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Manifolds of semi-negative curvature

Cristian Conde and Gabriel Larotonda

Dedicated to Mischa Cotlar (1913–2007), in Memoriam

Abstract

This paper studies the metric structure of manifolds of semi-negative curvature. Explicit estimates on the geodesic distance and sectional curvature are obtained in the setting of homogeneous spaces G/K of Banach–Lie groups, and a characterization of convex homogeneous submanifolds is given in terms of the Banach–Lie algebras. A splitting theorem via convex expansive submanifolds is proved, inducing the corresponding splitting of the Banach–Lie group G. The notion of nonpositive curvature in Alexandrov's sense is extended to include p-uniformly convex Banach spaces, and manifolds of semi-negative curvature with a p-uniformly convex tangent norm fall in this class of nonpositively curved spaces. Several well-known results, such as the existence and uniqueness of best approximations from convex closed sets, or the Bruhat–Tits fixed-point theorem, are shown to hold in this setting, without dimension restrictions. Finally, these notions are used to study the structure of the classical Banach–Lie groups of bounded linear operators acting on a Hilbert space, and the splittings induced by conditional expectations in such a setting.

1. *Introduction*

The present paper is a derivation from the study of the classical Banach–Lie groups of compact p-Schatten operators [**4**], where convexity methods have been applied to the study of the rectifiable distance in spaces of unitary operators, that is, the elliptic case. Our concern in the present paper are the cones of positive invertible operators derived from such operator ideals, that is, the hyperbolic case.

The study of nonpositively curved spaces began with the work of Hadamard in the early years of the last century and the work of Cartan about twenty years later. However, the foundations of the theory of metric spaces with upper curvature bounds were laid in the 1950s with the work of Alexandrov [**1**] and Busemann [**13**], who actually coined the term 'nonpositively curved space'. At the heart of their viewpoint (the use of conditions that are equivalent to nonpositive sectional curvature in the Riemannian case, rather than sectional curvature itself) is the work of Menger [**33**] and Wald [**43**], who introduced the notions and methods of curves in metric spaces, geodesic length spaces and comparison triangles. These methods have been used with great success in a wide variety of settings, especially since the work of Ballmann, Gromov and Schroeder [**9**]. The link with smooth manifolds is given by the following elementary fact: if M is a Riemannian–Hilbert manifold of semi-negative sectional curvature, then

$$\|(\exp_x)_{*v}(w)\| \geqslant \|w\|$$

for any $x \in M$ and $v, w \in T_xM$ (here $(\exp_x)_*$ denotes the differential of the exponential map of M). This condition is adopted in [**36**] by Neeb as a definition of semi-negative curvature in the context of Banach–Finsler manifolds, one of the main results in that paper being a Cartan–Hadamard theorem. In the special situation when $M = G/K$ is an homogeneous space of semi-negative curvature, a polar decomposition for G is also obtained that generalizes the

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usual polar decomposition for the group $\mathcal{B}(\mathcal{H})^{\times}$ of invertible bounded operators in a Hilbert space $\mathcal{H}$. In this paper, we translate to the setting $M = G/K$ several results on operator theory, particularly results on the group of invertibles of C*-algebras, that through time have been established using operator–theoretic techniques. Our major concern are the splitting theorems due to Porta and Recht [**37**], which can now be stated as splitting theorems for Banach–Lie groups (Corollary 4.39). To establish such results, we give a detailed characterization of the convex homogeneous submanifolds of M, which we think are interesting in their own right, since an infinite-dimensional theory is still lacking.

Nonpositive curvature in the sense of Alexandrov states that sufficiently small geodesic triangles in the inner metric space (X, d) are at least as thin as corresponding Euclidean triangles. Equivalently X verifies the so-called CN-inequality of Bruhat and Tits [**12**]: for any $x \in X$ and any geodesic segment $\gamma \in X$, we have

$$\tfrac{1}{4}L(\gamma)^2 \leqslant \tfrac{1}{2}(d(x,\gamma_0)^2 + d(x,\gamma_1)^2) - d(x,\gamma_{1/2})^2,$$

provided γ is sufficiently close to x. If X is a 2-uniformly convex Banach space (in the sense of Ball, Carlen and Lieb [**8**], that is there exists a positive constant C such that

$$2\left(\frac{1}{C}\|v\|^2 + \|w\|^2\right) \leqslant \|v-w\|^2 + \|v+w\|^2 \tag{1.1}$$

for any $v, w \in X$), then the nonpositive curvature condition of Alexandrov holds for X if $C \leqslant 1$.

It has been observed that Banach spaces with a p-uniformly convex norm ($p \geqslant 2$) share many of the nice properties of Hilbert spaces in spite of the fact that, generally speaking, they do not verify Alexandrov's definition of nonpositive curvature: in order to verify (1.1), a Banach space has to be necessarily Euclidean [**11**, II.1.14]. Thus it is only natural to consider such Banach spaces as a convenient generalization of Euclidean space, leading us to introduce the notion of Alexandrov p-space, which is a geodesic length space that verifies the following geodesic curvature condition:

$$\frac{1}{(2K)^p}L(\gamma)^p \leqslant \frac{1}{2}(d(x,\gamma_0)^p + d(x,\gamma_1)^p) - d(x,\gamma_{1/2})^p.$$

We show that if the Finsler norm of a manifold M of semi-negative curvature is p-uniformly convex, then M can be regarded as an Alexandrov p-space.

Is the distance map between two geodesics, in a manifold of semi-negative curvature, a convex function in this setting? This question was shown to have a positive answer by Lawson and Lim [**29**], as part of their studies on symmetric spaces. What other properties (of a Riemann–Hilbert manifold) can be translated to this context? for example, existence of best approximations from convex sets, or the Bruhat–Tits theorem for groups of isometries. One of the purposes of this paper is to answer some of the questions posed in Neeb's paper, assuming in some cases that the tangent norms of M are p-uniformly convex, thus dealing with the Alexandrov p-spaces just introduced.

This paper is organized as follows. In Section 2, the reader can find the basic definitions concerning Banach–Finsler manifolds with spray, and an account on the results in [**36**]. In Section 3, we study manifolds M of semi-negative curvature with a p-uniformly convex tangent norm, leading to the concept of Alexandrov p-space. We translate several results from the Riemannian context to this setting, and we establish metric splitting theorems for M via convex submanifolds C by means of the Birkhoff orthogonal to the tangent spaces T_xC, where $x \in C$. In Section 4 we drop the assumption on p-uniform convexity, and we establish some general metric results on homogeneous spaces $M = G/K$ of semi-negative curvature, such as formulas for the geodesic distance and estimates of sectional curvature, and a characterization of the different levels of convexity that arise in this setting. We conclude with a splitting theorem for the homogeneous space M via expansive reductive submanifolds, which gives the corresponding splitting of the Banach–Lie group G (Corollary 4.39), which is the main result of this paper.

The specialization of these results to the positive cones of the classical linear groups $G_p(\mathcal{H})$ of invertible p-Schatten operators, which were our original concern, are included in the Appendix, generalizing the typical scheme $G = \mathcal{B}(\mathcal{H})^{\times}$, $K =$ the group of unitary operators of $\mathcal{H}$. These constructions provide what we think are relevant examples of manifolds of semi-negative curvature with a p-uniformly convex tangent norm. Conditional expectations in $\mathcal{B}(\mathcal{H})$ provide a sufficient amount of expansive reductive submanifolds, inducing factorizations of linear operators via C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ (see Theorem A.3 in Appendix A.2).

2. *Background*

Let M be a Banach manifold with spray. Then M is a smooth manifold locally isomorphic to a fixed Banach space, provided with a second-order vector field $F : TM \to TTM$. A standard reference on the subject is the book of Lang [**25**, IV.4]. Recall that such a field verifies $\pi_* \circ F = \mathrm{id}_{TM}$, where $\pi : TM \to M$ is the projection map of the tangent bundle, and

$$F(sv) = (s_M)_* sF(v) \quad \text{for any } s \in \mathbb{R},\ v \in TM.$$

Here $s_M : TM \to TM$ denotes the multiplication map $v \mapsto sv$ by $s \in \mathbb{R}$, and throughout this paper $f_* : TX \to TY$ indicates the differential of the smooth map $f : X \to Y$. We use f_{*x} to indicate the differential of f at $x \in X$.

Let $v \in TM$ and let β_v be the unique integral curve of F with initial condition v, that is $\beta_v : I \to TM$, with $\beta_v(0) = v$ and

$$\frac{d}{dt}\beta_v = F(\beta_v).$$

Let $\mathcal{D}_{\exp} \subset TM$ stand for the set of vectors v such that β_v is defined at least on the interval $[0, 1]$. The exponential map $\exp : \mathcal{D}_{\exp} \to M$ is defined accordingly to

$$\exp(v) = \pi(\beta_v(1)),$$

and the restriction of exp to each T_xM will be denoted by $\exp_x$. The geodesics of M at x with initial speed $w \in T_xM$ are then given by $\alpha(t) = \pi(\beta_v(t))$, where $v = (x, w) \in TM$. Parallel translation along α will be denoted as follows:

$$P_s^t(\alpha) : T_{\alpha(s)}M \longrightarrow T_{\alpha(t)}M.$$

A *tangent norm* on M is a map $b : TM \mapsto \mathbb{R}^+$ whose restriction to each T_xM is a norm, and it is called a *compatible norm* if the topology induced by b on each T_xM matches the topology induced on it by the Banach space norm.

A *Finsler manifold* is a pair (M, b) of a Banach manifold M and a compatible norm b on TM. In this paper we identify b with the subjacent norm $\|\cdot\|_x = b(x)$ of the Banach space, and we measure the length of piecewise smooth curves $\gamma : [a, b] \to M$ with the usual rectifiable length given by

$$L_a^b(\gamma) = \int_a^b \|\dot{\gamma}\|_\gamma \, dt,$$

and when γ is defined in $I = [0, 1]$, we use $L(\gamma)$ for short.

In this paper the term smooth means C^1 and with nonzero derivative. The set of piecewise smooth curves in M joining two points $x, y \in M$ will be denoted by $\Omega_{x,y}$, and given by

$$\Omega_{x,y} = \{\gamma : [0, 1] \longrightarrow M,\ \gamma \text{ is piecewise smooth},\ \gamma(0) = x, \gamma(1) = y\},$$

and the distance between points in M is defined as the infimum of the lengths of the piecewise smooth curves joining them, given by

$$d(x,y)=\inf\{L(\gamma),\gamma\in\Omega_{x,y}\}.$$

Let $\mathrm{Aut}(M)=\mathrm{Aut}(M,b)$ stand for the group of compatible automorphisms of M, which is the set of diffeomorphisms φ of M such that $b\circ\varphi_*=b$. Then the distance defined above is compatible in the sense that the induced topology matches the topology of M, and it is invariant for the action of the automorphism group of M. See [**42**, Proposition 12.22] for a proof of these facts.

A *Finsler manifold with spray* is a Finsler manifold such that the tangent norm b is invariant under parallel transport along geodesics.

2.1. *Cartan–Hadamard manifolds*

In [**36**] Neeb established a definition of semi-negative curvature for Finsler manifolds with spray, which we recall here. A Finsler manifold M with spray has *semi-negative curvature* if, for any $x\in M$ and $v\in T_xM\cap\mathcal{D}_{\exp}$, then we have the following:

(1) $(\exp_x)_{*v}$ is invertible;

(2) for any $w\in T_xM$, we have

$$\|(\exp_x)_{*v}(w)\|_{\exp_x(v)}\geqslant\|w\|_x. \tag{2.1}$$

The following Cartan–Hadamard theorem can be found in [**36**, Theorem 1.10]:

THEOREM 2.1. *Let M be a connected Banach–Finsler manifold with spray with semi-negative curvature. Then M is geodesically complete if and only if M is complete*, *and in that case, for each $x\in M$, the exponential map* $\exp_x:T_xM\to M$ *is a surjective covering. In particular if M is simply connected* $\exp_x$ *is an isomorphism for each $x\in M$.*

REMARK 2.2. Since M has semi-negative curvature, if Γ is a lift (to T_xM) of a smooth curve $\gamma\in M$, then

$$L_{T_xM}(\Gamma)\leqslant L_M(\gamma). \tag{2.2}$$

Indeed, since $\exp_x(\Gamma)=\gamma$, then

$$\|\dot\gamma\|_\gamma=\|(\exp_x)_{*\Gamma}(\dot\Gamma)\|_{\exp_x(\Gamma)}\geqslant\|\dot\Gamma\|_x.$$

If γ is any smooth curve joining x to y in M, let $\Gamma\subset T_xM$ be the unique lift of γ such that $\Gamma(0)=0$. Then we have

$$L(\gamma)\geqslant L(\Gamma)\geqslant\|\Gamma(1)\|_x=L(\gamma_{x,y}),$$

where $\gamma_{x,y}=\exp_x(t\Gamma(1))$. In particular, given two points $x,y\in M$, there exists a smooth curve $\gamma_{x,y}$ (which is a geodesic) such that $\gamma_{x,y}$ is minimizing for the geodesic distance.

REMARK 2.3. Caution: in spite of the fact that the distance function is convex in a manifold of semi-negative curvature (Theorem 2.5), there might be other short (that is, distance minimizing) curves; see Remark A.2 in the Appendix.

However, provided that the norm of TM is *strictly convex*, we have that

$$\|v+w\|=\|v\|+\|w\| \text{ implies that } v=\lambda w \text{ for some } \lambda\in[0,+\infty),$$

the short curves being unique: Proposition 3.6 below proves this fact. We compare with Corollary 6.3 in [**25**, Chapter VIII], where uniqueness is proved via the Gauss lemma in the Riemann–Hilbert context.

Definition 2.4. A Cartan–Hadamard manifold is a simply connected complete Finsler manifold M of semi-negative curvature.

The question of whether the distance function is convex or not in a Cartan–Hadamard manifold was positively answered in [**29**]; we state this result.

Theorem 2.5. *Let M be a Cartan–Hadamard manifold; let α and β be two geodesics. Then the distance map $f : [0, 1] \to [0, +\infty)$ given by*

$$f(t) = d(\alpha(t), \beta(t))$$

is convex.

Remark 2.6. Let (X, d) be a metric space, which is also a *geodesic length space* in the sense that the distance of X can be computed via the infimum of the length of the rectifiable arcs joining given endpoints in X (see [**23**, Section 2.2]). A geodesic length space is *globally nonpositively curved in the sense of Busemann* if, for given geodesic arcs α and β starting at $x \in X$, the distance map

$$t \mapsto d(\alpha(t), \beta(t))$$

is a convex function. Then, by the theorem above, any Cartan–Hadamard manifold (M, d), where d is the rectifiable metric given by the Finsler norms, can be regarded as a metric space of nonpositive curvature in the sense of Busemann.

3. *Metric problems*

We begin this section with an elementary inequality (which can be found in the setting of Riemannian manifolds in [**25**, Chapter IX, Corollary 3.10]). It will be useful later; it compares the distance in M with the distance in the tangent linear space. We include a proof for the convenience of the reader. In the context of positive invertible operators (see the Appendix) it is known as the exponential metric increasing property.

Lemma 3.1. *Let M be a Cartan–Hadamard manifold, let $x \in M$ and let $v, w \in T_xM$. Then we have*

$$\|v - w\|_x \leqslant d(\exp_x(v), \exp_x(w)).$$

Proof. Let γ be any piecewise smooth curve in M joining $\exp_x(v)$ to $\exp_x(w)$. Then, by Theorem 2.1, there exists a piecewise smooth curve $\Gamma \subset T_xM$ such that $\gamma = \exp_x(\Gamma)$, with $\Gamma(0) = v$ and $\Gamma(1) = w$. Now, since the differential of the exponential map is an isomorphism, it follows that

$$\begin{aligned}\|w - v\|_x = \|\Gamma(0) - \Gamma(1)\|_x &= \left\|\int_0^1 \dot\Gamma(t)\,dt\right\|_x \\ &\leqslant \int_0^1 \|\dot\Gamma(t)\|_x\,dt = \int_0^1 \|(\exp_x)_{*\Gamma}^{-1}(\dot\gamma)\|_x\,dt.\end{aligned}$$

The last quantity inside the integral sign is, by (2.1), less than or equal to

$$\|\dot\gamma(t)\|_{\exp_x(\Gamma)} = \|\dot\gamma(t)\|_\gamma,$$

and hence $\|w - v\|_x \leqslant L(\gamma)$. Since γ is arbitrary, we obtain the asserted inequality. □

Problem 3.2. If one asks for equality to hold in the above lemma, then this imposes a rigidity condition. In Theorem 4.13 we study this problem, in the setting of homogeneous spaces. We would like to know if the following assertions hold in the general setting (here $R(\cdot,\cdot)$ indicates the curvature tensor of M derived from the spray):

(1) $R(v,w)|_{\mathrm{span}(v,w)} \equiv 0$ implies that equality holds in Lemma 3.1;
(2) If the tangent norms are strictly convex, and equality holds, then $R(v,w)$ restricted to $\mathrm{span}(v,w)$ vanishes.

This problem is closely related to [**36**, Problem 1.2].

Remark 3.3. Let $x \in M$. Given $v,w \in T_xM$, for $r>0$ let

$$s_x(r,v,w) = \frac{r\|v-w\|_x - d(\exp_x(rv),\exp_x(rw))}{r^2 d(\exp_x(v),\exp_x(w))}.$$

Milnor [**34**] observed that, in the Riemannian setting, sectional curvature can be obtained via the limiting procedure

$$s_x(v,w) = \frac{1}{6}\lim_{r\to 0^+} s_x(r,v,w).$$

Hence this limit (provided it exists) can be used as a suitable definition of curvature. In the present setting, by the inequality in Lemma 3.1, one has

$$s_x(r,v,w) \leqslant 0 \quad \text{for any } r>0.$$

Thus it seems only natural to ask if the limit exists, and if there are lower bounds. If $M=G/K$ is an homogeneous space, then the answer is affirmative; see Paragraph 4.1.3.

3.1. *Uniform convexity and minimizers*

Definition 3.4. Let $(E,\|\cdot\|)$ be a Banach space. The modulus of convexity of E is the nonnegative number

$$\delta_E(\varepsilon) = \inf\{1-\tfrac{1}{2}\|x+y\| : \ \|x\|,\|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\}.$$

A Banach space is uniformly convex if $\delta_E(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$. A uniformly convex Banach space is strictly convex (cf. Remark 2.3).

Remark 3.5. Assume that E is strictly convex. Then the unique short piecewise smooth curves of E are the straight segments [**18**, Lemma 2.10]. That is, if γ is a piecewise smooth curve in E joining 0 to v, and γ has length $\|v\|$, then $\gamma(t)=tv$. See [**18**] also for examples of infinitely many smooth curves joining given endpoints, in the setting of Banach spaces with a norm that is not strictly convex.

