

Scalar-Tensor Theories and Asymmetric Resonant Cavities

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Abstract—The recently published experimental results indicate the appearance of unusual forces on asymmetric, electromagnetic resonant cavities. It is argued here that a particular class of scalar-tensor theories of gravity could account for this effect.

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1. INTRODUCTION

Very recently, experimental results were published [1] on the measurement of forces on closed, asymmetric electromagnetic resonant cavities. These results add to previous claims in the same line by an independent researcher who did the first experiments [2]. Such claims were criticized by the scientific community mainly due to the proposed theoretical explanation since the Maxwell equations and special relativity (SR) clearly indicate that no force is possible without emission of radiation from the cavity. On the other hand, it appears that general relativity might allow for such kind of reactionless propulsion, as exemplified and noted for the first time in [3], where the low velocity limit of some warp drive spacetimes was analyzed. As indicated there, negative energy densities are required to accomplish that and, notably, some scalar fields present this possibility [4]. Of course, in order to have measurable effects similar to those reported, the coupling of the scalar field to matter or other fields acting as its source should be sufficiently strong, and this is precisely what has been proposed in [5] for the case of the electromagnetic field as a source of the scalar field to explain discordant measurements of Newton's gravitational constant. It is then only natural to wonder whether that theory (or a similar one) may account for the forces reported in resonant cavities. Of course, all this is highly speculative, and more prosaic explanations of these forces should be considered first. We proceed on the assumption that all spurious effects were accounted for, and on the belief that the possibility presented here is worth exploring.

The theory put forward by Mbelek and Lachièze-Rey in [5] (see also [6]) represents a reduction to four dimensions of a Kaluza-Klein theory coupled to an external scalar ψ , which in turn couples to matter. It is the source term of ψ which allows for a possible

strong coupling of the Kaluza-Klein scalar ϕ to other fields, in particular, to the electromagnetic field. The theory was applied in cosmological [7] and galactic situations [8] and, as mentioned, it was also used to investigate the possibility of the Earth's magnetic field influencing the measurements of Newton's gravitational constant. In all these applications of the theory only its weak-field limit was used, and since this limit is similar for a wide range of theories, we employ in this work rather a general scalar-tensor theory which incorporates an additional external scalar ψ .

In the next sections, the equations of the mentioned scalar-tensor theory are derived from its proposed action, along with the equation of motion of neutral matter. Some axisymmetric electromagnetic modes of a truncated conical cavity are then presented and used as a source in the weak-field approximation of the equations, previously obtained, to determine the force on the cavity. It is found that a coupling of the same magnitude as used in [5] between the scalar ϕ and the electromagnetic field results in a correct magnitude and sign of the forces reported in asymmetric resonant cavities. As expected, the solution for the cavity presents negative energy densities (more precisely, it violates the weak energy condition [9]). The theory, however, does not seem to be completely satisfactory because in its linearized version it also predicts strong gravitational effects by the Earth's magnetic field, which are clearly not observed. A possible resolution of this problem is considered in the last section.

2. SCALAR-TENSOR THEORY

We will consider a scalar-tensor theory of the Brans-Dicke type [10] with inclusion of Bekenstein's direct interaction of scalar and Maxwell fields [11],

and with an additional external scalar field ψ minimally coupled to gravity and universally coupled to matter, with the action given by (SI units are used)

$$\begin{aligned}
S = & \frac{c^3}{16\pi G_0} \int \sqrt{-g} d\Omega \left[-\phi R + \frac{\omega(\phi)}{\phi} \nabla^\nu \phi \nabla_\nu \phi \right] \\
& + \frac{c^3}{16\pi G_0} \int \sqrt{-g} \phi \left[\frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U(\psi) - J\psi \right] d\Omega \\
& - \frac{\varepsilon_0 c}{4} \int \sqrt{-g} \lambda(\phi) F_{\mu\nu} F^{\mu\nu} d\Omega - \frac{1}{c} \int \sqrt{-g} j^\nu A_\nu d\Omega \\
& + \frac{1}{c} \int \mathcal{L}_{\text{mat}}[\exp(\beta\psi) g_{\mu\nu}] d\Omega. \quad (1)
\end{aligned}$$

To have a dimensionless scalar field ϕ of values around unity, in the expression (1) the constant G_0 , representing Newton's gravitational constant, is included, c is the velocity of light in vacuum, and ε_0 is the vacuum permittivity. \mathcal{L}_{mat} is the lagrangian density of matter, which is assumed to couple to the scalar ψ . The other symbols are also conventional, R is the Ricci scalar, and g the determinant of the metric tensor $g_{\mu\nu}$. The Brans-Dicke parameter $\omega(\phi)$ is considered as a function of ϕ , as it usually results so in the 4D reduction of multidimensional theories [12]. The function $\lambda(\phi)$ in the electromagnetic field action is of the type appearing in Bekenstein's theory and other effective theories [7], it does not intervene in the weak field approximation ultimately employed, but is included for completeness. The electromagnetic tensor is $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, given in terms of the electromagnetic 4-vector A_ν , with sources given by the 4-current j^ν . U and J are the potential and source of the field ψ , respectively. The source J contains contributions from matter, the electromagnetic field and the scalar ϕ .

