Application of exterior calculus to waveguides

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Exterior calculus is a powerful tool for finding solutions to the electromagnetic field equations. Its strength can be better appreciated when applied to nontrivial configurations. We show how to exploit this tool to obtain the TM and TE modes in hollow cylindrical waveguides. The use of exterior calculus and Lorentz boosts leads straightforwardly to the solutions and their respective power transmitted along the waveguide.
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I. INTRODUCTION

Although most textbooks devoted to exterior calculus for physicists discuss some basic applications to electromagnetism,[1–4] none stresses the power of this language by applying it to Maxwell equations in waveguides and cavities. However, the application of relativity and exterior calculus in this context allows for a concise presentation of the subject and an easier way of determining the field configurations, which highlights the power of exterior calculus as a practical tool for solving difficult problems. Instead, the vector language, which is commonly used to treat the field configurations in waveguides (however, see Ref. 5), is not a natural language for electromagnetism, and thus the vector approach tends to be tedious. Frequently, the lack of an appropriate geometric language limits the derivation to a simple case such as a waveguide with a rectangular or circular cross-section (however, see Ref. 6).

In this paper we employ exterior calculus to treat propagating waves in hollow cylindrical waveguides of arbitrary cross-section. We start with TM (TE) (nonpropagating) stationary modes, whose field structure is an electric (magnetic) field along the waveguide and a magnetic (electric) field perpendicular to the waveguide. We turn these stationary solutions into propagating solutions by performing a Lorentz boost along the waveguide. Finally we calculate the transmitted power and emphasize its relativistic relation to the energy per unit length in the waveguide. The use of exterior derivatives, Hodge dualities, and the generalized Stokes theorem will provide a straightforward way for finding the solutions because all the vector equations become just one equation in the geometric language, which will show the power of this tool.

II. A BRIEF REVIEW OF EXTERIOR CALCULUS

Exterior calculus is the natural language for electrodynamics.[7,8] Developments in Hamiltonian mechanics,[4,9] thermodynamics,[1,4] Yang–Mills fields,[10,11] geometric (Berry) phases in quantum mechanics,[12] topological quantum fields such as the Chern–Simons theory,[10,11] gravity,[13] symplectic geometry,[14] and connections in fiber bundles[15] among other areas gain in clarity and depth when expressed in terms of exterior calculus. The reader is referred to Refs. 1–4 for an introduction to the subject. We will briefly review the main features of the exterior derivative d and the wedge product $\wedge$ between differential forms.

Any linear combination of coordinate differentials at each point of space is a *1-form* field (whatever the coordinates, Cartesian or not, and even if the geometry is non-Euclidean). An example of a 1-form is

$$\eta = 3x^2y^7dx + 5ydz. \tag{1}$$

If the space is three-dimensional and (x,y,z) are the chosen coordinates, we say that $\{dx,dy,dz\}$ is a coordinate basis for 1-forms. The *components* of η in this basis are $\eta_x=3x^2y^7$, $\eta_y=0$, and $\eta_z=5y$. Generically, a 1-form in a n-dimensional space is

$$\alpha = \alpha_\mu dx^\mu \tag{2}$$

(the Einstein convention is used). The superindex in dx^μ labels the n 1-forms of the coordinate basis, and the subindex in α_μ labels the components of the 1-form α, which are functions of the coordinates.

1-forms can be introduced geometrically as linear real valued functions on the space of (contravariant) vectors—the tangent space: They are covectors or covariant vectors.[1,4] Here we are not interested in the action of forms on vectors. Instead we will operate within the set of p *-forms*, which are defined as totally antisymmetric covariant tensors of p indices ($p \leqq n$). p-forms can be obtained from the (antisymmetrized) *wedge* tensor product $\wedge$ of 1-forms. For example, the wedge product between η and the 1-form $\xi = zdx+2dy$ is the 2-form

$$\varpi = \eta \wedge \xi = 6x^2y^7dx \wedge dy - 5yzdx \wedge dz. \tag{3}$$

Note the absence of the term $dx\wedge dx=0$ because the wedge product is antisymmetric. We also use the fact that $dx\wedge dz = -dz\wedge dx$. There are $\binom{n}{2}$ linearly independent 2-forms $dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$ ($\otimes$ is the tensor product), which span the coordinate basis of 2-forms. Any 2-form can be written as

$$\alpha = \frac{1}{2!}\alpha_{\mu\nu}dx^\mu \wedge dx^\nu, \tag{4}$$

where $\alpha_{\mu\nu} = -\alpha_{\nu\mu}$. In our example we have $\varpi_{xy}=6x^2y^7$, $\varpi_{yz}=0$, and $\varpi_{xz}=-5yz$. The factor $1/2!$ in Eq. (4) takes into account the fact that each independent element of the basis appears 2! times in the sum over μ and ν.

