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Optimal normal projections in Krein spaces

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ABSTRACT

Given a pseudo-regular subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ with fundamental symmetry J, if its isotropic part $\mathcal{S}^\circ$ is not trivial the set $\mathcal{Q}_\mathcal{S}$ of J-normal projections onto $\mathcal{S}$ has infinitely many elements. The aim of this work is to distinguish a projection $Q_0 \in \mathcal{Q}_\mathcal{S}$ according to a suitable criterion. When $\mathcal{H}$ is finite-dimensional, Q_0 turns out to be the unique minimal norm projection in $\mathcal{Q}_\mathcal{S}$ with respect to the Schatten p-norm for $1 \leq p < \infty$.

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1. Introduction

Given a Hilbert space $\mathcal{H}$, the Grassmannian of $\mathcal{H}$ (i.e. the family of closed subspaces of $\mathcal{H}$) can be naturally identified with the set of bounded normal (or equivalently, self-adjoint) projections acting on $\mathcal{H}$. However, if $\mathcal{H}$ is a Krein space with fundamental symmetry J, not every closed subspace of $\mathcal{H}$ is the range of a J-normal projection, i.e. a bounded projection which commutes with its J-adjoint. In fact, if $\mathcal{S}^{[\perp]}$ stands for the J-orthogonal subspace to $\mathcal{S}$, a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is the range of a J-normal projection if and only if $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed, see [22].

Subspaces satisfying this property are known as pseudo-regular subspaces, since they can be decomposed as the direct sum $\mathcal{S} = \mathcal{S}^\circ \dotplus \mathcal{M}$ of its isotropic part $\mathcal{S}^\circ$ and a regular subspace $\mathcal{M}$ of $\mathcal{H}$. Pseudo-regular subspaces [15] are relevant in Krein spaces theory since they allow to generalize some Pontryagin spaces arguments to general Krein spaces. They have been used as a technical tool for the study of spectral functions for particular classes of operators in Krein spaces [16,17,20,21,23] and to extend the Beurling–Lax theorem for shifts in indefinite metric spaces [6,7].

Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let $\mathcal{Q}_{\mathcal{S}}$ denote the set of J-normal projections onto $\mathcal{S}$:

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) : \; Q^2 = Q, \; Q^\# Q = QQ^\#, \; R(Q) = \mathcal{S}\},$$

where $Q^\#$ is the J-adjoint of Q and $R(Q)$ stands for the range of Q. If the isotropic part $\mathcal{S}^\circ$ of $\mathcal{S}$ is non-trivial, there are infinitely many J-normal projections onto $\mathcal{S}$. Therefore, it is necessary to establish some criterion in order to determine a one-to-one correspondence between pseudo-regular subspaces and J-normal projections.

For a fixed pseudo-regular subspace $\mathcal{S}$, the first attempt is to obtain a minimal norm projection Q_0 in the set $\mathcal{Q}_{\mathcal{S}}$. To do so it is necessary to consider a fixed norm on $\mathcal{H}$, or equivalently, to consider a fixed fundamental symmetry J which determines a definite inner-product on $\mathcal{H}$ via

$$\langle x, y\rangle = [Jx, y], \quad x, y \in \mathcal{H}.$$

Then, it is helpful to have a parametrization of $\mathcal{Q}_{\mathcal{S}}$, but the one proposed in [22] is not entirely satisfactory. Hence, the first part of this note is devoted to present an alternative parametrization of $\mathcal{Q}_{\mathcal{S}}$. In this sense, the operators acting on $\mathcal{H}$ are treated as 4×4 block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus (\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)) \oplus J(\mathcal{S}^\circ). \tag{1.1}$$

This new parametrization (Theorem 3.4) depends on three variables, r, A and B, which vary freely in $L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, $L(\mathcal{S}^\circ)$ and $L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, respectively. It is particularly interesting that there is a one-to-one correspondence between the set

$L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ in which varies the parameter r and the set of subspaces which are (regular) complements of $\mathcal{S}^\circ$ in $\mathcal{S}$, see Proposition 3.3. Therefore, fixing the parameter r, Theorem 3.4 also provides a parametrization for the set of J-normal projections with a prescribed J-selfadjoint part $E_{\mathcal{M}}$:

$$\mathcal{Q}_{\mathcal{S},\mathcal{M}} = \{Q \in \mathcal{Q}_{\mathcal{S}} : \ Q^{\#}Q = QQ^{\#} = E_{\mathcal{M}}\},$$

where $\mathcal{M}$ is a closed subspace such that $\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}$ and $E_{\mathcal{M}}$ denotes the J-selfadjoint projection onto $\mathcal{M}$.