To build a concrete model, we follow the proposal of [5] (convenient dimensional factors differing from those in [5] are employed here),

$$\begin{aligned}
J = & \frac{8\pi G_0}{c^4} \left[\beta_{\text{mat}}(\psi, \phi) T^{\text{mat}} \right. \\
& \left. + \beta_{EM}(\psi, \phi) c^2 \varepsilon_0 F_{\mu\nu} F^{\mu\nu} \right] + \beta_\phi(\psi, \phi) T^\phi, \quad (2)
\end{aligned}$$

where T^{mat} is the trace of the energy-momentum tensor of matter (note that this tensor is defined with respect to $g_{\mu\nu}$, not $\exp(\beta\psi)g_{\mu\nu}$),

$$T_{\mu\nu}^{\text{mat}} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{\mu\nu}},$$

and T^ϕ is the trace of the tensor

$$\begin{aligned}
T_{\mu\nu}^\phi = & \nabla_\mu \nabla_\nu \phi - \nabla^\gamma \nabla_\gamma \phi g_{\mu\nu} \\
& + \frac{\omega(\phi)}{\phi} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla^\gamma \phi \nabla_\gamma \phi g_{\mu\nu} \right).
\end{aligned}$$

Variation of (1) with respect to $g^{\mu\nu}$ results in ($T_{\mu\nu}^{EM}$ is the usual electromagnetic energy tensor)

$$\begin{aligned}
\phi \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = & \frac{8\pi G_0}{c^4} \left[\lambda(\phi) T_{\mu\nu}^{EM} + T_{\mu\nu}^{\text{mat}} \right] \\
& + T_{\mu\nu}^\phi + \frac{\phi}{2} \left(\nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} \nabla^\gamma \psi \nabla_\gamma \psi g_{\mu\nu} \right) \\
& + \frac{\phi}{2} (U + J\psi) g_{\mu\nu}. \quad (3)
\end{aligned}$$

Variation with respect to ϕ gives

$$\begin{aligned}
\phi R + 2\omega \nabla^\nu \nabla_\nu \phi = & \left(\frac{\omega}{\phi} - \frac{d\omega}{d\phi} \right) \nabla^\nu \phi \nabla_\nu \phi \\
& - \frac{4\pi G_0 \varepsilon_0}{c^2} \phi \frac{d\lambda}{d\phi} F_{\mu\nu} F^{\mu\nu} - \frac{\partial J}{\partial \phi} \psi \phi \\
& + \phi \left[\frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U(\psi) - J\psi \right],
\end{aligned}$$

which can be rewritten, using the contraction of (3) with $g^{\mu\nu}$ to replace R , as

$$\begin{aligned}
(2\omega + 3) \nabla^\nu \nabla_\nu \phi = & -\frac{d\omega}{d\phi} \nabla^\nu \phi \nabla_\nu \phi \\
& - \frac{4\pi G_0 \varepsilon_0}{c^2} \phi \frac{d\lambda}{d\phi} F_{\mu\nu} F^{\mu\nu} + \frac{8\pi G_0}{c^4} T^{\text{mat}} \\
& + \phi \left[\frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U(\psi) - J\psi \right] - \frac{\partial J}{\partial \phi} \psi \phi, \quad (4)
\end{aligned}$$

where we used that $T^{EM} = T_{\mu\nu}^{EM} g^{\mu\nu} = 0$.

The inhomogeneous Maxwell equations are obtained by varying (1) with respect to A_ν :

$$\nabla_\mu [\lambda(\phi) F^{\mu\nu}] = \mu_0 j^\nu \quad (5)$$

with μ_0 the vacuum permeability.

Variation with respect to ψ results in

$$\begin{aligned}
\nabla^\nu \nabla_\nu \psi + \frac{1}{\phi} \nabla^\nu \psi \nabla_\nu \phi = & -\frac{\partial U}{\partial \psi} - J \\
& - \frac{\partial J}{\partial \psi} \psi + \frac{\beta}{\phi} \frac{8\pi G_0}{c^4} T^{\text{mat}}. \quad (6)
\end{aligned}$$

Having included G_0 , it is understood that ϕ takes values around its vacuum expectation value (VEV) $\phi_0 = 1$. The scalar ψ is also dimensionless and has a VEV ψ_0 .