If α and β are 1-forms on a three-dimensional manifold, then the components of the product $\alpha\wedge\beta$ look like the Cartesian components of the vector product in Euclidean space,

$$\alpha \wedge \beta = (\alpha_x\beta_y - \alpha_y\beta_x)dx \wedge dy - (\alpha_x\beta_z - \alpha_z\beta_x)dz \wedge dx + (\alpha_y\beta_z - \alpha_z\beta_y)dy \wedge dz. \tag{5}$$

Any p-form α and q-form β satisfy

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha. \tag{6}$$

Thus 1-forms anticommute, but 2-forms commute, etc.

The exterior derivative d is a nilpotent operator ($d^2 \equiv 0$). If d acts on a function f (0-form), the result is the 1-form $df = (\partial f/\partial x^\mu)dx^\mu$. If d acts on a p-form α, then the result is a $(p+1)$-form $d\alpha$. Because $d(dx^\mu) \equiv 0$, $d\alpha$ is obtained by differentiating its components as exterior derivatives of functions,

$$d\alpha = d\left(\frac{1}{p!}\alpha_{\lambda\mu\nu\ldots}\right) \wedge dx^\lambda \wedge dx^\mu \wedge dx^\nu \ldots . \tag{7}$$

For instance

$$d\eta = -21x^2y^6 dx \wedge dy + 5dy \wedge dz. \tag{8}$$

If α is a p-form, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \tag{9}$$

A given p-form α is *closed* if $d\alpha = 0$, and it is *exact* if it can be written as the exterior derivative of a $(p-1)$-form. Each exact form is closed, and the inverse proposition is locally true, but its global validity depends on the topology of the space (see Poincaré's lemma[1,4,12]).

Finally, we will introduce the Hodge star operator $*$. This operator changes p-forms into $(n-p)$-forms and involves the components $g_{\mu\nu}$ of the metric tensor present in the space-time interval. If α is a p-form whose components are $\alpha_{\mu_1\ldots\mu_p}$, then

$$*\alpha_{\mu_{p+1}\ldots\mu_n} = \sqrt{|\det(g_{\mu\nu})|}\,\varepsilon_{\mu_{p+1}\ldots\mu_n\mu_1\ldots\mu_p}\alpha^{\mu_1\ldots\mu_p}, \tag{10}$$

where the indices are raised with the inverse metric tensor $g^{\mu\nu}$: $\alpha^{\mu_1\ldots\mu_p} = g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\alpha_{\nu_1\ldots\nu_p}$. ε is the Levi-Civita symbol, which is 1 (-1) for even (odd) permutations of the natural order of its indices and is zero if there are indices of equal value. The successive application of the Hodge star operator on a p-form α is[2,3]

$$**\alpha = (-1)^{p(n-p)+(n-\sigma)/2}\alpha, \tag{11}$$

where σ is the signature of the metric tensor (the difference between the numbers of positive and negative eigenvalues of the metric tensor).

III. POTENTIAL AND FIELD AS DIFFERENTIAL FORMS ON A MANIFOLD

The electromagnetic field is an exact 2-form $F = dA$, where the 1-form A is the potential. Given a set of non-necessarily Cartesian coordinates x, y, and z together with the time t, we can express these forms in the coordinate basis $\{dt, dx, dy, dz\}$,

$$A = A_\nu dx^\nu. \tag{12}$$

If x, y, and z are Cartesian coordinates, then the components A_ν will coincide with the scalar and vector potential: $A_\nu = (-\phi, \mathbf{A})$ (SI units). Therefore,

$$F = dA = dA_\nu \wedge dx^\nu = \partial_\mu A_\nu dx^\mu \wedge dx^\nu. \tag{13}$$

Because $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$, then the (antisymmetric) components of F are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For Cartesian coordinates, it is $F_{it} = E_i$ in SI units,[16] $F_{yz} = B_x$, $F_{zx} = B_y$, $F_{xy} = B_z$:

$$F = E \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \tag{14}$$

where $E = E_i dx^i$ is a 1-form. As Eq. (14) shows, in the three-dimensional Euclidean space the wedge product between 1-forms works as the vector product because $dy \wedge dz$ is the basis element for the x-component of the pseudovector $\mathbf{B}$ and so on (think of the wedge product as a vector product; see Eq. (5)).

