h_1, \dots, h_s\rangle$ *and* $l-2$ *of the others* a_j*'s in* $\widetilde{f}\langle h_1, \dots, h_s\rangle$.

1.1.1. Comparison with the normalized bar resolution

Let $(E \otimes \overline{E}^{\otimes^*} \otimes E, b'_*)$ be the normalized bar resolution of the K-algebra E. As it is well known, the complex

$$E \xleftarrow{\mu} E \otimes E \xleftarrow{b'_1} E \otimes \overline{E} \otimes E \xleftarrow{b'_2} E \otimes \overline{E}^{\otimes^2} \otimes E \xleftarrow{b'_3} \cdots$$

is contractible as a complex of (E, K)-bimodules, with contracting homotopy

$$\xi_0 : E \to E \otimes E, \qquad \xi_{n+1} : E \otimes \overline{E}^{\otimes^n} \otimes E \to E \otimes \overline{E}^{\otimes^{n+1}} \otimes E \quad (n \geqslant 0),$$

given by $\xi_n(\mathbf{x}) = (-1)^n \mathbf{x} \otimes 1$. Let

$$\phi_* : (X_*, d_*) \to \bigl(E \otimes \overline{E}^{\otimes^*} \otimes E, b'_*\bigr) \quad \text{and} \quad \psi_* : \bigl(E \otimes \overline{E}^{\otimes^*} \otimes E, b'_*\bigr) \to (X_*, d_*)$$

be the morphisms of E-bimodule complexes, recursively defined by

$$\phi_0 = \mathrm{id}, \qquad \psi_0 = \mathrm{id}, \qquad \phi_{n+1}(\mathbf{x} \otimes 1) = \xi_{n+1} \circ \phi_n \circ d_{n+1}(\mathbf{x} \otimes 1)$$

and

$$\psi_{n+1}(\mathbf{y} \otimes 1) = \overline{\sigma}_{n+1} \circ \psi_n \circ b'_{n+1}(\mathbf{y} \otimes 1).$$

Proposition 1.5. *(See [9].)* $\psi \circ \phi = \mathrm{id}$ *and* $\phi \circ \psi$ *is homotopically equivalent to the identity map. A homotopy* $\omega_{*+1} : \phi_* \circ \psi_* \to \mathrm{id}_*$ *is recursively defined by*

$$\omega_1 = 0 \quad \textit{and} \quad \omega_{n+1}(\mathbf{x}) = \xi_{n+1} \circ \bigl(\phi_n \circ \psi_n - \mathrm{id} - \omega_n \circ b'_n\bigr)(\mathbf{x}),$$

for $\mathbf{x} \in E \otimes \overline{E}^{\otimes^n} \otimes K$.

Remark 1.6. Since $\omega(E \otimes \overline{E}^{\otimes^{n-1}} \otimes K) \subseteq E \otimes \overline{E}^{\otimes^n} \otimes K$ and ξ vanishes on $E \otimes \overline{E}^{\otimes^n} \otimes K$,

$$\omega(\mathbf{x}_{0n} \otimes 1) = \xi\big(\phi \circ \psi(\mathbf{x}_{0n} \otimes 1) - (-1)^n \omega(\mathbf{x}_{0n})\big).$$

1.1.2. The filtrations of $(E \otimes \overline{E}^{\otimes^*} \otimes E, b'_*)$ and (X_*, d_*)

Let

$$F^i(X_n) = \bigoplus_{0 \leqslant s \leqslant i} E \otimes_A (E/A)^{\otimes_A^s} \otimes \overline{A}^{\otimes^{n-s}} \otimes E$$

and let $F^i(E \otimes \overline{E}^{\otimes^n} \otimes E)$ be the E-subbimodule of $E \otimes \overline{E}^{\otimes^n} \otimes E$ generated by the tensors $1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1$ such that at least $n - i$ of the x_j's belong to A. The normalized bar resolution $(E \otimes \overline{E}^{\otimes^*} \otimes E, b'_*)$ and the resolution (X_*, d_*) are filtered by

$$F^0\big(E \otimes \overline{E}^{\otimes^*} \otimes E\big) \subseteq F^1\big(E \otimes \overline{E}^{\otimes^*} \otimes E\big) \subseteq F^2\big(E \otimes \overline{E}^{\otimes^*} \otimes E\big) \subseteq \cdots$$

and

$$F^0(X_*) \subseteq F^1(X_*) \subseteq F^2(X_*) \subseteq F^3(X_*) \subseteq F^4(X_*) \subseteq F^5(X_*) \subseteq \cdots,$$

respectively. In [9, Proposition 1.2.2] it was proven that the maps ϕ_*, ψ_* and ω_{*+1} preserve filtrations. In Appendix A we improve this result.

1.2. Mixed complexes

In this subsection we recall briefly the notion of mixed complex. For more details about this concept we refer to [11] and [3].

A mixed complex (X, b, B) is a graded k-module $(X_n)_{n \geqslant 0}$, endowed with morphisms $b : X_n \to X_{n-1}$ and $B : X_n \to X_{n+1}$, such that

$$b \circ b = 0, \qquad B \circ B = 0$$

and

$$B \circ b + b \circ B = 0.$$

A morphism of mixed complexes $f : (X, b, B) \to (Y, d, D)$ is a family of maps $f : X_n \to Y_n$, such that $d \circ f = f \circ b$ and $D \circ f = f \circ B$. Let u be a degree 2 variable. A mixed complex $\mathcal{X} = (X, b, B)$ determines a double complex

$$
\mathrm{BP}(\mathcal{X}) = \begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
\cdots & \xleftarrow{B} & X_3u^{-1} & \xleftarrow{B} & X_2u^0 & \xleftarrow{B} & X_1u & \xleftarrow{B} & X_0u^2 \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
\cdots & \xleftarrow{B} & X_2u^{-1} & \xleftarrow{B} & X_1u^0 & \xleftarrow{B} & X_0u & & \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & & & \\
\cdots & \xleftarrow{B} & X_1u^{-1} & \xleftarrow{B} & X_0u^0 & & & & \\
 & & \downarrow{\scriptstyle b} & & & & & & \\
\cdots & \xleftarrow{B} & X_0u^{-1}, & & & & & &
\end{array}
$$

where $b(\mathbf{x}u^i) = b(\mathbf{x})u^i$ and $B(\mathbf{x}u^i) = B(\mathbf{x})u^{i-1}$. By deleting the positively numbered columns we obtain a subcomplex $\mathrm{BN}(\mathcal{X})$ of $\mathrm{BP}(\mathcal{X})$. Let $\mathrm{BN}'(\mathcal{X})$ be the kernel of the canonical surjection from $\mathrm{BN}(\mathcal{X})$ to (X, b). The quotient double complex $\mathrm{BP}(\mathcal{X})/\mathrm{BN}'(\mathcal{X})$ is denoted by $\mathrm{BC}(\mathcal{X})$. The homology groups $\mathrm{HC}_*(\mathcal{X})$, $\mathrm{HN}_*(\mathcal{X})$ and $\mathrm{HP}_*(\mathcal{X})$, of the total complexes of $\mathrm{BC}(\mathcal{X})$, $\mathrm{BN}(\mathcal{X})$ and $\mathrm{BP}(\mathcal{X})$ respectively, are called the cyclic, negative and periodic homology groups of $\mathcal{X}$. The homology $\mathrm{HH}_*(\mathcal{X})$, of (X, b), is called the Hochschild homology of $\mathcal{X}$. Finally, it is clear that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of mixed complexes induces a morphism from the double complex $\mathrm{BP}(\mathcal{X})$ to the double complex $\mathrm{BP}(\mathcal{Y})$.

As usual, given a K-bimodule M, we let $M\otimes$ denote the quotient $M/[M, K]$, where $[M, K]$ is the k-module generated by the commutators $m\lambda - \lambda m$, with $\lambda \in K$ and $m \in M$. Moreover $[m]$ will denote the class of an element $m \in M$ in $M\otimes$. Let C be a k-algebra and $K \subseteq C$ a subalgebra. The normalized mixed complex of the K-algebra C is the mixed complex $(C \otimes \overline{C}^{\otimes^*} \otimes, b, B)$, where b is the canonical Hochschild boundary map and the Connes operator B is given by

$$B\big([c_0 \otimes \cdots \otimes c_r]\big) = \sum_{i=0}^{r} (-1)^{ir} [1 \otimes c_i \otimes \cdots \otimes c_r \otimes c_0 \otimes \cdots \otimes c_{i-1}].$$

The cyclic, negative, periodic and Hochschild homology groups $\mathrm{HC}_*^K(C)$, $\mathrm{HN}_*^K(C)$, $\mathrm{HP}_*^K(C)$ and $\mathrm{HH}_*^K(C)$, of the K-algebra C, are the respective homology groups of $(C \otimes \overline{C}^{\otimes^*} \otimes, b, B)$.

1.3. The perturbation lemma

Next, we recall the perturbation lemma. We give the more general version introduced in [4]. A homotopy equivalence data

$$(Y, \partial) \underset{i}{\overset{p}{\rightleftarrows}} (X, d), \qquad h : X_* \to X_{*+1}, \tag{1.2}$$

consists of the following:

(1) Chain complexes (Y,∂), (X,d) and quasi-isomorphisms i, p between them.
(2) A homotopy h from $i\circ p$ to id.

A perturbation δ of (1.2) is a map $\delta: X_* \to X_{*-1}$ such that $(d+\delta)^2=0$. We call it small if $\mathrm{id}-\delta\circ h$ is invertible. In this case we write $\Lambda=(\mathrm{id}-\delta\circ h)^{-1}\circ\delta$ and we consider

$$\big(Y,\partial^1\big)\underset{i^1}{\overset{p^1}{\rightleftarrows}}(X,d+\delta),\qquad h^1: X_*\to X_{*+1},\tag{1.3}$$

with

$$\partial^1=\partial+p\circ\Lambda\circ i,\quad i^1=i+h\circ\Lambda\circ i,\quad p^1=p+p\circ\Lambda\circ h,\quad h^1=h+h\circ\Lambda\circ h.$$

A deformation retract is a homotopy equivalence data such that $p\circ i=\mathrm{id}$. A deformation retract is called special if $h\circ i=0$, $p\circ h=0$ and $h\circ h=0$.

In all the cases considered in this paper the map $\delta\circ h$ is locally nilpotent, and so $(\mathrm{id}-\delta\circ h)^{-1}=\sum_{n=0}^{\infty}(\delta\circ h)^n$.

Theorem 1.7. *(See [4].) If δ is a small perturbation of the homotopy equivalence data* (1.2)*, then the perturbed data* (1.3) *is a homotopy equivalence. Moreover, if* (1.2) *is a special deformation retract, then* (1.3) *is also.*

2. A mixed complex giving the cyclic homology of a crossed product

Recall that Υ is the family of all epimorphisms of E-bimodules which split as an (E,K)-bimodule map. Since (X_*,d_*) is an Υ-projective resolution of E, the Hochschild homology of the K-algebra E is the homology of $E\otimes_{E^e}(X_*,d_*)$. Write $\widehat{X}_{rs}=E\otimes_A (E/A)^{\otimes_A^s}\otimes\overline{A}^{\otimes^r}\otimes$. It is easy to check that $\widehat{X}_{rs}\simeq E\otimes_{E^e}X_{rs}$. Let $\widehat{d}^l_{rs}:\widehat{X}_{rs}\to\widehat{X}_{r+l-1,s-l}$ be the map induced by $\mathrm{id}_E\otimes_{E^e}d^l_{rs}$. Clearly $\widehat{d}^0_{rs}$ is $(-1)^s$ times the boundary map of the normalized chain Hochschild complex of the K-algebra A, with coefficients in $E\otimes_A(E/A)^{\otimes_A^s}$. Moreover, from Theorem 1.4, it follows easily that

$$\begin{aligned}\widehat{d}^1(\mathbf{x})&=\big[a_0\gamma(h_0)\gamma(h_1)\otimes_A\overline{\gamma(\mathbf{h}_{2s})}\otimes\mathbf{a}_{1r}\big]\\&\quad+\sum_{i=1}^{s-1}(-1)^i\big[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1,i-1})}\otimes_A\gamma(h_i)\gamma(h_{i+1})\otimes_A\overline{\gamma(\mathbf{h}_{i+2,s})}\otimes\mathbf{a}_{1r}\big]\\&\quad+(-1)^s\big[\gamma\big(h_s^{(2)}\big)a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1,s-1})}\otimes\mathbf{a}_{1r}^{h_s^{(1)}}\big]\end{aligned}$$

and

$$\widehat{d}^2(\mathbf{x})=(-1)^{s-1}\big[\gamma\big(h^{(2)}_{s-1}h^{(2)}_s\big)a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{0,s-2})}\otimes f\big(h^{(1)}_{s-1},h^{(1)}_s\big)\bar{*}\mathbf{a}_{1r}\big],$$

where $\mathbf{x}=[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1s})}\otimes\mathbf{a}_{1r}]$ and $f(h,l)\bar{*}\mathbf{a}_{1r}$ is as in Theorem 1.4. With the above identifications the complex $E\otimes_{E^e}(X_*,d_*)$ becomes $(\widehat{X}_*,\widehat{d}_*)$, where

$$\widehat{X}_n = \bigoplus_{r+s=n} \widehat{X}_{rs} \quad \text{and} \quad \widehat{d}_n := \sum_{l=1}^{n} \widehat{d}^l_{0n} + \sum_{r=1}^{n} \sum_{l=0}^{n-r} \widehat{d}^l_{r,n-r}.$$

Let

$$\widehat{\phi}_* : (\widehat{X}_*, \widehat{d}_*) \to \big(E \otimes \overline{E}^{\otimes^*} \otimes, b_*\big) \quad \text{and} \quad \widehat{\psi}_* : \big(E \otimes \overline{E}^{\otimes^*} \otimes, b_*\big) \to (\widehat{X}_*, \widehat{d}_*)$$

be the morphisms of complexes induced by ϕ and ψ, respectively. By Proposition 1.5, we have $\widehat{\psi} \circ \widehat{\phi} = \mathrm{id}$ and $\widehat{\phi} \circ \widehat{\psi}$ is homotopically equivalent to the identity map, being a homotopy $\widehat{\omega}_{*+1} : \widehat{\phi}_* \circ \widehat{\psi}_* \to \mathrm{id}_*$, the family of maps

$$\big(\widehat{\omega}_{n+1} : E \otimes \overline{E}^{\otimes^n} \otimes \to E \otimes \overline{E}^{\otimes^{n+1}} \otimes\big)_{n \geqslant 0},$$

induced by $(\omega_{n+1} : E \otimes \overline{E}^{\otimes^n} \otimes E \to E \otimes \overline{E}^{\otimes^{n+1}} \otimes E)_{n \geqslant 0}$.