Finally, we consider the motion of neutral test particles coupled to the scalar ψ as indicated in (1), which is then obtained requiring that

$$\delta \int mc \sqrt{\exp(\beta\psi) g_{\mu\nu} dx^\mu dx^\nu} = 0,$$

to give

$$\frac{Du^\gamma}{Ds} = \frac{\beta}{2} (g^{\gamma\nu} - u^\gamma u^\nu) \partial_\nu \psi. \quad (7)$$

It is important to mention that, in order to derive the previous equations, we have followed the prescription of not varying the trace of the energy-momentum tensors nor $F_{\mu\nu}F^{\mu\nu}$ in the source term (2), but only its coefficients β 's, as done in [5]. There is no clear reason for doing so, but on the one hand, inconsistent equations result if the mentioned variations are included. On the other hand, the exact source term may not depend explicitly on the tensors considered, and only after the equations are derived and substitutions made might it be expressible in terms of the said tensors.

3. WEAK-FIELD APPROXIMATION

In the weak field approximation, for values of $g_{\mu\nu}$ around $\eta_{\mu\nu}$ taken as those of flat Minkowski space with signature (1, -1, -1, -1), so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have

$$R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} = \frac{1}{2} \left(-\eta^{\gamma\delta}\partial_{\gamma\delta}\bar{h}_{\mu\nu} + \partial_{\gamma\mu}\bar{h}_{\nu}^{\gamma} + \partial_{\gamma\nu}\bar{h}_{\mu}^{\gamma} - \eta_{\mu\nu}\partial_{\gamma\delta}\bar{h}^{\gamma\delta} \right),$$

with

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu},$$

where

$$h \equiv \eta^{\gamma\delta}h_{\gamma\delta} = -\eta^{\gamma\delta}\bar{h}_{\gamma\delta}.$$

The system (3)–(5) can then be written, to lowest order in $\bar{h}_{\mu\nu}$ and in the perturbations around the VEV's of ϕ and ψ , as

$$-\eta^{\gamma\delta}\partial_{\gamma\delta}\bar{h}_{\mu\nu} = \frac{16\pi G_0}{c^4}T_{\mu\nu}^{\text{mat}} + 2(\partial_{\mu\nu}\phi - \eta^{\gamma\delta}\partial_{\gamma\delta}\phi\eta_{\mu\nu}), \quad (8)$$

with the Lorentz gauge

$$\partial_{\gamma}\bar{h}_{\nu}^{\gamma} = 0, \quad (9)$$

$$(2\omega_0+3)\eta^{\gamma\delta}\partial_{\gamma\delta}\phi = \frac{8\pi G_0}{c^4}T^{\text{mat}} - \left. \frac{\partial J}{\partial\phi} \right|_{\phi_0, \psi_0} \psi_0, \quad (10)$$

$$\partial_{\mu}F^{\mu\nu} = \mu_0 j^{\nu}, \quad (11)$$

$$\eta^{\gamma\delta}\partial_{\gamma\delta}\psi = \beta \frac{8\pi G_0}{c^4}T^{\text{mat}} - \left. \frac{\partial J}{\partial\psi} \right|_{\phi_0, \psi_0} \psi_0, \quad (12)$$

where $\omega_0 = \omega(\phi_0)$. It was used in these equations that, in order to recover the usual physics when the scalar fields are not excited, one must have $\lambda(\phi_0) = 1$, $U(\psi_0) = J(\psi_0, \phi_0) = 0$. Also, since according to [5] the contribution from the energy-momentum tensor of the electromagnetic field through the source J is much larger than all its other contributions, the latter were neglected in the above equations.

For slowly moving neutral masses, Eq. (7) corresponds to the action of a specific force (per unit mass) (Latin indices correspond to the spatial coordinates)

$$f_i = -\frac{c^2}{4} \frac{\partial}{\partial x_i} (\bar{h}_{00} + \bar{h}_{kk} + 2\beta\psi) + c \frac{\partial \bar{h}_{0i}}{\partial t}. \quad (13)$$

Introducing the D'Alembertian operator

$$\square = \eta^{\gamma\delta}\partial_{\gamma\delta} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2,$$

applying it to the force equation (13), and using Eq. (8), one easily obtains:

$$\square f_i = \frac{4\pi G_0}{c^2} \frac{\partial}{\partial x_i} (T_{00}^{\text{mat}} + T_{kk}^{\text{mat}}) - \frac{16\pi G_0}{c^3} \frac{\partial T_{0i}^{\text{mat}}}{\partial t} + \frac{c^2}{2} \frac{\partial}{\partial x_i} \left(\square\phi - \beta\square\psi - \frac{2}{c^2} \frac{\partial^2\phi}{\partial t^2} \right).$$

From Eqs. (10) and (12), and retaining only the most important component T_{00}^{mat} , this expression can be conveniently recast as

$$f_i = -\frac{\partial\chi}{\partial x_i},$$

where the ‘‘gravitational potential’’ χ satisfies

$$\square\chi = -\frac{4\pi G_0}{c^2}T_{00}^{\text{mat}} + \frac{4\pi G_0}{c^2} \left(\beta^2 - \frac{1}{2\omega_0+3} \right) T_{00}^{\text{mat}} + \frac{\partial^2\phi}{\partial t^2} + \frac{c^2\psi_0}{2} \left(\frac{1}{2\omega_0+3} \frac{\partial J}{\partial\phi} - \beta \frac{\partial J}{\partial\psi} \right)_{\phi_0, \psi_0}. \quad (14)$$