Because $F = dA$ is exact and d is nilpotent, we have the identity

$$dF = 0, \tag{15}$$

which is equivalent to the Maxwell equations that are used for defining the potentials: $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{B} = 0$.

The remaining Maxwell equations, $\nabla \times \mathbf{B} = -c^{-2}\partial \mathbf{E}/\partial t$, $\nabla \cdot \mathbf{E} = 0$, come from varying the electromagnetic action $S[A] = -(4\mu_0 c)^{-1}\int *F \wedge F$. Because F is a 2-form in a manifold of dimension $n = 4$ (the space-time), $*F$ is also a 2-form. $*F \wedge F$ is a 4-form (a *volume* in space-time). The resulting Euler–Lagrange equations are

$$d * F = 0. \tag{16}$$

IV. CYLINDRICAL WAVEGUIDES

We will solve Maxwell equations for the electromagnetic field in hollow cylindrical waveguides. Let z be the Cartesian coordinate along the waveguide and x and y represent the transverse coordinates. We will use Cartesian coordinates for a rectangular cross-section and polar coordinates for a circular cross-section. We begin by determining the stationary waves. We will then introduce propagation along the waveguide by means of a Lorentz boost in the z direction. Therefore we will start by proposing a solution independent of z.

A. Stationary TM modes

Consider a monochromatic potential having only a component along the waveguide,

$$A = e^{i\Omega t}\psi(x,y)dz. \tag{17}$$

The function ψ has units of magnetic field times length. Thus, the electromagnetic field is

$$F = dA = e^{i\Omega t}(i\Omega\psi dt + d\psi) \wedge dz \tag{18a}$$

$$= e^{i\Omega t}(i\Omega\psi dt \wedge dz + \partial_x\psi dx \wedge dz + \partial_y\psi dy \wedge dz). \tag{18b}$$

The first term is an electric field along z, and the other terms make up a magnetic field orthogonal to the z-axis. Therefore the proposed solution is a TM mode. For the moment this solution does not propagate along the waveguide because the components do not depend on z. The field in Eq. (18b) is a stationary wave. The function ψ must satisfy perfect conductor boundary conditions: The tangential electric field and the normal magnetic field must vanish on the boundary. Thus, ψ must be zero on the boundary so that the (pure) tangential electric field vanishes,

Table I. TM stationary modes with $d\ell$ the longitude element in the waveguide cross-section where ψ and ${}^{(2)}(*d\psi)$ are defined.

$F^{\mathrm{TM}}=e^{i\Omega_{mn}t}(i\Omega_{mn}\psi_{mn}dt+d\psi_{mn})\wedge dz$
$*F^{\mathrm{TM}}=e^{i\Omega_{mn}t}(-ic^{-1}\Omega_{mn}\sqrt{g_{xx}g_{yy}}\psi_{mn}dx\wedge dy+c\,{}^{(2)}(*d\psi_{mn})\wedge dt)$
${}^{(2)}\Delta\psi_{mn}(x,y)=-c^{-2}\Omega_{mn}^2\psi_{mn}(x,y),\quad \psi_{mn}\vert_{\mathrm{boundary}}=0$
$d\ell^2=g_{xx}(x,y)dx^2+g_{yy}(x,y)dy^2$

Table II. TM ψ's and Ω's for rectangular and circular cross-sections.

Rectangular cross-section
x,y are Cartesian coordinates, $0\le x\le a$, $0\le y\le b$, $g_{xx}=1$, and $g_{yy}=1$
$\psi_{mn}(x,y)=A_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\quad (m,n\in\mathbb{N})$
$c^{-2}\Omega_{mn}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$
Circular cross-section
x,y are polar coordinates r,φ: $0\le r\le R$, $0\le\varphi\le 2\pi$, $g_{rr}=1$, and $g_{\varphi\varphi}=r^2$
$\psi_{mn}(r,\varphi)=J_m(x_{mn}r/R)(A_{mn}\cos(m\varphi)+B_{mn}\sin(m\varphi))$
$\Omega_{mn}=\frac{x_{mn}c}{R}$
x_{mn} are zeros of Bessel functions J_m

$$\psi\vert_{\mathrm{boundary}}=0 \tag{19}$$

(Dirichlet condition for the potential $\psi(x,y)$). Because ψ is constant on the boundary, the 1-form $d\psi=(\partial\psi/\partial x^\mu)dx^\mu$ is *normal* to the boundary, and thus the magnetic part $d\psi\wedge dz$ is tangent to the boundary and perpendicular to the z-axis (think of the product as a vector product). Thus, the boundary condition for the electric field also guarantees the boundary condition for the magnetic field.