Proposition 3.6. *Let M be a Cartan–Hadamard manifold. If the norm of TM is strictly convex, then the geodesics of M are the unique piecewise smooth short paths in M.*

Proof. Let γ be a short curve in M, with $\gamma(0)=x$ and $\gamma(1)=y$. Let $\Gamma \subset T_xM$ be such that $\exp_x(\Gamma)=\gamma$ and $\Gamma(0)=0$. Then $L(\Gamma) \leqslant L(\gamma) = d(x,y)$ by (2.2). Let $v=\Gamma(1) \in T_xM$ and let $\alpha(t)=\exp_x(tv)$. Then we have

$$d(x,y) \leqslant L(\alpha) = \int_0^1 \|\dot\alpha\|_\alpha \, dt = \|v\|_x$$

since α is a geodesic, and then $L(\Gamma) \leqslant \|v\|_x$. Since Γ joins 0 to v in T_xM, by Remark 3.5, we obtain that $\Gamma(t) = tv$, or in other words $\gamma(t) = \exp_x(tv)$. □

DEFINITION 3.7. We call M a *p-uniformly convex* Cartan–Hadamard manifold if there exists a positive constant K_M and a number $p \geqslant 2$ such that

$$2\left(\frac{1}{K_M^p}\|v\|_x^p + \|w\|_x^p\right) \leqslant \|v+w\|_x^p + \|v-w\|_x^p, \tag{3.1}$$

for any $x \in M$ and any $v, w \in T_xM$.

By a result of Ball, Carlen and Lieb [**8**], a uniformly convex Banach space E has modulus of convexity of power type $p \geqslant 2$ (that is, $\delta_E(\varepsilon) \geqslant C\varepsilon^p$) if and only if there exists a constant $K_E > 0$ such that a weak Clarkson inequality like (3.1) holds. Hence we assume that all the tangent spaces of M are of power type p, with K_{T_xM} uniformly bounded by K_M. This condition guarantees uniform convexity and, in particular, strict convexity of the tangent norms.

This is a convenient generalization of the parallelogram law for the Riemannian metric of Riemann–Hilbert manifolds, since it induces a strong convexity result analogous to the Gauss lemma. Among the simplest examples of uniformly convex Banach spaces of power type p are the usual L^p measure spaces of functions that were the original concern of Clarkson [**14**], and their noncommutative counterpart, the $\mathcal{B}_p(\mathcal{H})$ spaces of compact Schatten operators.

In this section we prove the existence and uniqueness of minimizers in p-uniformly convex Cartan–Hadamard manifolds, and give a geometrical characterization of them. In what follows, for a given curve $\gamma : I \to M$, we define $\gamma(t) = \gamma_t$ for any $t \in I$. Then, if γ is a geodesic, $\gamma_{1/2}$ is the midpoint between γ_0 and γ_1.

THEOREM 3.8. *Let M be a p-uniformly convex Cartan–Hadamard manifold. Let $x, y, z \in M$ and let γ be the geodesic joining y to z in M. Then we have*

$$\frac{1}{(2K_M)^p} d(y,z)^p \leqslant \frac{1}{2}(d(x,y)^p + d(x,z)^p) - d(x,\gamma_{1/2})^p. \tag{3.2}$$

Proof. Let $a = \gamma_{1/2} \in M$. Note that $d(a,z) = \frac{1}{2}L(\gamma)$. Let $v, w \in T_aM$ be such that $y = \exp_a(-v)$, $z = \exp_a(v)$ and $x = \exp_a(w)$. Then, by Lemma 3.1, we have

$$d(x,z)^p = d(\exp_a(w), \exp_a(v))^p \geqslant \|v-w\|_a^p$$

and also

$$d(x,y)^p = d(\exp_a(w), \exp_a(-v))^p \geqslant \|v+w\|_a^p.$$

Adding these quantities and using the definition of p-uniform convexity, we obtain the stated inequality, since $\|v\|_a = d(a,z)$ and $\|w\|_a = d(a,x)$. □

REMARK 3.9. Let (X,d) be a geodesic length space [**23**]. Then X is said to be *nonpositively curved in the sense of Alexandrov* if, for any $x \in X$ and any geodesic segment $\gamma \in X$, we have

$$\tfrac{1}{4}L(\gamma)^2 \leqslant \tfrac{1}{2}(d(x,\gamma_0)^2 + d(x,\gamma_1)^2) - d(x,\gamma_{1/2})^2.$$

Nonpositive curvature in the sense of Alexandrov implies nonpositive curvature in the sense of Busemann (see Remark 2.6 above for the definition of Busemann nonpositive curvature).

Definition 3.10. If (X,d) is a geodesic length space and there exists a positive constant K such that (3.2) holds for any geodesic γ joining $y,z \in X$, then we say that X *is an Alexandrov p-space*.

Let (X,d) be an Alexandrov p-space. A set $C \subset X$ is called convex if, for given $x,y \in C$, the unique geodesic $\gamma_{x,y}$ of X joining x to y is fully contained in C.

Hence the semi-parallelogram law on M (Theorem 3.8) gives a link with the spaces of nonpositive curvature as studied by Alexandrov, Ballman, Busemann, Gromov and others; see [**1**, **13**, **23**]. Alexandrov p-spaces lie somewhere in between Busemann spaces and Alexandrov spaces, since the metric of the manifold fulfills a strong inequality *a la* Alexandrov, but one does not have the quadratic exponents.

Problem 3.11. Evidently Banach–Finsler manifolds M of semi-negative curvature with a p-uniformly convex tangent norm are Alexandrov p-spaces, and in that setting the distance between geodesics starting at a common point is a convex function. Is each Alexandrov p-space (X,d) nonpositively curved in the sense of Busemann, for $p>2$? The proof for $p=2$ (see [**23**, Corollary 2.3.1]) only gives

$$d(\alpha(t),\beta(t))^p \leqslant t^2 d(\alpha(1),\beta(1))^p + \left(1-\frac{1}{K^p}\right)(L(\alpha)^p + L(\beta)^p).$$

for two geodesics starting at $x \in X$. Even for $K=1$ this is not sufficient.

We now obtain the existence of (unique) minimizers from a convex set to any given point outside it in the same fashion as in [**23**, Chapter 3], where it is done for Alexandrov spaces.

Theorem 3.12. *Let (X,d) be an Alexandrov p-space. Let $C \subset X$ be a convex closed set in X and let $x \in X$. Then there exists a unique point $x_C \in C$ such that*

$$d(x_C,x) = \min_{y\in C} d(y,x) = d(C,x).$$

We call x_C the best approximation of x in C.

Proof. Let $D = d(C,x)$ be the distance between C and x. Let x_n be a decreasing minimizing sequence in C, that is $\lim_{n\to\infty} d(x_n,x) = D$ and

$$d(x_n,x) \geqslant d(x_{n+1},x)$$

We claim that $\{x_n\}$ is a Cauchy sequence in C. Let $\gamma_{n,m}:[0,1]\to M$ be the short geodesic joining x_n to x_m in M, which is contained in C. Let $m>n$ and let $x_{n,m} \in C$ be the middle point of $\gamma_{n,m}$. Then, by the semi-parallelogram law in Theorem 3.8, we have

$$\frac{1}{2}(d(x_n,x)^p + d(x_m,x)^p) - d(x_{n,m},x)^p \geqslant \frac{1}{(2K_M)^p} d(x_n,x_m)^p,$$

and $D \leqslant d(x_{n,m},x)$ since C is convex, and hence

$$\frac{1}{2}(d(x_n,x)^p + d(x_m,x)^p) - D^p \geqslant \frac{1}{(2K_M)^p} d(x_n,x_m)^p,$$

which proves the claim.

To prove uniqueness, assume that x^1 and x^2 are minimizers in C and let x^{12} be the middle point. If we replace them again in the semi-parallelogram law, then we obtain

$$0 = \frac{1}{2}(D^p + D^p) - D^p \geqslant \frac{1}{2}(d(x^1, x)^p + d(x^2, x)^p) - d(x^{12}, x)^p \geqslant \frac{1}{(2K_M)^p} d(x^1, x^2)^p. \qquad \square$$

Let (X, d) be an Alexandrov p-space, let $x_0 \in X$ and $\lambda > 0$ and let $F : X \to \mathbb{R} \cup \{\infty\}$ be any function. We define the Moreau–Yoshida approximation F^λ of F as follows:

$$F^\lambda = \inf_{y \in X} \{\lambda F(y) + d(x_0, y)^p\}.$$

It is not hard to see, using (3.2), that if F is convex, lower semi-continuous, bounded from below and not identically $+\infty$, then, for every $\lambda > 0$, there exists a unique $y_\lambda \in X$ such that

$$F^\lambda = \lambda\, F(y_\lambda) + d(x_0, y_\lambda)^p. \tag{3.3}$$

See [**23**, Lemma 3.1.2] for the details (done there for $p = 2$). Then the following result, using our semi-parallelogram laws, has a proof almost identical to that in [**23**, Theorem 3.1.1], and therefore we omit it.

THEOREM 3.13. *Let (X, d) be an Alexandrov p-space and let $F : X \to \mathbb{R} \cup \{\infty\}$ be a convex lower semi-continuous function that is bounded from below and not identically $+\infty$. Let y_λ be constructed as in (3.3). If $d(x_0, y_{\lambda_n})$ is bounded for some sequence $\lambda_n \to +\infty$, then $\{y_\lambda\}_{\lambda>0}$ converges to a minimizer of F as $\lambda \to \infty$.*

3.1.1. *Bruhat–Tits fixed point theorem.* The existence and uniqueness of minimal balls is guaranteed by the generalized semi-parallelogram laws (Theorem 3.8), and from there one obtains Bruhat–Tits fixed-point theorem and its usual corollaries. The proofs are straightforward and identical to the proofs of the case $p = 2$ (see for instance, [**26**, Section 3]), and therefore we omit them. The contents of this section are related to [**36**, Problem 1.3(2)]. In the following propositions M is an Alexandrov p-space (in particular, a Cartan–Hadamard manifold with a p-uniformly convex tangent norm).

PROPOSITION 3.14. *Let S be a bounded subset of M. Then there exists a unique closed ball $B_r(s_1) \subset M$ of minimal radius r containing S. The center $s_1 \in S$ is called the circumcenter of S.*

THEOREM 3.15 (Bruhat–Tits). *Let G be a group of isometries of M. Suppose that G has a bounded orbit* (*for instance, if G is finite*). *Then the orbit of G has a fixed point, for instance, the circumcenter.*

3.2. *Metric splittings via convex submanifolds*

In this section, we give a geometrical characterization of the best approximation $x_C \in C \subset M$, where C is a convex and closed submanifold of a Cartan–Hadamard manifold M, and we state a straightforward splitting of M via such submanifolds.

DEFINITION 3.16. Let X be a Banach space and let $S \subset X$ be a linear subspace. The Birkhoff orthogonal $S^\perp$ of S is given by

$$S^\perp = \{v \in X : \|v\| \leqslant \|v + s\| \ \text{ for any } \ s \in S\}.$$

The Birkhoff orthogonal is the analog of the usual orthogonal in Hilbert spaces. However the Birkhoff orthogonal does not necessarily have a linear structure [22].

REMARK 3.17. Let $C \subset M$ be a convex submanifold of a Cartan–Hadamard manifold. Let $z \in C$ and let $x = \exp_z(v)$, with v Birkhoff orthogonal to T_zC. Since (by virtue of the convexity of C), for any $y \in C$, we can write $y = \exp_z(s)$, with $s \in T_zC$, it follows that

$$d(x,z) = \|v\|_z \leqslant \|v-s\|_z \leqslant d(\exp_z(v), \exp_z(s)) = d(x,y)$$

by Lemma 3.1. Thus if $x \in M$ is reached by an orthogonal direction, then it has a closest point in C.

PROPOSITION 3.18. *Let $C \subset M$ be a submanifold of a Cartan–Hadamard manifold M. Let $x \in M$ and let $z \in C$. If z is the best approximation of x in C, then the initial speed of the geodesic α joining z to x in M is orthogonal to T_zC in the sense of Birkhoff. In addition, if C is a convex submanifold, then these conditions are equivalent.*

Proof. Assume that $d(z,x) \leqslant d(y,x)$ for any $y \in C$, and we consider $v = \dot{\alpha}(0)$, where α is the short geodesic joining z to x in M. Let $\tilde{\alpha}(t) = \alpha(1-t)$ be the geodesic joining x to z. Then $\tilde{\alpha}(t) = \exp_x(t\exp_x^{-1}(z))$ by the uniqueness of geodesics; thus $\exp_x^{-1}(z) = \dot{\tilde{\alpha}}(0) = -\dot{\alpha}(1)$, and also $P_z^x(\alpha)(v) = \dot{\alpha}(1)$ by parallel translation properties. Hence $P_z^x(\alpha)v = -\exp_x^{-1}(z)$.

Let $A_v : T_zM \to T_zM$ be the linear isomorphism given by $P_x^z(\exp_x^{-1})_{*z}$ (recall that $x = \exp_z(v)$). Then A_v is a contraction by the semi-negative curvature condition (2.1), and we claim that $A_v v = v$: the curve $\gamma(t) = \exp_x(P_z^x(t-1)v)$ is a geodesic of M with initial data $\gamma(0) = z$ and $\gamma(1) = x$, and hence $\gamma(t) = \alpha(t)$ and then we have

$$v = \dot{\gamma}(0) = (\exp_x)_{*-P_z^x v} P_z^x v,$$

which in turn implies that

$$P_x^z(\exp_x^{-1})_{*z} v = v.$$

Let $w \in T_zC$ and let $\beta \subset C$ be any smooth curve such that $\beta(0) = z$, and $\dot{\beta(0)} = w$. Consider the convex function $g(t) = d(\beta(t), x)$: if z is the best approximation of x in C, then $g'(0^+) \geqslant 0$. Let $v_t = \exp_x^{-1}(\beta(t))$. Then we have

$$g(t) = \|v_t\|_x = \|P_x^z(\tilde{\alpha})v_t\|_z = \|-v + A_v wt + o(t^2)\|_z \leqslant \|-v + A_v wt\|_z + o(t^2),$$

since $P_x^z(\tilde{\alpha})v_0 = P_x^z(\tilde{\alpha})\exp_x^{-1}(z) = -v$ and $\dot{v}_0 = (\exp_x^{-1})_{*z} w$. Now, since $g(0) = \|v\|_z$, it follows that

$$0 \leqslant g'(0^+) \leqslant \|-v + A_v w\|_z - \|v\|_z$$

by the convexity of the norm, and hence $\|v\|_z \leqslant \|-v + A_v w\|_z$ for any $w \in T_zC$. Then, since $v = A_v v$, it follows that

$$\|v\|_z \leqslant \|-A_v v + A_v w\|_z = \|A_v(-v+w)\|_z \leqslant \|v-w\|_z$$

because A_v is a contraction, and this shows that v is Birkhoff orthogonal to T_zC. The last assertion of the proposition follows from Remark 3.17. □

REMARK 3.19. In [37], Porta and Recht prove a splitting theorem for inclusions $N \subset M$ of C*-algebras. In their proof, a key element is the natural linear supplement of the tangent spaces of the submanifold, given by a conditional expectation $E : M \to N$. However, in the setting of p-uniformly convex Banach spaces, it is natural to replace linear supplements with Birkhoff orthogonals.

Since the orthogonal directions in the tangent bundle play a relevant role, we define *the normal of C* by

$$\mathfrak{N}_C = \{(x, v) : x \in C,\ v \in T_x C^{\perp}\} \subset TM.$$

We denote by $\exp : TM \to M$, with $\exp(x, v) = \exp_x(v)$ the exponential map of M.

THEOREM 3.20. *Let M be a p-uniformly convex Cartan–Hadamard manifold and let C be a convex closed submanifold. Then $\exp : \mathfrak{N}_C \to M$ is a bijection that induces a differentiable structure on $\mathfrak{N}_C$ which makes it diffeomorphic to M.*

Proof. Let $x \in M$ and let $z \in C$ be the unique minimizer (Theorem 3.12), with $D = d(x, C) = d(x, z)$. Let α be the unique geodesic in M joining z to x. Let v be the initial speed of α. Then $x = \exp_z(v)$. Note that $\|v\|_z = D$, and also that v is Birkhoff orthogonal to $T_z C$ by the previous proposition; hence $x = \exp(z, v)$ and the map exp is surjective. On the other hand, assume that $M \ni x = \exp_y(w) = \exp_z(v)$ with $(z, v), (y, w) \in \mathfrak{N}_C$. Let $D = d(x, C) = \|v\|_z = \|w\|_y$. Then by convexity $d(x, \gamma_{1/2}) \geqslant D$, and by inequality (3.2), we obtain

$$\begin{aligned}\frac{1}{(2K_M)^p} d(y, z)^p &\leqslant \frac{1}{2}(d(x, y)^p + d(x, z)^p) - d(x, \gamma_{1/2})^p \\ &= \frac{1}{2} D^p + \frac{1}{2} D^p - d(x, \gamma_{1/2})^p \leqslant 0,\end{aligned}$$

hence $y = z$ and thus exp is injective. With the induced differentiable structure, exp is a global isomorphism onto M, since its differential is everywhere invertible by hypothesis. □

COROLLARY 3.21. *Let $C \subset M$ be a convex closed submanifold of a p-uniformly convex Cartan–Hadamard manifold and let $x \in M$. Then there exists a unique $z \in C$ and $v \in T_z C^{\perp}$ such that $\|v\|_z = d(x, C)$ and $x = \exp_z(v)$.*

4. *Homogeneous spaces*

In this section we assume that $M \simeq G/K$ is an homogeneous reductive space, quotient of Banach–Lie groups. The assumption on p-uniform convexity of the tangent norms is dropped. First we recall basic facts, and then we include some considerations for the benefit of the reader.

A Banach–Lie group G with an involutive automorphism σ is called a *symmetric Lie group* in [**36**]. Let $\mathfrak{g}$ be the Banach–Lie algebra of G and let $K = G^{\sigma} = \{g \in G : \sigma(g) = g\}$ be the subgroup of σ-fixed points. Then the Banach–Lie algebra $\mathfrak{k}$ of K is a closed complemented subspace of $\mathfrak{g}$; the complement is given by the following closed subspace:

$$\mathfrak{p} = \{v \in \mathfrak{g} : \sigma_{*1} v = -v\},$$

since the Lie algebra $\mathfrak{k}$ matches the set of σ_{*1}-fixed points (here and in what follows, 1 is the neutral element of G). Hence K is a Banach–Lie subgroup of G, and the quotient space $M = G/K$ carries the structure of a Banach manifold. We indicate with $q : G \to M$, $g \mapsto gK$ the quotient map and with $\mathrm{Exp} : \mathfrak{g} \to G$ the exponential map of G. We use the short notation $e^v = \mathrm{Exp}(v)$ for $v \in \mathfrak{g}$ whenever possible. Then $q \circ \mathrm{Exp} : \mathfrak{p} \to M$ is the natural chart around $o = q(1) \in M$ given by the exponential map of G, $q \circ \mathrm{Exp} = \exp_o \circ q_{*1}$, and a general geodesic of $M = G/K$ is given by

$$\alpha(t) = g e^{tv} K = q(g e^{tv})$$

for some $v \in \mathfrak{p}$. In particular, note that M is geodesically complete.

Let $h \in G$ and let $\mu_h : M \to M$ stand for $\mu_h(q(g)) = q(hg) = q(L_h g)$. Then we have

$$(\mu_h)_{*q(g)} q_{*g} = q_{*hg}(L_h)_{*g}.$$

A generic point in M is denoted by $q(g)$ for $g \in G$, and we identify $\mathfrak{p}$ with T_oM, and hence a generic vector in $T_{q(g)}M$ is indicated by $(\mu_g)_{*o}v$ for $v \in \mathfrak{p}$.

We use Ad_k to denote both the automorphism of g given by $\mathrm{Ad}_k(g) = kgk^{-1}$, and also its differential $(\mathrm{Ad}_k)_{*1}$ which is an element of $\mathcal{B}(\mathfrak{g})$, the bounded linear operators acting on $\mathfrak{g}$. Note that $\sigma(\mathrm{Ad}_k\, e^{tv}) = \mathrm{Ad}_k e^{-tv}$ for any $v \in \mathfrak{p}$ and $k \in K$; thus $\sigma_{*1}\mathrm{Ad}_k v = -\mathrm{Ad}_k v$, and hence $\mathfrak{p}$ is Ad_K-invariant.