Making explicit the matter contribution to χ , using the expression (2), one has

$$\square\chi = -\frac{4\pi G_0}{c^2}T_{00}^{\text{mat}} + \frac{4\pi G_0}{c^2} \left[\beta \left(\beta - \psi_0 \frac{\partial\beta_{\text{mat}}}{\partial\psi} \right) + \frac{1}{2\omega_0+3} \left(\psi_0 \frac{\partial\beta_{\text{mat}}}{\partial\phi} - 1 \right) \right]_{\phi_0, \psi_0} T_{00}^{\text{mat}} + \dots,$$

where the dots represent non-matter terms. The first term corresponds to Newtonian gravity, while the second term, if one takes $\beta = 0$, corresponds to the matter contribution through the scalar ϕ , which is constrained by Solar System tests, requiring large values of ω_0 . An interesting conclusion (not to be explored further here) is that the inclusion of the external scalar ψ could thus allow $\omega_0 \sim 1$ if β is small enough (or, alternatively, if $\beta \simeq \psi_0 \partial\beta_{\text{mat}}/\partial\psi$), and

$$\psi_0 \frac{\partial\beta_{\text{mat}}}{\partial\phi} \Big|_{\phi_0, \psi_0} \simeq 1. \quad (15)$$

Note that the condition (15) does not invalidate the conclusions in [8] since only the term depending on the matter velocity of the ‘‘force’’ in the r.h.s. of Eq. (7) is used therein to explain the dynamics of rotating spiral galaxies.

Making now explicit the equation of the scalar ϕ , Eq. (10), with the expression of the source J , Eq. (2), one has

$$\square\phi = \frac{8\pi G_0}{(2\omega_0 + 3)c^4} \left(1 - \psi_0 \frac{\partial\beta_{\text{mat}}}{\partial\phi} \Big|_{\phi_0, \psi_0} \right) T^{\text{mat}} - \frac{8\pi G_0 \varepsilon_0}{(2\omega_0 + 3)c^2} \psi_0 \frac{\partial\beta_{EM}}{\partial\phi} \Big|_{\phi_0, \psi_0} (B^2 - E^2/c^2), \quad (16)$$

where it was used that, in terms of the modulus of the electric and magnetic vector fields, E and B , respectively, one has

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2/c^2),$$

and where the contribution from ϕ itself as its source was not considered because, even if it is present, in the weak-field approximation one has

$$T^\phi = -3\square\phi,$$

and so its effect amounts to a redefinition of the rest of the coefficients in the equations for ϕ and ψ .

A point worth noting is that the condition (15) refers so far to the motion of massive bodies, while the most stringent bounds on ω_0 come from propagation of electromagnetic waves near the Sun [13], not affected by the coupling of ψ to matter. According to the expression (16), these bounds can also be accommodated, always with $\omega_0 \sim 1$, if the same condition (15) holds.

With all this, the contributions other than the matter one to the potential χ can then be obtained from (14) as (we write $\chi = \chi_{\text{mat}} + \chi'$)

$$\square\chi' = \frac{\partial^2\phi}{\partial t^2} + 4\pi G_0 \varepsilon_0 \psi_0 \left(\frac{1}{2\omega_0 + 3} \frac{\partial\beta_{EM}}{\partial\phi} - \beta \frac{\partial\beta_{EM}}{\partial\psi} \Big|_{\phi_0, \psi_0} \right) (B^2 - E^2/c^2). \quad (17)$$

In [5] it is argued that in order to explain the discordant measurements of $G = G_0/\phi$ as due to the ϕ generated by the Earth's magnetic field according to (16), one must have

$$\frac{8\pi G_0 \varepsilon_0}{(2\omega_0 + 3)c^2} \psi_0 \frac{\partial\beta_{EM}}{\partial\phi} \Big|_{\phi_0, \psi_0} = -(5.4 \pm 0.6) \times 10^{-8} \frac{A^2}{N^2}, \quad (18)$$

while the value of $\beta\partial J/\partial\psi \Big|_{\phi_0, \psi_0}$ does not enter into the equation of ϕ and is thus left unspecified. In what follows we will evaluate the force predicted by (17) for the resonant electromagnetic field in a conical cavity, assuming that the coefficient in the brackets in (17) can be estimated from the value (18) alone.

4. NORMAL MODES IN A CONICAL CAVITY

As done in [14], we consider a conical cavity with side walls corresponding to a truncated cone, with spherical sections as end caps. The cone axis is taken as the z direction, the lateral wall corresponds to the spherical angle $\theta = \theta_0$ (half angle of the cone), and the spherical caps to the radii $r = r_{1,2}$, with $r_2 > r_1$.