The 2-form $*F$ involves the metric tensor. If orthogonal coordinates are used, then the space-time interval is

$$ds^2=-c^2dt^2+g_{xx}dx^2+g_{yy}dy^2+dz^2, \tag{20}$$

where the metric $\mathrm{diag}(g_{xx},g_{yy})$ in the transverse directions depends on the choice of the x and y coordinates. The determinant of the metric and the inverse metric tensor are

$$|\det(g_{\mu\nu})|=c^2g_{xx}g_{yy},\qquad g^{\mu\nu}=\mathrm{diag}(-c^{-2},1/g_{xx},1/g_{yy},1). \tag{21}$$

To write $*F$ we need a few results,

$$*(dt\wedge dz)=-c^{-1}\sqrt{g_{xx}g_{yy}}dx\wedge dy, \tag{22a}$$

$$*(dx\wedge dz)=-c\sqrt{g_{xx}g_{yy}}g^{xx}dt\wedge dy, \tag{22b}$$

$$*(dy\wedge dz)=c\sqrt{g_{xx}g_{yy}}g^{yy}dt\wedge dx. \tag{22c}$$

Thus,

$$*F=c\sqrt{g_{xx}g_{yy}}e^{i\Omega t}\{-ic^{-2}\Omega\psi dx\wedge dy + (g^{xx}\partial_x\psi dy-g^{yy}\partial_y\psi dx)\wedge dt\}, \tag{23}$$

$$d*F=c\sqrt{g_{xx}g_{yy}}e^{i\Omega t}(c^{-2}\Omega^2\psi+{}^{(2)}\Delta\psi)dt\wedge dx\wedge dy. \tag{24}$$

Equation (16) implies that function ψ must be an eigenfunction ψ_{mn} of the two-dimensional Laplacian operator, where $-c^{-2}\Omega_{mn}^2$ is its respective eigenvalue,

$${}^{(2)}\Delta\psi_{mn}\equiv\frac{1}{\sqrt{g_{xx}g_{yy}}}[\partial_x(\sqrt{g_{xx}g_{yy}}g^{xx}\partial_x\psi_{mn}) + \partial_y(\sqrt{g_{xx}g_{yy}}g^{yy}\partial_y\psi_{mn})]=-\frac{\Omega_{mn}^2}{c^2}\psi_{mn}. \tag{25}$$

Note that Eq. (23) contains the two-dimensional 1-form ${}^{(2)}(*d\psi)$. (The superscript (2) means that the Hodge star is applied in a $n=2$ submanifold; see the Appendix.) As can be seen, the Laplacian in the waveguide cross-section is ${}^{(2)}\Delta={}^{(2)}(-*d*d)$. Its eigenfunctions ψ_{mn}, which satisfy the boundary conditions, are identified by two discrete indices m,n. Table I summarizes the results for monochromatic TM stationary modes. Table II shows the solutions ψ of Eqs. (19) and (25) and their eigenvalues Ω (allowed frequencies for stationary TM modes) for typical waveguide cross-sections.

B. Stationary TE modes

Remarkably F and $*F$ are on an equal footing in Eqs. (15) and (16). This property will let us construct the stationary TE modes by exchanging their roles. In Eq. (14) this exchange amounts to the interchanging of **E** and **B**. Thus, we define

$$F^{\mathrm{TE}}=*F^{\mathrm{TM}}. \tag{26}$$

We will apply Eq. (11) to obtain $*F^{\mathrm{TE}}=**F^{\mathrm{TM}}$. In space-time it is $n=4$ and $\sigma=2$ (see Eq. (20)), and thus $**=(-1)^{p+1}$. Thus,

$$*F^{\mathrm{TE}}=-F^{\mathrm{TM}}. \tag{27}$$

The field (26) can be ascribed to the potential

$$A^{\mathrm{TE}}=\frac{e^{i\Omega t}c}{i\Omega}\sqrt{g_{xx}g_{yy}}(g^{yy}\partial_y\psi dx-g^{xx}\partial_x\psi dy) = \frac{e^{i\Omega t}c}{i\Omega}{}^{(2)}(*d\psi). \tag{28}$$

By differentiating the middle expression in Eq. (28) we recognize the appearance of the Laplacian defined in Eq. (25),

$$F^{\mathrm{TE}}=dA^{\mathrm{TE}}=e^{i\Omega t}c\sqrt{g_{xx}g_{yy}} \times\left\{\frac{i}{\Omega}{}^{(2)}\Delta\psi dx\wedge dy+(g^{xx}\partial_x\psi dy-g^{yy}\partial_y\psi dx)\wedge dt\right\} \tag{29a}$$

$$=e^{i\Omega t}c\sqrt{g_{xx}g_{yy}}\left\{\frac{i}{\Omega}{}^{(2)}\Delta\psi dx\wedge dy+{}^{(2)}(*d\psi)\wedge dt\right\}. \tag{29b}$$