Later on, a projection $Q_0 \in \mathcal{Q}_{\mathcal{S}}$ is distinguished, choosing the three parameters to be the null-operator in the corresponding spaces. It is shown that Q_0 has minimal norm among the projections in $\mathcal{Q}_{\mathcal{S}}$ (Proposition 4.1) and its norm is calculated in terms of the Friedrichs angle between $\mathcal{S}$ and its J-orthogonal subspace $\mathcal{S}^{[\perp]}$, see Proposition 4.2 and Corollary 4.4. The only drawback in this approach is that Q_0 is not the unique projection with minimal norm in $\mathcal{Q}_{\mathcal{S}}$, in general there are infinitely many projection in $\mathcal{Q}_{\mathcal{S}}$ with the same norm as Q_0.

If $\mathcal{H}$ is a finite-dimensional Krein space, every subspace is a pseudo-regular one. Hence, every element of the Grassmannian admits J-normal projections onto it. Also, in these spaces it is possible to consider different criteria, e.g. looking for a minimal norm projection with respect to different unitarily invariant norms. Thus, the last part of this note is devoted to explore in detail the finite-dimensional case.

The paper is organized as follows. In Section 2 the notation and terminology used along the paper are settled. It also contains a brief review on angles between subspaces and a short exposition on the basics of Krein spaces.

Section 3 presents the parametrization of $\mathcal{Q}_{\mathcal{S}}$ for a pseudo-regular subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$.

Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, Section 4 is devoted to study a particular J-normal projection Q_0 onto $\mathcal{S}$, which appears by setting the variables r, A and B of the parametrization to be the null-operator (in the corresponding spaces). Its norm is explicitly calculated, and it is shown that $\|Q_0\| \leq \|Q\|$ for every $Q \in \mathcal{Q}_{\mathcal{S}}$. But Q_0 is not the unique projection with this property, i.e. there exist J-normal projections onto $\mathcal{S}$ (different from Q_0) with the same norm as Q_0.

Finally, in Section 5 it is assumed that $\mathcal{H} = \mathbb{C}^n$, for some $n \in \mathbb{N}$. For a fixed subspace $\mathcal{S}$ of $\mathbb{C}^n$, it is shown that the singular values of Q_0 are (one-to-one) smaller than the singular values of any other J-normal projection onto $\mathcal{S}$, i.e. for $j = 1, \ldots, n$,

$$s_j(Q_0) \leq s_j(Q) \quad \text{for every } Q \in \mathcal{Q}_{\mathcal{S}},$$

where $s_j(A)$ stands for the j-th singular value of $A \in \mathbb{C}^{n\times n}$ (counted with multiplicities, arranged in non-increasing order). Furthermore, Q_0 is the unique projection in $\mathcal{Q}_{\mathcal{S}}$ with this minimal set of singular values, that is, $s_j(Q) = s_j(Q_0)$ for every $j = 1, \ldots, n$ implies that $Q = Q_0$.

As a consequence of Fan's Dominance Theorem, it follows that Q_0 has minimal norm in $\mathcal{Q}_{\mathcal{S}}$ not only with respect to the operator norm but also with respect to any other unitarily invariant norm defined on $\mathbb{C}^{n\times n}$. Moreover, Q_0 turns out to be the unique minimal norm projection in $\mathcal{Q}_{\mathcal{S}}$ with respect to the Schatten p-norm for $1 \leq p < \infty$.