Remark 4.1. Since σ is a group automorphism, it follows that σ_{*1} is a Lie algebra homomorphism, and the relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

follow. In particular, $\mathfrak{p}$ is $\mathrm{ad}_{\mathfrak{k}}$-invariant, as mentioned.

The bundle $G \times_K \mathfrak{p}$ identifies with TM via $(g, v) \mapsto (q(g), (\mu_g)_{*o}v)$, and the action of K is given by $(g, v) \mapsto (gk^{-1}, Ad_k v)$.

Assume that $f = gk$ for some $k \in K$, let $x = q(g) = q(f)$, and assume that $(\mu_g)_{*o}v = (\mu_{gk})_{*o}w \in T_xM$. From

$$\begin{aligned}(\mu_g)_{*o}v = (\mu_{gk})_{*o}w &= \frac{d}{dt}\Big|_{t=0} q(gke^{tw}) \\ &= \frac{d}{dt}\Big|_{t=0} q(g\mathrm{Ad}_k e^{tw}) = \frac{d}{dt}\Big|_{t=0} q(ge^{t\mathrm{Ad}_k w}) = (\mu_g)_{*o}\mathrm{Ad}_k w\end{aligned}$$

we obtain $v = \mathrm{Ad}_k w$. These considerations indicate that a natural way to make of M a Finsler manifold is by

$$\|(\mu_g)_{*o}v\|_{q(g)} := \|v\|_{\mathfrak{p}},$$

where $\|\cdot\|_{\mathfrak{p}}$ is any Ad_k-invariant norm on $\mathfrak{p}$. This definition makes parallel translation isometric, since from [**36**, p. 135] it follows that parallel translation along a geodesic $\alpha(t) = q(ge^{tv})$ is given by

$$P_0^t(\alpha) = (\mu_{ge^{tv}g^{-1}})_{*q(g)}.$$

Then the maps $\mu_h : M \to M$ are tautologically isometries since

$$(\mu_h)_{*q(g)}(\mu_g)_{*o} = (\mu_{hg})_{*o},$$

and the set $I(G) = \{\mu_g\}_{g\in G}$ is a subgroup of the path-component of the identity of $\mathrm{Aut}(M)$ which acts transitively on M.

Remark 4.2. Assume that G is connected. Then $M = G/K$ is a connected and geodesically complete Finsler manifold with spray. Assume that M has semi-negative curvature and let $\exp : TM \to M$, such that

$$(g, v) \longrightarrow \exp_{q(g)}((\mu_g)_{*o}v) = q(ge^v)$$

stand for the exponential map of M, where we identified TM with $G \times_K \mathfrak{p}$. In this context, Theorem 2.1 says that any element $x \in M$ can be written as $x = q(ge^v)$ for some $v \in \mathfrak{p}$.

Remark 4.3. From now on, whenever possible, we omit the isomorphism $(\mu_g)_{*o}$ that identifies $\mathfrak{p}$ with T_xM when $x = q(g)$, and we write $\exp_x(v) = q(ge^v)$ for $x \in M$ and $v \in \mathfrak{p}$ when there is no possibility of confusion. Let $\mathcal{B}(\mathfrak{p})$ stand for the bounded linear operators of $(\mathfrak{p}, \|\cdot\|_{\mathfrak{p}})$. In [**36**, Lemma 3.10], the formula for the differential of the exponential map is computed in a homogeneous space. Let $F(z) = z^{-1}\sinh z$ and recall the usual expression for the exponential of the differential map

$$\mathrm{Exp}_{*v} = (L_{e^v})_{*1}\left(\frac{1 - e^{-\operatorname{ad} v}}{\operatorname{ad} v}\right)$$

for $v \in \mathfrak{g}$ the Banach–Lie algebra of a Banach–Lie group G. Then we have

$$(\exp_o)_{*v} = (\mu_{e^v})_{*o}\frac{\sinh \operatorname{ad} v}{\operatorname{ad} v} = (\mu_{e^v})_{*o}F(\operatorname{ad} v)$$

for any $v \in \mathfrak{p}$, since $q_{*e^v} = (\mu_{e^v})_{*o}q_{*1}$, and q_{*1} is essentially the identity on $\mathfrak{p}$ and has kernel $\mathfrak{k}$.

We recall some related results for our general framework.

Remark 4.4. If Z is a Banach space, then an operator $A \in \mathcal{B}(Z)$ is called *dissipative* if

$$\Re e\ \varphi(Az) \leqslant 0$$

for some (or equivalently, any) $\varphi \in Z^*$ such that $\varphi(z) = \|z\|$ and $\|\varphi\| = 1$. This condition is equivalent to the fact that $1 - sA$ is expansive and invertible for any $s > 0$ (see [**31**]).

What follows is a useful semi-negative curvature criterion for homogeneous spaces, [**36**, Proposition 3.15, Theorem 2.2].

Proposition 4.5. *Let $M = G/K$ be a homogeneous space with a norm $\|\cdot\|_{\mathfrak{p}} : \mathfrak{p} \to \mathbb{R}_{\geqslant 0}$ that is Ad_K-invariant, and hence M can be regarded as a Finsler manifold. Then the following statements are equivalent:*
(i) *M has semi-negative curvature;*
(ii) *for each $v \in \mathfrak{p}$, the operator $T_v = -(\mathrm{ad}_v)^2|_{\mathfrak{p}}$ is dissipative;*
(iii) *for each $v \in \mathfrak{p}$, the operator $1 + (\mathrm{ad}_v)^2|_{\mathfrak{p}}$ is expansive and invertible;*
(iv) *for each $v \in \mathfrak{p}$, we have that $F(\operatorname{ad} v) = (\sinh \operatorname{ad} v/\operatorname{ad} v)|_{\mathfrak{p}}$ is expansive and invertible in $\mathfrak{p}$.*

Remark 4.6. By mimicking the proof of [**36**, Proposition 3.15], it is not hard to see that any entire function G, with purely imaginary roots and such that $G(0) = 1$, induces by functional calculus a bounded operator $G(\operatorname{ad} v) \in \mathcal{B}(\mathfrak{p})$, that is invertible and expansive and, in particular, its inverse a contraction. We use this fact repeatedly for $G(z) = \cosh(z)$. See [**28**] for further details on this technique.

We recall two more results on the fundamental group of M and polar decompositions from [**36**, Theorems 3.14, 5.1]

Theorem 4.7. *Let (G, σ) be a connected symmetric Banach–Lie group and let $K = G^\sigma$ be the subgroup of σ-fixed points. If $M = G/K$ has semi-negative curvature, then we have the following.*
(i) *The exponential map $q \circ \mathrm{Exp} : \mathfrak{p} \to M$ is a covering of Banach manifolds and*

$$\Gamma = \{z \in \mathfrak{p} : q(e^z) = q(1)\}$$

is a discrete additive subgroup of $\mathfrak{p} \cap Z(\mathfrak{g})$, with $\Gamma \simeq \pi_1(M)$ and $M \simeq \mathfrak{p}/\Gamma$. Here $Z(\mathfrak{g})$ denotes the center of the Banach–Lie algebra $\mathfrak{g}$. If $v, w \in \mathfrak{p}$ and $q(e^v) = q(e^w)$, then $v - w \in \Gamma$.

(ii) *The polar map* $m : \mathfrak{p} \times K \to G$*, given by* $(v, k) \mapsto e^v k$*, is a surjective covering map whose fibers are given by the sets* $\{(v - z, e^z k) : v \in \mathfrak{p},\ z \in \Gamma,\ k \in K\}$.

4.1. *Local metric structure and totally geodesic submanifolds*

In what follows we assume that $M = G/K$ is a complete and connected manifold of semi-negative curvature. This whole section is dedicated to the study of the local metric structure of M and the totally geodesic submanifolds of M.

4.1.1. *Local convexity of the geodesic distance.* First, by [**23**], we prove local convexity results for the geodesic distance (recall that Theorem 2.5 was proved in [**29**] in the context of simply connected manifolds).

Remark 4.8. Recall that $\Gamma = \exp_o^{-1}\{o\}$ is a discrete additive subgroup of $\mathfrak{p} \cap Z(\mathfrak{g})$, since the differential of the exponential map is an isomorphism. Let $\kappa_M \in (0, +\infty)$ stand for the maximum of the positive numbers r such that $0 \in \mathfrak{p}$ is the unique point of Γ in the open ball of radius r around it. Note that $\kappa_M = +\infty$ if and only if M is simply connected.

Note that $\|v - z_0\|_{\mathfrak{p}} < \kappa_M/2$ for some $z_0 \in \Gamma$ means that $\|v - z_0\|_{\mathfrak{p}} < \|v - z\|_{\mathfrak{p}}$ for any $z \in \Gamma - \{z_0\}$. This implies that, for any $x, y \in M$ and $d(x, y) < \kappa_M/2$, there exists a unique $v \in \mathfrak{p}$ such that $\|v\|_{\mathfrak{p}} = d(x, y)$ and $y = \exp_x(v)$. Indeed, take any v' such that $\exp_x(v') = y$ and then replace v' with $v = v' - z_0$, where z_0 is the element of Γ closer to v'.

Moreover, $\alpha(t) = \exp_x(tv)$ is the unique short geodesic joining x to y in M, for if $\beta(t) = \exp_x(tw)$ is another geodesic, we consider $z = v - w \in \Gamma$, and if $z \neq 0$, then we have

$$d(x, y) = L(\alpha) = \|v\|_{\mathfrak{p}} < \|v - z\|_{\mathfrak{p}} = \|w\|_{\mathfrak{p}} = L(\beta) = d(x, y),$$

which is a contradiction. Note that κ_M is the diameter of the geodesic balls of (M, d).

With similar argumentation one can show that, for any given $v, w \in \mathfrak{p}$, if we put $x = q(e^v)$, $y = q(e^v e^w)$, then $d(x, y)$ is given by $\|w - z_0\|_{\mathfrak{p}}$, where $z_0 \in \Gamma$ is one of the (possibly many, even infinite) elements of Γ closer to w. Then $\alpha(t) = q(e^v e^{t(w - z_0)})$ is a short geodesic joining x to y.

Proposition 4.9. *Let* $x, x' \in M$ *and let* $y = \exp_x(v)$, $y' = \exp_x(v') = \exp_{x'}(w)$, *such that* $d(x, y) = \|v\|_{\mathfrak{p}}$, $d(x, y') = \|v'\|_{\mathfrak{p}}$, $d(x', y') = \|w\|_{\mathfrak{p}}$. *Let* $0 < R < \kappa_M/4$. *Then we have the following.*

(i) *If* $z_0 \in \Gamma$ *is closer to* $v - v'$ *than any other* $z \in \Gamma$*, then*

$$\|v - v' - z_0\|_{\mathfrak{p}} \leqslant d(y, y').$$

In particular, if $y, y' \in B(x, R)$*, then*

$$\|v - v'\|_{\mathfrak{p}} \leqslant d(y, y').$$

(ii) *If* $y, y' \in B(x, R)$*, then* $f : [0, 1] \to [0, +\infty)$

$$f(t) = d(\exp_x(tv), \exp_x(tv')),$$

which gives the distance between the two geodesics starting at $x \in M$*, is a convex function.*

(iii) *The distance function between the two geodesics joining* x *to* y *and* x' *to* y'*, given by*

$$g(t) = d(\exp_x(tv), \exp_{x'}(tw)),$$

is also convex, provided that $y, y' \in B(x, R)$ *and* $d(x', y') < R$.

(iv) *In particular, if* γ *is the short geodesic joining* x' *to* y'*, then* $h(t) = d(x, \gamma(t))$ *is convex and* $\gamma \subset B(x, R)$*, provided that* $x', y' \in B(x, R)$.

Proof. We can assume that $x = o$. Let α be any piecewise curve joining y to y' in M. Let β be the piecewise smooth lift of α to $\mathfrak{p} \simeq T_oM$ such that $\beta(0) = v$. Then there exists $z_\alpha \in \Gamma$ such that $\beta(1) = v' - z_\alpha$. Hence we have

$$\|v - v' - z_0\|_{\mathfrak{p}} \leqslant \|v - v' + z_\alpha\|_{\mathfrak{p}} = \|\beta(1) - \beta(0)\|_{\mathfrak{p}} \leqslant L(\beta) \leqslant L(\alpha),$$

where the last inequality is due to Remark 2.2. This proves the first assertion, since if $y, y' \in B(x, R)$, then $\|v - v'\|_{\mathfrak{p}} \leqslant 2R < \kappa_M/2$ and also $z_0 = 0$, which proves (i).

We could prove (ii). Let α be a short geodesic joining y to y', namely $L(\alpha) = d(y, y') < 2R \leqslant \kappa_M/2$. If $\beta \subset T_xM$ is the lift of α such that $\beta(0) = v$, then

$$\|\beta(1) - v'\|_{\mathfrak{p}} \leqslant \|\beta(1) - \beta(0)\|_{\mathfrak{p}} + \|v - v'\|_{\mathfrak{p}} \leqslant 2d(y, y') < \kappa_M,$$

and hence $\beta(1) = v'$. It will suffice to prove statement (ii) for $t = 1/2$, since f is continuous and a standard argument with the dyadic numbers will complete the proof. Let $\overline{\alpha}(t) = q(e^{\beta/2})$. Then certainly $f(1/2) = d(q(e^{v/2}), q(e^{v'/2})) \leqslant L(\overline{\alpha})$ since $\overline{\alpha}$ joins the same endpoints. Note that

$$\dot{\overline{\alpha}} = \tfrac{1}{2} F(\operatorname{ad}\beta/2)\dot{\beta},$$

and on the other hand, we have

$$\dot{\alpha} = F(\operatorname{ad}\beta)\dot{\beta} = 2F(\operatorname{ad}\beta/2)\cosh(\operatorname{ad}\beta/2)\dot{\beta}.$$

Hence $\dot{\overline{\alpha}} = \frac{1}{2}\cosh(\operatorname{ad}\beta/2)^{-1}\dot{\alpha}$. By Remark 4.6, $\|\dot{\overline{\alpha}}\|_{\overline{\alpha}} \leqslant \frac{1}{2}\|\dot{\alpha}\|_\alpha$, and hence we have

$$L(\overline{\alpha}) \leqslant \tfrac{1}{2}L(\alpha) = \tfrac{1}{2}d(y, y') = \tfrac{1}{2}f(1),$$

which proves (ii).

To prove (iii), note that $g(t) \leqslant f(t) + f'(t)$, where f is the function of item (ii) and f' is the corresponding function for the geodesics starting at y' and ending at x, x', respectively. Then f and f' are convex functions and we have

$$g(1/2) \leqslant \tfrac{1}{2}(f(1) + f'(1)) = \tfrac{1}{2}(g(1) + g(0)).$$

The last statement follows choosing $y = x$, and then we have

$$h(t) = d(x, \gamma(t)) \leqslant td(x, x') + (1 - t)d(x, y') < R.$$

□

4.1.2. *A formula for the geodesic distance.* We use $\log : G \cap U \to \mathfrak{g}$ to denote the inverse function of the exponential map of G (restricted to a suitable neighborhood U of $1 \in G$ to obtain a diffeomorphism).

Since $d(\exp_x(rv), \exp_x(rw)) = d(o, q(e^{-rv}e^{rw}))$ for any $x \in M$ and $v, w \in \mathfrak{p}$, and for small $r \in \mathbb{R}$, we have

$$d(\exp_x(rv), \exp_x(rw)) = \tfrac{1}{2}\|\log(e^{-rv}e^{2rw}e^{-rv})\|_{\mathfrak{p}}.$$

Indeed, if $\gamma(r)$ is a continuous lift of $q(e^{-rv}e^{rw})$ to $\mathfrak{p}$ with $\gamma(0) = 0$, then $\|\gamma(r)\|_{\mathfrak{p}} = d(o, q(e^{-rv}e^{rw}))$ and on the other hand we have

$$e^{2\gamma(r)} = e^{-rv}e^{2rw}e^{-rv}.$$

Hence if r is small enough to ensure that the exponential is a local diffeomorphism, then we have

$$2\gamma(r) = \log(e^{-rv}e^{2rw}e^{-rv}).$$

Corollary 4.10. *Let $x \in M$ and $v, w \in \mathfrak{p}$. Let*

$$\mathcal{R}(v, w) = \tfrac{1}{12}[v + w, [w, v]] = \tfrac{1}{12}\left[\operatorname{ad}^2_w(v) - \operatorname{ad}^2_v(w)\right].$$

Then for small $r \in \mathbb{R}$, we have

$$d(\exp_x(rv), \exp_x(rw)) = \tfrac{1}{2}\| \log(e^{-rv}e^{2rw}e^{-rv})\|_{\mathfrak{p}} = \tfrac{1}{2}\| \log(e^{-rw}e^{2rv}e^{-rw})\|_{\mathfrak{p}}$$
$$= \|r(w-v) + r^3\mathcal{R}(v,w) + o(r^4)\|_{\mathfrak{p}},$$

where log *denotes the analytic inverse of the exponential map of G, defined in a suitable neighborhood of $1 \in G$.*

Proof. The first two equalities follow from the previous discussion. Iterating the Baker–Campbell–Hausdorff formula, one obtains

$$d(\exp_x(rv), \exp_x(rw)) = \tfrac{1}{2}\|2r(w-v) + r^3\tfrac{2}{12}[v+w,[w,v]] + o(r^4)\|_{\mathfrak{p}}$$
$$= \|r(w-v) + r^3\tfrac{1}{12}[v+w,[w,v]] + o(r^4)\|_{\mathfrak{p}},$$

which holds for r small enough. □

4.1.3. *Sectional curvature.* With the tools of the previous section we now return to the subject matter of Remark 3.3.

PROPOSITION 4.11. *Let $x \in M$ and $v, w \in \mathfrak{p}$. Let $r > 0$ and*

$$s_x(r,v,w) = \frac{r\|v-w\|_{\mathfrak{p}} - d(\exp_x(rv), \exp_x(rw))}{r^2 d(\exp_x(v), \exp_x(w))}.$$

Then $s_x(v,w) = \lim_{r\to 0^+} s_x(r,v,w)$ exists and we have

$$0 \geqslant s_x(v,w) \geqslant 1 - \frac{\|v-w+\mathcal{R}(v,w)\|_{\mathfrak{p}}}{\|v-w\|_{\mathfrak{p}}} \geqslant -\frac{\|\mathcal{R}(v,w)\|_{\mathfrak{p}}}{\|v-w\|_{\mathfrak{p}}}.$$

In particular if $\mathcal{R}(v,w) = 0$, then $s_x(v,w) = 0$ for any $x \in M$.