The resonant modes correspond to standing electromagnetic waves satisfying the vector wave equation (\mathbf{F} stands for either the electric field \mathbf{E} or to the magnetic induction \mathbf{B})

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} - \nabla^2 \mathbf{F} = 0.$$

The modes with rotational symmetry and \mathbf{B} transverse to the z direction \mathbf{e}_z (called the TM modes) that satisfy this equation are (spherical coordinates are employed, with unit vectors \mathbf{e}_r , \mathbf{e}_θ and \mathbf{e}_φ) (see [14] and references therein for details)

$$\mathbf{B} = -CkR(r)Q'(\theta) \cos(\omega t)\mathbf{e}_\varphi, \quad (19)$$

$$\frac{\mathbf{E}}{c} = C \left\{ \frac{R(r)}{r} n(n+1)Q(\theta)\mathbf{e}_r + \left[\frac{R(r)}{r} + R'(r) \right] Q'(\theta)\mathbf{e}_\theta \right\} \sin(\omega t), \quad (20)$$

where C is a global constant. The functions R and Q are defined as

$$Q(\theta) = P_n(\cos\theta), \\ R(r) = R_+(r) \cos\alpha + R_-(r) \sin\alpha, \\ R_\pm(r) = \frac{J_{\pm(n+1/2)}(kr)}{\sqrt{r}},$$

where P_n is the Legendre polynomial of order n , J_m the Bessel function of the first kind of order m , and α and k constants to be determined along with the order n . By construction, the magnetic field satisfies the boundary condition of zero normal component on the metallic walls, while in order to have zero tangential components of the electric field on the walls, the order n of the Legendre polynomial must satisfy

$$P_n(\cos\theta_0) = 0,$$

the wavenumber k should obey the condition

$$\left[\frac{R_+}{r} + R'_+ \right]_{r_2} \left[\frac{R_-}{r} + R'_- \right]_{r_1} = \left[\frac{R_+}{r} + R'_+ \right]_{r_1} \left[\frac{R_-}{r} + R'_- \right]_{r_2},$$

and α the condition

$$\tan\alpha = -\frac{R_+(r_2)/r_2 + R'_+(r_2)}{R_-(r_2)/r_2 + R'_-(r_2)}.$$

The resonant mode angular frequency is thus determined as $\omega = kc$.

There exists a complementary set of modes with \mathbf{E} transverse to the z direction (TE modes), but for concreteness we study only the lowest frequency TM modes.

An important parameter is the quality factor of the cavity, Q_{cav} , for each mode. It is conventionally defined as

$$Q_{\text{cav}} \equiv \frac{\omega \langle U \rangle}{\langle W \rangle}, \quad (21)$$

where ω is the angular frequency of the mode, $\langle U \rangle$ is the temporal average of its electromagnetic energy, and $\langle W \rangle$ is the average dissipated power in the walls of the cavities. As the average electric energy is equal to the average magnetic energy in the cavity, and the power loss can be obtained from the value of the magnetic field on the boundary, an explicit, practical expression of Q_{cav} can be obtained in terms of solely the magnetic field as [15]

$$Q_{\text{cav}} = \frac{2 \int \langle B^2 \rangle dV}{\delta \int \langle B^2 \rangle dS},$$

where the integrals are extended to the volume and the internal surface of the cavity, respectively, and δ is the penetration length in the metal wall of resistivity η ,

$$\delta = \sqrt{2\eta/(\mu_0\omega)}. \quad (22)$$

From (19) one thus can write

$$Q_{\text{cav}} = \frac{2 \int [R(r)Q'(\theta)]^2 dV}{\delta \int [R(r)Q'(\theta)]^2 dS}. \quad (23)$$

If the cavity is fed with an average electromagnetic power P , in the permanent regime one has $\langle W \rangle = P$, and so, from (19) and (21),

$$\begin{aligned} \langle U \rangle &= \frac{\int \langle B^2 \rangle dV}{\mu_0} = \frac{C^2 k^2}{2\mu_0} \int [R(r)Q'(\theta)]^2 dV \\ &= \frac{Q_{\text{cav}} P}{\omega}, \end{aligned} \quad (24)$$

which allows one to determine the global constant C , given the fed average power and the characteristics of the cavity for the relevant mode.