2.0.1. The filtrations of $(E \otimes \overline{E}^{\otimes^*} \otimes, b_*)$ and $(\widehat{X}_*, \widehat{d}_*)$

Let

$$F^i(\widehat{X}_n) = \bigoplus_{0 \leqslant s \leqslant i} \widehat{X}_{n-s,s}$$

and let $F^i(E \otimes \overline{E}^{\otimes^n} \otimes)$ be the k-submodule of $E \otimes \overline{E}^{\otimes^n} \otimes$ generated by the classes of the simple tensors $x_0 \otimes \cdots \otimes x_n$ such that at least $n - i$ of the elements $x_1, \ldots, x_n$ belong to A. The normalized Hochschild complex $(E \otimes \overline{E}^{\otimes^*} \otimes, b_*)$ and the complex $(\widehat{X}_*, \widehat{d}_*)$ are filtered by

$$F^0\big(E \otimes \overline{E}^{\otimes^*} \otimes\big) \subseteq F^1\big(E \otimes \overline{E}^{\otimes^*} \otimes\big) \subseteq F^2\big(E \otimes \overline{E}^{\otimes^*} \otimes\big) \subseteq \cdots$$

and

$$F^0(\widehat{X}_*) \subseteq F^1(\widehat{X}_*) \subseteq F^2(\widehat{X}_*) \subseteq \cdots,$$

respectively. From [9, Proposition 1.2.2] it follows immediately that the maps $\widehat{\phi}_*$, $\widehat{\psi}_*$ and $\widehat{\omega}_{*+1}$ preserve filtrations. In Appendix A we improve this result.

Let $\widehat{V}_n \subseteq \widehat{V}'_n$ be the k-submodules of $E \otimes \overline{E}^{\otimes^n} \otimes$ generated by the simple tensors $\mathbf{x}_{0n}$ such that $\#(\{j \geqslant 1\colon\ x_j \notin A \cup \mathcal{H}\}) = 0$ and $\#(\{j \geqslant 1\colon\ x_j \notin A \cup \mathcal{H}\}) \leqslant 1$, respectively.

Let $h_1, \ldots, h_i \in H$. Recall that $f\langle h_1, \ldots, h_i\rangle := f(\langle h_1, \ldots, h_i\rangle \otimes \langle h_1, \ldots, h_i\rangle)$ and $\widetilde{f}\langle h_1, \ldots, h_i\rangle$ is the minimal K-subbimodule of A including $f\langle h_1, \ldots, h_i\rangle$ and closed under the weak action of H. We will denote by $\widehat{C}_n(h_1, \ldots, h_i)$ the k-submodule of $E \otimes \overline{E}^{\otimes^n} \otimes$ generated by the classes of all the simple tensors $x_0 \otimes \cdots \otimes x_n$ with some $x_1, \ldots, x_n$ in $\widetilde{f}\langle h_1, \ldots, h_i\rangle$.

Proposition 2.1. *The map $\widehat{\phi}$ satisfies*

$$\widehat{\phi}\big(\big[a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{1,n-i}\big]\big) = \big[a_0\gamma(h_0) \otimes \gamma(\mathbf{h}_{1i}) * \mathbf{a}_{1,n-i}\big] + \big[a_0\gamma(h_0) \otimes_A \mathbf{x}\big],$$

where $[a_0\gamma(h_0) \otimes_A \mathbf{x}] \in F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes) \cap \widehat{V}_n \cap \widehat{C}_n(h_1, \ldots, h_i)$.

Proof. See Appendix A. □

Proposition 2.2. *If* $\mathbf{x} = [1 \otimes \mathbf{x}_{1n}] \in (F^i(E \otimes \overline{E}^{\otimes^n} \otimes) \cap \widehat{V}_n')$*, then*

$$\widehat{\omega}(\mathbf{x}) \in \big(K \otimes \overline{E}^{\otimes^{n+1}}\big) \cap F^i\big(E \otimes \overline{E}^{\otimes^{n+1}} \otimes\big) \cap \widehat{V}_{n+1}.$$

Proof. See Appendix A. □

Lemma 2.3. *Let* $B_* : E \otimes \overline{E}^{\otimes^*} \otimes \to E \otimes \overline{E}^{\otimes^{*+1}} \otimes$ *be the Connes operator. The composition* $B \circ \widehat{\omega} \circ B \circ \widehat{\phi}$ *is the zero map.*

Proof. Let $\mathbf{x} = [a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{1,n-i}] \in \widehat{X}_{n-i,i}$. By Proposition 2.1,

$$\widehat{\phi}(\mathbf{x}) \in F^i\big(E \otimes \overline{E}^{\otimes^n} \otimes\big) \cap \widehat{V}_n.$$

Hence $B \circ \widehat{\phi}(\mathbf{x}) \in (K \otimes \overline{E}^{\otimes^{n+1}}) \cap F^{i+1}(E \otimes \overline{E}^{\otimes^{n+1}} \otimes) \cap \widehat{V}_{n+1}'$, and so, by Proposition 2.2,

$$\widehat{\omega} \circ B \circ \widehat{\phi}(\mathbf{x}) \in \big(K \otimes \overline{E}^{\otimes^{n+1}} \otimes\big) \cap F^{i+1}\big(E \otimes \overline{E}^{\otimes^{n+1}} \otimes\big) \cap \widehat{V}_{n+2} \subseteq \ker B,$$

as desired. □

For each $n \geqslant 0$, let $\widehat{D}_n : \widehat{X}_n \to \widehat{X}_{n+1}$ be the map $\widehat{D} = \widehat{\psi} \circ B \circ \widehat{\phi}$.

Theorem 2.4. $(\widehat{X}, \widehat{d}, \widehat{D})$ *is a mixed complex giving the Hochschild, cyclic, negative and periodic homology of the K-algebra E. Moreover we have chain complexes maps*

$$\operatorname{Tot}\big(\mathrm{BP}(\widehat{X}, \widehat{d}, \widehat{D})\big) \underset{\widehat{\Phi}}{\overset{\widehat{\Psi}}{\rightleftarrows}} \operatorname{Tot}\big(\mathrm{BP}\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, B\big)\big),$$

given by

$$\widehat{\Phi}_n\big(\mathbf{x}u^i\big) = \widehat{\phi}(\mathbf{x})u^i + \widehat{\omega} \circ B \circ \widehat{\phi}(\mathbf{x})u^{i-1} \quad \text{and} \quad \widehat{\Psi}_n\big(\mathbf{x}u^i\big) = \sum_{j \geqslant 0} \widehat{\psi} \circ (B \circ \widehat{\omega})^j(\mathbf{x})u^{i-j}.$$

These maps satisfy $\widehat{\Psi} \circ \widehat{\Phi} = \mathrm{id}$ *and* $\widehat{\Phi} \circ \widehat{\Psi}$ *is homotopically equivalent to the identity map. A homotopy* $\widehat{\Omega}_{*+1} : \widehat{\Phi}_* \circ \widehat{\Psi}_* \to \mathrm{id}_*$ *is given by*

$$\widehat{\Omega}_{n+1}\big(\mathbf{x}u^i\big) = \sum_{j \geqslant 0} \widehat{\omega} \circ (B \circ \widehat{\omega})^j(\mathbf{x})u^{i-j}.$$

Proof. For each $i \geqslant 0$, let

$$\widehat{\phi}u^i : \widehat{X}_{n-2i}u^i \to \big(E \otimes \overline{E}^{\otimes^{n-2i}} \otimes\big)u^i,$$
$$\widehat{\psi}u^i : \big(E \otimes \overline{E}^{\otimes^{n-2i}} \otimes\big)u^i \to \widehat{X}_{n-2i}u^i$$

and

$$\widehat{\omega}u^i : \big(E \otimes \overline{E}^{\otimes^{n-2i}} \otimes\big)u^i \to \big(E \otimes \overline{E}^{\otimes^{n+1-2i}} \otimes\big)u^i,$$

be the maps defined by $\widehat{\phi}u^i(\mathbf{x}u^i) = \widehat{\phi}(\mathbf{x})u^i$, etcetera. By the comments preceding Lemma 2.3, we have a special deformation retract

$$\operatorname{Tot}\big(\mathrm{BC}(\widehat{X}, \widehat{d}, 0)\big) \underset{\bigoplus_{i\geqslant 0}\widehat{\phi}u^i}{\overset{\bigoplus_{i\geqslant 0}\widehat{\psi}u^i}{\rightleftarrows}} \operatorname{Tot}\big(\mathrm{BC}\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, 0\big)\big), \qquad \bigoplus_{i\geqslant 0}\widehat{\omega}u^i.$$

By applying the perturbation lemma to this datum endowed with the perturbation induced by B, and taking into account Lemma 2.3, we obtain the special deformation retract

$$\operatorname{Tot}\big(\mathrm{BC}(\widehat{X}, \widehat{d}, \widehat{D})\big) \underset{\widehat{\Phi}}{\overset{\widehat{\Psi}}{\rightleftarrows}} \operatorname{Tot}\big(\mathrm{BC}\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, B\big)\big), \qquad \widehat{\Omega}. \tag{2.1}$$

It is easy to see that $\widehat{\Phi}$, $\widehat{\Psi}$ and $\widehat{\Omega}$ commute with the canonical surjections

$$\operatorname{Tot}\big(\mathrm{BC}(\widehat{X}, \widehat{d}, \widehat{D})\big) \to \operatorname{Tot}\big(\mathrm{BC}(\widehat{X}, \widehat{d}, \widehat{D})\big)[2] \tag{2.2}$$

and

$$\operatorname{Tot}\big(\mathrm{BC}\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, B\big)\big) \to \operatorname{Tot}\big(\mathrm{BC}\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, B\big)\big)[2]. \tag{2.3}$$

A standard argument, from these facts, finishes the proof. □

Let $h_1, \dots, h_i \in H$. In the sequel we let $\widehat{J}_n(h_1, \dots, h_i)$ and $H\widehat{J}_{n+1}(h_1, \dots, h_i)$ denote the k-submodules of $\widehat{X}_n$ generated by all the classes of simple tensors $\mathbf{x}_{0s} \otimes \mathbf{a}_{1,n-s} \in E \otimes_A (E/A)^{\otimes_A^s} \otimes \overline{A}^{\otimes^{n-s}}$ with $0 \leqslant s < n$ and some a_j in $f\langle h_1, \dots, h_i\rangle$, and for all the classes of simple tensors $\mathbf{x}_{0s} \otimes \mathbf{a}_{1,n-s}$ with $0 \leqslant s < n$ and some a_j in $\widetilde{f}\langle h_1, \dots, h_i\rangle$, respectively.

Proposition 2.5. *Let $\widehat{R}_i = F^i(E \otimes \overline{E}^{\otimes^n} \otimes) \setminus F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes)$. The following equalities hold*:

(1) $\widehat{\psi}([a_0\gamma(h_0) \otimes \gamma(\mathbf{h}_{1i}) \otimes \mathbf{a}_{i+1,n}]) = [a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{i+1,n}]$.
(2) *If $\mathbf{x}_{0n} \in \widehat{R}_i \cap \widehat{V}_n$ and there is $1 \leqslant j \leqslant i$ such that $x_j \in A$, then $\widehat{\psi}(\mathbf{x}_{0n}) = 0$.*
(3) *If $\mathbf{x} = [a_0\gamma(h_0) \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes a_i\gamma(h_i) \otimes \mathbf{a}_{i+1,n}]$, then*

$$\begin{aligned}\widehat{\psi}(\mathbf{x}) \equiv{}& \big[a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes_A a_i\gamma(h_i) \otimes \mathbf{a}_{i+1,n}\big] \\ &+ \big[\gamma\big(h_i^{(2)}\big)a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes a_i \otimes \mathbf{a}_{i+1,n}^{h_i^{(1)}}\big],\end{aligned}$$

module $\bigoplus_{l=0}^{i-2}(\widehat{X}_{n-l,l} \cap \widehat{J}_n(h_1, \dots, h_i))$.