Proof. Note first that by Corollary 4.10, we have

$$\lim_{r\to 0^+} \frac{1}{r} d(\exp_x(rv), \exp_x(rw)) = \|w-v\|_{\mathfrak{p}}.$$

Since a norm is a convex function, then

$$\lim_{r\to 0^+} \frac{1}{r^2}\left(\|w-v\|_{\mathfrak{p}} - \|w-v+r^2\mathcal{R}(v,w) + o(r^2)\|_{\mathfrak{p}}\right)$$

exists and it is infact equal to $-J_{v-w}(\mathcal{R}(v,w))$, that is, (minus) the subdifferential of the norm at the point $v-w$, computed in the direction of $\mathcal{R}(v,w)$. Moreover, we have

$$\|x\|_{\mathfrak{p}} - \|x-y\|_{\mathfrak{p}} \leqslant J_x(y) \leqslant \|x+y\|_{\mathfrak{p}} - \|x\|_{\mathfrak{p}}.$$

For instance see [**7**, Proposition 4.1]. Then we have

$$\lim_{r\to 0^+} \frac{1}{r^2}\|w-v\|_{\mathfrak{p}} - \|w-v+r^2\mathcal{R}(v,w) + o(r^2)\|_{\mathfrak{p}} \geqslant \|v-w\|_{\mathfrak{p}} - \|w-v+\mathcal{R}(v,w)\|_{\mathfrak{p}},$$

and thus $s_x(v,w) = \lim_{r\to 0^+} s_x(r,v,w)$ exists, is nonpositive, and by the computation above

$$s_x(v,w) \geqslant 1 - \frac{\|v-w+\mathcal{R}(v,w)\|_{\mathfrak{p}}}{\|v-w\|_{\mathfrak{p}}}.$$

The right-hand inequality stated in the proposition follows straight from the triangle inequality. □

4.1.4. *On the distortion of the metric.* We now assume for convenience that $M \simeq G/K$ is simply connected. In our present setting, if we choose $x = o$, our concern now is the inequality stated as follows:

$$\|v - w\|_{\mathfrak{p}} \leqslant d(q(e^v), q(e^w)), \tag{4.1}$$

where $v, w \in \mathfrak{p}$. We have seen that it implies that sectional curvature in G/K is nonpositive. If $v, w \in \mathfrak{p}$ commute, then the exponential of the linear span of v, w is a two-dimensional flat in M, and clearly equality holds in (4.1); this condition $[v, w] = 0$ is equivalent (by Jacobi's theorem) to the commutativity of the local flows of the Jacobi fields V and W (induced by v and w, respectively). In the infinite-dimensional setting, one obtains a weaker notion made explicit in the following theorems. The definitions and considerations of Remark 3.5 are used here.

PROPOSITION 4.12. *Let $v, w \in \mathfrak{p}$. If $M = G/K$ is a Cartan–Hadamard manifold and the norm $\|\cdot\|_{\mathfrak{p}}$ is strictly convex, then*

$$\|v - w\|_{\mathfrak{p}} = d(q(e^v), q(e^w))$$

implies that $\operatorname{ad}_v^2(w) = \operatorname{ad}_w^2(v) = 0$.

Proof. Let α be the short geodesic of M joining $q(e^v)$ with $q(e^w)$, $\alpha(t) = q(e^v e^{tz})$ and $q(e^v e^z) = q(e^w)$, where z is the unique lift to $\mathfrak{p}$ of $q(e^{-v} e^w)$; note that $\|z\|_{\mathfrak{p}} = d(q(e^v), q(e^w)) = \|v - w\|_{\mathfrak{p}}$. Let γ be the unique lift to $\mathfrak{p}$ of α, $\gamma(0) = v$ and $\gamma(1) = w$; by Remark 2.2 we have

$$L(\gamma) \leqslant L(\alpha) = \|v - w\|_{\mathfrak{p}}.$$

Since the norm of $\mathfrak{p}$ is strictly convex, it must be $\gamma(t) = (1-t)v + tw$, and hence we have

$$q(e^{(1-t)v + tw}) = q(e^v e^{tz}).$$

Differentiating at $t = 0$ we obtain

$$(\mu_{e^v})_{*o} q_{*1} \frac{1 - e^{-\operatorname{ad} v}}{\operatorname{ad} v}(w - v) = (\mu_{e^v})_{*o} q_{*1} z$$

by Remark 4.3, that is

$$F(\operatorname{ad} v)(w - v) = z,$$

where F denotes the entire function $F(z) = z^{-1}\sinh(z)$ as before. Then $\|F(\operatorname{ad} v)(w - v)\|_{\mathfrak{p}} = \|z\|_{\mathfrak{p}} = \|w - v\|_{\mathfrak{p}}$. If $\varphi \in \mathfrak{p}^*$ is the unique norming functional of $w - v$, since $-\operatorname{ad}_v^2$ is dissipative by Proposition 4.5, it follows that

$$2\|w - v\|_p = 2\varphi(w - v) \leqslant \varphi(2(w - v) + \frac{1}{\pi^2}\operatorname{ad}_v^2(w - v)) \leqslant \left\| 2(w - v) + \frac{1}{\pi^2}\operatorname{ad}_v^2(w - v) \right\|_{\mathfrak{p}},$$

that is

$$2\|w - v\|_{\mathfrak{p}} \leqslant \left\| w - v + \left(1 + \frac{1}{\pi^2}\operatorname{ad}_v^2\right)(w - v) \right\|_{\mathfrak{p}}.$$

On the other hand, we have

$$\|w - v\|_{\mathfrak{p}} \leqslant \left\| \left(1 + \frac{1}{\pi^2}\operatorname{ad}_v^2\right)(w - v) \right\|_{\mathfrak{p}} \leqslant \|F(\operatorname{ad} v)(w - v)\|_{\mathfrak{p}} = \|w - v\|_{\mathfrak{p}}$$

since

$$F(z) = \prod_{n \geqslant 1}\left(1 + \frac{z^2}{n^2\pi^2}\right) = \left(1 + \frac{z^2}{\pi^2}\right)\prod_{n \geqslant 2}\left(1 + \frac{z^2}{n^2\pi^2}\right),$$

and each factor is an expansive operator, thus we have

$$\|w-v\|_{\mathfrak{p}} = \left\| \left(1+\frac{1}{\pi^2}\mathrm{ad}\,_v^2\right)(w-v) \right\|_{\mathfrak{p}}.$$

Then

$$\left\| w-v+\left(1+\frac{1}{\pi^2}\mathrm{ad}\,_v^2\right)(w-v) \right\|_{\mathfrak{p}} = \|w-v\|_{\mathfrak{p}} + \left\| \left(1+\frac{1}{\pi^2}\mathrm{ad}\,_v^2\right)(w-v) \right\|_{\mathfrak{p}},$$

and since the norm is strictly convex and both elements have the same norm, it must be

$$w-v = \left(1+\frac{1}{\pi^2}\mathrm{ad}\,_v^2\right)(w-v) = w-v+\frac{1}{\pi^2}\mathrm{ad}\,_v^2(w-v).$$

Interchanging w and v also gives $\mathrm{ad}\,_w^2(v)=0$. □

THEOREM 4.13. *Let $v,w\in\mathfrak{p}$. Let $M=G/K$ be a Cartan–Hadamard manifold and we assume the following:*

(i) $[v,[v,w]]=[w,[v,w]]=0$*;*

(ii) $\|v-w\|_{\mathfrak{p}} = d(q(e^v),q(e^w))$*.*

Then (i) *implies* (ii)*, and if the norm of M is strictly convex, then* (ii) *is equivalent to* (i)*.*

Proof. The previous proposition gives (ii) $\Rightarrow$ (i). On the other hand, if $[v,[v,w]] = [w,[v,w]]=0$, then by the Baker–Campbell–Hausdorff formula, we have

$$e^{-v}e^w = e^{w-v-(1/2)[v,w]} = e^{w-v}e^{-(1/2)[v,w]}$$

since higher-order commutators vanish. Thus $q(e^{-v}e^w)=q(e^{w-v})$, and if $\alpha(t)=q(e^v e^{t(w-v)})$, then α is the unique geodesic joining $q(e^v)$ to $q(e^w)$ in M, and hence $d(q(e^v),q(e^w))=\|w-v\|_{\mathfrak{p}}$. □

REMARK 4.14. In the finite-dimensional setting, if $[v,[v,w]]=[w,[v,w]]$ and $B:\mathfrak{g}\times\mathfrak{g}$ denotes the Killing form of $\mathfrak{g}$ (that is, $B(x;y)=\mathrm{Tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$, where Tr denotes the usual trace of $\mathcal{B}(\mathfrak{g})$), then we have

$$B([v,w];[v,w]) = B(v;[w,[v,w]) = B(v;[v,[v,w]]) = B([v,w];[v,v]) = 0.$$

Thus if $\mathfrak{g}$ is semi-simple, the condition $[v,[v,w]]=[w,[v,w]]$ implies that $[v,w]=0$. From Proposition 4.12 it follows that such a condition is guaranteed if

$$\|v-w\|_{\mathfrak{p}} = d(q(e^v),q(e^w)),$$

and thus in this setting the (apparently weaker) metric condition is equivalent to the commutativity of local flows and then to the presence of a two-dimensional flat. This line of reasoning can be extended to the infinite-dimensional setting in the presence of a trace (Hilbert–Schmidt operators or L^*-algebras); see [**5**] for full details.

PROBLEM 4.15. Find necessary and sufficient conditions on the norm of $\mathfrak{p}$ in order to ensure that if $v,w\in\mathfrak{p}$ and $[v,[v,w]]=0$, then $[v,w]=0$.

4.1.5. *Totally geodesic submanifolds.* Some of the results in the following proposition can be originally found in [**35**], in the setting of the group of positive invertible $n\times n$ matrices. They express the standard relation between totally geodesic submanifolds and Lie triple systems. In the finite-dimensional (Riemannian) setting, the standard reference would be the book of

Helgason [**21**]. In [**38**], the authors study exponential sets in C*-algebras with similar techniques and recently, the results in [**35**] were extended to Hilbert–Schmidt operators [**27**].

PROPOSITION 4.16. *[(Exponential sets)] Let $M = G/K$ be a connected manifold of semi-negative curvature, where $T_oM \simeq \mathfrak{p}$. Let $\mathfrak{s} \subset \mathfrak{p}$ be a closed linear space and let $C = q(e^{\mathfrak{s}})$. Then we call C an exponential set because the following conditions are equivalent:*
(i) $[[v, w], s] \in \mathfrak{s}$ *for any* $v, w, s \in \mathfrak{s}$;
(ii) $\mathrm{ad}\,_s^2(\mathfrak{s}) \subset \mathfrak{s}$ *for any* $s \in \mathfrak{s}$;
(iii) $F(\mathrm{ad}\, v) = (\sinh \mathrm{ad}\, v/\mathrm{ad}\, v) \in \mathcal{B}(\mathfrak{p})$ *is an isomorphism of* $\mathfrak{s}$ *for any* $v \in \mathfrak{s}$;
(iv) *if* $v, w \in \mathfrak{s}$ *and* $\beta \subset T_oM \simeq \mathfrak{p}$ *is a lift of* $\alpha(t) = q(e^v e^{tw})$ *such that* $\beta(0) \in \mathfrak{s}$, *then* $\beta \subset \mathfrak{s}$.

Proof. Let $v, w, s \in \mathfrak{s}$. Then we have

$$[[v, w], s] = -\mathrm{ad}^2_{v-w}(s) + \mathrm{ad}^2_v(s) + \mathrm{ad}^2_w(s)$$

by the Jacobi identity. This shows that (ii) is equivalent to (i).

Assume that (ii) holds; then certainly (iii) holds since the series expansion of $F(z) = z^{-1}\sinh(z)$ has only even powers of z. If (iii) holds, then replacing v with tv yields

$$\mathfrak{s} \ni s_t = F(\mathrm{ad}\, tv)w = w + \tfrac{1}{6}t^2 \mathrm{ad}\,_v^2\, w + o(t^4),$$

and hence $\frac{1}{6}\mathrm{ad}\,_v^2 w = \lim_{t\to 0}((s_t - w)/t^2) \in \mathfrak{s}$.

Assume that (ii) holds and let $v, w \in \mathfrak{s}$. Consider the flow $F_{v,w} : \mathfrak{p} \to \mathfrak{p}$ given by

$$F_{v,w}(z) = \frac{\mathrm{ad}\, z}{\sinh(2\,\mathrm{ad}\, z)} \cosh \mathrm{ad}\, v(w).$$

Then $F_{v,w}$ is a Lipschitz map, and if (ii) holds, then $F_{v,w}(\mathfrak{s}) \subset \mathfrak{s}$. We claim that if $\beta(t) \in \mathfrak{p}$ is the smooth lift of $q(e^v e^{tw})$ with $\beta(0) = v$, then $\dot\beta = F_{v,w}(\beta)$, and this will prove that $\beta \subset \mathfrak{s}$ by the uniqueness of the solution of the differential equation $\dot x = F_{v,w}(x)$ in the Banach space $(\mathfrak{s}, \|\cdot\|_{\mathfrak{p}})$. To prove the claim $\dot\beta = F_{v,w}(\beta)$, we write $e^\gamma = e^v e^{tw} k$ for some $k(t) \in K$. The derivative of $q(e^\beta)$ gives

$$(\mu_{e^\beta})_{*o} q_{*1} \frac{1 - e^{-\mathrm{ad}\,\beta}}{\mathrm{ad}\,\beta}\dot\beta,$$

and the derivative of $q(e^v e^{tw})$ gives

$$(\mu_{e^v e^{tw}})_{*o} q_{*1} w = (\mu_{e^v e^{tw}})_{*o} w = (\mu_{e^\beta})_{*o}(\mathrm{Ad}_{k^{-1}} w).$$

Then we have

$$q_{*1} \frac{1 - e^{-\mathrm{ad}\,\beta}}{\mathrm{ad}\,\beta}\dot\beta = \mathrm{Ad}_{k^{-1}} w,$$

or since $1 - e^x = 1 - \cosh(x) + \sinh(x)$ and $q_{*1}(\mathfrak{k}) = \{0\}$ (and q_{*1} is the identity on $\mathfrak{p}$), it follows that

$$\frac{\sinh(\mathrm{ad}\,\beta)}{\mathrm{ad}\,\beta}\dot\beta = e^{-\beta} e^v w e^{-v} e^\beta = e^{-\mathrm{ad}\,\beta} e^{\mathrm{ad}\, v} w.$$

Multiplying by $e^{\mathrm{ad}\,\beta}$ we obtain

$$\frac{e^{2\,\mathrm{ad}\,\beta} - 1}{\mathrm{ad}\,\beta}\dot\beta = e^{\mathrm{ad}\, v} w,$$

and applying q_{*1} to both sides, we obtain

$$\frac{\sinh(2\,\mathrm{ad}\,\beta)}{\mathrm{ad}\,\beta}\dot\beta = \cosh(\mathrm{ad}\, v)w,$$

showing that $\dot\beta = F_{v,w}(\beta)$.

Assume that (iv) holds and let $\gamma_s \subset \mathfrak{s}$ be as above; then $q(e^{\gamma_s}) = q(e^{sv}e^{tw})$. Then by the computation above, with $t \to 0$, we obtain

$$\mathfrak{s} \ni \dot{\gamma}_s(0) = \frac{\operatorname{ad} sv}{\sinh \operatorname{ad} sv} w = w - \frac{4}{3}s^2 \operatorname{ad}_v^2 w + o(s^4).$$

Then $-\frac{4}{3}\operatorname{ad}_v^2 w = \lim_{s\to 0}((\dot{\gamma}_s(0) - w)/s^2) \in \mathfrak{s}$, showing that (ii) holds. □

COROLLARY 4.17. *Let $C = q(e^{\mathfrak{s}})$ be an exponential set in M and let $V \in \mathfrak{p}$ be an open ball of radius strictly less than $\kappa_M/2$. Then we have the following:*

(i) *the charts $(V \cap \mathfrak{s}, \exp_x|_{V\cap\mathfrak{s}})$, for $x \in C$, give an atlas of C which makes of C an immersed differentiable manifold $C \subset M$, with a topology that is possibly finer than the topology of M;*

(ii) *$T_xC = (\mu_{e^s})_{*o}\mathfrak{s}$ for any $x = q(e^s) \in C$. In particular $\exp_x(T_xC) = C$ for any $x \in C$, that is, C is totally geodesic in M.*

Proof. For the first statement note that $\exp_x(\mathfrak{s}) \subset C$ by Proposition 4.16, and that $\exp_x|_V$ gives an isomorphism $\exp_x|_V : V \to \exp_x(V) \subset M$ by Remark 4.8. Then the proposed charts are bijective and, moreover, the transition maps give isomorphisms between open neighborhoods of $\mathfrak{s}$ since the exponential of M is a local isomorphism and $\mathfrak{s}$ is a closed linear subspace of $\mathfrak{p}$ that (by Proposition 4.16) is stable for the action of the differential of the exponential map at $x = q(e^v)$, given by $F(\operatorname{ad} v) = (\sinh \operatorname{ad} v/\operatorname{ad} v)$ by Remark 4.2. Then C with the topology and differentiable structure induced by the atlas is an immersed submanifold since $\mathfrak{s} \subset \mathfrak{p}$ is closed.

The second assertion is elementary, and its proof follows combining (i) with Proposition 4.16. □

DEFINITION 4.18. Let $[\mathfrak{s}, \mathfrak{s}]$ stand for the closure of the linear span of the elements $[v, w] \in \mathfrak{g}$, where $v, w \in \mathfrak{s}$. Then $\mathfrak{s} \cap [\mathfrak{s}, \mathfrak{s}] = \{0\}$ since $\mathfrak{s} \subset \mathfrak{p}$ and $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$. We agree to call a Banach–Lie algebra $\mathfrak{g}_C \subset \mathfrak{g}$ *involutive* if $\sigma_{*1}\mathfrak{g}_C = \mathfrak{g}_C$, and a connected Banach–Lie group $G_C \subset G$ involutive if $\sigma(G_C) = G_C$, or equivalently, if its Lie algebra is involutive.

Let $p \in \mathcal{B}(\mathfrak{p})$ be an idempotent, i.e. $p^2 = p$. Let $\mathfrak{s} = \operatorname{Ran}(p)$ and $\mathfrak{s}' = \operatorname{Ran}(1 - p)$, and hence $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}'$. In this case, we say that $\mathfrak{s}$ is *split* in $\mathfrak{p}$. We say that $C = q(e^{\mathfrak{s}})$ is a reductive submanifold if C is totally geodesic and, in addition, $\operatorname{ad}_{\mathfrak{s}}^2(\mathfrak{s}') \subset \mathfrak{s}'$.

See Remark 4.37 for a brief discussion on these definitions in the classical (Riemannian, finite-dimensional) setting; see also item (vi) in the following proposition.

REMARK 4.19. If $G_C \subset G$ is a connected involutive Banach–Lie group, with Banach–Lie algebra $\mathfrak{g}_C \subset \mathfrak{g}$, then σ allows us to write $\mathfrak{g}_C = \mathfrak{p}_C \oplus \mathfrak{k}_C$, where $\mathfrak{p}_C = \mathfrak{p} \cap \mathfrak{g}_C$ and $\mathfrak{k}_C = \mathfrak{k} \cap \mathfrak{g}_C$. Then $q(G_C) = q(e^{\mathfrak{p}_C}) \subset M$ is a totally geodesic immersed submanifold.

