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Charge dynamics and "in plane" magnetic field I: Rashba–Dresselhauss interaction, Majorana fermions and Aharonov–Casher theorems

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The two-dimensional charge transport with parallel (in plane) magnetic field is considered from the physical and mathematical point of view. To this end, we start with the magnetic field parallel to the plane of charge transport, in sharp contrast to the configuration described by the theorems of Aharonov and Casher where the magnetic field is perpendicular. We explicitly show that the specific form of the arising equation enforces the respective field solution to fulfill the Majorana condition. Consequently, as soon any physical system is represented by this equation, the rise of fields with Majorana type behavior is immediately explained and predicted. In addition, there exists a quantized particular phase that removes the action of the vector potential producing interesting effects. Such new effects are able to explain due to the geometrical framework introduced, several phenomenological results recently obtained in the area of spintronics and quantum electronic devices. The quantum ring as spin filter is worked out in this framework and also the case of the quantum Hall effect.

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1. Introduction

In 1937 Majorana propose a new representation to the celebrated Dirac equation, where the components of the spinor solution are related themselves by complex conjugation [15]. Due to his personal problems, he could not have foreseen the whirlwind of activity that would follow: not only in particle physics but also in nanoscience and condensed matter physics (for a review about this issue condensed matter see [18]). The recent storm of activity in condensed matter physics has focused on the "Majorana zero modes," i.e. emergent Majorana-like states occurring at exactly zero energy, that have a remarkable property of, if they are considered as particles, being their own antiparticles (self-conjugated). Sometimes, this property is expressed as an equality between the particle's creation and annihilation operators. As we will see below, there exists the general idea that any ordinary fermion can be thought of as composed of two Majorana fermions: this is only a partial picture, the real fact is that there exists a particular representation where a fermion effectively can be represented as bilinear combination of two states of fractionary spin [8–13]. For example, in condensed matter physics the considered "Majorana zero modes" are believed to exhibit the so-called non-abelian exchange statistics [4, 5] which endows them with a technological potential as building blocks of future quantum memory immune against many sources of decoherence. Recent advances in our understanding of solids with strong spin–orbit coupling, combined with the progress in nanofabrication, put the physical realization of the Majorana states to be considered as possible. In fact signatures consistent with their existence in quantum wires coupled to conventional superconductors and other type of devices have been reported by several groups [16].

On the other hand and with other motivations, Aharonov and Casher [1] proved two theorems for the case of a two-dimensional magnetic field. The first theorem states that an electron moving in a plane under the influence of a perpendicular inhomogeneous magnetic field has N ground-energy states, where N is the integral part of the total flux in units of the flux quantum $\Phi_0 = 2\pi/e \equiv hc/e$ ($m = 1$). The corresponding Dirac equation for the Aharonov–Casher theorem (ACT) configuration is[a]

$$[\sigma_x(\partial_x - ieA_x) + \sigma_y(\partial_y - ieA_y)]\varphi = 0. \tag{A}$$

We introduce the transformation

$$\psi = e^{e\phi\sigma_z}\varphi. \tag{B}$$

This transformation (phase) permits us to eliminate the magnetic field explicitly from the Dirac equation where ϕ satisfies the relations

$$\partial_x\phi = A_y, \quad \partial_y\phi = -A_x,$$

and φ is eigenfunction of σ_z ($\sigma_z\varphi_s = s\varphi_s$). Having into account that $B(x,y) = \partial_x A_y - \partial_y A_x$ we arrive to

$$B(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi.$$

[a]We denote the fixed reference system as X, Y, Z and the coordinates in plane by x_1, x_2, x_3.

It is easy to see that (asymptotically) for $r \to \infty$ we have

$$\phi(x, y) = \frac{\Phi}{2\pi} \ln\left(\frac{r}{r_0}\right),$$

where

$$\Phi = \int B(x, y) dx dy$$

is the total magnetic flux through the (x, y)-plane, r_0 is some real constant. Consequently, we immediately obtain

$$\varphi_s = \left(\frac{r_0}{r}\right)^{\frac{\Phi s}{\Phi_0}} \psi_s(w),$$

where $w = x + isy$ and $\psi_s(w)$ is an entire function of w because after the elimination of the magnetic field from Eq. (A) it takes the simplest form

$$(\partial_x + is\partial_y)\psi_s(w) = 0.$$

In order that φ_s to be square integrable function we should consider $\Phi s > 0$ and ψ_s has to be a polynomial whose degree is not greater than $N - 1$, where $N = \{\Phi/\Phi_0\}$, obtaining N independent solutions for ψ_s : $1, w, w^2, \ldots, w^{N-1}$. Throughout this paper the same procedure as for the ACT configuration will be performed but in the case of "in plane" (parallel) magnetic field [2–5].