5. THE FORCE ON THE CAVITY

In the permanent regime of the established resonant mode, sustained against decay by a continuous power input P , the electromagnetic field (19)–(20) corresponds to

$$B^2 - \frac{E^2}{c^2} = F_B(r, \theta) \cos^2(\omega t) - F_E(r, \theta) \sin^2(\omega t)$$

$$= \frac{1}{2}(F_B - F_E) + \frac{1}{2}(F_B + F_E) \cos(2\omega t), \quad (25)$$

where

$$F_B(r, \theta) = C^2 k^2 [R(r)Q'(\theta)]^2, \quad (26)$$

$$\begin{aligned} F_E(r, \theta) &= C^2 \left\{ \left[\frac{R(r)}{r} n(n+1)Q(\theta) \right]^2 \right. \\ &\quad \left. + \left[\frac{R(r)}{r} + R'(r) \right]^2 Q'^2(\theta) \right\}. \end{aligned} \quad (27)$$

From (16), the expression (25) then leads to a constant plus a contribution to ϕ harmonic in time, which, together with (25) in (17), result in χ' also having a constant plus a harmonic part. The latter has a zero contribution to the time average of the force, and so we consider only the constant part, χ'_0 , whose equation is, from (17),

$$\nabla^2 \chi'_0 = \varkappa(F_B - F_E), \quad (28)$$

where, due to (18),

$$\begin{aligned} \varkappa &= -2\pi G_0 \varepsilon_0 \psi_0 \left(\frac{1}{2\omega_0 + 3} \frac{\partial \beta_{EM}}{\partial \phi} - \beta \frac{\partial \beta_{EM}}{\partial \psi} \right)_{\phi_0, \psi_0} \\ &\simeq -\frac{2\pi G_0 \varepsilon_0 \psi_0}{2\omega_0 + 3} \frac{\partial \beta_{EM}}{\partial \phi} \Big|_{\phi_0, \psi_0} \\ &\simeq 1.2 \times 10^9 \left(\frac{Am}{Ns} \right)^2. \end{aligned} \quad (29)$$

Note that the magnetic field in the r.h.s. of (17) is the total field, which includes the contribution from the Earth's magnetic field. The latter, although of much smaller magnitude than that of the cavity, cannot be neglected due to its large spatial scale. However, one has for the time average (denoted by $\langle \dots \rangle$)

$$\begin{aligned} \langle B_{\text{Earth}}^2 + B_{\text{cavity}}^2 \rangle &= B_{\text{Earth}}^2 + \langle B_{\text{cavity}}^2 \rangle \\ + 2\mathbf{B}_{\text{Earth}} \cdot \langle \mathbf{B}_{\text{cavity}} \rangle &= B_{\text{Earth}}^2 + F_B/2, \end{aligned}$$

since $\langle \mathbf{B}_{\text{cavity}} \rangle = 0$. In this way, the contributions from the magnetic fields of the Earth and of the cavity to the potential χ'_0 can be separated, and that of the cavity alone is correctly described by (28).

Eq. (28) is solved taking into account that its r.h.s. is zero outside the cavity, so that, using the axial symmetry, the solution of the Poisson equation (28) is

$$\begin{aligned} \chi'_0(r, \theta) &= -\frac{\varkappa}{\pi} \int \frac{F_B(r', \theta') - F_E(r', \theta')}{\sqrt{r^2 + r'^2 - 2rr' \cos(\theta' - \theta)}} \\ &\quad \times K \left(-\frac{4rr' \sin \theta \sin \theta'}{r^2 + r'^2 - 2rr' \cos(\theta' - \theta)} \right) \\ &\quad \times r'^2 \sin \theta' dr' d\theta', \end{aligned} \quad (30)$$

where K is the complete elliptic integral of the first kind, and the integral is extended to the interior of the cavity. Note that since the volume integral of the r.h.s. of (28) is equal to zero, $\nabla\chi'_0$ decays rapidly outside the cavity.

Assuming a cavity with thin walls (but much thicker than the penetration depth δ , in order that the boundary conditions used be correct) of mass surface density σ , the force on the cavity is finally evaluated as

$$\mathbf{F} = -\sigma \int \nabla\chi'_0 dS, \quad (31)$$

where the integral is taken over the internal surface of the cavity. Due to the axial symmetry, the force has only a z component,

$$\begin{aligned} F_z &= -\sigma \int \frac{\partial\chi'_0}{\partial z} dS = -\sigma \int \nabla\chi'_0 \cdot \mathbf{e}_z dS \\ &= -\sigma \int \left(\cos\theta \frac{\partial\chi'_0}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\chi'_0}{\partial\theta} \right) dS, \end{aligned}$$

which is explicitly written as

$$\begin{aligned} -\frac{F_z}{2\pi\sigma} &= r_2^2 \int_0^{\theta_0} \left(\cos\theta \frac{\partial\chi'_0}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\chi'_0}{\partial\theta} \right) \Big|_{r_2} \sin\theta d\theta \\ &+ r_1^2 \int_0^{\theta_0} \left(\cos\theta \frac{\partial\chi'_0}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\chi'_0}{\partial\theta} \right) \Big|_{r_1} \sin\theta d\theta \\ &+ \sin\theta_0 \int_{r_1}^{r_2} \left(\cos\theta \frac{\partial\chi'_0}{\partial r} - \frac{\sin\theta}{r} \frac{\partial\chi'_0}{\partial\theta} \right) \Big|_{\theta_0} r dr. \quad (32) \end{aligned}$$

There are no details in the literature as to the precise dimensions of the cavities used in the experiments, so that an example roughly similar to the overall dimension reported and with the proportions observed in the published photographs will be used. Assuming a wall of thickness 1 mm, and the copper mass density of $8.9 \times 10^3 \text{ kg/m}^3$, we have $\sigma = 8.9 \text{ kg/m}^2$.