This result coincides with field in Eq. (23) when the pair Ω,ψ is chosen among the solutions Ω_{mn},ψ_{mn} of the eigenvalue Eq. (25). Although F^{TE} solves both Eqs. (15) and (16), the boundary condition should be changed so that the electric field is normal to the boundary to satisfy the perfect conductor boundary condition. Because ${}^{(2)}(*d\psi)$ in Eq. (29b) is a 1-form proportional to the electric field in the waveguide cross-section, we require that

Table III. TE stationary modes with $d\ell$ the longitude element in the waveguide cross-section where ψ and ${}^{(2)}(*d\psi)$ are defined.

$F^{\mathrm{TE}}=e^{i\Omega_{mn}t}(-ic^{-1}\Omega_{mn}\sqrt{g_{xx}g_{yy}}\psi_{mn}dx\wedge dy+c^{(2)}(*d\psi_{mn})\wedge dt)$
$*F^{\mathrm{TE}}=-e^{i\Omega_{mn}t}(i\Omega_{mn}\psi_{mn}dt+d\psi_{mn})\wedge dz$
${}^{(2)}\Delta\psi_{mn}(x,y)=-c^{-2}\Omega_{mn}^2\psi_{mn}(x,y),\ \ \mathbf{n}\cdot\nabla\psi\|_{\mathrm{boundary}}=0$
$d\ell^2=g_{xx}(x,y)dx^2+g_{yy}(x,y)dy^2$

$$ {}^{(2)}(*d\psi)\wedge n|_{\mathrm{boundary}}=0, \tag{30}$$

where $n=n_x dx+n_y dy$ is a 1-form normal to the boundary (think of the product as a vector product). In terms of vector language this requirement means

$$\mathbf{n}\cdot\nabla\psi|_{\mathrm{boundary}}=0 \tag{31}$$

(Neumann boundary condition for the potential $\psi(x,y)$). Table III summarizes the monochromatic stationary TE modes. Table IV shows the functions ψ for typical waveguide cross-sections.

V. PROPAGATING MODES

The solutions given in Sec. IV do not propagate energy along the waveguide. Because the Poynting vector is proportional to $\mathbf{E}\times\mathbf{B}$, there must exist transverse components of $\mathbf{E}$ and $\mathbf{B}$ so that energy can propagate along the waveguide. However, the field F^{TM} in Eq. (18b) has only a perpendicular component of the magnetic field, and F^{TE} in Eq. (29b) has only a perpendicular component of the electric field. Thus, the Poynting vector in Eqs. (18b) and (29b) is orthogonal to the waveguide axis, and thus the solutions (18b) and (29b) are stationary waves and energy does not propagate along the waveguide. It is easy to prove that the time-averaged Poynting vector vanishes, which means that the fields (18b) and (29b) are solutions in their *proper frame*.

The stationary solutions (18b) and (29b) can be transformed into solutions that propagate energy along the waveguide by performing a Lorentz boost in the z-direction. To do this we use

$$t=\gamma(V)\left(t'-\frac{V}{c^2}z'\right),\quad dt=\gamma(V)\left(dt'-\frac{V}{c^2}dz'\right),$$

Table IV. TE ψ's and Ω's for rectangular and circular cross-sections.

Rectangular cross-section
x,y are Cartesian coordinates, $0\leq x\leq a$, $0\leq y\leq b$, $g_{xx}=1$, and $g_{yy}=1$
$\psi_{mn}(x,y)=A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$
$c^{-2}\Omega_{mn}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$
Circular cross-section
x,y are polar coordinates r,φ: $0\leq r\leq R$, $0\leq\varphi\leq 2\pi$, $g_{rr}=1$, and $g_{\varphi\varphi}=r^2$
$\psi_{mn}(r,\varphi)=J_m(y_{mn}r/R)(A_{mn}\cos(m\varphi)+B_{mn}\sin(m\varphi))$
$\Omega_{mn}=\frac{y_{mn}c}{R}$
y_{mn} are zeros of the derivatives of Bessel functions J_m

$$dz=\gamma(V)(dz'-Vdt'), \tag{32}$$

where $\gamma(V)=(1-V^2/c^2)^{-1/2}$. Thus,

$$dt\wedge dz=\gamma(V)^2(1-V^2/c^2)dt'\wedge dz'=dt'\wedge dz'. \tag{33}$$

A. TM modes

Equation (33) means that the longitudinal electric field remains invariant in Eq. (18b). In contrast, the transverse term $d\psi(x,y)\wedge dz$ changes to

$$d\psi(x,y)\wedge dz=\gamma(V)d\psi(x,y)\wedge(dz'-Vdt'), \tag{34}$$

which not only changes the transverse magnetic field but also leads to a nonzero transverse electric field. This result is nothing but the usual rule for transforming electric and magnetic fields (see, for example, Ref. 17). The geometric language shows it more elegantly.