2. Preliminaries

Notation and terminology. Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H},\mathcal{K})$ is the vector space of bounded linear transformations from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H},\mathcal{H})$. The group of linear invertible operators acting on $\mathcal{H}$ is denoted by $GL(\mathcal{H})$. Also, $L(\mathcal{H})^+$ denotes the cone of positive (semidefinite) operators acting on $\mathcal{H}$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H},\mathcal{K})$ then $T^* \in L(\mathcal{K},\mathcal{H})$ denotes the adjoint operator of T and the modulus of T is the unique positive square root of T^*T, $|T| := (T^*T)^{1/2}$. Given $T \in L(\mathcal{H},\mathcal{K})$, $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, $\mathcal{S}\dot{+}\mathcal{T}$ denotes the direct sum of them and $\mathcal{S}\oplus\mathcal{T}$ stands for their (direct) orthogonal sum. The orthogonal difference of $\mathcal{S}$ and $\mathcal{T}$ is defined by $\mathcal{S}\ominus\mathcal{T} := \mathcal{S}\cap(\mathcal{S}\cap\mathcal{T})^\perp$. In particular, if $\mathcal{T}\subseteq\mathcal{S}$ then $\mathcal{S}\ominus\mathcal{T} = \mathcal{S}\cap\mathcal{T}^\perp$.

If $\mathcal{H} = \mathcal{S}\dot{+}\mathcal{T}$, there exists a (unique) bounded projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. Hereafter, it is denoted by $P_{\mathcal{S}//\mathcal{T}}$. In the particular case of $\mathcal{T} = \mathcal{S}^\perp$, the *orthogonal projection onto* $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$.

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $T \in L(\mathcal{H})$, T can be treated as a block-operator matrix according to the orthogonal decomposition $\mathcal{H} = \mathcal{S}\oplus\mathcal{S}^\perp$:

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix},$$

where $t_{11} = P_{\mathcal{S}}T|_{\mathcal{S}}$, $t_{12} = P_{\mathcal{S}}T|_{\mathcal{S}^\perp}$, $t_{12} = P_{\mathcal{S}^\perp}T|_{\mathcal{S}}$ and $t_{22} = P_{\mathcal{S}^\perp}T|_{\mathcal{S}^\perp}$ are bounded transformation acting between the corresponding subspaces of $\mathcal{H}$.

Angles between subspaces. Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, assume that neither $\mathcal{S}\subseteq\mathcal{T}$ nor $\mathcal{T}\subseteq\mathcal{S}$.

Then, the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is the unique $\theta\in[0,\frac{\pi}{2}]$ whose cosine is given by

$$c(\mathcal{S},\mathcal{T}) = \sup\{|\langle x,y\rangle| : x\in\mathcal{S}\ominus\mathcal{T},\ \|x\|=1,\ y\in\mathcal{T}\ominus\mathcal{S},\ \|y\|=1\}.$$

It is well known that

$$c(\mathcal{S},\mathcal{T}) < 1 \quad \Leftrightarrow \quad \mathcal{S}+\mathcal{T} \text{ is closed.}$$

On the other hand, the *Dixmier (or minimal) angle* between $\mathcal{S}$ and $\mathcal{T}$ is the unique $\theta_1\in[0,\frac{\pi}{2}]$ whose cosine is given by

$$c_0(\mathcal{S},\mathcal{T}) = \sup\{|\langle x,y\rangle| : x \in \mathcal{S},\ \|x\|=1,\ y\in\mathcal{T},\ \|y\|=1\}.$$

It is clear that $c(\mathcal{S},\mathcal{T}) \leq c_0(\mathcal{S},\mathcal{T})$, and if $\mathcal{S}\cap\mathcal{T}=\{0\}$ then $c(\mathcal{S},\mathcal{T}) = c_0(\mathcal{S},\mathcal{T})$.

Remark 2.1. If $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then

$$c_0(\mathcal{S},\mathcal{T}) = \|P_{\mathcal{S}}P_{\mathcal{T}}\|.$$

Also, $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ if and only if $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| < 1$. See [13] for further details.