We further consider the copper cavity to have $r_1 = 18 \text{ cm}$, $r_2 = 36 \text{ cm}$, and $\theta_0 = 22^\circ$. For this cavity, the lowest TM mode corresponds to the order $n = 5.75632$ of the Legendre polynomial, with a resonant frequency $\nu = 1.05 \text{ GHz}$. For the resistivity $\eta = 1.72 \times 10^{-8} \Omega \text{ m}$, the quality factor for this mode is $Q_{\text{cav}} = 3.13 \times 10^4$. The next two TM modes have the same order $n = 5.75632$ and the resonant frequencies $\nu = 2.05 \text{ GHz}$ and $\nu = 2.76 \text{ GHz}$, with the quality factors $Q_{\text{cav}} = 3.11 \times 10^4$ and $Q_{\text{cav}} = 5.24 \times 10^4$, respectively.

For an average power $P = 1 \text{ kW}$, the constant C is evaluated for each mode using (24), and (26) and (27) are used in (30) to obtain by numerical integration the

values of $\chi'_0(r, \theta)$ needed in the numerical evaluation of (32).

Note that, from (24), the force on the cavity is proportional to the fed power and to the quality factor Q_{cav} .

For the lowest TM mode ($\nu = 1.05 \text{ GHz}$) the value obtained is $F_z = 7.7 \text{ N}$, while for the next two TM modes, with $\nu = 2.05 \text{ GHz}$ and $\nu = 2.76 \text{ GHz}$, we obtain $F_z = -1.4 \text{ N}$ and $F_z = -0.9 \text{ N}$, respectively. The values reported in [1] are not easy to compare with since the power of the microwave source is distributed over rather a wide range of frequencies, so that the actual power into the resonant mode is not precisely determined. Using a spectrum analysis of the power source, the authors evaluate, for instance, that when $F_z = -0.3 \text{ N}$, the actual power into the resonant mode is $P = 0.12 \text{ kW}$, which would correspond to $F_z = -2.5 \text{ N}$ at $P = 1 \text{ kW}$. The last two modes considered are closer to the reported value of the resonance, $\nu = 2.45 \text{ GHz}$, and give theoretical results with the correct sign and similar magnitude. According to the model, the force is proportional to the thickness of the wall and also depends on the precise geometry of the cavity (neither of them are reported in the literature), and as the value (29) is only an estimate, since the contribution from $\beta\partial\beta_{EM}/\partial\psi|_{\phi_0, \psi_0}$ cannot be ascertained independently, the results seem to be consistent with the measured force being due to the studied effect.

Note that the lowest mode ($\nu = 1.05 \text{ GHz}$) leads to a force much larger in magnitude and of opposite direction to that of the next two modes. This and other dependencies of the predicted force, such as the proportionality to the cavity wall thickness (within certain limits since $\nabla\chi'_0$ rapidly decays outside the cavity), can be explored experimentally with relative ease to test the theory.

Finally, it is worth noting that the weak energy condition (WEC) [9] is violated for the cavity, as is the case in other models of a propellantless drive [3]. In effect, from (8), the WEC is written for the cavity as

$$(\partial_{\mu\nu}\phi - \eta^{\gamma\delta}\partial_{\gamma\delta}\phi\eta_{\mu\nu})U^\mu U^\nu \geq 0, \quad (33)$$

for any timelike four-vector U^μ . By taking $U^\mu = (1, 0, 0, 0)$ one has the particular WEC

$$\nabla^2\phi \geq 0,$$

which is seen from (16) and (25) to be violated at different times and in different regions inside the cavity.

6. DISCUSSION

It has been shown that the weak field approximation of rather a general scalar-tensor theory of gravity, which includes an additional scalar with strong coupling to the electromagnetic field, as proposed in [5], could account for the forces reported on asymmetric resonant cavities. Although highly speculative, it is interesting that this was done using the same coupling coefficient adjusted by [5] to explain discordant measurements of Newton’s gravitational constant. It is also of interest that the inclusion of the external scalar ψ can help one to reconcile the Solar System tests with values of the Brans-Dicke parameter ω close to unity (see Eq. (15)). The weakest part of the theory seems to be that there is no clear way of preventing large gravitational effects due to the magnetic field of the Earth, as predicted by Eq. (17). These effects can be seen by considering the solution of Eq. (17) for a static magnetic dipole of scales and magnitude similar to that of the Earth:

$$\mathbf{B} = \frac{2B_0 R_E^3}{r^3} \left(\cos \theta \mathbf{e}_r + \frac{1}{2} \sin \theta \mathbf{e}_\theta \right),$$