B. TE modes

In this case Eq. (33) means that the longitudinal magnetic field remains invariant in Eq. (29b). Instead, the Lorentz boost changes the transverse electric field of F^{TE} to

$$c^{(2)}(*d\psi_{mn})\wedge dt=c\,\gamma(V)^{(2)}(*d\psi_{mn})\wedge(dt'-Vc^{-2}dz') \tag{35a}$$

$$=c\,\gamma(V)\sqrt{g_{xx}g_{yy}}(g^{yy}\partial_y\psi dx - g^{xx}\partial_x\psi dy)\wedge(dt'-Vc^{-2}dz'). \tag{35b}$$

Therefore, not only the transverse electric field is changed by the boost, but a nonzero transverse magnetic field appears in the new frame.

In conclusion, in any frame differing from the proper frame where the solutions (18b) and (29b) were determined, the propagating TM and TE modes display both transverse electric and transverse magnetic fields, and thus energy propagates along the waveguide.

VI. TRANSMITTED POWER

The existence of both magnetic and electric transverse fields in the new frame produces a nonzero Poynting vector along the waveguide. Thus, in the new frame there is energy propagating in the waveguide. The velocity V is the velocity relative to the proper frame. In this sense V can be called the *energy velocity*, because no energy propagates along the waveguide in the original proper frame (there is just an energy flux orthogonal to the waveguide axis whose time average vanishes).

The time-averaged energy flux along the waveguide, in the frame moving with velocity V relative to the proper frame, results from the $t'z'$ component of the electromagnetic energy-momentum tensor $T_{\mu\nu}$,[6]

$$\mu_0 T_{\mu\nu}=F_{\mu\rho}F_{\nu}{}^{\rho}-\tfrac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}. \tag{36}$$

Only real fields should be considered, and thus we have to average products of trigonometric functions. We have that $\langle\sin^2(\Omega t)\rangle=1/2=\langle\cos^2(\Omega t)\rangle$ and $\langle\sin(\Omega t)\cos(\Omega t)\rangle=0$. Thus we obtain

$$\mu_0\langle T_{t'z'}\rangle = g^{xx}\langle F_{t'x}F_{z'x}\rangle + g^{yy}\langle F_{t'y}F_{z'y}\rangle$$
$$= -\tfrac{1}{2}\gamma(V)^2 V|\nabla\psi|^2. \tag{37}$$

(Note that the Lorentz boost does not change the components of the metric tensor.) Result (37) is shared by TM and TE modes (although it is more difficult to obtain it for TE modes).

The energy per unit of time and area going through the waveguide cross-section at z' is $c^2T^{t'z'}=-T_{t'z'}$. The transmitted power results from integrating this quantity over the area S of the cross-section. Because ψ vanishes on the boundary of the cross-section (its normal derivative vanishes at the boundary for TE modes), Green's first identity implies that[18]

$$\int \nabla\psi\cdot\nabla\psi dS = -\int \psi^{(2)}\Delta\psi dS \tag{38}$$

(for both TM and for TE modes). If we use Eq. (25), we obtain

$$\int |\nabla\psi|^2 dS = \int c^{-2}\Omega^2\psi^2 dS. \tag{39}$$

Therefore the transmitted power is

$$P_{mn} = \int c^2 T^{t'z'} dS = \frac{\gamma^2 V\Omega_{mn}^2}{2\mu_0 c^2}\int \psi_{mn}^2 dS. \tag{40}$$

This result can be written in terms of the frequency ω and the wavenumber $k_{z'}$. The transformation of the coordinate t implies that the phase of the wave becomes

$$\Omega t = \Omega\gamma(V)(t' - Vc^{-2}z'). \tag{41}$$

Hence,

$$\omega = \gamma(V)\Omega, \quad k_{z'} = \gamma(V)\Omega V c^{-2}, \tag{42}$$

which leads to the dispersion relation

$$\omega = \sqrt{c^2k_{z'}^2 + \Omega^2} \tag{43}$$

(Ω_{mn} is the cut-off frequency for each mode). Thus the energy velocity V written in terms of ω and $k_{z'}$ is

$$V = \frac{c^2k_{z'}}{\omega} = \frac{\partial\omega}{\partial k_{z'}} < c. \tag{44}$$

As expected, the energy velocity coincides with the group velocity $\partial\omega/\partial k_{z'}$.