(4) *If* $\mathbf{x}=[a_0\gamma(h_0)\otimes\gamma(\mathbf{h}_{1,j-1})\otimes a_j\gamma(h_j)\otimes\gamma(\mathbf{h}_{j+1,i})\otimes\mathbf{a}_{i+1,n}]$ *with* $j<i$, *then*

$$\widehat{\psi}(\mathbf{x})\equiv\big[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1,j-1})}\otimes_A a_j\gamma(h_j)\otimes_A\overline{\gamma(\mathbf{h}_{j+1,i})}\otimes\mathbf{a}_{i+1,n}\big],$$

module $\bigoplus_{l=0}^{i-2}(\widehat{X}_{n-l,l}\cap\widehat{J}_n(h_1,\dots,h_i))$.

(5) *If* $\mathbf{x}=[a_0\gamma(h_0)\otimes\gamma(\mathbf{h}_{1,i-1})\otimes\mathbf{a}_{i,j-1}\otimes a_j\gamma(h_j)\otimes\mathbf{a}_{j+1,n}]$ *with* $j>i$, *then*

$$\widehat{\psi}(\mathbf{x})\equiv\big[\gamma\big(h_j^{(2)}\big)a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1,i-1})}\otimes\mathbf{a}_{ij}\otimes\mathbf{a}_{j+1,n}^{h_j^{(1)}}\big],$$

module $\bigoplus_{l=0}^{i-2}(\widehat{X}_{n-l,l}\cap\widehat{J}_n(h_1,\dots,h_{i-1},h_j))$.

(6) *If* $\mathbf{x}_{0n}\in\widehat{R}_i\cap\widehat{V}'_n$ *and there exists* $1\leqslant j_1<j_2\leqslant n$ *such that* $x_{j_1}\in A$ *and* $x_{j_2}\in\mathcal{H}$, *then* $\widehat{\psi}(\mathbf{x}_{0n})=0$.

Proof. See Appendix A. □

Let $\widehat{\eta}_n:\widehat{X}_n\to\widehat{X}_{n+1}$, $\widehat{t}_{H,n}:\widehat{X}_n\to\widehat{X}_n$ and $\widehat{t}_{A,n}:\widehat{X}_{n+1}\to\widehat{X}_{n+1}$ be the k-linear maps defined by

$$\widehat{\eta}\big(\big[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1i})}\otimes\mathbf{a}_{1,n-i}\big]\big)=\big[\overline{\gamma(\mathbf{h}_{0i})}\otimes\mathbf{a}_{1,n-i}\otimes a_0\big],$$

$$\widehat{t}_H\big(\big[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1i})}\otimes\mathbf{a}_{1,n-i}\big]\big)=\big[\gamma\big(h_i^{(2)}\big)\otimes_A a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1,i-1})}\otimes\mathbf{a}_{1,n-i}^{h_i^{(1)}}\big]$$

and

$$\widehat{t}_A\big(\big[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1i})}\otimes\mathbf{a}_{1,n-i+1}\big]\big)=\big[\overline{\gamma\big(\mathbf{h}_{0i}^{(2)}\big)}\otimes\mathbf{a}_{2,n-i+1}\otimes a_0a_1^{\mathbf{h}_{0i}^{(1)}}\big],$$

respectively.

Proposition 2.6. *The Connes operator* $\widehat{D}$ *satisfies*:

(1) *If* $\mathbf{x}=[a_0\otimes_A\overline{\gamma(\mathbf{h}_{1i})}\otimes\mathbf{a}_{1,n-i}]$, *then*

$$\widehat{D}(\mathbf{x})=\sum_{j=0}^{n-i}(-1)^{j(n-i)+n}\widehat{t}_A^{\,j}\circ\widehat{\eta}(\mathbf{x}),$$

module $F^{i-1}(\widehat{X}_{n+1})\cap H\widehat{J}_{n+1}(h_1,\dots,h_i)$.

(2) *If* $\mathbf{x}=[a_0\gamma(h_0)\otimes_A\overline{\gamma(\mathbf{h}_{1i})}\otimes\mathbf{a}_{1,n-i}]$ *with* $a_0\gamma(h_0)\notin A$, *then*

$$\widehat{D}(\mathbf{x})=\sum_{j=0}^{i}(-1)^{ji}1\otimes_A\widehat{t}_H^{\,j}(\mathbf{x})+\sum_{j=0}^{n-i}(-1)^{j(n-i)+n}\widehat{t}_A^{\,j}\circ\widehat{\eta}(\mathbf{x})$$

module $F^{i}(\widehat{X}_{n+1})\cap H\widehat{J}_{n+1}(h_1,\dots,h_i)$.

Proof. It is a direct consequence of the definition of B, Propositions 2.1 and 2.5. We leave the details to the reader. □

3. The cyclic homology of a crossed product with invertible cocycle

Let $E = A\#_f H$. Assume that the cocycle f is invertible. Then, the map γ is convolution invertible and its inverse is given by $\gamma^{-1}(h) = f^{-1}(S(h^{(2)}), h^{(3)})\#S(h^{(1)})$. In [9] it was proven that under this hypothesis the complex $(\widehat{X}_*, \widehat{d}_*)$ of Section 2 is isomorphic to a simpler complex $(\overline{X}_*, \overline{d}_*)$. In this section we obtain a similar result for the mixed complex $(\widehat{X}, \widehat{d}, \widehat{D})$.

For each $r, s \geqslant 0$, let

$$\overline{X}_{rs} = \big(E \otimes \overline{A}^{\otimes^r} \otimes\big) \otimes_k \overline{H}^{\otimes^s_k}.$$

The map $\theta_{rs} : \widehat{X}_{rs} \to \overline{X}_{rs}$, defined by

$$\theta_{rs}(\mathbf{x}) = (-1)^{rs}\big[a_0\gamma(h_0)a_1\gamma\big(h_1^{(1)}\big)\cdots a_s\gamma\big(h_s^{(1)}\big) \otimes \mathbf{a}_{s+1,s+r}\big] \otimes_k \mathbf{h}_{1s}^{(2)},$$

where $\mathbf{x} = [a_0\gamma(h_0) \otimes_A \cdots \otimes_A a_s\gamma(h_s) \otimes \mathbf{a}_{s+1,s+r}]$, is an isomorphism. The inverse map of θ_{rs} is the map given by

$$\big[a_0\gamma(h_0) \otimes \mathbf{a}_{1r}\big] \otimes_k \mathbf{h}_{1s} \mapsto (-1)^{rs}\big[a_0\gamma(h_0)\gamma^{-1}\big(h_s^{(1)}\big)\cdots\gamma^{-1}\big(h_1^{(1)}\big) \otimes_A \overline{\gamma\big(\mathbf{h}_{1s}^{(2)}\big)} \otimes \mathbf{a}_{1r}\big].$$

Let $\overline{d}^l_{rs} : \overline{X}_{rs} \to \overline{X}_{r+l-1,s-l}$ be the map $\overline{d}^l_{rs} := \theta_{r+l-1,s-l} \circ \widehat{d}^l_{rs} \circ \theta_{rs}^{-1}$. In the absolute case the following result was obtained in [9]. The generalization to the relative context is direct.

Theorem 3.1. *The Hochschild homology of the K-algebra E is the homology of $(\overline{X}_*, \overline{d}_*)$, where*

$$\overline{X}_n = \bigoplus_{r+s=n} \overline{X}_{rs} \quad \text{and} \quad \overline{d}_n := \sum_{l=1}^{n} \overline{d}^l_{0n} + \sum_{r=1}^{n}\sum_{l=0}^{n-r} \overline{d}^l_{r,n-r}.$$

Moreover $\overline{d}^0_{rs}$ is the boundary map of the normalized chain Hochschild complex of the K-algebra A, with coefficients in E, tensored on the right over k with $\mathrm{id}_{\overline{H}^{\otimes^s}}$,

$$\begin{aligned}\overline{d}^1_{rs}(\mathbf{x}) &= (-1)^{r+s}\big[\gamma\big(h_s^{(3)}\big)a_0\gamma(h_0)\gamma^{-1}\big(h_s^{(1)}\big) \otimes \mathbf{a}_{1r}^{h_s^{(2)}}\big] \otimes_k \mathbf{h}_{1,s-1}\\ &\quad+ \sum_{i=1}^{s-1}(-1)^{r+i}\big[a_0\gamma(h_0) \otimes \mathbf{a}_{1r}\big] \otimes_k \mathbf{h}_{1,i-1} \otimes_k h_ih_{i+1} \otimes_k \mathbf{h}_{i+2,s}\\ &\quad+ (-1)^r\big[a_0\gamma(h_0)\epsilon(h_1) \otimes \mathbf{a}_{1r}\big] \otimes_k \mathbf{h}_{2s}\end{aligned}$$

and

$$\begin{aligned}\overline{d}^2_{rs}(\mathbf{x}) &= \sum_{i=0}^{r}(-1)^{i-1}\big[\gamma\big(h_{s-1}^{(5)}h_s^{(5)}\big)a_0\gamma(h_0)\gamma^{-1}\big(h_s^{(1)}\big)\gamma^{-1}\big(h_{s-1}^{(1)}\big)\\ &\quad\otimes \big(\mathbf{a}_{1i}^{h_s^{(2)}}\big)^{h_{s-1}^{(2)}} \otimes f\big(h_{s-1}^{(3)}, h_s^{(3)}\big) \otimes \mathbf{a}_{i+1,r}^{h_{s-1}^{(4)}h_s^{(4)}}\big] \otimes_k \mathbf{h}_{1,s-2},\end{aligned}$$

where $\mathbf{x} = [a_0\gamma(h_0) \otimes \mathbf{a}_{1r}] \otimes_k \mathbf{h}_{1s}$.

For each $n \geqslant 0$, let $\overline{D}_n = \theta_n \circ \widehat{D}_n \circ \theta_n^{-1}$.

Theorem 3.2. *$(\overline{X}, \overline{d}, \overline{D})$ is a mixed complex giving the Hochschild, cyclic, negative and periodic homology of E. More precisely, the mixed complexes $(\overline{X}, \overline{d}, \overline{D})$ and $(E \otimes \overline{E}^{\otimes^*}, b, B)$ are homotopically equivalent.*

Proof. Clearly $(\overline{X}, \overline{d}, \overline{D})$ is a mixed complex and $\theta : (\widehat{X}, \widehat{d}, \widehat{D}) \to (\overline{X}, \overline{d}, \overline{D})$ is an isomorphism of mixed complexes. So the result follows from Theorem 2.4. □

We are now going to obtain a formula for $\overline{D}$. To do this we need to introduce a map $T : H^{\otimes_k^{i+1}} \to A$ such that

$$\gamma(h_0)\gamma^{-1}(h_i)\cdots\gamma^{-1}(h_1) = T\big(h_0^{(1)}, S(h_1)^{(1)}, \dots, S(h_i)^{(1)}\big)\gamma\big(h_0^{(2)} S(h_i)^{(2)} \cdots S(h_1)^{(2)}\big).$$

To abbreviate notations we set

$$\zeta = \gamma^{-1} \circ S^{-1} \quad \text{and} \quad U(\mathbf{h}_{0i}) = T\big(h_0, S(h_1), \dots, S(h_i)\big).$$

Since

$$\gamma(h_0)\gamma^{-1}(h_i)\cdots\gamma^{-1}(h_1) = \gamma(h_0)\zeta\big(S(h_i)\big)\cdots\zeta\big(S(h_1)\big),$$

we can solve

$$\begin{aligned} U(\mathbf{h}_{0i}) &= \gamma\big(h_0^{(1)}\big)\zeta\big(S(h_i)^{(1)}\big)\cdots\zeta\big(S(h_1)^{(1)}\big)\gamma^{-1}\big(h_0^{(2)} S(h_1\cdots h_i)^{(2)}\big) \\ &= \gamma\big(h_0^{(1)}\big)\zeta\big(S\big(h_i^{(2)}\big)\big)\cdots\zeta\big(S\big(h_1^{(2)}\big)\big)\gamma^{-1}\big(h_0^{(2)} S\big(h_1^{(1)}\cdots h_i^{(1)}\big)\big) \\ &= \gamma\big(h_0^{(1)}\big)\gamma^{-1}\big(h_i^{(2)}\big)\cdots\gamma^{-1}\big(h_1^{(2)}\big)\gamma^{-1}\big(h_0^{(2)} S\big(h_1^{(1)}\cdots h_i^{(1)}\big)\big). \end{aligned}$$

We must check that $T(h_0, S(h_1), \dots, S(h_i)) \in A$. For this purpose it suffices to see that this element is coinvariant under the coaction $\nu = \mathrm{id} \otimes \Delta$ of $A\#_f H$, which follows easily because $\nu(\gamma^{-1}(h)) = \gamma^{-1}(h^{(2)}) \otimes S(h^{(1)})$ and $A\#_f H$ is a comodule algebra. Note that

$$a_0\gamma(h_0)\gamma^{-1}(h_i)\cdots\gamma^{-1}(h_1) = a_0 U\big(h_0^{(1)}, \mathbf{h}_{1i}^{(2)}\big)\gamma\big(h_0^{(2)} S\big(h_1^{(1)}\cdots h_i^{(1)}\big)\big).$$

For each $0 \leqslant i \leqslant n$, let $F^i(\overline{X}_n) = \bigoplus_{0\leqslant s\leqslant i} \overline{X}_{n-s,s}$. The complex $(\overline{X}_*, \overline{d}_*)$ is filtered by $F^0(\overline{X}_*) \subseteq F^1(\overline{X}_*) \subseteq F^2(\overline{X}_*) \subseteq \cdots$.