PROPOSITION 4.20. *Let $M = G/K$ be a connected manifold with semi-negative curvature. Let $\mathfrak{s} \subset \mathfrak{p}$ be a closed linear space. Assume that $\operatorname{ad}_{\mathfrak{s}}^2(\mathfrak{s}) \subset \mathfrak{s}$ and let $\mathfrak{g}_{\mathfrak{s}} = \mathfrak{s} \oplus [s, s]$. Then we have the following:*

(i) *$\mathfrak{g}_{\mathfrak{s}}$ is an involutive Banach–Lie algebra and it can be enlarged to a connected involutive Banach–Lie group $G_{\mathfrak{s}} \hookrightarrow G$;*

(ii) *let $K_{\mathfrak{s}} = K \cap G_{\mathfrak{s}}$. If $C = q(e^{\mathfrak{s}})$, then $G_{\mathfrak{s}}/K_{\mathfrak{s}} \simeq C$, and C is a totally geodesic immersed submanifold of M;*

(iii) *the group $G_{\mathfrak{s}}$ acts isometrically and transitively on C;*

(iv) *M-parallel transport along geodesics in C preserves tangent vectors of C;*
(v) *C is a split submanifold if and only if $\mathfrak{s}$ is split in $\mathfrak{p}$;*
(vi) *let $\mathfrak{k}_{\mathfrak{s}} = [\mathfrak{s}, \mathfrak{s}]$ and let $K_C \hookrightarrow G_{\mathfrak{s}}$ stand for the Banach–Lie group generated by $\mathfrak{k}_{\mathfrak{s}}$. Then C is reductive if and only if Ad_{K_C} is a group of isometries of both $\mathfrak{s}$ and $\mathfrak{s}'$;*
(vii) *if C is an embedded submanifold of M, then we find that $K_{\mathfrak{s}}$ is a Banach–Lie subgroup of $G_{\mathfrak{s}}$, that K_C is the connected component of the identity of $K_{\mathfrak{s}}$ and that $G_{\mathfrak{s}}/K_{\mathfrak{s}} \simeq C$ as homogeneous spaces.*

Proof. That $\mathfrak{g}_C$ is a Lie algebra follows from the Jacobi identity. Since it is a subalgebra of $\mathfrak{g}$, which is the Banach–Lie algebra of the Banach–Lie group G, it can be integrated as claimed [**40**], and this settles (i).

To prove (ii), note that if $g \in G_{\mathfrak{s}}$, then $g = \prod e^{s_i} e^{k_i}$, where $s_i \in \mathfrak{s}$ and $k_i \in [\mathfrak{s}, \mathfrak{s}]$. Then $q(g) = q(\prod e^{s_i'})$, where $s_i' \in \mathfrak{s}$ since

$$e^{k_i} e^{s_{i+1}} e^{k_{i+1}} = e^{\mathrm{Ad}_{e^{k_i}}(s_{i+1})} e^{k_i} e^{k_{i+1}},$$

and on the other hand $\mathrm{Ad}_{e^{[v,w]}} s = e^{ad_{[v,w]}} s \in \mathfrak{s}$ if $v, w \in \mathfrak{s}$ by Proposition 4.16. Then there exists $s \in \mathfrak{s}$ such that $q(g) = q(e^s) \in C$ by Proposition 4.16. Then $q|_{G_{\mathfrak{s}}}$ gives the isomorphism of $G_{\mathfrak{s}}/K_{\mathfrak{s}}$ with C. That C is a totally geodesic immersed submanifold follows from Corollary 4.17.

To prove (iii), note that the transitive and isometric action of $G_{\mathfrak{s}}$ is given by the maps μ_g, with $g \in G_{\mathfrak{s}}$: if $v \in \mathfrak{s}$, then we have

$$\mu_g(q(e^v)) = q(ge^v) = q\left(\prod e^{s_i} e^{k_i} e^v\right) = q(e^{s_i'} e^{v'})$$

by the argument above, where $s_i', v' \in \mathfrak{s}$, and then $\mu_g(q(e^v)) \in C$ by Proposition 4.16.

To prove (iv), recall (Remark 4.1) that M-parallel transport along $\alpha(t) = q(e^s e^{tv})$ is given by

$$(\mu_{e^s e^v e^{-s}})_{*q(e^s)}.$$

Then, if $s, v \in \mathfrak{s}$, parallel transport along α from $\alpha(0) = q(e^s)$ to $\alpha(1) = q(e^s e^v)$ of a vector $(\mu_{e^s})_{*o} w \in T_x C$ gives $(\mu_{e^s e^v})_{*o} w$. By Proposition 4.16, there exists $l \in \mathfrak{s}$ and $k \in K$ such that $e^l = e^s e^v$, and then we have

$$P_0^1(\alpha)(\mu_{e^s})_{*o} w = (\mu_{e^l})_{*o} \mathrm{Ad}_k w.$$

However, $\mathrm{Ad}_k w = e^{-\mathrm{ad}\, l} e^{\mathrm{ad}\, s} e^{\mathrm{ad}\, v} w \in \mathfrak{p} \cap \mathfrak{g}_{\mathfrak{s}}$, and hence $\mathrm{Ad}_k w \in \mathfrak{s}$, which proves that $P_0^1(\alpha)$ maps $T_{\alpha(0)}C$ to $T_{\alpha(1)}C$.

Item (v) is obvious: C is a split submanifold if and only if $\mathfrak{s}$ is split in $\mathfrak{p}$.

To prove (vi), note that each $k \in K_C$ can be written as a finite product $k = \prod e^{l_i}$, with $l_i \in [\mathfrak{s}, \mathfrak{s}]$. Then C is reductive if and only if $\mathfrak{s}$ and $\mathfrak{s}'$ are $\mathrm{ad}_{[\mathfrak{s},\mathfrak{s}]}$-invariant.

Finally, if C is an embedded submanifold of M, then $q^{\mathfrak{s}} = q|_{\mathfrak{s}}$ gives the topological identification $G_{\mathfrak{s}}/K_{\mathfrak{s}} = C$, and inspection of the action of $q_{*1}^{\mathfrak{s}}$ shows that $K_{\mathfrak{s}}$ is a Banach–Lie subgroup of $G_{\mathfrak{s}}$ with Banach–Lie algebra $[\mathfrak{s}, \mathfrak{s}]$. □

PROPOSITION 4.21 (Locally convex sets). *Let $C = q(e^{\mathfrak{s}})$ be an exponential set in M. Then we call C a locally convex set because the following statements are equivalent:*

(i) *There exists $0 < \varepsilon < \kappa_M/2$ such that if $x, y \in C$, $d(x, y) < \varepsilon$ and $\alpha(t) = q(e^v e^{tz})$ is the unique short geodesic of M joining x to y, then $z \in \mathfrak{s}$ and, moreover, $\alpha \subset C$.*

(ii) *There exists $0 < \delta < \kappa_M/2$ such that $d(\Gamma - \Gamma \cap \mathfrak{s}, \mathfrak{s}) \geqslant \delta$.*

(iii) *There exists $0 < R < \kappa_M/2$ such that if $U = \{v \in \mathfrak{p} : \|v\|_{\mathfrak{p}} < R\}$, then $\exp_x(U) \cap C = \exp_x(U \cap \mathfrak{s})$ for any $x \in C$.*

Proof. Assume that (ii) does not hold. Then, given $0<\varepsilon<\kappa_M/2$, there exists $z_0\in\Gamma-\mathfrak{s}$ such that $d(z_0,\mathfrak{s})<\varepsilon/2$. Take $s\in\mathfrak{s}$ such that $\|s-z_0\|_{\mathfrak{p}}\leqslant\varepsilon$. Let $w=s-z_0\notin\mathfrak{s}$, $x=o$ and $y=q(e^w)=q(e^s)\in C$. Then $d(x,y)=\|w\|_{\mathfrak{p}}=\varepsilon$ by Remark 4.8, and hence $\alpha(t)=q(e^{tw})$ is the unique short geodesic of M joining x to y. However, α does not have initial speed in $\mathfrak{s}$, and thus (i) does not hold.

Now assume that (ii) holds for some $0<\delta<\kappa_M/2$ and let $x=q(e^s)\in C$. Take $R=\delta$, and note that the inclusion $\exp_x(U\cap\mathfrak{s})\subset\exp_x(U)\cap C$ always holds due to Proposition 4.16. Let $v\in U$, and assume that $q(e^se^v)\in C$, namely $q(e^se^v)=q(e^w)$, with $w\in\mathfrak{s}$. Then there exists $s'\in\mathfrak{s}$ (again due to Proposition 4.16) such that $q(e^v)=q(e^{-s}e^w)=q(e^{s'})$. Then there exists $z\in\Gamma$ such that $s'-v=z$. If $z\in\mathfrak{s}$, then we are done since $q(e^se^v)=q(e^se^{s'-z})\in\exp_x(U\cap\mathfrak{s})$. If $z\notin\mathfrak{s}$, then $\delta\leqslant\|s'-z\|_{\mathfrak{p}}=\|v\|_{\mathfrak{p}}<R=\delta$, which is absurd, and hence $z\in\mathfrak{s}$. This shows that (ii) implies (iii).

Assume that (iii) holds for some $R>0$ and let $\varepsilon=R$. Let $x=q(e^v)$ and $y=q(e^w)\in C$ with $d(x,y)<\varepsilon$, let $\alpha(t)=q(e^ve^{tz})$ be the unique short geodesic of M joining x to y, namely $\|z\|_{\mathfrak{p}}=d(x,y)$ and let $q(e^ve^z)=q(e^w)$. Then, due to (iii), there exists $s\in U\cap\mathfrak{s}$ such that $q(e^ve^z)=q(e^ve^l)$, and hence there exists $z_0\in\Gamma$ such that $z-l=z_0$. Since $\|z_0\|_{\mathfrak{p}}\leqslant\|z\|_{\mathfrak{p}}+\|l\|_p<2R$, it follows that $z_0=0$ and $z=l\in\mathfrak{s}$. That $\alpha\subset C$ follows from Proposition 4.16, and thus we have shown that (iii) implies (i). □

COROLLARY 4.22. *Let $C=q(e^{\mathfrak{s}})$ be a locally convex set in M and let $U\subset\mathfrak{p}$ be an open ball around 0 of radius R, where R is as in Proposition 4.21. Then we have the following:*

(i) *The set C is an embedded submanifold of M, and $\exp_x|_{U\cap\mathfrak{s}}:U\cap\mathfrak{s}\to C\cap\exp_x(U)$ is a topological isomorphism when C is given the subspace topology. It is also a diffeomorphism that gives an atlas that makes of M an immersed embedded submanifold of C.*

(ii) *With the induced spray and metric, C is a Banach–Finsler manifold with spray of semi-negative curvature, with the exponential map $\exp_x^C=\exp_x|_{\mathfrak{s}}$ given by restriction. The fundamental group of C is given by $\Gamma_{\mathfrak{s}}=\Gamma\cap\mathfrak{s}$, and $C\subset M$ is a closed metric subspace.*

(iii) *If $K_{\mathfrak{s}}=K\cap G_{\mathfrak{s}}$, then $K_{\mathfrak{s}}$ is a Banach–Lie subgroup of $G_{\mathfrak{s}}$, and $C\simeq G_{\mathfrak{s}}/K_{\mathfrak{s}}$ as homogeneous spaces.*

Proof. That C is an embedded submanifold follows from the fact that if $V\subset U$ is open in $\mathfrak{p}$, then $\exp_x(V)\cap C=\exp_x(V\cap\mathfrak{s})$, because $\exp_x(V)\subset\exp_x(U)$ and then (consider $x=q(e^v)$ with $v\in\mathfrak{s}$) $q(e^ve^z)\in C$ for $z\in V$ implies that $q(e^z)=q(e^s)$ for some $s\in U\cap\mathfrak{s}$, and thus $z=s$ since $z,s\in U$ and $2R<\kappa_M$.

That $\exp_x|_{U\cap\mathfrak{s}}:U\cap\mathfrak{s}\to C\cap\exp_x(U)$ is a diffeomorphism follows from Proposition 4.21.

The second assertion follows from the fact that the norm of C is compatible since C and M share the topology, and the exponential map of C is just the restriction of the exponential map of M, and then at each point its differential is an invertible expansive operator. Then Theorem 4.7 applies.

Now we prove that $C\subset M$ is a closed subspace. If $x_n\to x$ with $x_n\in C$, then we take n_0 such as $d(x_n,x)<R/2$ for any $n\geqslant n_0$. Let $x_{n_0}=q(e^{v_{n_0}})$, and consider $z_n=\mu_{x_{n_0}}^{-1}x_n$, and $z=\mu_{x_{n_0}}^{-1}x$. Since $d(x_n,x_{n_0})<R$, it follows that there exists $v_n\in\mathfrak{s}\cap U$ such that $x_n=q(e^{v_{n_0}}e^{v_n})$ and $\|v_n\|_{\mathfrak{p}}=d(z_n,o)<R$. Then $z_n=q(e^{v_n})\in C$, $d(z_n,z)\to 0$ and then $d(z_n,z_m)<R$. Hence $\|v_n-v_m\|_{\mathfrak{p}}\leqslant d(z_n,z_m)$ by Proposition 4.9. Since $\mathfrak{s}$ is complete, it follows that there exists $v_0\in\mathfrak{s}$ such that $v_n\to v_0$. Let $z_0=q(e^{v_0})\in C$. Then we have

$$d(z,z_0)\leqslant d(z,z_n)+d(z_n,z_0)=d(x,x_n)+d(q(e^{v_n}),q(e^{v_0})),$$

and hence $z=z_0\in C$; thus $x=\mu_{x_{n_0}}(z)\in C$.

The last assertion follows from Proposition 4.20, since C is an embedded submanifold. □

Proposition 4.23 (Convex sets). *Let $C = q(e^{\mathfrak{s}})$ be a locally convex set in M. Then we call C a convex set because the following statements are equivalent:*

(i) *C is geodesically convex: if $x, y \in C$, then any geodesic of M joining x to y is entirely contained in C;*
(ii) *Γ is an additive subgroup of $\mathfrak{s}$;*
(iii) *For any $x \in C$, we see that $\exp_x(v) \in C$ implies that $v \in \mathfrak{s}$. In particular $\exp_x|_{\mathfrak{s}}$ is a global chart of C, and C is an immersed embedded submanifold of M.*

Proof. First assume that C is convex and let $z \in \Gamma$. Then $\alpha(t) = q(e^{tz})$ joins o to o, and hence $\alpha \subset C$. In particular, since $T_oC = \mathfrak{s}$ by Corollary 4.22, we have $\dot{\alpha}(0) = z \in \mathfrak{s}$, and thus $\Gamma \subset \mathfrak{s}$.

Assume now that $\Gamma \subset \mathfrak{s}$, let $x = q(e^s) \in C$ and let $v \in \mathfrak{p}$ such that $q(e^s e^v) \in C$, namely there exists $w \in \mathfrak{s}$ such that $q(e^s e^v) = q(e^w)$. Then, by Proposition 4.16, there exists $s' \in \mathfrak{s}$ such that $q(e^v) = q(e^{-s} e^w) = q(e^{s'})$. Since $v - s' \in \Gamma \subset \mathfrak{s}$, it follows that $v \in \mathfrak{s}$.

Let $x, y \in C$ and let $\alpha(t) = q(e^v e^{tz})$ be a geodesic of M joining $x = q(e^v)$ to y. If (iii) holds, then at $t = 1$ we obtain $z \in \mathfrak{s}$ and then we have $\alpha \subset C$ by Proposition 4.16. □

Corollary 4.24. *Let $C = q(e^{\mathfrak{s}})$ be a convex submanifold in M. Then if $v, w \in \mathfrak{s}$ and $\beta \subset T_oM \simeq \mathfrak{p}$ is any lift of $\alpha(t) = q(e^v e^{tw})$, we have $\beta \subset \mathfrak{s}$.*

Proof. Let $\beta \in \mathfrak{p}$ be any lift of $\alpha(t) = q(e^v e^{tw})$. If $v, w \in \mathfrak{s}$, then $\alpha \subset C$ by Proposition 4.16 and, moreover, $q(e^{\beta(0)}) = q(e^v) \in C$. If C is convex, then (iii) holds in the above proposition, and if we put $x = o$, then we obtain $\beta(0) \in \mathfrak{s}$, and we have $\beta \subset \mathfrak{s}$ by Proposition 4.16. □

4.2. *Splitting theorems for expansive submanifolds*

In this section, we prove straightforward generalizations of the results due to Corach, Porta and Recht in [**17**, **37**, **38**] for C*-algebras, and hence we would like to refer to these splitting results as CPR splittings.

In what follows, we assume that $M = G/K$ is connected, complete and of semi-negative curvature. We also assume that $C = q(e^{\mathfrak{s}})$ is a locally convex reductive submanifold of M.

Definition 4.25. If $C = q(e^{\mathfrak{s}})$ is a locally convex reductive submanifold, and in addition $\|p\| = 1$, then we say that C is an *expansive reductive* submanifold of M.

Remark 4.26. Let $p \in \mathcal{B}(\mathfrak{p})$ be an idempotent with $\|p\| = 1$. Then $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}'$, where $\mathfrak{s} = \operatorname{Ran}(p)$, $\mathfrak{s}' = \operatorname{Ker}(p)$ and

$$\|s\|_{\mathfrak{p}} = \|p(s + s')\|_{\mathfrak{p}} \leqslant \|s + s'\|_{\mathfrak{p}}$$

for any $s \in \mathfrak{s}$ and $s' \in \mathfrak{s}'$. This shows that $\|p\| = 1$ if and only if $\mathfrak{s}$ is a subset of the Birkhoff orthogonal of $\mathfrak{s}'$, and there is a Banach space isometric isomorphism $\mathfrak{p}/\mathfrak{s}' \simeq \mathfrak{s}$ when $\mathfrak{p}/\mathfrak{s}'$ is given the quotient norm. Moreover, it easy to check that the following statements are equivalent:

(1) $\|p\| = 1$;
(2) $\mathfrak{s}$ is the Birkhoff orthogonal of $\mathfrak{s}'$;
(3) $1 - p = Q_{\mathfrak{s}}$, where $Q_{\mathfrak{s}}$ indicates the metric projection to $\mathfrak{s}$.

Obviously the same assertions hold if we replace p with $1 - p$ and $\mathfrak{s}$ with $\mathfrak{s}'$.

Definition 4.27. Vectors in $\mathfrak{s}'$ are *normal directions*, and a geodesic $\exp_x(tv)$ starting at $x \in C$ is a *normal geodesic* if $v \in \mathfrak{s}'$.

Lemma 4.28. *Let $0 < R \leqslant \kappa_M/8$ and let $x_0 \in C$. Let $x, y \in B(x_0, R) \cap C$ and let $v, w \in \mathfrak{s}'$ such that $\|v\|_{\mathfrak{p}}, \|w\|_{\mathfrak{p}} < R$. Let $f : [0, +\infty) \to [0, +\infty)$ be the distance between the two normal geodesics, given by*

$$f(t) = d(\exp_x(tv), \exp_y(tw)).$$

Then, if C is expansive, f is increasing. If f is increasing for any such $x, y \in C$ and $v, w \in \mathfrak{s}'$, then C is expansive.

Proof. As always we assume that $x_0 = o$. Consider $x = q(e^r)$ and $y = q(e^s)$, with $\|s\|_{\mathfrak{p}}, \|r\|_{\mathfrak{p}} < R$. Then $f(t) = d(q(e^r e^{tv}), q(e^s e^{tw}))$ is a convex function by Proposition 4.9, and it is increasing if and only if

$$f'(0^+) = \lim_{t \to 0^+} \frac{f(t) - f(0)}{t} \geqslant 0.$$

Let $l_0 \in \mathfrak{s}$ be such that $q(e^{l_0}) = q(e^{-r} e^s)$ and let $\|l_0\|_{\mathfrak{p}} = d(x, y)$ (such element exists by Proposition 2.1). Let $k \in K$ be such that $e^{l_0} k = e^{-r} e^s$ and let

$$\beta(t) = q(e^{-tv} e^{-r} e^s e^{tw}) = q(e^{-tv} e^{l_0} e^{tw'}),$$

where $w' = \mathrm{Ad}_k\, w \in \mathfrak{s}'$. Note that $d(o, \beta(t)) = f(t) \leqslant t\|v\|_{\mathfrak{p}} + R + t\|w\|_{\mathfrak{p}} < \kappa_M/2$. Then, if we consider $l_t \in \mathfrak{p}$ the smooth lift of $\beta(t)$ to the ball $B(0, \kappa_M/2)$ in $\mathfrak{p}$, we have $q(e^{l_t}) = q(e^{-tv} e^{l_0} e^{tw'})$, and $\|l_t\|_{\mathfrak{p}} = d(o, \beta(t)) = f(t)$ since $\|l_t\|_{\mathfrak{p}} < \kappa_M/2$.