Given two finite dimensional subspaces $\mathcal{M}$ and $\mathcal{W}$ of $\mathcal{H}$, assume that

$$\dim(\mathcal{M}) \leq \dim(\mathcal{W}).$$

Then, there exist $k := \dim(\mathcal{M})$ angles $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k \leq \pi/2$ called the *principal angles between* $\mathcal{M}$ *and* $\mathcal{W}$. The first one is θ_1, the minimal angle between $\mathcal{M}$ and $\mathcal{W}$. A pair of normalized vectors $(u_1, v_1) \in \mathcal{M}\times\mathcal{W}$ such that

$$\cos\theta_1 = c_0(\mathcal{M},\mathcal{W}) = |\langle u_1, v_1\rangle|$$

is a pair of corresponding *principal vectors*. The other principal angles and vectors are then defined recursively via

$$\theta_i := \min\left\{\arccos\left(|\langle u,v\rangle|\right) : u \in \mathcal{M}\ominus\mathcal{M}_i,\ v\in\mathcal{W}\ominus\mathcal{W}_i,\ \|u\|=\|v\|=1\right\},$$

where $\mathcal{M}_i = \text{span}\{u_1,\ldots,u_{i-1}\}$ and $\mathcal{W}_i = \text{span}\{v_1,\ldots,v_{i-1}\}$ for $i=2,\ldots,k$.

This means that the principal angles $\theta_1,\ldots,\theta_k$ form a set of minimal angles between the two subspaces, and the principal vectors in each subspace are orthogonal to each other. See e.g. [1] for further details on principal angles.

Krein spaces. Throughout this paper, J is a fixed symmetry acting on a Hilbert space $(\mathcal{H},\langle\,,\,\rangle)$ i.e. $J\in L(\mathcal{H})$ satisfies $J = J^* = J^{-1}$. This symmetry induces an indefinite inner-product $[\,,\,]$ on $\mathcal{H}$, given by

$$[x,y] = \langle Jx, y\rangle, \quad x,y\in\mathcal{H}. \tag{2.1}$$

Furthermore, given $\mathcal{H}_\pm = N(J\mp I)$, it defines a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+\oplus\mathcal{H}_-$ which turns $(\mathcal{H},[\,,\,])$ into a Krein space.

Usually, a Krein space is defined as an indefinite inner-product space $(\mathcal{H},[\,,\,])$ that admits a fundamental decomposition, i.e. there exists a pair of (orthogonal) subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ such that $(\mathcal{H}_+,[\,,\,])$ and $(\mathcal{H}_-,-[\,,\,])$ are Hilbert spaces. Each fundamental decomposition determines a *fundamental symmetry* J given by

$$J(x_+ + x_-) = x_+ + x_-, \quad x_\pm \in \mathcal{H}_\pm.$$

It also defines a definite inner-product on $\mathcal{H}$ by means of (2.1), which turns $(\mathcal{H}, \langle\, , \,\rangle)$ into a Hilbert space. Moreover, the norms associated to different fundamental decompositions are equivalent.

The approach used in this paper is different from the classical approach to Krein spaces, but it is equivalent. The authors prefer to consider a Krein space with a fixed fundamental symmetry J because it is important to make clear which is the norm considered. The notions defined below are easily seen to be intrinsic to the indefinite inner-product of the Krein space and do not depend on the fixed fundamental symmetry used to construct it. For a detailed exposition on these facts, and a deeper discussion on Krein spaces see [2,5,9].

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H}, \langle\, , \,\rangle_\mathcal{H})$ and $(\mathcal{K}, \langle\, , \,\rangle_\mathcal{K})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the J-adjoint operator of T is the unique $T^\# \in L(\mathcal{K}, \mathcal{H})$ such that

$$[Tx, y] = [x, T^\# y], \quad x \in \mathcal{H},\ y \in \mathcal{K}.$$

An operator $T \in L(\mathcal{H})$ is J-selfadjoint if $T = T^\#$.