Given elements $h_1, \dots, h_i \in H$, we let $H\overline{J}_n(h_1, \dots, h_i)$ denote the k-submodule of $\overline{X}_n$ generated by all the elements $[a_0\gamma(h_0) \otimes \mathbf{a}_{1r}] \otimes_k \mathbf{h}_{1s}$, with $r > 0$ and some $a_j \in \widetilde{f}\langle h_1, \dots, h_i\rangle$ (for the definition of this last expression see the discussion above Theorem 1.4).

Let $\overline{\eta}_n : \overline{X}_n \to \overline{X}_{n+1}$ and $\overline{t}_{H,n} : \overline{X}_{n+1} \to \overline{X}_{n+1}$ be the k-linear maps defined by

$$\overline{\eta}(\mathbf{x}) = \big[a_0\gamma\big(h_0^{(1)}\big) \otimes \mathbf{a}_{1,n-i}\big] \otimes_k h_0^{(2)} S\big(h_1^{(1)}\cdots h_i^{(1)}\big) \otimes_k \mathbf{h}_{1i}^{(2)}$$

and

$$\bar{t}_H(\mathbf{y}) = \left[\gamma\left(h_{i+1}^{(3)}\right)a_0\gamma(h_0)\gamma^{-1}\left(h_{i+1}^{(1)}\right) \otimes \mathbf{a}_{1,n-i}^{h_{i+1}^{(2)}}\right] \otimes_k h_{i+1}^{(4)} \otimes_k \mathbf{h}_{1i},$$

where

$$\mathbf{x} = \left[a_0\gamma(h_0) \otimes \mathbf{a}_{1,n-i}\right] \otimes_k \mathbf{h}_{1i} \quad \text{and} \quad \mathbf{y} = \left[a_0\gamma(h_0) \otimes \mathbf{a}_{1,n-i}\right] \otimes_k \mathbf{h}_{1,i+1},$$

respectively.

Theorem 3.3. *If* $\mathbf{x} = [a_0\gamma(h_0) \otimes \mathbf{a}_{1,n-i}] \otimes_k \mathbf{h}_{1i}$, *then*

$$\begin{aligned}\overline{D}(\mathbf{x}) = {} & \sum_{j=0}^{i}(-1)^{ji+n-i}\bar{t}_H^{\,j} \circ \bar{\eta}(\mathbf{x}) \\ & + \sum_{j=0}^{n-i}(-1)^{(j+1)(n-i)}\left[\gamma\left(h_0^{(3)}S\left(h_1^{(1)}\cdots h_i^{(1)}\right)\right)\gamma\left(h_1^{(5)}\right)\cdots\gamma\left(h_i^{(5)}\right)\right. \\ & \left.\otimes \mathbf{a}_{j+1,n-i} \otimes a_0U\left(h_0^{(1)}, \mathbf{h}_{1i}^{(3)}\right) \otimes \left(\mathbf{a}_{1j}^{\mathbf{h}_{1i}^{(4)}}\right)^{h_0^{(2)}S(h_1^{(2)}\cdots h_i^{(2)})}\right] \otimes_k \mathbf{h}_{1i}^{(6)},\end{aligned}$$

module $F^i(\overline{X}_{n+1}) \cap H\overline{J}_{n+1}(h_1, \dots, h_i)$.

Proof. It follows straightforwardly from Proposition 2.6, the equality $\overline{D} = \theta \circ \widehat{D} \circ \theta^{-1}$, and the formulas of θ and θ^{-1}. □

3.1. First spectral sequence

Arguing as in [9, Proposition 3.2] we see that, for each $h \in H$, there is a morphism of complexes

$$\vartheta_*^h : \left(E \otimes \overline{A}^{\otimes^*} \otimes, b_*\right) \to \left(E \otimes \overline{A}^{\otimes^*} \otimes, b_*\right),$$

which is given by $\vartheta_r^h([a_0\gamma(h_0) \otimes \mathbf{a}_{1r}]) = [\gamma(h^{(3)})a_0\gamma(h_0)\gamma^{-1}(h^{(1)}) \otimes \mathbf{a}_{1r}^{h^{(2)}}]$ and that, for each $h, l \in H$, the endomorphisms of $\mathrm{H}_*^K(A, E)$ induced by $\vartheta_*^h \circ \vartheta_*^l$ and by ϑ_*^{hl} coincide. So, $\mathrm{H}_*^K(A, E)$ is a left H-module. Let

$$\widetilde{d}_s : \mathrm{H}_r^K(A, E) \otimes_k \overline{H}^{\otimes^s} \to \mathrm{H}_r^K(A, E) \otimes_k \overline{H}^{\otimes^{s-1}}$$

and

$$\widetilde{D}_s : \mathrm{H}_r^K(A, E) \otimes_k \overline{H}^{\otimes^s} \to \mathrm{H}_r^K(A, E) \otimes_k \overline{H}^{\otimes^{s+1}}$$

be the maps induced by $\bar{d}_{rs}^1$ and $\sum_{j=0}^{s}(-1)^{js+r}\bar{t}_H^{\,j} \circ \bar{\eta}_{r+s}$, respectively.

Proposition 3.4. *Assume that $\overline{H}$ is a flat k-module. For each $r \geqslant 0$,*

$$\widetilde{\mathrm{H}_r^K(A,E)} := \left(\mathrm{H}_r^K(A,E) \otimes_k \overline{H}^{\otimes^*}, \widetilde{d}_*, \widetilde{D}_*\right)$$

is a mixed complex and there is a convergent spectral sequence

$$E_{sr}^2 = \mathrm{HC}_s\left(\widetilde{\mathrm{H}_r^K(A,E)}\right) \quad \Rightarrow \quad \mathrm{HC}_{r+s}^K(E).$$

Proof. Consider the spectral sequence $(E_{sr}^v, d_{sr}^v)_{v \geqslant 0}$, associated with the filtration

$$F^0\left(\mathrm{Tot}\left(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D})\right)\right) \subseteq F^1\left(\mathrm{Tot}\left(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D})\right)\right) \subseteq F^2\left(\mathrm{Tot}\left(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D})\right)\right) \subseteq \cdots$$

of the complex $\mathrm{Tot}(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D}))$, given by

$$F^i\left(\mathrm{Tot}\left(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D})\right)_n\right) = \bigoplus_{j \geqslant 0} F^{i-2j}(\overline{X}_{n-2j})u^j.$$

A straightforward computation shows that:

- $E_{sr}^0 = \bigoplus_{j \geqslant 0}((E \otimes \overline{A}^{\otimes^r} \otimes) \otimes_k \overline{H}^{\otimes^{s-2j}})u^j$ and d_{sr}^0 is $\bigoplus_{j \geqslant 0} \bar{d}_{r,s-2j}^0 u^j$,
- $E_{sr}^1 = \bigoplus_{j \geqslant 0}(\mathrm{H}_r(A,E) \otimes_k \overline{H}^{\otimes^{s-2j}})u^j$ and d_{sr}^1 is $\widetilde{d} + \widetilde{D}$.

From this it follows easily that $\widetilde{\mathrm{H}_r^K(A,E)}$ is a mixed complex and

$$E_{sr}^2 = \mathrm{HC}_s\left(\widetilde{\mathrm{H}_r^K(A,E)}\right).$$

In order to finish the proof it suffices to note that the filtration of $\mathrm{Tot}(\mathrm{BC}(\overline{X}, \bar{d}, \overline{D}))$ introduced above is canonically bounded, and so, by Theorem 3.2, the spectral sequence $(E_{sr}^v)_{v \geqslant 0}$ converges to the cyclic homology of the K-algebra E. □

Corollary 3.5. *If $\mathrm{H}_i^K(A,E) = 0$ for all $i > 0$, then $\mathrm{HC}_n^K(E) = \mathrm{HC}_n(\widetilde{\mathrm{H}_0^K(A,E)})$.*

Proposition 3.6. *Assume H is a separable algebra and let t be the integral of H satisfying $\epsilon(t) = 1$. Then*

$$E_{sr}^2 = \begin{cases} \mathrm{H}_0(H, \widetilde{\mathrm{H}_r^K(A,E)}) & \text{if } s \text{ is even,} \\ 0 & \text{if } s \text{ is odd,} \end{cases}$$

and for s even the map $d_{sr}^2 : E_{sr}^2 \to E_{s-2,r+1}^2$ is given by

$$d^2\left(\overline{\sum [a_0\gamma(h) \otimes \mathbf{a}_{1r}]}\right) = \sum_{j=0}^{r} \overline{\sum (-1)^{(j+1)r}\left[\gamma\left(h^{(2)}\right) \otimes a_{j+1,r} \otimes a_0 \otimes a_{1j}^{h^{(1)}}\right]}$$

$$+\sum_{j=0}^{r}(-1)^j\overline{\sum\left[\gamma\left(t^{(5)}h^{(4)}\right)a_0\gamma^{-1}\left(t^{(1)}\right)\otimes\left(\mathbf{a}_{1j}^{h^{(1)}}\right)^{t^{(2)}}\otimes f\left(t^{(3)},h^{(2)}\right)\otimes\mathbf{a}_{j+1,r}^{t^{(4)}h^{(3)}}\right]},$$

where $\sum[a_0\widetilde{\gamma(h)\otimes}\mathbf{a}_{1r}]$ *is an* r*-cycle of* $(E\otimes\overline{A}^{\otimes^*}\otimes, b_*)$ *and* $\overline{\sum[a_0\gamma(h)\otimes\mathbf{a}_{1r}]}$ *denotes its class in* $\mathrm{H}_0(H,\mathrm{H}_r^K(A,E))$*, and similarly for the other terms.*

Proof. The first assertion is trivial and the second one follows from a direct computation using the construction of the spectral sequence of a filtrated complex. To prove this it is convenient to note that

$$\bar{t}_H\circ\bar{\eta}\left(\left[a_0\gamma(h)\otimes\mathbf{a}_{1r}\right]\right)-\bar{d}^1\left(\left[a_0\gamma\left(h^{(1)}\right)\otimes\mathbf{a}_{1r}\right]\otimes_k t\otimes_k h^{(2)}\right)\in\mathrm{Im}(\widetilde{d}_s).$$

We leave the details to the reader. □

3.2. Second spectral sequence

In this subsection we assume that f takes values in K. Under this hypothesis the maps $\overline{d}^l$ vanish for all $l\geqslant 2$ and we obtain a spectral sequence that generalizes those given in [1] and [12].

For each $r\geqslant 0$, we define a map

$$\begin{aligned} H\otimes_k\left(E\otimes\overline{A}^{\otimes^r}\otimes\right)&\longrightarrow E\otimes\overline{A}^{\otimes^r}\otimes,\\ h\otimes\mathbf{x}&\longmapsto h\blacktriangleright\mathbf{x}\end{aligned}$$

by $h\blacktriangleright[a\gamma(u)\otimes\mathbf{a}_{1r}]=[\gamma(h^{(3)})a\gamma(u)\gamma^{-1}(h^{(1)})\otimes\mathbf{a}_{1r}^{h^{(2)}}]$.

Proposition 3.7. *For each* $r\geqslant 0$ *the map* $\blacktriangleright$ *is an action of* H *on* $E\otimes\overline{A}^{\otimes^r}\otimes$.