Let $\varphi_0 \in \mathfrak{p}^*$ be a linear functional such that $\|\varphi_0\| = 1$ and $\varphi_0(l_0) = \|l_0\|_{\mathfrak{p}} = d(x, y)$, and let $\varphi = \varphi_0 \circ p$. Then $\varphi(\mathfrak{s}') = \{0\}$. Let $g(t) = \varphi(l_t)$. Note that $g(0) = \varphi(l_0) = f(0)$. If C is expansive, then $\varphi(l_t) \leqslant f(t)$. Then we have

$$\frac{f(t) - f(0)}{t} \geqslant \frac{g(t) - g(0)}{t},$$

for $t > 0$, and we will show that $g'(0) = 0$ to prove that f is increasing. From $q(e^{l_t}) = q(e^{-tv} e^{l_0} e^{tw'})$ we obtain

$$\frac{\sinh \mathrm{ad}\, l_0}{\mathrm{ad}\, l_0} \dot{l}_0 = q_{*1}(-e^{-\mathrm{ad}\, l_0} v + w') = w' - \cosh(\mathrm{ad}\, l_0) v,$$

and hence we have

$$\dot{l}_0 = F^{-1}(\mathrm{ad}\, l_0) w' - H(\mathrm{ad}\, l_0) v,$$

with $F(z) = z^{-1} \sinh(z)$ and $H(z) = z \coth(z)$, which are both series in z^2. Then we have

$$g'(0) = \varphi(\dot{l}_0) = \sum \alpha_k \varphi(\mathrm{ad}\,_{l_0}^{2k} w') - \sum \beta_k \varphi(\mathrm{ad}\,_{l_0}^{2k} v) = 0$$

since $\mathrm{ad}\,_{l_0}^{2}(\mathfrak{s}') \subset \mathfrak{s}'$.

Assume now that f is increasing for $x = o$ and $v = 0$, and for given $l_0 \in \mathfrak{s}$ and $w_0 \in \mathfrak{s}'$, consider $y = \exp_x(l_0) \in C$. Assume first that $\|l_0\|_{\mathfrak{p}}, \|w_0\|_{\mathfrak{p}} < R$. Consider $w = \mathrm{Ad}_{k^{-1}} w_0$. Then, in the notation of the first part of the proof, $w' = w_0$ and

$$f(t) = \|l_t\|_{\mathfrak{p}} = \|l_0 + t\dot{l}_0 + o(t^2)\|_{\mathfrak{p}} \leqslant \|l_0 + t\dot{l}_0\|_{\mathfrak{p}} + o(t^2),$$

and if f is increasing we have

$$0 \leqslant f'(0^+) \leqslant \lim_{t \to 0^+} \frac{\|l_0 + t\dot{l}_0\|_{\mathfrak{p}} - \|l_0\|_{\mathfrak{p}}}{t} \leqslant \|l_0 + \dot{l}_0\|_{\mathfrak{p}} - \|l_0\|_{\mathfrak{p}}$$

by the convexity of the norm. By the computation above, $\dot{l}_0 = F^{-1}(\operatorname{ad} l_0)w_0$. Then

$$\|l_0\|_{\mathfrak{p}} \leqslant \|l_0 + F^{-1}(\operatorname{ad} l_0)w_0\|_{\mathfrak{p}} = \|F^{-1}(\operatorname{ad} l_0)(l_0 + w_0)\|_{\mathfrak{p}} \leqslant \|l_0 + w_0\|_{\mathfrak{p}},$$

since F^{-1} is a contraction. If now $l \in \mathfrak{s}$ and $w \in \mathfrak{s}'$, replacing them with a convenient positive multiple, we obtain that $\|l\|_{\mathfrak{p}} \leqslant \|l + w\|_{\mathfrak{p}}$, and this shows that $\|p\| = 1$. $\square$

LEMMA 4.29. *The sets*

$$\mathfrak{s}_R \oplus \mathfrak{s}'_R = \{v \in \mathfrak{p} : v = s + s', s \in \mathfrak{s}, s' \in \mathfrak{s}', \|s\|_{\mathfrak{p}}, \|s'\|_{\mathfrak{p}} < R\}$$

are open neighborhoods of $0 \in \mathfrak{p}$.

Proof. Let $s + s' \in \mathfrak{s}_R \oplus \mathfrak{s}'_R$ with $\|s\|_{\mathfrak{p}} = R - \delta$ and $\|s'\|_{\mathfrak{p}} = R - \delta'$, and let

$$\varepsilon = \min\left\{\frac{\delta}{\|p\|}, \frac{\delta'}{1 + \|p\|}\right\}.$$

We claim that $B(s + s', \varepsilon) \subset \mathfrak{s}_R \oplus \mathfrak{s}'_R$. Let $t + t' \in B(s + s', \varepsilon)$; then

$$\|t\|_{\mathfrak{p}} \leqslant \|t - s\|_{\mathfrak{p}} + \|s\|_{\mathfrak{p}} \leqslant \|p\|\|t - s + t' - s'\|_{\mathfrak{p}} + R - \delta < \|p\|\varepsilon + R - \delta < R,$$

and on the other hand we have

$$\begin{aligned}\|t'\|_{\mathfrak{p}} &\leqslant \|t' - s'\|_{\mathfrak{p}} + R - \delta' \leqslant \|t' - s' + t - s\|_{\mathfrak{p}} + \|t - s\|_{\mathfrak{p}} + R - \delta' \\ &< \varepsilon + \|p\|\varepsilon + R - \delta' < R.\end{aligned}$$

$\square$

LEMMA 4.30. *Let* $x_0 = q(e^{s_0}) \in C$, $R > 0$ *and*

$$\Omega^R_{x_0} = \{\exp_y(v), y \in C, d(x_0, y) < R, v \in \mathfrak{s}', \|v\|_{\mathfrak{p}} < R\}.$$

Let $E_{x_0} : \mathfrak{p} \to M$ *be given by*

$$E_{x_0}(s + s') = q(e^{s_0}e^{s}e^{s'}) = \exp_y((\mu_g)_{*o}s'),$$

where $y = q(e^{s_0}e^{s}) \in C$ *and* $g = e^{s_0}e^{s}$. *Then there exists* $\varepsilon > 0$ (*and strictly smaller than* $\kappa_M/8$) *such that* $E_{x_0} : \mathfrak{s}_\varepsilon \oplus \mathfrak{s}'_\varepsilon \to \Omega^\varepsilon_{x_0}$ *is a diffeomorphism, and in particular* $\Omega^\varepsilon_{x_0} \subset M$ *is open. The set*

$$NC^\varepsilon = \{\exp_y(v) : y \in C, v \in \mathfrak{s}', \|v\|_{\mathfrak{p}} < \varepsilon\}$$

is an open neighborhood of C *in* M.

Proof. Let $\alpha(t) = t(s + s')$ with $s + s' \in \mathfrak{p}$. Then $E_{x_0} \circ \alpha(t) = q(e^{s_0}e^{ts}e^{ts'})$, and hence we have

$$(E_{x_0})_{*0}(s + s') = s + s',$$

and thus by the inverse function theorem there exists an open neighborhood U of $0 \in \mathfrak{p}$ and an open neighborhood V of $x_0 \in M$ such that E_{x_0} restricted to them is a diffeomorphism. Shrinking, we can assume that $U = \mathfrak{s}_\varepsilon \oplus \mathfrak{s}'_\varepsilon$ and then $\Omega^\varepsilon_{x_0} = E_{x_0}(U)$. The last statement is due to the fact that $NC^\varepsilon = \bigcup_{x_0 \in C} \Omega^\varepsilon_{x_0}$. $\square$

REMARK 4.31. Assume that C is locally convex, reductive and expansive. Let $\exp_y(v) = \exp_{y'}(v') \in \Omega^\varepsilon_{x_0}$, with $y, y' \in B(x_0, R)$ and $v, v' \in \mathfrak{s}'_\varepsilon$. If $\varepsilon < \kappa_M/8$, then by Lemma 4.28, it is clear that $y = y'$. Moreover $v - v' \in \Gamma$ but since $\|v - v'\|_{\mathfrak{p}} \leqslant 2\varepsilon$, it follows that $v = v'$.

Let $\pi_{x_0} : \Omega^\varepsilon_{x_0} \to C \cap B(x_0, \varepsilon)$ be the local projection to C, such that $\pi_{x_0}(\exp_y(v)) = y$. Then by Lemma 4.30, if $\alpha \subset \Omega^\varepsilon_{x_0}$ is the short geodesic starting at $z = q(e^{s_0}e^{s}e^{s'})$ with initial speed

$w \in \mathfrak{p}$ and $\|w\|_p = L(\alpha)$, then $\alpha(t) = q(e^{s_0}e^{s_t}e^{v_t})$ for some smooth curves $s_t \in \mathfrak{s}$ and $v_t \in \mathfrak{s}'$, with $s_0 = s$ and $v_0 = s'$. Hence $\pi_{x_0} \circ \alpha(t) = q(e^{s_t})$, and it follows that $(\pi_{x_0})_{*z}w = F(\operatorname{ad} s)\dot{s}_0$. Since π_{x_0} is a contraction by Lemma 4.28, it follows that

$$d(q(e^{s_0}e^{s}), q(e^{s_0}e^{s_t})) = d(\pi_{x_0}(z), \pi_{x_0}(\alpha(t))) \leqslant d(z, \alpha(t)) = L_0^t(\alpha) = t\|w\|_{\mathfrak{p}}.$$

If γ is a smooth curve in $\mathfrak{s}$ such that $\gamma(0) = 0$, $q(e^{\gamma}) = q(e^{-s}e^{s_t})$ and $\|\gamma\|_{\mathfrak{p}} = d(q(e^{-s}), q(e^{s_t}))$, then from $t\|w\|_{\mathfrak{p}} \geqslant \|\gamma(t)\|_{\mathfrak{p}}$ it follows that $\|F(\operatorname{ad} s)\dot{s}_0\|_{\mathfrak{p}} \leqslant \|w\|_{\mathfrak{p}}$, or equivalently, $\|(\pi_{x_0})_{*z}\| \leqslant 1$.

Theorem 4.32. *Let $x_0 \in C$, let ε be as in Lemma* 4.30 *and consider*

$$\Omega_{x_0} = \{\exp_y(v) : y \in C,\ d(y, x_0) < \varepsilon,\ v \in \mathfrak{s}'\}.$$

Let $k \in \mathbb{N}_0$ and let $\eta_k : \Omega_{x_0}^{\varepsilon} \to \Omega_{x_0}$ be $\eta_k(\exp_y(v)) = \exp_y(2^k v)$. Then the differential of η_k is an expansive invertible operator. In particular, η_k is a local isomorphism.

Proof. As always we assume that $x_0 = o$. Let $z = \exp_y(v) = q(e^s e^v) \in \Omega_{x_0}^{\varepsilon}$ and let $\alpha(t) = q(e^s e^v e^{tw})$ for $s + v \in \mathfrak{s}_{\varepsilon} \oplus \mathfrak{s}'_{\varepsilon}$. Then for t small enough, $\alpha(t) \in \Omega_{x_0}^{\varepsilon}$, and so we consider $\beta = \eta \circ \alpha$ to compute $\eta_{*z}w = \dot{\beta}(0)$. Let $s_t + v_t \in \mathfrak{s}_{\varepsilon} \oplus \mathfrak{s}'_{\varepsilon}$ be such that $\alpha(t) = q(e^{s_t}e^{v_t})$, with $s_0 = s$ and $v_0 = v$. Then a straightforward but tedious computation yields

$$w = \frac{\sinh \operatorname{ad} v}{\operatorname{ad} v}\dot{v}_0 + \cosh(\operatorname{ad} v)\frac{\sinh \operatorname{ad} s}{\operatorname{ad} s}\dot{s}_0 - \sinh(\operatorname{ad} v)\left(\frac{1 - \cosh \operatorname{ad} s}{\operatorname{ad} s}\right)\dot{s}_0.$$

Replacing v_t with $2^k v_t$ yields

$$(\eta_k)_{*z}w = \frac{\sinh 2^k \operatorname{ad} v}{\operatorname{ad} v}\dot{v}_0 + \cosh(2^k \operatorname{ad} v)\frac{\sinh \operatorname{ad} s}{\operatorname{ad} s}\dot{s}_0 - \sinh(2^k \operatorname{ad} v)\left(\frac{1 - \cosh \operatorname{ad} s}{\operatorname{ad} s}\right)\dot{s}_0.$$

Using the trigonometric identities $\sinh(2z) = 2\sinh(z)\cosh(z)$, $\sinh^2(z) + 1 = \cosh^2(z)$ and $\cosh(2z) = \cosh^2(z) + \sinh^2(z)$, we obtain

$$(\eta_k)_{*z}w = 2\cosh(2^{k-1}\operatorname{ad} v)(\eta_{k-1})_{*z}w - F(\operatorname{ad} s)\dot{s}_0.$$

From the previous remark, the last term matches with $(\pi_{x_0})_{*z}w$. Now by Remark 4.6, $\cosh(2^{k-1}\operatorname{ad} v)$ is an expansive invertible operator of $\mathfrak{p}$, and hence we have

$$(\eta_k)_{*z}w = \cosh(2^{k-1}\operatorname{ad} v)\left[2(\eta_{k-1})_{*z} - \cosh^{-1}(2^{k-1}\operatorname{ad} v)(\pi_{x_0})_{*z}\right]w.$$

The proof is on induction on k. If $k = 0$, then there is nothing to prove since $\eta_0 = \mathrm{id}$. Then assume that $(\eta_{k-1})_{*z}$ is expansive and invertible for any $z \in \Omega_{x_0}^{\varepsilon}$. Then we have

$$(\eta_k)_{*z}w = \cosh(2^{k-1}\operatorname{ad} v)(\eta_{k-1})_{*z}\left[2 - (\eta_{k-1})_{*z}^{-1}\cosh^{-1}(2^{k-1}\operatorname{ad} v)(\pi_{x_0})_{*z}\right]w.$$

If $u \in \mathfrak{p}$ and $\varphi \in \mathfrak{p}^*$ is any unit norming functional for u, then if we consider

$$A_k = (\eta_{k-1})_{*z}^{-1}\cosh^{-1}(2^{k-1}\operatorname{ad} v)(\pi_{x_0})_{*z},$$

we obtain

$$\varphi(A_k u - u) \leqslant \|A_k u\|_{\mathfrak{p}} - \varphi(u) \leqslant \|u\|_{\mathfrak{p}} - \|u\|_{\mathfrak{p}} = 0,$$

which shows that $A_k - 1$ is a dissipative operator on $\mathfrak{p}$, and by Remark 4.4, the operator

$$1 - (A_k - 1) = 2 - (\eta_{k-1})_{*z}^{-1}\cosh^{-1}(2^{k-1}\operatorname{ad} v)(\pi_{x_0})_{*z}$$

is expansive and invertible in $\mathfrak{p}$. Then $(\eta_k)_{*z}$ is also expansive and invertible. In particular η_k is a local isomorphism by the inverse function theorem. □

Theorem 4.33. *Let $C = q(e^{\mathfrak{s}})$ be a locally convex expansive reductive submanifold in M. Then Ω_{x_0} is an open neighborhood of $\exp_{x_0}(\mathfrak{s}')$ in M and*

$$NC = \{\exp_x(v) : x \in C, v \in \mathfrak{s}'\}$$

is an open neighborhood of C in M. The inequality $d(\eta_k x, \eta_k y) \geqslant d(x, y)$ holds for $x, y \in \Omega_{x_0}$ sufficiently close.

Proof. Since $\Omega_{x_0} = \cup_{k \in \mathbb{N}_0} \eta_k \Omega^{\varepsilon}_{x_0}$, it follows that Ω_{x_0} is an open set in M. Clearly NC contains C, and on the other hand NC is the union of open sets $NC = \cup_{x_0 \in C} \Omega_{x_0}$.

If α is a short geodesic joining $\eta_k x$ to $\eta_k y$, and x and y are close enough, then $\alpha \subset \Omega_{x_0}$ and $\beta = \eta_k^{-1} \circ \alpha$ is a smooth curve in $\Omega^{\varepsilon}_{x_0}$ (for some $\varepsilon > 0$) joining x to y. Then we have

$$d(x, y) \leqslant L(\beta) \leqslant L(\alpha) = d(\eta_k x, \eta_k y). \qquad \square$$

Theorem 4.34 (CPR splittings for Cartan–Hadamard manifolds). *Let $C = q(e^{\mathfrak{s}})$ be a reductive expansive submanifold in M and assume that M is simply connected. Then, if $v, w \in \mathfrak{s}'$ and $x, y \in C$, the distance function $f : [0, +\infty) \to [0, +\infty)$ given by*

$$f(t) = d(\exp_x(tv), \exp_y(tw))$$

is increasing. For each $k \in \mathbb{N}_0$, the map $\eta_k : NC^{\varepsilon} \to NC$ given by $\eta_k \exp_x(v) = \exp_x(2^k v)$ is injective, and it is an isomorphism onto its image, with expansive differential. Moreover, $NC = M$, namely

$$M = \{\exp_x(v) : x \in C, v \in \mathfrak{s}'\},$$

and hence for any $v \in \mathfrak{p}$ there exists a unique $s \in \mathfrak{s}$ and a unique $s' \in \mathfrak{s}'$ such that $q(e^v) = q(e^s e^{s'})$. The projection map $\pi : M \to C$ is contractive for the geodesic distance.

Proof. If M is simply connected, then C is a closed, convex, embedded immersed submanifold of M by Corollary 4.22. In Lemma 4.28, we can take $R = +\infty$ since $\kappa_M = +\infty$. This proves the first assertion, and moreover, it shows that η_k is injective. Then $NC = \cup_{k \in \mathbb{N}_0} \eta^k NC^{\varepsilon} \subset M$ is an open set in M, and moreover $\pi : NC \to M$ is contractive by Remark 4.31, since $\pi(\exp_y(v)) = \pi(\exp_y(\lambda v))$ for real λ, and then the argument in that remark applies. To finish, we claim that NC is closed in M, considering $x_n \in NC$ such that $x_n \to x \in M$. Then any x_n can be uniquely written as $x_n = \exp_{y_n}(v_n)$, with $y_n \in C$ and $v_n \in \mathfrak{s}'$. Since π is a contraction it follows that y_n is a Cauchy sequence in C, and since C is closed in M, it follows that there exists $y_0 \in C$ such that $\lim y_n = y_0$. Then, by Lemma 3.1, we have

$$\|v_n - v_0\|_{\mathfrak{p}} \leqslant d(q(e^{v_n}), q(e^{v_0})) = d(\exp_{y_n}(v_n), \exp_{y_n}(v_0)) = d(x_n, \exp_{y_n}(v_0)).$$

Letting $n \to \infty$ gives $v_n \to v_0 \in \mathfrak{s}'$, and then $x = \lim x_n = \lim \exp_{y_n}(v_n) = \exp_{y_0}(v_0) \in NC$. $\square$

Problem 4.35. We extend the results of Theorem 4.34 to arbitrary Cartan–Hadamard manifolds (that is, the general setting of Subsection 3.1).