A vector $x \in \mathcal{H}$ is *J-positive* if $[x, x] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. J-nonnegative, J-neutral, J-negative and J-nonpositive vectors and subspaces are defined analogously.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal subspace* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [x, s] = 0 \text{ for every } s \in \mathcal{S}\}.$$

The isotropic part of $\mathcal{S}$ is the (not necessarily trivial) subspace $\mathcal{S}^\circ := \mathcal{S} \cap \mathcal{S}^{[\perp]}$. It holds that

$$\mathcal{H} = \overline{\mathcal{S} + \mathcal{S}^{[\perp]}} \oplus J(\mathcal{S}^\circ),$$

see [5, Prop. 1.7.6]. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-non-degenerated* if $\mathcal{S}^\circ = \{0\}$. Otherwise, it is a J-degenerated subspace of $\mathcal{H}$.

A (closed) subspace $\mathcal{S}$ of $\mathcal{H}$ is *regular* if $\mathcal{S} \dotplus \mathcal{S}^{[\perp]} = \mathcal{H}$. Equivalently, $\mathcal{S}$ is regular if and only if there exists a (unique) J-selfadjoint projection E onto $\mathcal{S}$, see e.g. [5, Thm. 1.7.16].

On the other hand, a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Equivalently, $\mathcal{S}$ is pseudo-regular if there exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}$, where $[\dotplus]$ stands for the J-orthogonal direct sum of the subspaces, see [15].

Also, $\mathcal{S}$ is pseudo-regular if and only if $\mathcal{S}$ is the range of a J-normal projection, i.e. if there exists a projection $Q \in L(\mathcal{H})$ with $R(Q) = \mathcal{S}$ such that $QQ^\# = Q^\# Q$, see [22, Thm. 4.3]. However, if $\mathcal{S}^\circ \neq \{0\}$ then there are infinitely many J-normal projections Q

satisfying $R(Q) = \mathcal{S}$. In what follows, $\mathcal{Q}_\mathcal{S}$ stands for the set of J-normal projections onto the pseudo-regular subspace $\mathcal{S}$, i.e.

$$\mathcal{Q}_\mathcal{S} = \{Q \in L(\mathcal{H}) : \ Q^2 = Q, \ QQ^\# = Q^\# Q \text{ and } R(Q) = \mathcal{S}\}.$$

The following results belong to [22]. Their statements are included in order to make the paper self-contained.

Proposition 2.2. *A bounded projection Q acting on $\mathcal{H}$ is J-normal if and only if there exist a J-selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ such that*

$$Q = E + P.$$

The projections E and P are uniquely determined by Q. More precisely, $E = QQ^\#$ and $P = Q(I - Q^\#)$.

Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, consider the family of (regular) complements of $\mathcal{S}^\circ$ in $\mathcal{S}$:

$$\mathcal{R} = \{\mathcal{M} \text{ is a subspace of } \mathcal{H} \, : \, \mathcal{S} = \mathcal{M}[\dot{+}]\mathcal{S}^\circ\}. \tag{2.2}$$

In [22, Lemma 6.4] it was shown that $\mathcal{Q}_\mathcal{S}$ can be written as a disjoint union

$$\mathcal{Q}_\mathcal{S} = \bigcup_{\mathcal{M} \in \mathcal{R}} \mathcal{Q}_{\mathcal{S},\mathcal{M}},$$

where $\mathcal{Q}_{\mathcal{S},\mathcal{M}} = \{Q \in \mathcal{Q}_\mathcal{S} : \ QQ^\# = E_\mathcal{M}\}$ and $E_\mathcal{M}$ stands for the J-selfadjoint projection onto $\mathcal{M}$. Furthermore, in [11, Thm. 5.6] is was proved that $\mathcal{Q}_\mathcal{S}$ is a covering space over any of the decks $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$.

3. Parametrization of J-normal projections in Krein spaces

The following paragraphs are devoted to parametrize the set $\mathcal{Q}_\mathcal{S}$ of J-normal projections onto a fixed pseudo-regular subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ with fundamental symmetry J. Along this work operators are frequently treated as block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus (\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)) \oplus J(\mathcal{S}^\circ). \tag{3.1}$$

The results in this section are the main tools for the rest of the paper. Since their long proofs may deviate the attention of the reader, they are postponed until Appendix A.