The plan of this paper is as follows. In Sec. 2 we obtain the conditions whether the magnetic field parallel to the charge transport can be "removed" as in the ACT. In Sec. 3 we obtain the conditions fulfilled by the solution: types of spinors and flux quantization. In Sec. 3, the origin and conditions whether the quantum Hall effect appears from the "in plane" magnetic field are explicitly shown. Magnetic field parallel and the quantum ring as an application (e.g. spin-filter) are the focus of Sec. 5. The remnant section is devoted to discuss, give some concluding remarks and perspectives.

2. Magnetic Field "In Plane"

Now the magnetic field B, in contrast to the ACT configuration described before, is parallel to the plane defined by x-, y-axes (usually denominated: "B in plane") where the particle lives. Explicitly the Dirac equation with the magnetic field parallel takes the following form:

$$[\sigma_B \partial_B + \sigma_\perp(\partial_\perp - ieA_\perp) - ie\sigma_z A_z]\varphi = 0. \tag{1}$$

Here, the subscripts B, $\perp$ and z denote the direction of the B field in the plane, the direction of the component of the potential vector in the plane (obviously, perpendicular to the B direction) and the direction of component of the potential vector coincident with the z-axis, respectively.

Defining ω the angle of the magnetic field with respect the x-axis in the plane x–y, the transformation (B) takes, in this case, the following general form:

$$\psi = e^{i(\alpha\sigma_x+\beta\sigma_y)}\varphi = e^{ie\phi\cdot\sigma_B}\varphi \tag{2}$$

with

$$\alpha = \lambda\cos\omega, \quad \beta = \lambda\sin\omega, \tag{3}$$

$$|\phi|^2 = \lambda^2(\cos^2\omega + \sin^2\omega) = \lambda^2 \Rightarrow |\phi| = \pm|\lambda|, \tag{4}$$

e.g. the projection of the field ϕ. Equation (1) explicitly written, having account of (2), as

$$[\sigma_x\partial_x + \sigma_y\partial_y - ieA_\perp(\sigma_x\sin^2\omega + \sigma_y\cos^2\omega) - ie\sigma_z A_z]\varphi = 0. \tag{5}$$

It is easily seen that, when $\omega = 0$, B coincides with x-axis and when $\omega = \pi/2$, B coincides with the y-axis. Also the Lie algebraic relation holds

$$\sigma_B\sigma_\perp = (\cos\omega\sigma_x + \sin\omega\sigma_y)(-\sin\omega\sigma_x + \cos\omega\sigma_y) = i\sigma_z \tag{6}$$

as expected.

Operating analogically as in the ACT configuration but having into account the new transformation and the physical situation, we obtain the conditions where the magnetic field can be eliminated. Precisely using expression (2) in (1) we obtain explicitly the following non-trivial conditions in order to remove the magnetic field

$$-\partial_\perp\phi = iA_z \quad \partial_B\phi = -A_\perp\sigma_\perp. \tag{7}$$

The first equation is precisely as in the ACT case but for the second one the interpretation is more involved and suggest, in principle, a complex structure for the field ϕ in a doublet form. Knowing that the doublet can be written as

$$\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{8}$$

the previous expressions take the following explicit form:

$$-\partial_\perp\phi_1 = -\partial_\perp\phi_2 = iA_z \quad \text{and} \quad \partial_B\phi_1 = -\partial_B\phi_2 = iA_\perp. \tag{9a}$$

Note that the above condition, in general, suggests the introduction of two real functions u and v as

$$\phi \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_1^*\end{pmatrix} = \begin{pmatrix}u(x_\perp) + iv(x_B)\\ (u(x_\perp) + iv(x_B))^*\end{pmatrix} \tag{9b}$$

in such a manner that the conditions to remove the magnetic field are automatically fulfilled as

$$-\partial_\perp\phi = iA_z \quad \text{and} \quad \partial_B\phi = A_\perp. \tag{10}$$

Equation (9b) is nothing more that the Majorana condition over φ that appear as consequence of the existence of a parallel magnetic field.