Proof. It is trivial that $\blacktriangleright$ is unitary. Next we verify the associative property. By definition

$$l\blacktriangleright\left(h\blacktriangleright\left[a\gamma(u)\otimes\mathbf{a}_{1r}\right]\right)=\left[\gamma\left(l^{(3)}\right)\gamma\left(h^{(3)}\right)a\gamma(u)\gamma^{-1}\left(h^{(1)}\right)\gamma^{-1}\left(l^{(1)}\right)\otimes\left(\mathbf{a}_{1r}^{h^{(2)}}\right)^{l^{(2)}}\right].$$

Since

$$\left(\mathbf{a}_{1r}^{h}\right)^l=f\left(l^{(1)},h^{(1)}\right)\mathbf{a}_{1r}^{l^{(2)}h^{(2)}}f^{-1}\left(l^{(3)},h^{(3)}\right),\qquad\gamma(l)\gamma(h)=f\left(l^{(1)},h^{(1)}\right)\gamma\left(l^{(2)}h^{(2)}\right)$$

and f^{-1} is the convolution inverse of f, we have

$$\begin{aligned}&l\blacktriangleright\left(h\blacktriangleright\left[a\gamma(u)\otimes\mathbf{a}_{1r}\right]\right)\\&\quad=\left[\gamma\left(l^{(4)}h^{(4)}\right)a\gamma(u)\gamma^{-1}\left(h^{(1)}\right)\gamma^{-1}\left(l^{(1)}\right)\otimes f\left(l^{(2)},h^{(2)}\right)\mathbf{a}_{1r}^{l^{(3)}h^{(3)}}\right].\end{aligned}$$

By the twisted module condition applied twice,

$$\begin{aligned}\gamma^{-1}(h)\gamma^{-1}(l) &= f^{-1}\big(S\big(h^{(2)}\big),h^{(3)}\big)\gamma\big(S\big(h^{(1)}\big)\big)f^{-1}\big(S\big(l^{(2)}\big),l^{(3)}\big)\gamma\big(S\big(l^{(1)}\big)\big)\\ &= f^{-1}\big(S\big(h^{(3)}\big),h^{(4)}\big)f^{-1}\big(S\big(l^{(3)}\big),l^{(4)}\big)f\big(S\big(h^{(2)}\big),S\big(l^{(2)}\big)\big)\gamma\big(S\big(l^{(1)}h^{(1)}\big)\big)\\ &= f^{-1}\big(S\big(h^{(3)}\big),h^{(4)}\big)f^{-1}\big(S\big(h^{(2)}\big)S\big(l^{(2)}\big),l^{(3)}\big)\gamma\big(S\big(l^{(1)}h^{(1)}\big)\big)\\ &= f^{-1}\big(S\big(l^{(3)}h^{(3)}\big)l^{(4)},h^{(4)}\big)f^{-1}\big(S\big(l^{(2)}h^{(2)}\big),l^{(5)}\big)\gamma\big(S\big(l^{(1)}h^{(1)}\big)\big)\\ &= f^{-1}\big(S\big(l^{(2)}h^{(2)}\big),l^{(3)}h^{(3)}\big)f^{-1}\big(l^{(4)},h^{(4)}\big)\gamma\big(S\big(l^{(1)}h^{(1)}\big)\big).\end{aligned}$$

Combining the precedent identity with the fact that f^{-1} is the convolution inverse of f, we obtain

$$\begin{aligned}l \blacktriangleright \big(h \blacktriangleright \big[a\gamma(u)\otimes \mathbf{a}_{1r}\big]\big) &= \big[\gamma\big(v^{(5)}\big)a\gamma(u)f^{-1}\big(S\big(v^{(2)}\big),v^{(3)}\big)\gamma\big(S\big(v^{(1)}\big)\big)\otimes \mathbf{a}_{1r}^{v^{(4)}}\big]\\ &= \big[\gamma\big(v^{(3)}\big)a\gamma(u)\gamma^{-1}\big(v^{(1)}\big)\otimes \mathbf{a}_{1r}^{v^{(2)}}\big],\end{aligned}$$

where $v = lh$. Since the last expression equals $(lh) \blacktriangleright [a\gamma(u)\otimes \mathbf{a}_{1r}]$, this finishes the proof. □

For each $r \geqslant 0$, let $\mathcal{M}_r$ be $E\otimes \overline{A}^{\otimes^r}\otimes$, endowed with the left H-module structure given by $\blacktriangleright$. For each $r,s\geqslant 0$, let $\mathcal{B}_{rs}:\mathcal{M}_r\otimes_k \overline{H}^{\otimes_k^s}\to \mathcal{M}_{r+1}\otimes_k \overline{H}^{\otimes_k^s}$ be the map defined by

$$\begin{aligned}\mathcal{B}(\mathbf{x}) = \sum_{j=0}^{r}(-1)^{(j+1)r}\big[&\gamma\big(h_0^{(3)}S\big(h_1^{(1)}\cdots h_s^{(1)}\big)\big)\gamma\big(h_1^{(5)}\big)\cdots\gamma\big(h_s^{(5)}\big)\\ &\otimes \mathbf{a}_{j+1,r}\otimes a_0U\big(h_0^{(1)},\mathbf{h}_{1s}^{(3)}\big)\otimes\big(\mathbf{a}_{1j}^{\mathbf{h}_{1s}^{(4)}}\big)^{h_0^{(2)}S(h_1^{(2)}\cdots h_s^{(2)})}\big]\otimes_k \mathbf{h}_{1s}^{(6)},\end{aligned}$$

where $\mathbf{x} = [a_0\gamma(h_0)\otimes \mathbf{a}_{1r}]\otimes_k \mathbf{h}_{1s}$. For each $r,s\geqslant 0$, let

$$\partial_r:\mathrm{H}_s(H,\mathcal{M}_r)\to \mathrm{H}_s(H,\mathcal{M}_{r-1})\quad\text{and}\quad \mathcal{D}_r:\mathrm{H}_s(H,\mathcal{M}_r)\to \mathrm{H}_s(H,\mathcal{M}_{r+1})$$

be the maps induced by $\overline{d}_{rs}^0$ and $\mathcal{B}_{rs}$, respectively.

Proposition 3.8. *For each $s\geqslant 0$,*

$$\widetilde{\mathrm{H}_s^K(H,E)} := \big(\mathrm{H}_s(H,\mathcal{M}_*),\partial_*,\mathcal{D}_*\big)$$

is a mixed complex and there is a convergent spectral sequence

$$\mathcal{E}_{rs}^2 = \mathrm{HC}_r\big(\widetilde{\mathrm{H}_s^K(H,E)}\big)\quad\Rightarrow\quad \mathrm{HC}_{r+s}^K(E).$$

Proof. Consider the spectral sequence $(\mathcal{E}_{rs}^v,\delta_{rs}^v)_{v\geqslant 0}$, associated with the filtration

$$\mathcal{F}^0\big(\mathrm{Tot}\big(\mathrm{BC}(\overline{X},\overline{d},\overline{D})\big)\big)\subseteq \mathcal{F}^1\big(\mathrm{Tot}\big(\mathrm{BC}(\overline{X},\overline{d},\overline{D})\big)\big)\subseteq \mathcal{F}^2\big(\mathrm{Tot}\big(\mathrm{BC}(\overline{X},\overline{d},\overline{D})\big)\big)\subseteq\cdots$$

of the complex $\operatorname{Tot}(\operatorname{BC}(\overline{X}, \overline{d}, \overline{D}))$, given by

$$\mathcal{F}^i\big(\operatorname{Tot}\big(\operatorname{BC}(\overline{X}, \overline{d}, \overline{D})\big)_n\big) = \bigoplus_{j\geqslant 0} \mathcal{F}^{i-2j}(\overline{X}_{n-2j})u^j,$$

where $\mathcal{F}^l(\overline{X}_m) = \bigoplus_{0\leqslant r\leqslant l} \overline{X}_{r,m-r}$. A straightforward computation shows that:

- $\mathcal{E}^0_{rs} = \bigoplus_{j\geqslant 0}(\mathcal{M}_{r-2j} \otimes_k \overline{H}^{\otimes^s})u^j$ and δ^0_{rs} is $\bigoplus_{j\geqslant 0}\overline{d}^1_{r-2j,s}u^j$,
- $\mathcal{E}^1_{rs} = \bigoplus_{j\geqslant 0} \operatorname{H}_s(H, \mathcal{M}_{r-2j})u^j$ and δ^1_{rs} is $\partial + \mathcal{D}$.

From this it is easy to see that $\widetilde{\operatorname{H}^K_s(H, E)}$ is a mixed complex and

$$\mathcal{E}^2_{rs} = \operatorname{HC}_r\big(\widetilde{\operatorname{H}^K_s(H, E)}\big).$$

In order to finish the proof it suffices to note that the filtration of $\operatorname{Tot}(\operatorname{BC}(\overline{X}, \overline{d}, \overline{D}))$ introduced above is canonically bounded, and so, by Theorem 3.2, the spectral sequence $(\mathcal{E}^v_{rs}, \delta^v_{rs})_{v\geqslant 0}$ converges to the cyclic homology of the K-algebra E. □

Corollary 3.9. *If H is separable, then* $\operatorname{HC}^K_n(E) = \operatorname{HC}_n(\widetilde{\operatorname{H}^K_0(H, E)})$.

4. Some decompositions of the mixed complexes

Let $[H, H]$ be the k-submodule of H spanned by the set of all elements $hl - lh$ with $h, l \in H$. It is easy to see that $[H, H]$ is a coideal in H. Let $\breve{H}$ be the quotient coalgebra $H/[H, H]$. In this section we study decompositions of the mixed complexes $(E \otimes \overline{E}^{\otimes^*}\otimes, b, B)$, $(\widehat{X}, \widehat{d}, \widehat{D})$ and $(\overline{X}, \overline{d}, \overline{D})$ induced by decompositions of $\breve{H}$.

For $h \in H$, we let $\overline{h}$ denote the class of h in $\breve{H}$. Given a subcoalgebra C of $\breve{H}$ and a right $\breve{H}$-comodule (N, ρ), we set $N^C = \{n \in N\colon\ \rho(n) \in N \otimes C\}$. It is well known that if $\breve{H}$ decomposes as a direct sum of a family $(C_i)_{i\in I}$ of subcoalgebras, then $N = \bigoplus_{i\in I} N^{C_i}$.

For each $n \geqslant 0$, the module $E \otimes \overline{E}^{\otimes^n}\otimes$ is an $\breve{H}$-comodule via

$$\rho_n\big(\big[a_0\gamma(h_0) \otimes\cdots\otimes a_n\gamma(h_n)\big]\big) = \big[a_0\gamma\big(h^{(1)}_0\big) \otimes\cdots\otimes a_n\gamma\big(h^{(1)}_n\big)\big] \otimes_k \overline{h^{(2)}_0\cdots h^{(2)}_n},$$

and the map $\rho_*\colon E \otimes \overline{E}^{\otimes^*}\otimes \to (E \otimes \overline{E}^{\otimes^*}\otimes) \otimes_k \breve{H}$ is a morphism of mixed complexes. This last result implies that if C is a subcoalgebra of $\breve{H}$, then

$$b\big(E \otimes \overline{E}^{\otimes^n}\otimes^C\big) \subseteq E \otimes \overline{E}^{\otimes^{n-1}}\otimes^C \quad\text{and}\quad B\big(E \otimes \overline{E}^{\otimes^n}\otimes^C\big) \subseteq E \otimes \overline{E}^{\otimes^{n+1}}\otimes^C.$$

Let $(E \otimes \overline{E}^{\otimes^*}\otimes^C, b^C, B^C)$ be the mixed subcomplex of $(E \otimes \overline{E}^{\otimes^*}\otimes, b, B)$, with modules $E \otimes \overline{E}^{\otimes^n}\otimes^C$. We let $\operatorname{HH}^{K,C}_*(E)$, $\operatorname{HC}^{K,C}_*(E)$, $\operatorname{HP}^{K,C}_*(E)$ and $\operatorname{HN}^{K,C}_*(E)$ denote its Hochschild, cyclic, periodic and negative homology groups, respectively.

Similarly, for each $n \geqslant 0$, the module $\widehat{X}_n$ is an $\breve{H}$-comodule via

$$\rho_n\big(\big[a_0\gamma(h_0) \otimes_A \overline{\gamma(\mathbf{h}_{1s})} \otimes \mathbf{a}_{1,n-s}\big]\big) = \big[a_0\gamma\big(h^{(1)}_0\big) \otimes_A \overline{\gamma\big(\mathbf{h}^{(1)}_{1s}\big)} \otimes \mathbf{a}_{1,n-s}\big] \otimes \overline{h^{(2)}_0\cdots h^{(2)}_s},$$

and the map $\rho_*: \widehat{X}_* \to \widehat{X}_* \otimes \breve{H}$ is a morphism of mixed complexes. Consequently, if C is a subcoalgebra of $\breve{H}$, then

$$\widehat{d}_n\big(\widehat{X}_n^C\big) \subseteq \widehat{X}_{n-1}^C \quad \text{and} \quad \widehat{D}_n\big(\widehat{X}_n^C\big) \subseteq \widehat{X}_{n+1}^C.$$

Let $(\widehat{X}^C, \widehat{d}^C, \widehat{D}^C)$ be the mixed subcomplex of $(\widehat{X}, \widehat{d}, \widehat{D})$ with modules $\widehat{X}_n^C$. The homotopy equivalent data introduced in Theorem 2.4 induces by restriction a homotopy equivalent data between $(\widehat{X}^C, \widehat{d}^C, \widehat{D}^C)$ and $(E \otimes \overline{E}^{\otimes^*} \otimes^C, b^C, B^C)$. So, $\mathrm{HH}_*^{K,C}(E)$, $\mathrm{HC}_*^{K,C}(E)$, $\mathrm{HP}_*^{K,C}(E)$ and $\mathrm{HN}_*^{K,C}(E)$ are the Hochschild, cyclic, periodic and negative homology of $(\widehat{X}^C, \widehat{d}^C, \widehat{D}^C)$, respectively.