From Eq. (42) $\omega k_{z'}$ equals the expression contained in the transmitted power in Eq. (40). Thus,

$$P_{mn} = \frac{\omega_{mn}k_{z'}}{2\mu_0}\int \psi_{mn}^2 dS. \tag{45}$$

If the waveguide is filled with a homogeneous linear medium, then Eq. (45) for the transmitted power can be used in the frame where the medium is at rest by replacing μ_0 by the permeability μ (the constitutive relations are only valid in the media proper frames).

To finish the study of the energy transmission we calculate the time-averaged energy density $c^2T^{t't'}=c^{-2}T_{t't'}$. The results are common to TM and TE modes, but it will be easier to calculate them for TM modes. We start from

$$\mu_0\langle T_{t't'}\rangle = g^{xx}\langle F_{t'x}F_{t'x}\rangle + g^{yy}\langle F_{t'y}F_{t'y}\rangle + \langle F_{t'z'}F_{t'z'}\rangle$$
$$+ \frac{c^2}{4}\langle F_{\lambda\rho}F^{\lambda\rho}\rangle. \tag{46}$$

The invariant $\langle F_{\lambda\rho}F^{\lambda\rho}\rangle$ can be calculated with the components of the stationary field (it is invariant under a Lorentz boost). The field of stationary TM modes has only three independent components. The real fields are

$$F_{tz} = -\Omega_{mn}\psi_{mn}\sin(\Omega_{mn}t), \tag{47a}$$

$$F_{xz} = \partial_x\psi_{mn}\cos(\Omega_{mn}t), \tag{47b}$$

$$F_{yz} = \partial_y\psi_{mn}\cos(\Omega_{mn}t). \tag{47c}$$

Therefore, the time-averaged scalar invariant $\langle F_{\lambda\rho}F^{\lambda\rho}\rangle$ is

$$\langle F_{\lambda\rho}F^{\lambda\rho}\rangle = 2\langle F_{tz}F^{tz} + F_{xz}F^{xz} + F_{yz}F^{yz}\rangle \tag{48a}$$

$$= -c^{-2}\Omega_{mn}^2\psi_{mn}^2 + g^{xx}(\partial_x\psi_{mn})^2 + g^{yy}(\partial_y\psi_{mn})^2 \tag{48b}$$

$$= -c^{-2}\Omega_{mn}^2\psi_{mn}^2 + |\nabla\psi_{mn}|^2. \tag{48c}$$

Then

$$\mu_0\langle T_{t't'}\rangle = \frac{1}{2}\gamma(V)^2V^2|\nabla\psi|^2 + \frac{1}{2}\Omega^2\psi^2 - \frac{1}{4}\Omega^2\psi^2 + \frac{c^2}{4}|\nabla\psi|^2 \tag{49a}$$

$$= \frac{1}{4}\Omega^2\psi^2 + \frac{c^2}{4}\frac{1+\dfrac{V^2}{c^2}}{1-\dfrac{V^2}{c^2}}|\nabla\psi|^2. \tag{49b}$$

By performing the integral in the waveguide cross-section, we obtain the energy per unit length,

$$U_{mn} = \int c^2T^{t't'}dS = \frac{1}{4\mu_0c^2}$$
$$\times\int\left\{\Omega_{mn}^2\psi_{mn}^2 + c^2\frac{1+\dfrac{V^2}{c^2}}{1-\dfrac{V^2}{c^2}}|\nabla\psi_{mn}|^2\right\}dS \tag{50a}$$

$$= \frac{\gamma(V)^2\Omega_{mn}^2}{2\mu_0c^2}\int \psi_{mn}^2 dS, \tag{50b}$$

where we have used Eq. (39). By comparing with Eq. (45), we obtain

$$P_{mn} = VU_{mn}. \tag{51}$$

Again we find the velocity V in the expected role of the energy velocity: In this case it is the ratio of transmitted power to energy per unit length as is usually defined.[6] Note that $T^{z't'}$ is the density of the z'-component of the momentum. Therefore, because the energy-momentum tensor is symmetric, Eq. (51) can also be interpreted as that the momentum per unit length is equal to the energy per unit length over c^2 times the energy velocity (the usual relativistic relation for massive particles). Even though the electromagnetic field is massless, its energy in the waveguide behaves like that of a massive field as a consequence of the boundary conditions imposed by the waveguide. The same feature emerges in theories with compactified dimensions, which impose periodic boundary conditions to the (otherwise massless) fields.[19]