2.1. *Structure of the magnetic field: Conditions over A and ϕ*

The magnetic field that can be effectively generated ($B = \nabla \wedge A$) from the vector potential components of our problem, namely A_z and $A_\perp$.

The "in plane" magnetic field is consequently

$$B_B = (\partial_\perp A_z - \partial_z A_\perp), \tag{11}$$

where the simplest possibility was took: $A \neq A(x_B)$, e.g. the vector potential does not depend on the direction of the magnetic field, only on the plane defined by $x_\perp$ and x_z. From (10) we have

$$B = i\partial_\perp^2 \phi = \frac{\Phi}{x_\perp}, \tag{12}$$

where the total transversal flux to the plane per unit of longitude was used. Then, ϕ is immediately obtained

$$\phi \cdot \sigma_B = -i(\Phi\sigma_B)x_\perp \left[\ln\left|\frac{x_\perp}{l_0}\right| - \frac{C - x_\perp}{x_\perp}\right]. \tag{13}$$

Putting the arbitrary constant $C = 0$ for simplicity, the behavior of the exponential function in (2) for φ is determined

$$e^{-ie\phi\cdot\sigma_B} = \left|\frac{l_0}{x_\perp}\right|^{\frac{e\Phi}{l_0}\sigma_B x_\perp} e^{-\frac{e\Phi}{l_0}\sigma_B x_\perp} \tag{14}$$

with l_0 some real constant with units of length (its physical meaning will be analyzed later). As in the ACT case, the following condition must be fulfilled in order that φ be normalizable and square integrable

$$\Phi s_B \geq 0 \tag{15}$$

(s_B is the spin in the B-direction) due to

$$\varphi = e^{-ie\phi\cdot\sigma_B}\psi(s, w). \tag{16}$$

In the above expression, the function ψ depends on the spin and some complex variable w to determine from the simple Dirac–Weyl equation obtained after the procedure of explicit elimination of the magnetic field.

2.2. *Majorana, Dirac–Weyl states and discrete coordinates: Conditions over $\psi(s, z)$*

The simple Dirac–Weyl equation obtained after "removing the magnetic field" is

$$(e^{-ie\phi\cdot\sigma_B}\sigma_B\partial_B + e^{ie\phi\cdot\sigma_B}\sigma_\perp\partial_\perp)\psi(s, z) = 0. \tag{17}$$

To solve the equation, a quantization should be imposed on the flow (strictly on the product $\phi \cdot \sigma_B$). This fact will induce an automatic discretization over the "in plane" transverse coordinate $x_\perp$:

$$\phi \cdot \sigma_B = n\pi, \quad n = 0, 1, 2, \ldots. \tag{18}$$

If the above condition holds, we obtain

$$(\sigma_B \partial_B + \sigma_\perp \partial_\perp)\psi(s,z) = 0. \tag{19}$$

This expression is very important: this is a simple two-dimensional Dirac equation *without* A_μ. The particular phase introduced as an ansatz plus a quantization condition indicates that the effect of the magnetic field (due the potential vector) can be removed.

2.3. *Analysis of the solution*

Looking the specific form of the above equations, there are two possibilities over the spin behavior of ψ

(i) $\sigma_B \psi(s,z) = s\psi(s,z)$ (eigenspinor of σ_B).

This case is compatible with the assumption that the state is eigenvector of the spin in the magnetic field direction. The Dirac equation is reduced to

$$\left(\partial_B + \frac{i\mathbb{C}}{s}\partial_\perp\right)\psi(s,z) = 0 \tag{20}$$

with $\mathbb{C}$ the charge conjugation operator. Then, $\psi(s,z)$ and, for instance, $\varphi(s,z)$ must fulfill the Majorana condition:

$$\mathbb{C}\varphi(s,z) = \pm c\varphi(s,z). \tag{21}$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + \frac{ic}{s}x_\perp$ but the states solution is of Majorana type.