Suppose now the cocycle f is invertible. A direct computation shows that the $\breve{H}$-coaction of $(\overline{X}, \overline{d}, \overline{D})$, obtained by transporting the one of $(\widehat{X}, \widehat{d}, \widehat{D})$ through $\theta: (\widehat{X}, \widehat{d}, \widehat{D}) \to (\overline{X}, \overline{d}, \overline{D})$, is given by

$$\big[a_0\gamma(h_0) \otimes \mathbf{a}_{1r}\big] \otimes_k \mathbf{h}_{1s} \mapsto \big[a_0\gamma\big(h_0^{(1)}\big) \otimes \mathbf{a}_{1r}\big] \otimes_k \mathbf{h}_{1s}^{(2)} \otimes h_0^{(2)} S\big(h_1^{(1)} \cdots h_s^{(1)}\big) h_1^{(3)} \cdots h_s^{(3)}.$$

This implies that if $\breve{H}$ is cocommutative, then

$$\overline{X}_n^C = \bigoplus_{r+s=n} \overline{X}_{rs}^C = \bigoplus_{r+s=n} E^C \otimes \overline{A}^{\otimes^r} \otimes \overline{H}^{\otimes^s}.$$

For each subcoalgebra C of $\breve{H}$, we consider the mixed subcomplex $(\overline{X}^C, \overline{d}^C, \overline{D}^C)$ of $(\overline{X}, \overline{d}, \overline{D})$ with modules $\overline{X}_n^C$. It is clear that θ induces an isomorphism

$$\theta^C: \big(\widehat{X}^C, \widehat{d}^C, \widehat{D}^C\big) \to \big(\overline{X}^C, \overline{d}^C, \overline{D}^C\big).$$

So, $\mathrm{HH}_*^{K,C}(E)$, $\mathrm{HC}_*^{K,C}(E)$, $\mathrm{HP}_*^{K,C}(E)$ and $\mathrm{HN}_*^{K,C}(E)$ are the Hochschild, cyclic, periodic and negative homology of $(\overline{X}^C, \overline{d}^C, \overline{D}^C)$, respectively.

By the discussion at the beginning of this subsection, if $\breve{H}$ decomposes as a direct sum of a family $(C_i)_{i\in I}$ of subcoalgebras, then

$$\big(E \otimes \overline{E}^{\otimes^*} \otimes, b, B\big) = \bigoplus_{i\in I} \big(E \otimes \overline{E}^{\otimes^*} \otimes^{C_i}, b^{C_i}, B^{C_i}\big)$$

$$(\widehat{X}, \widehat{d}, \widehat{D}) = \bigoplus_{i\in I} \big(\widehat{X}^{C_i}, \widehat{d}^{C_i}, \widehat{D}^{C_i}\big)$$

and

$$(\overline{X}, \overline{d}, \overline{D}) = \bigoplus_{i\in I} \big(\overline{X}^{C_i}, \overline{d}^{C_i}, \overline{D}^{C_i}\big).$$

In particular $\mathrm{HH}_*^K(E) = \bigoplus_{i\in I} \mathrm{HH}_*^{K,C_i}(E)$, etcetera.

In the sequel we use the notations introduced in Sections 3.1 and 3.2.

Lemma 4.1. *Assume that $\breve{H}$ is cocommutative and $\overline{H}$ is a flat k-module. If C is a subcoalgebra of $\breve{H}$, then for each $r, s \geqslant 0$,*

$$\widetilde{d}\left(\mathrm{H}_r^K\left(A, E^C\right) \otimes_k \overline{H}^{\otimes^n}\right) \subseteq \mathrm{H}_r^K\left(A, E^C\right) \otimes_k \overline{H}^{\otimes^{n-1}}$$

and

$$\widetilde{D}\left(\mathrm{H}_r^K\left(A, E^C\right) \otimes_k \overline{H}^{\otimes^n}\right) \subseteq \mathrm{H}_r^K\left(A, E^C\right) \otimes_k \overline{H}^{\otimes^{n+1}}.$$

Proof. Left to the reader. □

Proposition 4.2. *Assume that $\breve{H}$ is cocommutative and $\overline{H}$ is a flat k-module. Let C be a subcoalgebra of $\breve{H}$ and let*

$$\mathrm{H}_r^K\widetilde{\left(A, E^C\right)} := \left(\mathrm{H}_r^K\left(A, E^C\right) \otimes_k \overline{H}^{\otimes^*}, \widetilde{d}_*^C, \widetilde{D}_*^C\right)$$

be the submixed complex of $\mathrm{H}_r^K\widetilde{(A, E)}$ with modules $\mathrm{H}_r^K(A, E^C) \otimes_k \overline{H}^{\otimes^n}$. There is a convergent spectral sequence

$$E_{sr}^2 = \mathrm{HC}_s\left(\mathrm{H}_r^K\widetilde{\left(A, E^C\right)}\right) \quad \Rightarrow \quad \mathrm{HC}_{r+s}^{K,C}(E).$$

Proof. Left to the reader. □

Lemma 4.3. *Assume that $\breve{H}$ is cocommutative. If C is a subcoalgebra of $\breve{H}$, then $\mathcal{M}_n^C = E^C \otimes \overline{A}^{\otimes^n} \otimes$ is an H-submodule of $\mathcal{M}_n$ for each $n \geqslant 0$. Moreover*

$$\partial\left(\mathrm{H}_s(H, \mathcal{M}_n)\right) \subseteq \mathrm{H}_s(H, \mathcal{M}_{n-1}) \quad \text{and} \quad \mathcal{D}\left(\mathrm{H}_s(H, \mathcal{M}_n)\right) \subseteq \mathrm{H}_s(H, \mathcal{M}_{n+1}).$$

Proof. Left to the reader. □

Proposition 4.4. *Assume that $\breve{H}$ is cocommutative. Let C be a subcoalgebra of $\breve{H}$ and let*

$$\mathrm{H}_s^K\widetilde{\left(H, E^C\right)} := \left(\mathrm{H}_s\left(H, \mathcal{M}_*^C\right), \partial_*, \mathcal{D}_*\right)$$

be the submixed complex of $\mathrm{H}_s^K\widetilde{(H, E)}$ with modules $\mathrm{H}_s(H, \mathcal{M}_n^C)$. There is a convergent spectral sequence

$$\mathcal{E}_{rs}^2 = \mathrm{HC}_r\left(\mathrm{H}_s^K\widetilde{\left(H, E^C\right)}\right) \quad \Rightarrow \quad \mathrm{HC}_{r+s}^{K,C}(E).$$

Proof. Left to the reader. □

Appendix A

This appendix is devoted to prove Propositions 2.1, 2.2 and 2.5.

Lemma A.1. *We have*

$$\overline{\sigma}_{n+1} = -\sigma^0_{0,n+1} \circ \sigma^{-1}_{n+1} \circ \mu_n + \sum_{r=0}^{n} \sum_{l=0}^{n-r} \sigma^l_{r+l+1,n-r-l}.$$

Proof. By the definition of μ, σ^{-1} and $\overline{\sigma}$ it suffices to prove that

$$\sigma^l\big(E \otimes_A (E/A)^{\otimes_A^{n+1}} \otimes_A A\big) = 0 \quad \text{for all } l \geqslant 1.$$

Assume the result is false and let $l \geqslant 1$ be the minimal upper index for which the above equality is wrong. Let $\mathbf{x} \in E \otimes_A (E/A)^{\otimes_A^{n+1}} \otimes_A A$. Then

$$\sigma^l(\mathbf{x}) = -\sum_{i=0}^{l-1} \sigma^0 \circ d^{l-i} \circ \sigma^i(\mathbf{x}) = -\sigma^0 \circ d^l \circ \sigma^0(\mathbf{x}).$$

But, because $\sigma^0(\mathbf{x}) \in E \otimes_A (E/A)^{\otimes_A^{n+1}} \otimes K$, from the definition of d^l it follows that $d^l \circ \sigma^0(\mathbf{x}) \in \operatorname{Im}(\sigma_0)$. Since $\sigma^0 \circ \sigma^0 = 0$, this implies that $\sigma^l(\mathbf{x}) = 0$, which contradicts the assumption. □

Lemma A.2. *The contracting homotopy $\overline{\sigma}$ satisfies $\overline{\sigma} \circ \overline{\sigma} = 0$.*

Proof. By Lemma A.1 it will be sufficient to see that $\sigma^0 \circ \sigma^{-1} \circ \mu \circ \sigma^0 \circ \sigma^{-1} \circ \mu = 0$ and $\sigma^l \circ \sigma^{l'} = 0$ for all $l, l' \geqslant 0$. The first equality follows from the equalities $\mu \circ \sigma^0 = \mathrm{id}$ and $\sigma^{-1} \circ \sigma^{-1} = 0$. We now prove the last one. An inductive argument shows that there exists a map γ^l such that $\sigma^l = \sigma^0 \circ \gamma^l \circ \sigma^0$ for all $l \geqslant 1$. So $\sigma^{l'} \circ \sigma^l = 0$, since clearly $\sigma^0 \circ \sigma^0 = 0$. □

Remark A.3. The previous lemma implies that $\psi_n(\mathbf{y} \otimes 1) = (-1)^n \overline{\sigma} \circ \psi(\mathbf{y})$ for all $n \geqslant 1$.

Let $L_{rs} \subseteq U_{rs}$ be the k-submodules of $E \otimes_A (E/A)^{\otimes_A^s} \otimes \overline{A}^{\otimes^r} \otimes E$ generated by the simple tensors of the form

$$1 \otimes_A \overline{\gamma(\mathbf{h}_{1s})} \otimes \mathbf{a}_{1r} \otimes 1 \quad \text{and} \quad 1 \otimes_A \overline{\gamma(\mathbf{h}_{1s})} \otimes \mathbf{a}_{1r} \otimes \gamma(h),$$

respectively.

Note that under the identification $X_{rs} \simeq E \otimes_k \overline{H}^{\otimes_k^s} \otimes \overline{A}^{\otimes^r} \otimes E$, the subspaces and L_{rs} and U_{rs} of X_{rs} correspond to $k \otimes_k \overline{H}^{\otimes_k^s} \otimes \overline{A}^{\otimes^r} \otimes k$ and $k \otimes_k \overline{H}^{\otimes_k^s} \otimes \overline{A}^{\otimes^r} \otimes \mathcal{H}$, respectively.

Lemma A.4. *It always holds that $d^l(L_{rs}) \subseteq U_{r+l-1,s-l}$, for each $l \geqslant 2$. Moreover*

$$d^1(L_{rs}) \subseteq E L_{r,s-1} + U_{r,s-1}.$$

Proof. We proceed by induction on l and r. For $l=1$ and $r \geqslant 0$, the result follows immediately from Theorem 1.4. Assume that $s \geqslant l > 1$, $r=0$ and that the result for $l \geqslant 2$ is true for every $d^j_{r's'}$'s with arbitrary r', s' and $j < l$. Let $\mathbf{x} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1s})} \otimes 1$. By the very definition of d^l, the above inclusion of $d^1(L_{rs})$, and the inductive hypothesis

$$
\begin{aligned}
d^l(\mathbf{x}) &= -\sum_{j=1}^{l-1} \sigma^0 \circ d^{l-j} \circ d^j(\mathbf{x}) \\
&\in \sigma^0 \circ d^{l-1}(EL_{0,s-1}) + \sum_{j=1}^{l-1} \sigma^0 \circ d^{l-j}(U_{j-1,s-j}) \\
&= \sum_{j=1}^{l-1} \sigma^0 \circ d^{l-j}(U_{j-1,s-j}),
\end{aligned}
$$

where the last equality follows from the fact that

$$\operatorname{Im}(\sigma^0) \subseteq \ker(\sigma^0) \quad \text{and} \quad d^{l-1}(EL_{0,s-1}) \subseteq \operatorname{Im}(\sigma^0),$$

by the definition of d^{l-1}. Now, by the inductive hypothesis,

$$d^{l-j}(U_{j-1,s-j}) \subseteq L_{l-2,s-l}E \quad \text{for } l-j > 1,$$

and

$$d^1(U_{l-2,s-l+1}) \subseteq EU_{l-2,s-l} + L_{l-2,s-l}E.$$

Thus, by the definition of σ^0, we have $d^l(\mathbf{x}) \in U_{l-1,s-l}$. Suppose now that $r > 0$ and the result is true for all the $d^j_{r's'}$'s with arbitrary r', s' and $j < l$, and for all the $d^l_{r's'}$'s with arbitrary s' and $r' < r$. Let $\mathbf{x} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1s})} \otimes \mathbf{a}_{1r} \otimes 1$. Arguing as above we see that

$$d^l(\mathbf{x}) \equiv -\sigma^0 \circ d^l \circ d^0(\mathbf{x}) \pmod{U_{r+l-1,s-l}}.$$

Finally, by the definition of d^0 and the inductive hypothesis,

$$
\begin{aligned}
\sigma^0 \circ d^l \circ d^0(\mathbf{x}) &\in \sigma^0 \circ d^l(AL_{r-1,s} + L_{r-1,s}A) \\
&\subseteq \sigma^0(AU_{r+l-2,s-l} + U_{r+l-2,s-l}A) \\
&\subseteq U_{r+l-1,s-l},
\end{aligned}
$$

which finishes the proof. □

We recursively define $\gamma(\mathbf{h}_{1s}) * \mathbf{a}_{1r}$ by:

- $\gamma(\mathbf{h}_{1s}) * \mathbf{a}_{1r} = \mathbf{a}_{1r}$ if $s=0$ and $\gamma(\mathbf{h}_{1s}) * \mathbf{a}_{1r} = \gamma(\mathbf{h}_{1s})$ if $r=0$.
- If $r, s \geqslant 1$, then $\gamma(\mathbf{h}_{1s}) * \mathbf{a}_{1r} = \sum_{i=0}^{r} (-1)^i \gamma(\mathbf{h}_{1,s-1}) * \mathbf{a}_{1i}^{h_s^{(1)}} \otimes \gamma(h_s^{(2)}) \otimes \mathbf{a}_{i+1,r}$.