The relationship between this last result and Theorem 3.20 of Subsection 3.2 is presented in the following theorem.

Theorem 4.36. *Let $C = q(e^{\mathfrak{s}}) \subset M$ be an expansive reductive submanifold, let $z \in NC$, $z = \exp_x(v)$ for some $x \in C$, $v \in \mathfrak{s}'$ and assume that $\|v\|_{\mathfrak{p}} = d(x, z) \leqslant \kappa_M/8$. Then x is (locally)* *the best approximation to z in C if and only if $\|1-p\|=1$. In that situation, clearly we find that*

$$d(z, C\cap B(z,\kappa_M/8)) = \|v\|_{\mathfrak{p}}.$$

Proof. Assume first that $\|1-p\|=1$. Since the action of $G_{\mathfrak{s}}$ is transitive and isometric on C, we can assume that $x=o$, and hence $z=q(e^v)$. Let $y=q(e^r)\in C$, with $r\in\mathfrak{s}$ such that $\|r\|_{\mathfrak{p}}=d(x,y)\leqslant \kappa_M/8$. Then $d(z,y)\leqslant\kappa_M/4$ and

$$d(x,z)=\|v\|_{\mathfrak{p}}=\|(1-p)(v-r)\|_{\mathfrak{p}}\leqslant\|v-r\|_{\mathfrak{p}}\leqslant d(q(e^v),q(e^r))=d(x,y),$$

where the last inequality follows from Proposition 4.9.

On the other hand, if $d(x,o)\leqslant d(x,y)$ for any $y\in C\cap B(x,\kappa_M/8)$, consider the function $f(t)=d(q(e^v),q(e^{ts}))$, with $f:[0,+\infty)\to(0,+\infty)$ and $s\in\mathfrak{s}$, with $\|s\|_{\mathfrak{p}}\leqslant\kappa_M/8$. Then the claim implies that f has a local minimum at $t=0$. In particular, $f'(0^+)\geqslant 0$. As in the proof of Lemma 4.28, we obtain $f(t)=\|\gamma(t)\|_{\mathfrak{p}}$, where $q(e^{\gamma})=q(e^{-v}e^{ts})$, with $\gamma(0)=-v$ and $\dot\gamma(0)=G(\operatorname{ad} v)s$. Hence we have

$$0\leqslant f'(0^+)\leqslant\|-v+G(\operatorname{ad} v)s\|_{\mathfrak{p}}-\|v\|_{\mathfrak{p}},$$

and then $\|v\|_{\mathfrak{p}}\leqslant\|-v+s\|_{\mathfrak{p}}$ for any $s\in\mathfrak{s}$ small enough, and thus replacing v and s with convenient multiples, we obtain $\|1-p\|=1$. □

Remark 4.37. In the setting of finite-dimensional (Riemannian) symmetric spaces $M=G/K$, a *symmetric* submanifold $C\subset M$ is a submanifold such that there exists an involutive isometry ε_0 of M such that $\varepsilon_0(K)=K$, $\varepsilon_0(C)=C$ and $(\varepsilon_0)_*(v)=(-1)^j v$, with $j=0$ if $v\in T_oC^{\perp}$ and $j=-1$ if $v\in T_oC$. In this context, it is easy to see that a submanifold is symmetric if the supplement $\mathfrak{s}'$ of its tangent space $\mathfrak{s}$ at $o=q(1)$ is a Lie triple system, that is $\operatorname{ad}_{\mathfrak{s}'}(\mathfrak{s}')\subset\mathfrak{s}'$. A submanifold $C\subset M$ is called *reflective* if it is both totally geodesic and symmetric. In the Riemannian setting, if $\mathfrak{s}'=\mathfrak{s}^{\perp}$, one also has the dual relations given by

$$\operatorname{ad}^2_{\mathfrak{s}'}(\mathfrak{s})\subset\mathfrak{s},\quad \operatorname{ad}^2_{\mathfrak{s}}(\mathfrak{s}')\subset\mathfrak{s}'$$

due to the fact that ad^2_v is self-adjoint for any $v\in\mathfrak{p}$. Hence any reflective submanifold is reductive.

In our infinite-dimensional setting, it is natural to consider, given a Cartan–Hadamard manifold $M=G/K$, a second involutive automorphism τ of G which commutes with σ. Let

$$\mathfrak{u}_+=\{v\in\mathfrak{g}:\tau_{*1}v=v\},\quad \mathfrak{u}_-=\{v\in\mathfrak{g}:\tau_{*1}v=-v\}.$$

Then if we consider $\mathfrak{s}=\mathfrak{p}\cap\mathfrak{u}_-$ and $\mathfrak{s}'=\mathfrak{p}\cap\mathfrak{u}_+$, the conditions

$$\operatorname{ad}^2_{\mathfrak{s}}(\mathfrak{s})\subset\mathfrak{s},\quad \operatorname{ad}^2_{\mathfrak{s}}(\mathfrak{s}')\subset\mathfrak{s}' \tag{4.2}$$

are automatically fulfilled, and thus $C=q(e^{\mathfrak{s}})$ is a reductive submanifold according to our previous Definition 4.25.

If we define $\tau_0:M\to M$ as the involution given by $\tau_0(q(g))=q(\tau^{-1}(g))$, then if M is simply connected, we can compute $\tau_0(q(e^v))=q(e^{-\tau_{*1}v})$ for any $v\in\mathfrak{p}$, and C is the set of τ_0-fixed points. If τ_0 is an isometry of M, since this is equivalent to the fact that $\tau_{*1}|_{\mathfrak{p}}$ is an isometry of $\mathfrak{p}$, it follows that the reductive submanifold $C\subset M$ is expansive according to Definition 4.25, due to the fact that the projection p onto $\mathfrak{s}$ is given by $p=(1-\tau_{*1}|_{\mathfrak{p}})/2$. Moreover, since $1-p=(1+\tau_{*1}|_{\mathfrak{p}})/2$, it implies that the normal bundle gives the best approximation from C. Hence isometric involutions τ that commute with σ induce reductive submanifolds for which Theorems 4.34 and 4.36 apply, inducing a metric splitting as in Corollary 3.21 of Subsection 3.2.

Remark 4.38. If $C = q(e^{\mathfrak{s}})$ is a reflective submanifold, in the sense that

$$\operatorname{ad}^2_{\mathfrak{s}}(\mathfrak{s}) \subset \mathfrak{s}, \quad \operatorname{ad}^2_{\mathfrak{s}}(\mathfrak{s}') \subset \mathfrak{s}', \quad \operatorname{ad}^2_{\mathfrak{s}'}(\mathfrak{s}) \subset \mathfrak{s}, \quad \operatorname{ad}^2_{\mathfrak{s}'}(\mathfrak{s}') \subset \mathfrak{s}',$$

then one obtains that $NC = \{\exp_x(v) : x \in C,\ v \in \mathfrak{s}'\}$ is open in M with a more direct proof. One has to observe that if $v = s + s' \in \mathfrak{p}$ and $w = t + t'$ (here $s, t \in \mathfrak{s}$ and $t, t' \in \mathfrak{s}'$ as usual), then the map $E : \mathfrak{p} \to M$, of Lemma 4.30 given by $E(v) = q(e^s e^{s'})$ has its differential in the form of a block matrix relative to $\mathfrak{s} \oplus \mathfrak{s}'$ given by

$$E_{*v} w = \begin{pmatrix} \cosh(\operatorname{ad} s') \dfrac{\sinh(\operatorname{ad} s)}{\operatorname{ad} s} & 0 \\ \sinh(\operatorname{ad} s') \dfrac{(\cosh(\operatorname{ad} s) - 1)}{\operatorname{ad} s} & \dfrac{\sinh(\operatorname{ad} s')}{\operatorname{ad} s'} \end{pmatrix} \begin{pmatrix} t \\ t' \end{pmatrix}.$$

Then E is a local isomorphism at any $v \in \mathfrak{p}$, and thus $E(\mathfrak{p})$ is open in M.

4.2.1. *CPR splittings for Banach–Lie groups.* Let (G, σ) be an involutive Banach–Lie group. Let $\tau = \sigma_{*1}$, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the τ-decomposition of $\mathfrak{g}$. Assume that the Banach–Lie algebra $\mathfrak{g}$ has a compatible norm b that makes $-\operatorname{ad}^2_v|_{\mathfrak{p}}$ dissipative for each $v \in \mathfrak{p}$. We say that (G, τ) satisfies semi-negative curvature. According to Proposition 4.5, this last condition is equivalent to the fact that $M = G/K$ is a Banach–Finsler manifold with spray of semi-negative curvature.

Combining Neeb's result on the polar map (Theorem 4.7) with Theorem 4.34, we obtain our fundamental result on polar decompositions relative to reductive submanifolds.

Corollary 4.39. *Let $C = q(e^{\mathfrak{s}})$ be an expansive reductive submanifold of a Cartan–Hadamard homogeneous space $M = G/K$. Then the map*

$$(q(e^s), s', k) \longmapsto e^s e^{s'} k$$

induces an isomorphism $C \times \mathfrak{s}' \times K \simeq G$.

Assume that C is also reflective. If we put $C' = q(e^{\mathfrak{s}'})$, then C' is a reductive submanifold of M and we obtain an isomorphism as follows:

$$G \simeq C \times C' \times K.$$

If $g = e^s e^{s'} k \in G$, then one obtains $\|s\|_{\mathfrak{p}} = d(q(g), C')$. Moreover $\|s'\|_{\mathfrak{p}} = d(q(g), C)$ if and only if $\|1 - p\| = 1$, that is, if and only if C' is also expansive.

4.3. *Positive elements*

For a symmetric Banach–Lie group (G, σ) one has the natural involution $^* : G \to G$ given by $g^* = \sigma(g^{-1}) = \sigma(g)^{-1}$. It allows one to write down the quotient map in a concrete way as $P : G \to G$, such that $P(g) = gg^*$ (note that the isotropy of $1 \in G$ is just $K =$ the fixed-point set of σ). Thus $M := P(G) \simeq G/K$ has a natural structure of Finsler manifold with spray, under the usual hypothesis.

The set $P(G)$ is the set of positive invertible elements when G is one of the so-called classical Banach–Lie groups (see the Appendix). In this picture, the geodesics of M are given by

$$\alpha(t) = e^v e^{2tz} e^v.$$

Let G^s stand for the set of invertible self-adjoint elements, $g^* = g$, that is

$$G^s = \{g \in G : \sigma(g) = g^{-1}\}.$$

Then the natural action of G on G^s is $a \mapsto gag^*$.

If $G = \mathcal{B}(\mathcal{H})^{\times}$ is the subgroup of invertible elements of $\mathcal{B}(\mathcal{H})$ (the bounded linear operators on a Hilbert space $\mathcal{H}$), then this action defines Banach homogeneous spaces $G^{s,a}$, the orbits of $a \in G^s$; the existence of smooth local sections is essentially given by the square root of $\mathcal{B}(\mathcal{H})$; see [**17**, Proposition 1.1] for the details. Via polar decomposition, one has the projection $\pi : G^s \to K^s$, where K^s is the set of reflections of G, that is, the set of self-adjoint elements of K. That is, we write $g = e^v k$, where $v \in \mathfrak{p}$ and $k \in K$ (see Theorem 4.7), and for $g \in G^s$, we consider $\pi(g) = k$. If $G = \mathcal{B}(\mathcal{H})^{\times}$, then this fibration π has very nice properties, for instance, its differential is a contraction [**17**, Theorem 5.1], a fact related to the geodesic structure of the group of reflections K^s.

Appendix. *Examples and applications*

Here we indicate some applications to operator theory. We concentrate on operators ideals and we omit other relevant examples such as bounded symmetric domains and JB^*-algebras. See [**36**, Section 6] for a further discussion on these topics. An account of the applications for finite von Neumann algebras analogous to those studied in [**3**] in the Riemannian situation, can be found in [**16**].

A.1. *Operator algebras*

Let $\mathcal{B}(\mathcal{H})$ stand for the set of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$, with the uniform norm denoted by $\|\cdot\|$. Let $\|\cdot\|_{\mathcal{I}} : \mathcal{B}(\mathcal{H}) \to \mathbb{R}_+ \cup \{\infty\}$ be a norm and let $\mathcal{I}$ stand for the set of operators with finite norm, that is

$$\mathcal{I} = \{x \in \mathcal{B}(\mathcal{H}) : \|x\|_{\mathcal{I}} < \infty\}.$$

Further it is assumed that

(1) $\|xyz\|_{\mathcal{I}} \leqslant \|x\|\, \|y\|_{\mathcal{I}}\, \|z\|$ for any $y \in \mathcal{I}$ and $x, z \in \mathcal{B}(\mathcal{H})$.

(2) $(\mathcal{I}, d_{\mathcal{I}})$ is a complete metric space, where $d_{\mathcal{I}}(x, y) = \|x - y\|_{\mathcal{I}}$.

Then $\mathcal{I}$ is a complex self-adjoint ideal of compact operators in $\mathcal{B}(\mathcal{H})$; the standard reference on the subject is the book of Gohberg and Krein [**19**]. If $y \mapsto y^*$ denotes the usual involution of $\mathcal{B}(\mathcal{H})$, then it is easy to check whether $\|y^*\|_{\mathcal{I}} = \|y\|_{\mathcal{I}}$ and further whether the norm is unitarily invariant in the sense that

$$\|uyv\|_{\mathcal{I}} = \|y\|_{\mathcal{I}}$$

for any $y \in \mathcal{I}$ and $u, v \in \mathcal{B}(\mathcal{H})$ unitary operators.

Remark A.1. Elementary examples of symmetrically normed ideals are given by the Schatten ideals $\mathcal{B}_p(\mathcal{H})$ of operators, defined by the p-norms in $\mathcal{B}(\mathcal{H})$ $(1 \leqslant p < \infty)$ by

$$\|v\|_p^p = \operatorname{tr}|v|^p = \operatorname{tr}((v^*v)^{p/2}),$$

where tr is the infinite trace of $\mathcal{B}(\mathcal{H})$. Elements of $\mathcal{B}_p(\mathcal{H})$ are compact operators whose spectra are l^p summable. One has the inequalities

$$\|v\| \leqslant \|v\|_p \leqslant \|v\|_q \leqslant \cdots \leqslant \|v\|_1$$

for $p \geqslant q$, and the inclusions

$$\mathcal{B}_1(\mathcal{H}) \subset \cdots \subset \mathcal{B}_q(\mathcal{H}) \subset \mathcal{B}_p(\mathcal{H}) \subset \cdots \subset \mathcal{K}(\mathcal{H}),$$

where $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators. The trace map $(v, w) \mapsto \operatorname{tr}(vw^*)$ induces the duality $\mathcal{B}_p(\mathcal{H})^* = \mathcal{B}_q(\mathcal{H})$ for $1/p + 1/q = 1$ and $1 < p < \infty$. Moreover, $\mathcal{K}(\mathcal{H})^* = \mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$. The $\mathcal{B}_p(\mathcal{H})$ spaces are 2-uniformly convex for $p \in (1, 2]$ and p-uniformly convex for $p \in [2, +\infty)$, due to McCarthy's inequalities [**32**].

Let $G^{\mathcal{I}}$ stand for the group of invertible operators in the unitized ideal, that is

$$G^{\mathcal{I}} = \{1 + x : \ x \in \mathcal{I}, \ \mathrm{Sp}(1+x) \subset \mathbb{R}^*\},$$

where Sp denotes the usual spectrum of an element in $\mathcal{B}(\mathcal{H})$. Equivalently

$$G^{\mathcal{I}} = \{g \in \mathcal{B}(\mathcal{H})^{\times} : \ g - 1 \in \mathcal{I}\}.$$

Then $G^{\mathcal{I}}$ is a Banach–Lie group (one of the so-called classical Banach–Lie groups [**20**]), open in $\mathcal{I}$ with the inherited topology, and $\mathcal{I}$ identifies with its Banach–Lie algebra; it suffices to prove that a neighborhood of $1 \in G^{\mathcal{I}}$ is isomorphic to $\mathcal{I}$. To prove these statements, consider the usual analytic logarithm: for $\|g-1\|_{\mathcal{I}} < 1$ consider $\log(g) = \sum_k (1-g)^n$. Then if $g \in G^{\mathcal{I}}$ is such that $\|g-1\|_{\mathcal{I}} < 1$, we have $x = \log(g) \in \mathcal{I}$ and $e^x = g$.

Let $\mathcal{I}_h$ stand for the set of self-adjoint elements in $\mathcal{I}$ and consider $M^{\mathcal{I}}$ the cone of positive invertible elements in the unitized ideal, that is, elements with positive spectrum given by

$$M^{\mathcal{I}} = \{1 + x : \ x \in \mathcal{I}_h, \ \mathrm{Sp}(1+x) \subset (0, +\infty)\}.$$

Consider the involutive automorphism $\sigma : G^{\mathcal{I}} \to G^{\mathcal{I}}$ given by $g \mapsto (g^*)^{-1}$. Let $U^{\mathcal{I}} \subset G^{\mathcal{I}}$ stand for the unitary subgroup of fixed points of σ. Its Banach–Lie algebra is the set of skew-adjoint elements of $\mathcal{I}$, and $\mathcal{I} = \mathcal{I}_h \oplus i\mathcal{I}_h$. The quotient space $G^{\mathcal{I}}/U^{\mathcal{I}}$ can be identified with $M^{\mathcal{I}}$ via $q : G^{\mathcal{I}} \to M^{\mathcal{I}}$ given by $q(g) = gg^*$ as in Subsection 4.3.

We claim that the unitarily invariant norm of $\mathcal{I}$ makes of $(G^{\mathcal{I}}, \sigma)$ a semi-negative curvature group. We use the criteria of Proposition 4.5 as follows:

$$\|e^{i\,\mathrm{ad}\,x} v\|_{\mathcal{I}} = \|\mathrm{Ad}_{e^{ix}} v\|_{\mathcal{I}} = \|e^{ix} v e^{-ix}\|_{\mathcal{I}} = \|v\|_{\mathcal{I}}$$

for any $x, v \in \mathcal{I}_h$; then $1 - it\,\mathrm{ad}\,x$ is expansive and invertible for any $t > 0$, and hence $1 + t\,\mathrm{ad}_x^2$ is expansive and invertible for any $t > 0$, proving that $-\mathrm{ad}_x^2$ is dissipative for any $x \in \mathcal{I}_h$.

Thus the positive cone $M^{\mathcal{I}} \simeq G^{\mathcal{I}}/U^{\mathcal{I}}$ can be regarded as a complete manifold of semi-negative curvature, since it is geodesically complete. Moreover, since $Z(\mathcal{I}) = \{0\}$ for a proper ideal $\mathcal{I}$, it follows that $M^{\mathcal{I}}$ is simply connected and $\exp : \mathcal{I}_h \to M^{\mathcal{I}}$ is an isomorphism.