(ii) $\sigma_z \psi(s,z) = s\psi(s,z)$ (eigenspinor of σ_Z).

In this case the spin remains as in the ACT situation (e.g. in the z-direction). Now the Dirac equation is reduced to

$$(\partial_B + is\partial_\perp)\psi(s,z) = 0. \tag{22}$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + isx_\perp$, and the state solution is Dirac–Weyl.

The specific form of Eq. (20) shows that the result is not causality: the states are Majorana. The inclusion of the charge conjugation operator $\mathbb{C}$ due to the symmetry of the physical scenario, enforces obviously, the Majorana condition over the states solution.

3. Quantum Hall Effect and the "In Plane" Magnetic Field

It is not difficult to see that, if the plane where the charges are moving is finite an "in plane" current transversal to the magnetic field B must appear (e.g. in the $x_\perp$-direction). This current will be quantized due to the condition (18). This condition explicitly can be written

$$\phi \cdot \sigma_B = (\Phi\sigma_z)\widetilde{x}_\perp \left[\ln\left|\frac{x_\perp}{l_0}\right| - 1\right] = n\pi, \quad n = 0,1,2,\ldots, \tag{23}$$

where $\widetilde{x}_\perp = \sigma_\perp x_\perp$ is a new matrix-valued coordinate that its meaning will be analyzed later.

The explicit formula for the Hall current is coming from the expression for the surface current

$$n \times B = K_{\text{surface}}. \tag{24}$$

(Here n is unitary vector normal to the interface surface.) This current is obviously perpendicular to the "in plane" magnetic field (e.g. $x_\perp$-direction). Due to the existence of a quantization condition, the "emergent" Hall current is also quantized leading the QHE

$$\frac{\Phi}{x_\perp} x_\perp^v = \frac{2\pi N \hbar c}{e x_\perp} x_\perp^v = K_{\text{surface}}, \tag{25}$$

where $x_\perp^\nu$ is the unimodular vector in the $x_\perp$-direction.

3.1. *Generalized momentum operators and Majorana conditions*

The interpretation of the non-standard Dirac equation

$$[\sigma_B \partial_B + \sigma_\perp(\partial_\perp - ieA_\perp) - ie\sigma_z A_z]\varphi = 0 \tag{26}$$

can be elucidated as follows:

$$\Bigg[\sigma_B \underbrace{(\partial_B - ie\sigma_B\sigma_z A_z)}_{\widetilde{\Pi}_B} + \sigma_\perp \underbrace{(\partial_\perp - ieA_\perp)}_{\Pi_\perp}\Bigg]\varphi = 0 \Rightarrow [\sigma_B \widetilde{\Pi}_B + \sigma_\perp \Pi_\perp]\varphi = 0. \tag{27}$$

Then, the question that immediately appears is: who is the operator $\widetilde{\Pi}_B$? The answer is obvious, using the algebra (6) $\sigma_B\sigma_z = i\sigma_\perp$ and the definition of the charge conjugation operator as function of the sigma matrices. It is easy to see that

$$(\partial_B - ie\sigma_B\sigma_z A_z) = (\partial_B + ie\mathbb{C}A_z). \tag{28}$$

As in ordinary non-abelian gauge theories, the operator $\widetilde{\Pi}_B$ seems as equipped with a *non-abelian* vector potential $\widetilde{A_B} \equiv -\mathbb{C}A_z$. This conceptual interpretation will be utilized in the next section for the analysis of the quantum ring.

4. Magnetic Field Parallel and the Quantum Ring

As recently pointed out [6], the Rashba and Dresselhaus spin–orbit interactions in two dimensions can be regarded as a non-abelian gauge field reminiscent of the standard Yang–Mills in QFT [17]. The explanation given in such references is that the physical field generated by the gauge field brings to the electron wave function a spin-dependent phase. This phase generally is called the Aharonov–Casher phase. In [6] the authors showed that applying on an AB ring this non-abelian field, together with the usual vector potential, is certainly possible make the interference condition completely destructive for one component of the spin while completely constructive for the other component of the spin over the entire energy range. This enables us to construct a perfect spin filter. However in [6] the magnetic field was *perpendicular* to the plane of the ring. Now we will proceed analogously but considering the *in plane* magnetic field to see the physical consequences over the physical states and over the spin control. In order to perform the analysis and to compare with the case described in [6], the same method and definitions will be used it remiting to the reader to [6] for more details.