Let V_n be the k-submodule of $E \otimes \overline{E}^{\otimes^n} \otimes E$ generated by the simple tensors $1 \otimes \mathbf{x}_{1n} \otimes 1$ such that $x_i \in A \cup \mathcal{H}$ for $1 \leqslant i \leqslant n$.

Recall that $H \cdot \mathrm{Im}(f)$ denotes the minimal k-submodule of A that includes $\mathrm{Im}(f)$ and it is closed under the weak action of H. We will denote by C_n the E-subbimodule of $E \otimes \overline{E}^{\otimes^n} \otimes E$ generated by all the simple tensors $1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1$ with some x_i in $H \cdot \mathrm{Im}(f)$.

Proposition A.5. *The map ϕ satisfies*

$$\phi\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{1,n-i} \otimes 1\big) \equiv 1 \otimes \gamma(\mathbf{h}_{1i}) * \mathbf{a}_{1,n-i} \otimes 1$$

module $F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n \cap C_n$.

Proof. We proceed by induction on n. Let $\mathbf{x} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{1,n-i} \otimes 1$. By item (2) of Theorem 1.4, the fact that $d^l(\mathbf{x}) \in U_{n-i+l-1,i-l}$ (by Lemma A.4), and the inductive hypothesis

$$\xi \circ \phi \circ d^l(\mathbf{x}) \in F^{i-l+1}\big(E \otimes \overline{E}^{\otimes^n} \otimes E\big) \cap V_n \cap C_n \quad \text{for all } l > 1.$$

So,

$$\phi(\mathbf{x}) \equiv \xi \circ \phi \circ d^0(\mathbf{x}) + \xi \circ \phi \circ d^1(\mathbf{x}) \quad \big(\mathrm{mod}\ F^{i-1}\big(E \otimes \overline{E}^{\otimes^n} \otimes E\big) \cap V_n \cap C_n\big).$$

Moreover, by the definition of d^0 and Theorem 1.4

$$\xi \circ \phi \circ d^0(\mathbf{x}) = (-1)^n \xi \circ \phi\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{1,n-i}\big),$$

and

$$\xi \circ \phi \circ d^1(\mathbf{x}) = (-1)^i \xi \circ \phi\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{1,n-i}^{h_i^{(1)}} \otimes \gamma\big(h_i^{(2)}\big)\big),$$

since $\phi(EL_{n-s-1,s}) \subseteq E \otimes \overline{E}^{\otimes^{n-1}} \otimes K \subseteq \ker(\xi)$. The proof can be now easily finished using the inductive hypothesis. □

In the sequel we let J_n denote the E-subbimodule of X_n generated by all the simple tensors

$$1 \otimes_A x_1 \otimes_A \cdots \otimes_A x_s \otimes a_1 \otimes \cdots \otimes a_r \otimes 1 \quad (r + s = n),$$

with some a_i in the image of the cocycle f.

The proof of [9, Proposition 1.2.2] is divided in several parts. The first item of the following result improves part (b).

Lemma A.6. *We have*:

(1) *Let $\mathbf{x} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{i+1,n}$. If $i < n$, then*

$$\overline{\sigma}(\mathbf{x}) = \sigma^0(\mathbf{x}) = (-1)^n \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{i+1,n} \otimes 1.$$

(2) *If* $\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{i,n-1} \otimes a_n\gamma(h_n)$, *then* $\sigma^l(\mathbf{z}) \in U_{n-i+l+1,i-1-l}$ *for* $l \geqslant 0$ *and* $\sigma^l(\mathbf{z}) \in J_n$ *for* $l \geqslant 1$.
(3) *If* $\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1}) \otimes \mathbf{a}_{i,n-1}} \otimes \gamma(h_n)$, *then* $\sigma^l(\mathbf{z}) = 0$ *for* $l \geqslant 0$.
(4) *If* $\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{i,n-1} \otimes a_n\gamma(h_n)$ *and* $i < n$, *then* $\overline{\sigma}(\mathbf{z}) \equiv \sigma^0(\mathbf{z})$, *module* $\bigoplus_{l=0}^{i-2}(U_{n-l,l} \cap J_n)$.
(5) *If* $\mathbf{y} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-1})} \otimes a_n\gamma(h_n)$, *then* $\overline{\sigma}(\mathbf{y}) \equiv -\sigma^0 \circ \sigma^{-1} \circ \mu(\mathbf{y}) + \sigma^0(\mathbf{y})$, *module* $\bigoplus_{l=0}^{n-2}(U_{n-l,l} \cap J_n)$.
(6) *If* $\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-1})} \otimes \gamma(h_n)$, *then* $\overline{\sigma}(\mathbf{z}) = -\sigma^0 \circ \sigma^{-1} \circ \mu(\mathbf{z})$.
(7) *If* $\mathbf{z} = 1 \otimes_A \gamma(\mathbf{h}_{1,i-1}) \otimes \mathbf{a}_{i,n-1} \otimes \gamma(h_n)$ *and* $i < n$, *then* $\overline{\sigma}(\mathbf{z}) = 0$.

Proof. We first claim that if $l \geqslant 1$, then $\sigma^l(\mathbf{x}) = 0$. We proceed by induction on l. By the recursive definition of σ^l and the inductive hypothesis

$$\sigma^l(\mathbf{x}) = -\sum_{i=0}^{l-1} \sigma^0 \circ d^{l-i} \circ \sigma^i(\mathbf{x}) = -\sigma^0 \circ d^l \circ \sigma^0(\mathbf{x}) = (-1)^{n-1}\sigma^0 \circ d^l(\mathbf{x} \otimes 1).$$

In order to finish the proof of the claim it is sufficient to note that $\sigma^0 \circ \sigma^0 = 0$ and that, by the very definition, $d^l(\mathbf{x} \otimes 1) \in \operatorname{Im}(\sigma^0)$. When $i < n-1$ item (1) follows clearly from the claim. When $i = n-1$ it is necessary to see also that $\sigma^l \circ \sigma^{-1} \circ \mu(\mathbf{x}) = 0$, which is immediate, since $\sigma^{-1} \circ \mu(\mathbf{x}) = 0$ by the definitions of μ and σ^{-1}. We will next prove the first part of item (2). By definition this is clear for σ^0. Assume the result is valid for σ^i with $i < l$. Then, by Lemma A.4,

$$\begin{aligned}
\sigma^l(\mathbf{z}) &= -\sum_{j=0}^{l-1} \sigma^0 \circ d^{l-j} \circ \sigma^j(\mathbf{z}) \\
&\subseteq \sum_{j=0}^{l-1} \sigma^0 \circ d^{l-j}(U_{n-i+j+1,i-1-j}) \\
&\subseteq \sigma^0(EU_{n-i+l,i-1-l}) + \sigma^0(U_{n-i+l,i-1-l}E) \\
&= U_{n-i+l+1,i-1-l},
\end{aligned}$$

as desired. We now prove the second part. By Theorem 1.4, the recursive definition of σ^l and the definition of σ^0, we know that

$$\sigma^l(\mathbf{z}) = -\sum_{j=0}^{l-1} \sigma^0 \circ d^{l-j} \circ \sigma^j(\mathbf{z}) \equiv -\sigma^0 \circ d^1 \circ \sigma^{l-1}(\mathbf{z}) \pmod{J_n}.$$

Since $\sigma^0 \circ d^1 \circ \sigma^{l-1}(\mathbf{z}) \in \sigma^0 \circ d^1(U_{n-i+l,i-l})$, in order to finish the proof it suffices to see that $\sigma^0 \circ d^1(U_{n-i+l,i-l}) \subseteq J_n$, which is a direct consequence of Theorem 1.4 and the definition of σ^0. Item (3) follows immediately by induction on l. Items (4) and (5) follow easily from the definition of $\overline{\sigma}$, item (2) and Lemma A.1. Finally, items (6) and (7) follow from the definition of $\overline{\sigma}$, item (3) and Lemma A.1. □

Let V_n' be the k-submodule of $E \otimes \overline{E}^{\otimes^n} \otimes E$ generated by the simple tensors $1 \otimes \mathbf{x}_{1n} \otimes 1$ such that $\#(\{j\colon x_j \notin A \cup \mathcal{H}\}) \leqslant 1$. (Note that $V_n \subseteq V_n'$.)

Let $R_i = F^i(E \otimes \overline{E}^{\otimes^n} \otimes E) \setminus F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E)$.

Proposition A.7. *The following equalities hold*:

(1) $\psi(1 \otimes \gamma(\mathbf{h}_{1i}) \otimes \mathbf{a}_{i+1,n} \otimes 1) = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{i+1,n} \otimes 1$.

(2) *If* $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n$ *and there exists* $1 \leqslant j \leqslant i$ *such that* $x_j \in A$, *then* $\psi(\mathbf{x}) = 0$.

(3) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes a_i\gamma(h_i) \otimes \mathbf{a}_{i+1,n} \otimes 1$, *then*

$$\psi(\mathbf{x}) \equiv 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes_A a_i\gamma(h_i) \otimes \mathbf{a}_{i+1,n} \otimes 1 + 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes a_i \otimes \mathbf{a}_{i+1,n}^{h_i^{(1)}} \otimes \gamma\big(h_i^{(2)}\big),$$

module $\bigoplus_{l=0}^{i-2}(U_{n-l,l} \cap J_n)$.

(4) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1,j-1}) \otimes a_j\gamma(h_j) \otimes \gamma(\mathbf{h}_{j+1,i}) \otimes \mathbf{a}_{i+1,n} \otimes 1$ *with* $j < i$, *then*

$$\psi(\mathbf{x}) \equiv 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,j-1})} \otimes_A a_j\gamma(h_j) \otimes_A \overline{\gamma(\mathbf{h}_{j+1,i})} \otimes \mathbf{a}_{i+1,n} \otimes 1,$$

module $\bigoplus_{l=0}^{i-2}(U_{n-l,l} \cap J_n)$.

(5) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes \mathbf{a}_{i,j-1} \otimes a_j\gamma(h_j) \otimes \mathbf{a}_{j+1,n} \otimes 1$ *with* $j > i$, *then*

$$\psi(\mathbf{x}) \equiv 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{ij} \otimes \mathbf{a}_{j+1,n}^{h_j^{(1)}} \otimes \gamma\big(h_j^{(2)}\big),$$

module $\bigoplus_{l=0}^{i-2}(U_{n-l,l} \cap J_n)$.

(6) *If* $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n'$ *and there exists* $1 \leqslant j_1 < j_2 \leqslant i$ *such that* $x_{j_1} \in A$ *and* $x_{j_2} \in \mathcal{H}$, *then* $\psi(\mathbf{x}) = 0$.

Proof. 1) We proceed by induction on n. The case $n = 0$ is trivial. Suppose $n > 0$ and the result is valid for $n-1$. Assume first that $i < n$. By Remark A.3 and the inductive hypothesis,

$$\begin{aligned}\psi\big(1 \otimes \gamma(\mathbf{h}_{1i}) \otimes \mathbf{a}_{i+1,n} \otimes 1\big) &= (-1)^n\overline{\sigma} \circ \psi\big(1 \otimes \gamma(\mathbf{h}_{1i}) \otimes \mathbf{a}_{i+1,n}\big)\\ &= (-1)^n\overline{\sigma}\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1i})} \otimes \mathbf{a}_{i+1,n}\big),\end{aligned}$$

and the result follows from item (1) of Lemma A.6. Assume now that $i = n$. By Remark A.3, the inductive hypothesis and item (6) of Lemma A.6,

$$\begin{aligned}\psi\big(1 \otimes \gamma(\mathbf{h}_{1n}) \otimes 1\big) &= (-1)^n\overline{\sigma} \circ \psi\big(1 \otimes \gamma(\mathbf{h}_{1n})\big)\\ &= (-1)^{n+1}\sigma^0 \circ \sigma^{-1} \circ \mu\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-1})} \otimes \gamma(h_n)\big).\end{aligned}$$

The result follows now immediately from the definitions of μ, σ^{-1} and σ^0.

2) We proceed by induction on n. Assume first that there exist $j_1 < j_2 < n$ such that $x_{j_1} \in A$ and $x_{j_2} \in \mathcal{H}$. By Remark A.3 and the inductive hypothesis,

$$\psi(\mathbf{x}) = (-1)^n \overline{\sigma} \circ \psi(1 \otimes \mathbf{x}_{1n}) = (-1)^n \overline{\sigma}(0) = 0.$$

Assume now that $\mathbf{x}_{1n} = \gamma(\mathbf{h}_{1,i-1}) \otimes \mathbf{a}_{i,n-1} \otimes \gamma(h_n)$. By Remark A.3 and item (1),

$$\psi(\mathbf{x}) = (-1)^n \overline{\sigma} \circ \psi(1 \otimes \mathbf{x}_{1n}) = (-1)^n \overline{\sigma}\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{i,n-1} \otimes \gamma(h_n)\big),$$

and the result follows from item (7) of Lemma A.6.