The unique geodesic of $M^{\mathcal{I}}$ joining positive invertible $a, b \in M^{\mathcal{I}}$ is short and is given by

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2} b a^{-1/2})^t a^{1/2}.$$

Its length is given by $\|\ln(a^{-1/2} b a^{-1/2})\|_{\mathcal{I}}$, and the exponential map at $a \in M^{\mathcal{I}}$ is given by

$$\exp_a(v) = a^{1/2} \exp(a^{-1/2} v a^{-1/2}) a^{1/2},$$

whenever $v \in T_a M^{\mathcal{I}} \simeq \mathcal{I}_h$. In particular we have

$$d(a,b) = \|\ln(a^{1/2} b^{-1} a^{1/2})\|_{\mathcal{I}}.$$

The semi-negative curvature condition is the (well known for matrices; for instance see [**10**]) *exponential metric increasing property* as follows:

$$\|\ln(a^{-1/2} b a^{-1/2})\|_{\mathcal{I}} \geqslant \|\ln(a) - \ln(b)\|_{\mathcal{I}}.$$

The convexity of the geodesic distance between two geodesics starting at $a = 1$ apparently is given by the following inequality:

$$\|\ln(a^{-t/2} b^t a^{-t/2})\|_{\mathcal{I}} \leqslant t\|\ln(a^{-1/2} b a^{-1/2})\|_{\mathcal{I}}.$$

This inequality seems to be new in this general context, but for $\mathcal{B}(\mathcal{H})$ it was extensively studied and is known as one of the equivalent forms of the Löwner–Heinz theorem on monotone operator maps [**30**]. For the p-norms of $\mathcal{B}(\mathcal{H})$ it is stated as follows:

$$\mathrm{tr}((B^{1/2} A B^{1/2})^{rp}) \leqslant \mathrm{tr}((B^{r/2} A^r B^{r/2})^p), \quad r \geqslant 1,$$

an inequality due to Araki [**6**]. As it is, it was generalized to the noncommutative $L^p(\mathcal{M}, \tau)$-spaces of a semi-finite von Neumann algebra $\mathcal{M}$ by Kosaki [**24**]. In the context of the uniform norm, the relation between this inequality and the convexity of the geodesic distance in the positive cone of $\mathcal{B}(\mathcal{H})$ was studied in [**2**].

When $\mathcal{I} = \mathcal{B}_p(\mathcal{H})$, and $p > 1$, we can apply the results of Subsection 3.1 to convex closed sets. In particular, if $C \subset M$ is a convex submanifold, one obtains splittings as in Corollary 3.21. These examples were studied for $p = 2$ (the Riemann–Hilbert situation) in [**27**, **41**]. The nonuniformly convex situation, when $p = 1$, was studied in [**15**].

In this setting, the standard example of a convex submanifold is given by $C = q(e^{\mathfrak{s}})$, where $\mathfrak{s}$ equals the real Banach–Lie algebra of self-adjoint diagonal operators (relative to a fixed orthonormal basis $\{e_i\}$ of $\mathcal{H}$), and $\mathfrak{s}'$ are the co-diagonal self-adjoint operators.

Remark A.2. If we decompose a tangent vector $v \in \mathcal{I}_h$, $v = w + z$ and $\|v\|_{\mathcal{I}} = \|w\|_{\mathcal{I}} + \|z\|_{\mathcal{I}}$, then the curve

$$\delta(t) = \begin{cases} e^{2tw}, & t \in [0, 1/2], \\ e^{w} e^{(2t-1)z}, & t \in [1/2, 1] \end{cases}$$

is piecewise smooth and joins 1 to e^v in $M_{\mathcal{I}}$; moreover $L(\delta) = \|w\|_{\mathcal{I}} + \|y\|_{\mathcal{I}} = L(\exp(tv))$; thus δ is a minimizing piecewise smooth curve joining 1 to e^v, and it is not a geodesic unless w and z are aligned. This shows that Proposition 3.6 is false for $p = 1$ and $p = \infty$ (whose norm is not strictly convex). For example, consider

$$v = \tfrac{1}{2} p_1 + \tfrac{1}{2}(p_1 + p_2)$$

with p_i mutually orthogonal one-dimensional projections in $\mathcal{B}(\mathcal{H})$. Then $x = \frac{1}{2} p_1$ and $y = \frac{1}{2}(p_1 + p_2)$ commute, and $\|v\|_\infty = 1 = 1/2 + 1/2 = \|x\|_\infty + \|y\|_\infty$.

A.2. *Inclusions of* C*-*algebras*

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-subalgebra and let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ be a conditional expectation with range $\mathcal{A}$. Let H stand for the linear supplement of $\mathcal{A}$ given by $\mathcal{E}$, that is, $H = \ker \mathcal{E}$. Then $\|\mathcal{E}\| = 1$ and $\mathcal{E}$ is a bi-module map, that is, $\mathcal{E}(nmn') = n\mathcal{E}(m)n'$ for any $n, n' \in \mathcal{A}$.

In [**37**], the authors studied inclusions of C*-algebras $N \subset M$ with a conditional expectation $\mathcal{E} : M \to N$. In that setting, one has the inclusions $P_N \subset P_M$ of cones of positive invertible elements; the tangent spaces are the sets of self-adjoint elements of N and M, respectively. The projection $p = \mathcal{E}|_{M_h} : M_h \to N_h$ provides a reductive supplement $H = \ker p$ for N_h, and moreover $\|p\| = 1$. The exponential map provides a splitting of the positive cone P_M of M via the positive cone P_N of N as a convex submanifold and H as the normal bundle. In such a situation, the norm of $1 - \mathcal{E}$ can be as large as 2. The purpose of this short section is to extend this situation to the Finsler norms of the p-Schatten ideals, applying the results of the previous sections.

Let $p \geqslant 1$ and consider $\mathcal{A}_p = \mathcal{A} \cap \mathcal{B}_p(\mathcal{H})$, and $\mathcal{E}_p = \mathcal{E}|_{\mathcal{B}_p(\mathcal{H})}$. In certain situations one can ensure that $\mathcal{E}(\mathcal{B}_1(\mathcal{H})) \subset \mathcal{B}_1(\mathcal{H})$. A sufficient condition is that $\mathcal{E}$ maps finite rank operators into finite rank operators, a condition that is easy to check in most situations. Throughout, it is assumed that the expectation is compatible with the trace, that is $\mathrm{Tr}(\mathcal{E}x) = \mathrm{Tr}(x)$ for any $x \in \mathcal{B}_1(\mathcal{H})$. The example to have in mind is that of a maximal abelian subalgebra $\mathcal{A}$ given by the diagonal operators in some fixed orthonormal base of $\mathcal{H}$. In this case the conditional expectation is given by compression to the diagonal.

Note that by duality (since $\|\mathcal{E}\| = 1$) we have

$$\begin{aligned} \|\mathcal{E}_1(x)\|_1 &= \sup_{\|z\| \leqslant 1} |\mathrm{tr}(\mathcal{E}(x)z)| = \sup_{\|z\| \leqslant 1} |\mathrm{tr}(\mathcal{E}(x)\mathcal{E}(z))| = \sup_{\|w\| \leqslant 1, w \in \mathcal{A}} |\mathrm{tr}(\mathcal{E}(x)w)| \\ &= \sup_{\|w\| \leqslant 1, w \in \mathcal{A}} |\mathrm{tr}(xw)| \leqslant \sup_{\|w\| \leqslant 1} |\mathrm{tr}(xw)| = \|x\|_1. \end{aligned}$$

Thus $\|\mathcal{E}_1\| \leqslant 1$, and since $\mathcal{E}_1$ is a projection, it follows that $\|\mathcal{E}_1\| = 1$. The essence of the argument is the fact that $\mathcal{E}$ (as a Banach space linear operator) is self-dual. Then $1-\mathcal{E}$ is also self-dual, and with the same proof, one also has $\|1-\mathcal{E}_1\| \leqslant \|1-\mathcal{E}\|$.

From the fact that $\mathcal{B}_p(\mathcal{H})$ can be obtained via complex interpolation from the pair $(\mathcal{B}_1(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ (for instance see [**39**]) and that $\mathcal{B}_1(\mathcal{H})$ is dense in each $\mathcal{B}_p(\mathcal{H})$ (since finite rank operators are dense), it follows that the restriction $\mathcal{E}_p$ defined above matches the interpolated conditional expectation.

Now we observe that, for $p=2$, this restriction is an orthogonal projection: indeed, we have

$$\begin{aligned}\|\mathcal{E}_2(z)\|_2^2 &= \mathrm{Tr}(\mathcal{E}(z)\mathcal{E}(z)^*) = \mathrm{Tr}(\mathcal{E}(z)\mathcal{E}(z^*)) = \mathrm{Tr}(z^*\mathcal{E}(z)) \\ &\leqslant \mathrm{Tr}(z^*z)^{1/2}\mathrm{Tr}(\mathcal{E}(z)\mathcal{E}(z)^*)^{1/2} = \|z\|_2\|\mathcal{E}_2(z)\|_2\end{aligned}$$

by the Cauchy–Schwarz inequality for the trace inner product. With the same argument, one obtains $\|1-\mathcal{E}_2\| = 1$.

From these facts (using interpolation again), for any $p \in [1,2]$, it follows that

$$\|\mathcal{E}_p\| = 1 \quad \text{and} \quad \|1-\mathcal{E}_p\| \leqslant \|1-\mathcal{E}\|^{2/p-1}.$$

Then by duality, we see that

$$\|\mathcal{E}_p\| = 1 \quad \text{and} \quad \|1-\mathcal{E}_p\| \leqslant \|1-\mathcal{E}\|^{1-2/p}$$

holds for any $p \in [2,\infty)$.

Certainly $[\mathcal{A},\mathcal{A}] \subset \mathcal{A}$ since $\mathcal{A}$ is an associative subalgebra, and hence evidently

$$\mathrm{ad}\,^2_{\mathcal{A}_h}(\mathcal{A}_h) \subset \mathcal{A}_h.$$

However, also note that, since $\mathcal{E}$ is a bi-module map, $[\mathcal{A}, \ker\mathcal{E}] \subset \ker\mathcal{E}$. Thus $C = \exp(\mathcal{A}_h)$ is a reductive submanifold of the positive cone of $\mathcal{B}(\mathcal{H})$. Hence by restricting the conditional expectation to the self-adjoint part of the p-Schatten ideals, one sees that the positive cone of the (unitized) subalgebra $\mathcal{A}_p$ has a natural structure of a reductive expansive submanifold in $\mathcal{B}_p(\mathcal{H})$. Thus one obtains splittings of the respective classical Banach–Lie groups invoking Corollary 4.39:

THEOREM A.3. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra with a conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{A}$ compatible with the trace, such that $\mathcal{E}(\mathcal{B}_1(\mathcal{H})) \subset \mathcal{B}_1(\mathcal{H})$. Let $p \geqslant 1$ and let $\mathcal{A}_p = \mathcal{A} \cap \mathcal{B}(\mathcal{H})$. Then, for each invertible element g, with*

$$g \in G_p(\mathcal{H}) = \{g \in \mathcal{B}(\mathcal{H})^\times : g-1 \in \mathcal{B}_p(\mathcal{H})\},$$

there exist unique operators $u, g_{\mathcal{A}}, v_p$ such that we have the following

(i) *u is a unitary operator and*

$$u \in U_p(\mathcal{H}) = \{u \in \mathcal{U}(\mathcal{H}) : u-1 \in \mathcal{B}_p(\mathcal{H})\};$$

(ii) *$g_{\mathcal{A}}$ is invertible and*

$$g_{\mathcal{A}} \in \mathcal{A}_p^\times = \{g \in \mathcal{B}(\mathcal{H})^\times : g-1 \in \mathcal{A}_p\};$$

(iii) *$v_p \in \mathcal{B}_p(\mathcal{H})_h$ and $\mathcal{E}(v_p) = 0$;*

(iv) *the operator g can be decomposed as follows:*

$$g = g_{\mathcal{A}} e^{v_p} u,$$

which gives the isomorphism

$$G_p(\mathcal{H}) \simeq \mathcal{A}_p^\times \times (\ker\mathcal{E} \cap \mathcal{B}_p(\mathcal{H})_h) \times U_p(\mathcal{H}).$$

If $\|1-\mathcal{E}\| = 1$, then $\|v_p\|_p = d(\sqrt{gg^*}, \mathcal{A}_p^+)$, where d indicates the geodesic distance in the positive cone, and $\mathcal{A}_p^+$ denotes the positive cone of the (unitized) subalgebra $\mathcal{A}_p$ of $\mathcal{B}_p(\mathcal{H})$.

Equivalently, if we write $g = e^v$ *with* $v \in \mathcal{B}_p(\mathcal{H})_h$ *and* $g_{\mathcal{A}} = e^{Z_{\mathcal{A}}}$ *with* $Z_{\mathcal{A}}$ *in the self-adjoint part of* $\mathcal{A}_p$*, then* $Z_{\mathcal{A}}$ *is the unique minimizer of the nonlinear functional* $\varphi : (\mathcal{A}_p)_h \to \mathbb{R}_+$ *given by*

$$\varphi : z \longmapsto \| \ln(e^{v/2} e^{-z} e^{v/2}) \|_p.$$

These factorizations, in the context of $n \times n$ real matrices, for the Riemannian metric induced by the trace, stem back to Mostow [**35**], where he uses the semi-parallelogram law to obtain the best approximant, bringing new light on the real linear group.

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References

1. A. D. Alexandrov, 'A theorem on triangles in a metric space and some of its applications', *Tr. Mat. Inst. Steklova* 38 (1951) 5–23 (Russian).
2. E. Andruchow, G. Corach and D. Stojanoff, 'Geometrical significance of Löwner–Heinz inequality', *Proc. Amer. Math. Soc.* 128 (2000) 1031–1037.
3. E. Andruchow and G. Larotonda, 'Nonpositively curved metric in the positive cone of a finite von Neumann algebra', *J. London Math. Soc.* (2) 74 (2006) 205–218.
4. E. Andruchow, G. Larotonda and L. Recht, 'Finsler geometry and actions of the p-Schatten unitary groups', *Trans. Amer. Math. Soc.* 362 (2010) 319–344.
5. E. Andruchow and L. Recht, 'Sectional curvature and commutation of pairs of selfadjoint operators', *J. Operator Theory* 55 (2006) 225–238.
6. H. Araki, 'On an inequality of Lieb and Thirring', *Lett. Math. Phys.* 19 (1990) 167–170.
7. J. Aubin, *Optima and equilibria. An introduction to nonlinear analysis*, 2nd edn, Graduate Texts in Mathematics 140 (Springer, Berlin, 1998).
8. K. Ball, E. Carlen and E. Lieb, 'Sharp uniform convexity and smoothness inequalities for trace norms', *Invent. Math.* 115 (1994) 463–482.
9. W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics 61 (Birkhäuser, Boston, 1985).
10. R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics 169 (Springer, New York, 1997).
11. M.R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences) 319 (Springer, Berlin, 1999).
12. F. Bruhat and J. Tits, 'Groupes réductifs sur un corps local', *Publ. Math. Inst. Hautes Études Sci.* 41 (1972) 5–251 (French).
13. H. Busemann, 'Spaces with non-positive curvature', *Acta Math.* 80 (1948) 259–310.
14. J. Clarkson, 'Uniformly convex spaces', *Trans. Amer. Math. Soc.* 40 (1936) 396–414.
15. C. Conde, 'Differential geometry for nuclear positive operators', *Integral Equations Operator Theory* 57 (2007) 451–471.
16. C. Conde and G. Larotonda, 'Spaces of nonpositive curvature arising from a finite algebra', Preprint, 2009, arXiv:0908.3725v1.
17. G. Corach, H. Porta and L. Recht, 'The geometry of the space of selfadjoint invertible elements in a C^*-algebra', *Integral Equations Operator Theory* 16 (1993) 333-359.
18. A. Duci and A. C. Mennucci, 'Banach-like metrics and metrics of compact sets', Preprint, 2007, arXiv:0707.1174v1.
19. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs 18 (American Mathematical Society, Providence, RI, 1969) Translated from the Russian by A. Feinstein.
20. P. de la Harpe, *Classical Banach–Lie algebras and Banach–Lie groups of operators in Hilbert space*, Lecture Notes in Mathematics 285 (Springer, Berlin, 1972).
21. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics 80 (Academic Press, New York, 1978).
22. R. James, 'Orthogonality and linear functionals in normed linear spaces', *Trans. Amer. Math. Soc.* 61 (1947) 265–292.
23. J. Jost, *Nonpositive curvature: geometric and analytic aspects*, Lectures in Mathematics ETH Zürich (Birkhäuser, Basel, 1997).

24. H. Kosaki, 'An inequality of Araki–Lieb–Thirring', *Proc. Amer. Math. Soc.* 114 (1992) 477–481.
25. S. Lang, 'Differential and Riemannian manifolds', 3rd edn, Graduate Texts in Mathematics 160 (Springer, New York, 1995).
26. S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics 191 (Springer, New York, 1999).
27. G. Larotonda, 'Nonpositive curvature: a geometrical approach to Hilbert–Schmidt operators', *Differential Geom. Appl.* 25 (2007) 679–700.
28. G. Larotonda, 'Norm inequalities in operator ideals', *J. Funct. Anal.* 255 (2008) 3208–3228.
29. J. Lawson and Y. Lim, 'Symmetric spaces with convex metric', *Forum Math.* 19 (2007) 571–602.
30. K. Löwner, 'Über monotone Matrixfunktionen', *Math. Z.* 38 (1934) 177–216 (German).
31. G. Lumer and R. S. Phillips, 'Dissipative operators in a Banach space', *Pacific J. Math.* 11 (1961) 679–698.
32. C. McCarthy, 'c_p', *Israel J. Math.* 5 (1967) 249–271.
33. K. Menger, 'Zur Dimensions-und Kurventheorie', *Monatsh. Math. Phys.* 36 (1929) 411–432 (German). Unveröffentlichte Aufsätze aus den Jahren 1921-23.
34. J. Milnor, *Morse theory*. Annals of Mathematical Studies 51 (Princeton University Press, Princeton, NJ, 1963) Based on lecture notes by M. Spivak and R. Wells.
35. G. D. Mostow, 'Some new decomposition theorems for semi-simple groups', *Mem. Amer. Math. Soc.* (1955) 31–54.
36. K.-H. Neeb, 'A Cartan–Hadamard theorem for Banach–Finsler manifolds', *Geom. Dedicata* 95 (2002) 115–156.
37. H. Porta and L. Recht, 'Conditional expectations and operator decompositions', *Ann. Global Anal. Geom.* 12 (1994) 335–339.
38. H. Porta and L. Recht, 'Exponential sets and their geometric motions', *J. Geom. Anal.* 6 (1996) 277–285.
39. B. Simon, *Trace ideals and their applications*, London Mathematical Society Lecture Note Series 35 (Cambridge University Press, Cambridge, 1979).
40. S. Swierczkowski, 'Embedding theorems for local analytic groups', *Acta Math.* 114 (1965) 207–235.
41. A. B. Tumpach, 'Mostow's decomposition theorem for L^*-groups and applications to affine coadjoint orbits and stable manifolds', Preprint, 2006, arXiv:math-ph/0605039v1.
42. H. Upmeier, *Banach manifolds and Jordan C^*-algebras*, North-Holland Mathematics Studies 104, Notas de Matemática (Mathematical Notes), 96 (North-Holland, Amsterdam, 1985).
43. A. Wald, 'Axiomatik des Zwischenbegriffes in metrischen Räumen', *Math. Ann.* 104 (1931) 476–484 (German).

Cristian Conde and Gabriel Larotonda
Instituto Argentino de Matemática
Alberto Calderón, CONICET
Saavedra 15, 3er piso
C1083ACA Buenos Aires
Argentina

condecm@gmail.com
gabriel.larotonda@gmail.com

Insitituto de Ciencias
Universidad Nacional de General Sarmiento
J. M. Gutierrez 1150
B1613GSX Los Polvorines
Argentina

cconde@ungs.edu.ar
glaroton@ungs.edu.ar