The general Dirac equation with the magnetic field parallel ("in plane") in cylindrical coordinates takes the form

$$\left[\sigma_\rho\partial_\rho + \frac{1}{\rho}\sigma_\varphi\partial_\varphi - ie\underbrace{(-\sigma_\rho A_\rho \sin(\omega-\varphi) + \sigma_\varphi A_\varphi \cos(\omega-\varphi))}_{\propto\sigma_\perp A_\perp} - ie\sigma_z A_z\right]\widehat{\varphi} \tag{29}$$

$$= \left[\sigma_\rho\partial_\rho + \frac{1}{\rho}\sigma_\varphi\left(\partial_\varphi - ie\sigma_\rho\underbrace{\widetilde{A_z}}_{2iA_z}\right) - ie(-\sigma_\rho A_\rho \sin(\omega-\varphi)\right.$$
$$\left. + \sigma_\varphi A_\varphi\cos(\omega-\varphi))\right]\widehat{\varphi} = 0, \tag{30}$$

where in the last equation the properties of the algebra described in the previous paragraph have been used in order to introduce the "non-abelian" potential. Notice that in our case it is not only a trick in sharp contrast with other references in the literature. The explicit forms of the Pauli matrices in the configuration that we are interested in are

$$\sigma_\rho = \sigma_x\cos\varphi + \sigma_y\sin\varphi, \tag{31}$$
$$\sigma_\varphi = \sigma_y\cos\varphi - \sigma_x\sin\varphi, \tag{32}$$
$$\sigma_B = \sigma_\rho\cos(\omega-\varphi) + \sigma_\varphi\sin(\omega-\varphi), \tag{33}$$
$$\sigma_\perp = \sigma_\varphi\cos(\omega-\varphi) - \sigma_\rho\sin(\omega-\varphi). \tag{34}$$

However, φ is the angular cylindrical coordinate, ω is the angle of the magnetic field parallel to the plane measured from the axis x (e.g. $\varphi = 0$) and the state is denoted as $\widehat{\varphi}$.

For the ring configuration, and having account the condition that the potential does not depend on the direction of the magnetic field, the Dirac equation takes this non-abelian form

$$\left[\frac{1}{\rho}\sigma_\varphi\left(\partial_\varphi - ie\sigma_\rho\widetilde{A_z}\right)\right]\widehat{\varphi} = 0. \tag{35}$$

Then, the corresponding second-order equation suggests the Hamiltonian for the magnetic field "in plane" as

$$\mathcal{H}_{\text{ring}} = \frac{1}{\rho^2}(\partial_\varphi - ie\sigma_\rho\widetilde{A_z})^2. \tag{36}$$

Note that the non-abelian character of the above equation was described in the previous paragraph.

4.1. *Screening of Rashba term and "in plane" magnetic field*

When the Rashba spin–orbit interaction is introduced, the following Hamiltonian is obtained

$$\mathcal{H}_{\text{ring}}|_{B_{n\text{-plane}}} = \frac{\hbar^2}{2m^* R^2}\left(-i\partial_\varphi - \sigma_\rho\left(\underbrace{\widetilde{A_z}}_{\text{potential}} + \underbrace{\frac{\theta R}{2}}_{\text{Rashba}}\right)\right)^2, \tag{37}$$

where $\theta \equiv \frac{2m^*\alpha}{\hbar}$ plays the role of coupling constant of the Rashba term (the same units as in [6]). The main point is that the vector potential corresponding to the "in plane" magnetic field (perpendicular to the plane of the ring) is at the same *non-abelian* level of the Rashba term.

In the case treated by the authors in [6], the magnetic field is perpendicular having the Hamiltonian for the ring in the following fashion

$$\mathcal{H}_{\text{ring}}|_{B_z} = \frac{\hbar^2}{2m^* R^2}\left(-i\partial_\varphi - \underbrace{\phi_B}_{\frac{e\pi R^2 B_z}{h}} - \left(\sigma_\rho \underbrace{\frac{\theta R}{2}}_{\text{Rashba}}\right)\right)^2, \tag{38}$$

where it is easily seen that ϕ_B is not at the same "non-abelian" level of the Rashba term. Consequently, this is the explanation of the screening of the Rashba interaction by the "in plane" magnetic field.