APPENDIX: GREEN'S FIRST IDENTITY AND THE PROOF OF EQ. (39)

Let ϕ, ψ be two differentiable functions (0-forms). We have the following identity among volumes (n-forms):

$$d\phi \wedge * d\psi + \phi d * d\psi = d(\phi * d\psi), \tag{A1}$$

which can be integrated to become

$$\int_S (d\phi \wedge * d\psi + \phi d * d\psi) = \int_{\partial S} \phi * d\psi. \tag{A2}$$

(Stokes theorem[1–4] has been used on the right-hand side.) The n-form $d*d\psi$ is connected with $\Delta\psi = -*d*d\psi$. According to Eq. (11), $*\Delta\psi = -d*d\psi$ whenever ψ is a 0-form, and the space has signature $\sigma = n$. Thus we obtain Green's first identity

$$\int_S (d\phi \wedge * d\psi - \phi * \Delta\psi) = \int_{\partial S} \phi * d\psi. \tag{A3}$$

To prove Eq. (39) we apply Green's first identity to the case $\phi = \psi$ in the waveguide cross-section, which is a two-dimensional manifold with coordinates x, y and signature $\sigma = 2$. For any differentiable function (0-form) ψ, we have

$$d\psi = \partial_x \psi dx + \partial_y \psi dy, \tag{A4a}$$

$$*d\psi = \sqrt{g_{xx} g_{yy}} g^{yy} \partial_y \psi dx - g^{xx} \partial_x \psi dy \tag{A4b}$$

$$d\psi \wedge * d\psi = -\sqrt{g_{xx} g_{yy}} (g^{xx} (\partial_x \psi)^2 + g^{yy} (\partial_y \psi)^2) dx \wedge dy. \tag{A4c}$$

The 2-form $d\psi \wedge * d\psi$ is a volume in the two-dimensional surface S and contains the surface element $dS = \sqrt{g_{xx} g_{yy}} dx dy$. The integral in Eq. (39) becomes

$$\int |\nabla \psi|^2 dS = -\int_S d\psi \wedge * d\psi. \tag{A5}$$

Thus, we can calculate it by using Eq. (A3) for $\phi = \psi$. We are interested in functions ψ such that they vanish on the boundary (TM modes), or their normal derivatives vanish on the boundary (TE modes) (that is, $*d\psi$ restricted to the boundary is zero; see Eq. (30)). Therefore, the right-hand side of Eq. (A3) is zero in these cases. In addition, $*1 = \sqrt{g_{xx} g_{yy}} dx \wedge dy$, which will be used to calculate the Hodge star of the 0-form $\Delta\psi$. Moreover, we are working with functions that are solutions to the eigenvalue equation $\Delta\psi = -\Omega^2 c^{-2} \psi$, and thus $*\Delta\psi = -\Omega^2 c^{-2} \psi \sqrt{g_{xx} g_{yy}} dx \wedge dy$. The result is

$$\int |\nabla \psi|^2 dS = -\int_S d\psi \wedge * d\psi = -\int_S \psi * \Delta\psi \tag{A6a}$$

$$= \int_S c^{-2} \Omega^2 \psi^2 \sqrt{g_{xx} g_{yy}} dx \wedge dy = \int c^{-2} \Omega^2 \psi^2 dS. \tag{A6b}$$

We emphasize that expressions such as the ones in Eqs. (A4a)–(A4c) depend only on the *normalized* basis of forms and vectors,

$$\mathbf{e}^{\hat{x}} = \sqrt{g_{xx}} dx, \quad \mathbf{e}^{\hat{y}} = \sqrt{g_{yy}} dy, \tag{A7}$$

$$\mathbf{e}_{\hat{x}} = \sqrt{g^{xx}} \frac{\partial}{\partial x}, \quad \mathbf{e}_{\hat{y}} = \sqrt{g^{yy}} \frac{\partial}{\partial y}. \tag{A8}$$

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[16]It is $A_\nu = (-c\phi, \mathbf{A})$ and $F_{it} = cE_i$ in Gaussian units.

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[18]Green's first identity can be stated as $\int_S (\phi \Delta\psi + \nabla\phi \cdot \nabla\psi) dS = \int_{\partial S} \phi \frac{\partial\psi}{\partial n} d\ell$, where $d\ell$ is the length element on the boundary ∂S and $\partial/\partial n$ is the normal derivative. See the Appendix for a proof using exterior calculus.

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