3) We proceed by induction on n. Assume first that $i < n$. Let

$$\mathbf{y} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes_A a_i \gamma(h_i) \otimes \mathbf{a}_{i+1,n},$$
$$\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes a_i \otimes \mathbf{a}_{i+1,n-1}^{h_i^{(1)}} \otimes \gamma\big(h_i^{(2)}\big) a_n.$$

By Remark A.3 and the inductive hypothesis,

$$\psi(\mathbf{x}) = (-1)^n \overline{\sigma} \circ \psi\big(1 \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes a_i \gamma(h_i) \otimes \mathbf{a}_{i+1,n}\big) \equiv (-1)^n \overline{\sigma}(\mathbf{y} + \mathbf{z}),$$

module $\overline{\sigma}(\bigoplus_{l=0}^{i-2} (U_{n-1-l,l} \cap J_{n-1})A)$. So, by items (1) and (4) of Lemma A.6,

$$\psi(\mathbf{x}) \equiv (-1)^n \sigma^0(\mathbf{y} + \mathbf{z}),$$

module $\bigoplus_{l=0}^{i-2} (U_{n-l,l} \cap J_n) + \sigma^0(\bigoplus_{l=0}^{i-2} (U_{n-1-l,l} \cap J_{n-1})A)$. Using the definition of σ^0 we obtain immediately the desired expression for $\psi(\mathbf{x})$. Assume now that $i = n$. Let

$$\mathbf{y} = 1 \otimes \gamma(\mathbf{h}_{1,n-1}) \otimes a_n \gamma(h_n) \quad \text{and} \quad \mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-1})} \otimes a_n \gamma(h_n).$$

By Remark A.3, item (1) of the present proposition and item (5) of Lemma A.6,

$$\psi(\mathbf{x}) = (-1)^n \overline{\sigma} \circ \psi(\mathbf{y}) = (-1)^n \overline{\sigma}(\mathbf{z}) \equiv (-1)^{n+1} \sigma^0 \circ \sigma^{-1} \circ \mu(\mathbf{z}) + (-1)^n \sigma^0(\mathbf{z}),$$

module $\bigoplus_{l=0}^{n-2} (U_{n-l,l} \cap J_n)$. The established formula for $\psi(\mathbf{x})$ follows now easily from the definitions of μ, σ^{-1} and σ^0.

4) We proceed by induction on n. When $i < n$ the same argument that in item (3) works. Assume now that $j < i - 1$ and $i = n$. Let

$$\mathbf{y} = 1 \otimes \gamma(\mathbf{h}_{1,j-1}) \otimes a_j \gamma(h_j) \otimes \gamma(\mathbf{h}_{j+1,n}),$$
$$\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,j-1})} \otimes_A a_j \gamma(h_j) \otimes_A \overline{\gamma(\mathbf{h}_{j+1,n-1})} \otimes \gamma(h_n).$$

By Remark A.3 and the inductive hypothesis,

$$\psi(\mathbf{x}) = (-1)^n \overline{\sigma} \circ \psi(\mathbf{y}) \equiv (-1)^n \overline{\sigma}(\mathbf{z}),$$

module $\overline{\sigma}(\bigoplus_{l=0}^{n-3} (U_{n-1-l,l} \cap J_{n-1})E)$. So, by items (4) and (6) of Lemma A.6,

$$\psi(\mathbf{x}) \equiv (-1)^{n+1}\sigma^0 \circ \sigma^{-1} \circ \mu(\mathbf{z}),$$

module $\bigoplus_{l=0}^{n-4}(U_{n-l,l} \cap J_n) + \sigma^0(\bigoplus_{l=0}^{n-3}(U_{n-1-l,l} \cap J_{n-1})E)$. The formula for $\psi(\mathbf{x})$ follows now easily from the definitions of μ, σ^{-1} and σ^0. Assume finally that $j = i-1$ and $i = n$. Let

$$\mathbf{y} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-2})} \otimes_A a_{n-1}\gamma(h_{n-1}) \otimes \gamma(h_n),$$
$$\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,n-2})} \otimes a_{n-1} \otimes \gamma(h_{n-1})\gamma(h_n).$$

By Remark A.3 and item (3),

$$\psi(\mathbf{x}) = (-1)^n\overline{\sigma} \circ \psi\big(1 \otimes \gamma(\mathbf{h}_{1,n-2}) \otimes a_{n-1}\gamma(h_{n-1}) \otimes \gamma(h_n)\big) \equiv (-1)^n\overline{\sigma}(\mathbf{y}+\mathbf{z}),$$

module $\overline{\sigma}(\bigoplus_{l=0}^{n-3}(U_{n-1-l,l} \cap J_{n-1})E)$. So, by the fact that $\sigma^0(\mathbf{z}) \in U_{2,n-2} \cap J_n$, and items (4) and (6) of Lemma A.6, we know that

$$\psi(\mathbf{x}) \equiv (-1)^{n+1}\sigma^0 \circ \sigma^{-1} \circ \mu(\mathbf{y}),$$

module $\bigoplus_{l=0}^{n-2}(U_{n-l,l} \cap J_n) + \sigma^0(\bigoplus_{l=0}^{n-3}(U_{n-1-l,l} \cap J_{n-1})E)$. The formula for $\psi(\mathbf{x})$ follows now easily from the definitions of μ, σ^{-1} and σ^0.

5) We proceed by induction on n. Let

$$\mathbf{y} = 1 \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes \mathbf{a}_{i,j-1} \otimes a_j\gamma(h_j) \otimes \mathbf{a}_{j+1,n},$$
$$\mathbf{z} = 1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes \mathbf{a}_{ij} \otimes \mathbf{a}_{j+1,n-1}^{h_j^{(1)}} \otimes \gamma\big(h_j^{(2)}\big)a_n.$$

By Remark A.3 and item (1) or the inductive hypothesis (depending on $j = n$ or $j < n$),

$$\psi(\mathbf{x}) = (-1)^n\overline{\sigma} \circ \psi(\mathbf{y}) \equiv (-1)^n\overline{\sigma}(\mathbf{z}),$$

module $\overline{\sigma}(\bigoplus_{l=0}^{i-2}(U_{n-l-1,l} \cap J_{n-1})A)$. Thus, by item (4) of Lemma A.6,

$$\psi(\mathbf{x}) = (-1)^n\sigma^0 \circ \psi(\mathbf{y}) \equiv (-1)^n\sigma^0(\mathbf{z}),$$

module $\bigoplus_{l=0}^{i-2}(U_{n-l,l} \cap J_n) + \sigma^0(\bigoplus_{l=0}^{i-2}(U_{n-l-1,l} \cap J_{n-1})A)$. The result is obtained now immediately using the definition of σ^0.

6) We proceed by induction on n. By Remark A.3 and item (2) or the inductive hypothesis (depending on $x_n \notin A \cup \mathcal{H}$ or $x_n \in A \cup \mathcal{H}$),

$$\psi(\mathbf{x}) = (-1)^n\overline{\sigma} \circ \psi(1 \otimes \mathbf{x}_{1n}) = (-1)^n\overline{\sigma}(0) = 0,$$

as desired. □

Lemma A.8. *The following equalities hold*:

(1) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1i}) \otimes \mathbf{a}_{1,n-i} \otimes 1$, *then*

$$\phi \circ \psi(x) \equiv 1 \otimes \gamma(\mathbf{h}_{1i}) * \mathbf{a}_{1,n-i} \otimes 1$$

module $F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n$.

(2) *If* $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n$ *and there exists* $1 \leqslant j \leqslant i$ *such that* $x_i \in A$, *then* $\phi \circ \psi(\mathbf{x}) = 0$.

(3) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1,i-1}) \otimes a_i \gamma(h_i) \otimes \mathbf{a}_{i+1,n} \otimes 1$, *then*

$$\phi \circ \psi(\mathbf{x}) \equiv a_i^{\mathbf{h}_{1,i-1}^{(1)}} \otimes \big(\gamma\big(\mathbf{h}_{1,i-1}^{(2)}\big) \otimes \gamma(h_i)\big) * \mathbf{a}_{i+1,n} \otimes 1$$
$$+ 1 \otimes \gamma(\mathbf{h}_{1,i-1}) * \big(a_i \otimes \mathbf{a}_{i+1,n}^{h_i^{(1)}}\big) \otimes \gamma\big(h_i^{(2)}\big),$$

module $F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap AV_n + F^{i-2}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n\mathcal{H}$.

(4) *If* $\mathbf{x} = 1 \otimes \gamma(\mathbf{h}_{1,j-1}) \otimes a_j \gamma(h_j) \otimes \gamma(\mathbf{h}_{j+1,i}) \otimes \mathbf{a}_{i+1,n} \otimes 1$ *with* $j < i$, *then*

$$\phi \circ \psi(\mathbf{x}) \equiv a_j^{\mathbf{h}_{1,j-1}^{(1)}} \otimes \big(\gamma\big(\mathbf{h}_{1,j-1}^{(2)}\big) \otimes \gamma(\mathbf{h}_{ji})\big) * \mathbf{a}_{i+1,n} \otimes 1,$$

module $F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap AV_n + F^{i-2}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n\mathcal{H}$.

(5) *If* $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n'$ *and there exists* $1 \leqslant j \leqslant i$ *such that* $x_j \in A$, *then* $\phi \circ \psi(\mathbf{x}) \in F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n\mathcal{H}$.

Proof. Item (1) follows from item (1) of Proposition A.7 and Proposition A.5, and item (2) follows from item (2) of Proposition A.7. We will next prove item (3). By item (3) of Proposition A.7,

$$\phi \circ \psi(\mathbf{x}) \equiv \phi\big(a_i^{\mathbf{h}_{1,i-1}^{(1)}} \otimes_A \overline{\gamma\big(\mathbf{h}_{1,i-1}^{(2)}\big)} \otimes_A \gamma(h_i) \otimes \mathbf{a}_{i+1,n} \otimes 1\big)$$
$$+ \phi\big(1 \otimes_A \overline{\gamma(\mathbf{h}_{1,i-1})} \otimes a_i \otimes \mathbf{a}_{i+1,n}^{h_i^{(1)}} \otimes \gamma\big(h_i^{(2)}\big)\big),$$

module $\phi(\bigoplus_{l=0}^{i-2} U_{n-l,l})$. So, by Proposition A.5,

$$\phi \circ \psi(\mathbf{x}) \equiv a_i^{\mathbf{h}_{1,i-1}^{(1)}} \otimes \big(\gamma\big(\mathbf{h}_{1,i-1}^{(2)}\big) \otimes \gamma(h_i)\big) * \mathbf{a}_{i+1,n} \otimes 1$$
$$+ 1 \otimes \gamma(\mathbf{h}_{1,i-1}) * \big(a_i \otimes \mathbf{a}_{i+1,n}^{h_i^{(1)}}\big) \otimes \gamma\big(h_i^{(2)}\big),$$

module $F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap AV_n + F^{i-2}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n\mathcal{H}$. We leave the demonstrations of items (4) and (5) to the reader. □

Proposition A.9. *If* $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n'$, *then* $\omega(\mathbf{x}) \in F^i(E \otimes \overline{E}^{\otimes^{n+1}} \otimes E) \cap V_{n+1}$.

Proof. We first claim that if $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n$, then $\omega(\mathbf{x}) = 0$. For $n = 1$ this is immediate, since $\omega_1 = 0$ by definition. Assume that $n > 1$ and the claim holds for $n - 1$. Then,

$$\omega(\mathbf{x}) = \xi\bigl(\phi \circ \psi(\mathbf{x}) - (-1)^n \omega(1 \otimes \mathbf{x}_{1n})\bigr) = \xi \circ \phi \circ \psi(\mathbf{x}) = 0,$$

where the last equality follows from the facts that $\phi \circ \psi(\mathbf{x}) \in V_n$ (by items (1) and (2) of Lemma A.8) and $V_n \subseteq \ker(\xi)$. We now prove the proposition by induction on n. This is trivial for $n = 1$ since $w_1 = 0$. Assume that $n > 1$ and the proposition is true for $n - 1$. Let $\mathbf{x} = 1 \otimes \mathbf{x}_{1n} \otimes 1 \in R_i \cap V_n'$. Since

$$\omega(\mathbf{x}) = \xi\bigl(\phi \circ \psi(\mathbf{x}) - (-1)^n \omega(1 \otimes \mathbf{x}_{1n})\bigr),$$

and, by items (3)–(5) of Lemma A.8,

$$\xi \circ \phi \circ \psi(\mathbf{x}) \in F^i\bigl(E \otimes \overline{E}^{\otimes^{n+1}} \otimes E\bigr) \cap V_{n+1},$$

in order to finish the proof it suffices to check that

$$\xi \circ \omega(1 \otimes \mathbf{x}_{1n}) \in F^i\bigl(E \otimes \overline{E}^{\otimes^{n+1}} \otimes E\bigr) \cap V_{n+1}.$$

By the inductive hypothesis and the claim:

- If $x_n \in A$, then $\omega(1 \otimes \mathbf{x}_{1n}) \in F^i(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n A$.
- If $x_n \in \mathcal{H}$, then $\omega(1 \otimes \mathbf{x}_{1n}) \in F^{i-1}(E \otimes \overline{E}^{\otimes^n} \otimes E) \cap V_n \mathcal{H}$.
- If $x_n \notin A \cup \mathcal{H}$, then $\omega(1 \otimes \mathbf{x}_{1n}) = 0$.

In all these cases the required inclusion is true. □

Proofs of Propositions 2.1, 2.2 and 2.5. They follow immediately from Propositions A.5, A.9 and A.7, respectively. □

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