4.2. *Physical consequences and effects*

We know from Sec. 2 that we can select solutions that are eigenfunctions of σ_z. Besides this issue, the potential vector A_z must come perpendicularly to the ring plane ($\widehat{z}$ direction) in accordance with the ACT situation. Notice that, in sharp contrast with previous references, *the effective Hamiltonian arises from the true Dirac equation with minimal coupling.* As we can assume in general that we know the magnetic field (B_{pl}) in the plane: $e\widetilde{A_z} \equiv 2ieA_z = 2ieB_{pl}R/h$, then,

$$\mathcal{H}_{\text{ring}}|_{B_{n\text{-plane}}} = \frac{\hbar^2}{2m^* R^2}\left[-i\partial_\varphi - \sigma_\rho\left(\frac{(\theta - 4eB_{\text{pl}})\,R}{2}\right)\right]^2, \tag{39}$$

the interplay between the magnetic field parallel to the plane of the ring and the Rashba interaction is clearly seen.

Assuming free interaction into the two leads, in response to

$$\mathcal{H}_{\text{lead}} = -\frac{\hbar^2}{2m^*}\partial_x^2, \tag{40}$$

we obtain (same units and notation that in [6]) for the ring Hamiltonian the wave functions

$$\Psi_{\pm\pm} = e^{i(\pm k_\varphi \pm \phi_T)\varphi} e^{-i\beta\sigma_\varphi/2}\chi_\pm \tag{41}$$

($\chi_\pm$ eigenfunctions of σ_z) with the eigenvalues

$$E = \frac{\hbar^2 k_\varphi^2}{2m^* \rho^2}, \tag{42}$$

where now the total phase is

$$\phi_T = \sqrt{1 + (\theta - 4eB_{\mathrm{pl}})^2 \rho^2} - 1 \tag{43}$$

and

$$\beta = \arctan \xi, \tag{44}$$

with

$$(\theta - 4eB_{\mathrm{pl}}) \equiv \xi. \tag{45}$$

Notice the important fact that the total phase ϕ_T is *identically zero* if the following condition holds

$$\theta = 4eB_{\mathrm{pl}}. \tag{46}$$

As in [6] the first sign of $\Psi_{\pm\pm}$ denotes the sign of the momentum, and the second one denotes the spin. In the phase corresponding to Rashba interaction, a small radius ρ of the ring was considered. Following similar task that in [6] in order to realize a perfect spin filter, the wave function (41) at $\varphi = 2\pi$ is

$$\Psi_{\pm\pm}(2\pi, k_\varphi) = e^{\pm 2i\pi k_\varphi} U_{\mathrm{phase}} \chi_\pm; \quad U_{\mathrm{phase}} = e^{\pm 2\pi i \phi_T} e^{-i\beta\sigma_y/2} \tag{47}$$

realizing the spin filter by adjusting parameters:

$$2\pi\phi_T = (2n+1)\pi.$$

At this point, two important cases must be considered:

Case (a). Considering (43) and small radius ρ, the above condition is translated to

$$\begin{aligned} \xi\rho &= \sqrt{n + 3/2}, \quad n \in \mathbb{Z} \\ &= \sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}, \sqrt{\frac{7}{2}}, \ldots . \end{aligned} \tag{48}$$

Case (b). Case (a) must be complemented with a condition that only appears as an effect of the existence of the magnetic field "in plane", that is

$$\xi = 0 \to \theta = 4eB_{\mathrm{pl}}.$$

Both conditions realize the perfect spin filter being the second condition possible *only* in the case when the magnetic field is "in plane", and it importance will be more evident in the coefficient transmission description, as follows.