where spherical coordinates are used, B_0 is a typical magnetic field value at the Earth’s surface, $B_0 \simeq 5 \times 10^{-5}$ T, and R_E is the Earth’s radius. Using the notation in (29), we then have

$$\nabla^2 \chi' = \frac{2\kappa B_0^2 R_E^6}{r^6} (1 + 3 \cos^2 \theta),$$

whose solution is

$$\chi' = \frac{\kappa B_0^2 R_E^6}{r^4} \cos^2 \theta,$$

with the corresponding gravitational force per unit mass

$$\begin{aligned} \mathbf{f} &= -\nabla \chi' \\ &= -\frac{4\kappa B_0^2 R_E^6}{r^5} \left(\cos^2 \theta \mathbf{e}_r + \frac{1}{2} \sin \theta \cos \theta \mathbf{e}_\theta \right). \end{aligned}$$

Using the value (29), one would thus have a specific force of huge magnitude on the Earth’s surface, due to the large scales involved, $f \sim 4\kappa B_0^2 R_E \simeq 7.7 \times 10^7 \text{ ms}^{-2}$.

A possible solution can be sought for in nonlinear effects, such as those due to the second terms in the l.h.s. of Eqs. (4) and (6). In effect, their inclusion would modify (14) to

$$\begin{aligned} \square \chi &= (\square \chi)_{\text{original}} \\ &+ \frac{c^2}{2} \left(\frac{d\omega}{d\phi} \Big|_{\phi_0, \psi_0} \partial^\nu \phi \partial_\nu \phi - \beta \partial^\nu \phi \partial_\nu \psi \right). \end{aligned}$$

For the stationary case of the Earth’s magnetic field one would then have

$$\begin{aligned} \nabla^2 \chi &= \frac{4\pi G_0}{c^2} T_{00}^{\text{mat}} - \frac{4\pi G_0}{c^2} \left(\beta^2 - \frac{1}{2\omega_0 + 3} \right) T_{00}^{\text{mat}} \\ &- \frac{c^2 \psi_0}{2} \left(\frac{1}{2\omega_0 + 3} \frac{\partial J}{\partial \phi} - \beta \frac{\partial J}{\partial \psi} \right)_{\phi_0, \psi_0} \\ &+ \frac{c^2}{2} \left(\frac{d\omega}{d\phi} \Big|_{\phi_0, \psi_0} \nabla \phi \cdot \nabla \phi - \beta \nabla \phi \cdot \nabla \psi \right). \end{aligned} \quad (34)$$

If the terms $\nabla \phi \cdot \nabla \phi$ and $\nabla \phi \cdot \nabla \psi$ were to dominate over $\nabla^2 \phi$ and $\nabla^2 \psi$, respectively, Eqs. (4) and (6) would result in

$$\begin{aligned} \frac{d\omega}{d\phi} \Big|_{\phi_0, \psi_0} \nabla \phi \cdot \nabla \phi &\simeq -\frac{8\pi G_0}{c^4} T^{\text{mat}} + \frac{\partial J}{\partial \phi} \Big|_{\phi_0, \psi_0} \psi_0, \\ \nabla \phi \cdot \nabla \psi &\simeq -\beta \frac{8\pi G_0}{c^4} T^{\text{mat}} + \frac{\partial J}{\partial \psi} \Big|_{\phi_0, \psi_0} \psi_0, \end{aligned}$$

which clearly cancel the terms in (34) leading to large values of the force. To put it more plainly, $\nabla^2 \phi$ and $\nabla^2 \psi$ are the sources of the potential χ' , and so situations where they are small or even zero would reduce or even nullify the gravitational effect of the electromagnetic field. Note that in the case of a static magnetic field outside its sources one can write $\mathbf{B} = \nabla \Psi$, with $\nabla^2 \Psi = 0$, so it is possible that equations like (4) and (6) for the static case

$$\begin{aligned} (2\omega_0 + 3) \nabla^2 \phi + \frac{d\omega}{d\phi} \Big|_{\phi_0, \psi_0} \nabla \phi \cdot \nabla \phi \\ \propto B^2 = \nabla \Psi \cdot \nabla \Psi, \\ \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \propto B^2 = \nabla \Psi \cdot \nabla \Psi, \end{aligned}$$

have the solutions $\nabla \phi \propto \nabla \psi \propto \nabla \Psi$, and $\nabla^2 \phi = \nabla^2 \psi = 0$, which is in general impossible for the case of the cavity, where the constant part of $B^2 - E^2/c^2$ cannot be written as $\nabla \Psi \cdot \nabla \Psi$.

If this is what happens in the case of the Earth’s magnetic field, it would seem to invalidate the derivations in [5], where the solution with $\nabla^2 \phi \neq 0$ was used. However, it can be shown that if $d\omega/d\phi|_{\phi_0, \psi_0}/(2\omega_0 + 3) \sim 1$, the two solutions are numerically similar.

Along these lines note finally that $\partial^2 \phi / \partial t^2$ is also a source of the potential χ' , which would contribute in transient situations.

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