The eigenvectors of the phase factor $U_{\mathrm{phase}} = e^{\pm 2\pi i \phi_T} e^{-i\beta\sigma_y/2}$ can be exactly computed,

$$\widetilde{\chi}_+ = \begin{pmatrix} \dfrac{\sqrt{\xi^2+1}+1}{2} \\ \xi/2 \end{pmatrix}, \tag{49}$$

$$\widetilde{\chi}_{-} = \begin{pmatrix} \xi/2 \\ -\frac{\sqrt{\xi^2+1}+1}{2} \end{pmatrix}. \tag{50}$$

Notice that when the critical value $\theta = 4eB_{\rm pl}$ holds, then $\xi = 0$ consequently the eigenvectors $\widetilde{\chi}_{\pm}$ goes automatically to $\chi_{\pm}$ (eigenvectors of σ_z) as expected in sharp contrast with similar results in [6] that the correctness is doubtful. Although there are several effective manners to compute the transmission coefficients, we follow [6] in order to compare the results with other works involving the similar devices. We first assume the amplitudes of the left-going and right-going wave functions separately for the left lead, the portion $0 < \varphi < \pi$ of the ring, the portion $\pi < \varphi < 2\pi$ of the ring, and the right lead. This amounts to 16 amplitudes in total when we take the spin degree of freedom into account. The continuation of the wave function at $\varphi = 0$ and $\varphi = \pi$ will give eight conditions and the conservation of the generalized momentum at $\varphi = 0$ and $\varphi = 0$ will give four conditions. Then, four degrees of freedom finally remain. The S-matrix of the quantum ring is obtained by expressing the four amplitudes of the outgoing waves (the left-going wave on the left lead and the right-going wave on the right lead with spin up and down) in terms of the four amplitudes of the incoming waves (the right-going wave on the left lead and the left-going wave on the right lead with spin up and down). The off-diagonal 2×2 blocks of the 4×4 S-matrix give the transmission coefficients. In our case, the transmission coefficients are proportional to

$$T_{\widetilde{\uparrow\uparrow}}, T_{\widetilde{\uparrow\downarrow}} \propto |1 + e^{2\pi i\phi_T}|^2, \tag{51}$$

$$T_{\widetilde{\downarrow\uparrow}}, T_{\widetilde{\downarrow\downarrow}} \propto |1 + e^{-2\pi i\phi_T}|^2, \tag{52}$$

where $\widetilde{\uparrow}$ and $\widetilde{\downarrow}$ denote respectively the spin up (49) and spin down (50) diagonalizing the phase factor $U_{\rm phase} = e^{\pm 2\pi i\phi_T} e^{-i\beta\sigma_y/2}$.

Summarizing, in the case (a) evidently $T_{\widetilde{\uparrow\uparrow}}, T_{\widetilde{\uparrow\downarrow}}, T_{\widetilde{\downarrow\uparrow}}, T_{\widetilde{\downarrow\downarrow}} = 0$, and in the case (b) the transmission coefficients are constant, realizing together the perfect spin filter [6, 7].

5. Concluding Remarks and Outlook

In this paper the two-dimensional charge transport with parallel (in plane) magnetic field was considered. The starting point was reminiscent as the described in the Aharonov and Casher theorems but with the magnetic field parallel to the plane of charge transport. In this first paper, several important results were found of which we can conclude enumerating the following issues:

(i) the specific forms of the arising equation enforce the respective field solution to fulfill the Majorana condition;

(ii) when any physical system is represented by this equation the rise of fields with Majorana type behavior is immediately explained and predicted;

(iii) there exists a quantized particular phase that removes the action of the vector potential and this produces interesting effects being the quantum Hall effect (QHE) one of them that is straightforwardly explained;
(iv) the interpretation of Dirac equation with a non-abelian electromagnetic field appears in consequence of the conditions imposed on the fields, not as an added "by hand".

Also the quantum ring was treated as example where these new effects must appear.

As was mentioned in some references, the term non-abelian was introduced "by hand" in order to reproduce the effects of the interaction of type RD. In our case, the "non-abelian" term appears due to the presence of the field parallel to the transport plane of the charges. Therefore, and as we saw in the problem of the ring in Sec. 4, there is competition or screening between the effects produced by the interaction RD and from the parallel magnetic field. This competition brings two important consequences, namely:

(1) new spin filter effects (different in essence to [6]), as we have analyzed from Sec. 4,
(2) measurable effects of screening that could give a clear explanation of the new effects observed in planar nanostructures described in [14].

In the second part of this paper, the relationship between the hidden symmetries of the particular physical systems described by this equation and the (nanostructures, composite particle states etc.) will be discussed and elucidated with a clear explanation about Majorana, zero modes and supersymmetry.

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