First, it is necessary to characterize the block-operator matrix representation of the fundamental symmetry J.

Proposition 3.1. *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, the fundamental symmetry J is represented as the block-operator matrix*

$$J = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & u(I - cc^*)^{1/2} & c & 0 \\ 0 & c^* & v(I - c^*c)^{1/2} & 0 \\ a^* & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S}^\circ \\ \mathcal{S} \ominus \mathcal{S}^\circ \\ \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ) \\ J(\mathcal{S}^\circ) \end{matrix}, \tag{3.2}$$

where $a \in L(J(\mathcal{S}^\circ), \mathcal{S}^\circ)$ is an isometric isomorphism, $c \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ is a uniform contraction (i.e. $\|c\| < 1$), and $u \in L(\mathcal{S} \ominus \mathcal{S}^\circ)$, $v \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ))$ are symmetries commuting with $|c^*|$, $|c|$, respectively, satisfying

$$uc = -cv. \tag{3.3}$$

It is important to stress that the pseudo-regularity of $\mathcal{S}$ cannot be dropped in the above proposition. Later (Corollary 4.4) it is shown that the norm of the contraction c coincides with $c(\mathcal{S}, \mathcal{S}^{[\perp]})$, the cosine of the Friedrichs angle between $\mathcal{S}$ and $\mathcal{S}^{[\perp]}$. Therefore, c is a uniform contraction if and only if $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed, i.e. if $\mathcal{S}$ is pseudo-regular.

Remark 3.2. The operators a and c appearing in Proposition 3.1 are systematically used in the rest of the paper. Also, the invertible selfadjoint operators $b := u(I - cc^*)^{1/2} \in L(\mathcal{S} \ominus \mathcal{S}^\circ)$ and $d := v(I - c^*c)^{1/2} \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ))$ are considered. Observe that (3.3) can be rewritten in terms of b and d as

$$bc = -cd,$$

because $(I - cc^*)^{1/2}c = c(I - c^*c)^{1/2}$.

If Q is a J-normal projection onto $\mathcal{S}$ then $E = QQ^\#$ is a J-selfadjoint projection onto some (regular) complement of $\mathcal{S}^\circ$ in $\mathcal{S}$. So, given a subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dot{+}] \mathcal{M}$, it is necessary to describe the J-selfadjoint projection $E_\mathcal{M}$ onto $\mathcal{M}$ according to the decomposition given in (3.1).

Proposition 3.3. *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, $E \in L(\mathcal{H})$ is the J-selfadjoint projection onto a regular complement of $\mathcal{S}^\circ$ in $\mathcal{S}$ if and only if there exists a (unique) $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ such that*

$$E = \begin{pmatrix} 0 & ar^*b & ar^*c & ar^*br \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

where a, b and c were defined above. Moreover, its range is given by

$$R(E) = \left\{ \begin{pmatrix} ar^*by \\ y \\ 0 \\ 0 \end{pmatrix} : y \in \mathcal{S} \ominus \mathcal{S}^\circ \right\}.$$

In particular, the J-selfadjoint projection onto $\mathcal{S} \ominus \mathcal{S}^\circ$ is given by

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & b^{-1}c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It corresponds to consider $r = 0$.

The following is an improvement of the parametrization given in [22, Thm. 6.9]. Note that r, A and B are variables, whereas a, b, c, d are the operators considered in Proposition 3.1 and Remark 3.2:

Theorem 3.4. *Let $Q \in L(\mathcal{H})$ be a projection onto a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$. Then, $Q \in \mathcal{Q}_{\mathcal{S}}$ if and only if there exist (unique) $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, $A \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$ such that*

$$Q = \begin{pmatrix} I & 0 & B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Observe that, according to Proposition 3.3, there is a one-to-one correspondence between the operators $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ and the complements $\mathcal{M} \in \mathcal{R}$, see (2.2). Moreover, for a fixed $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, the above theorem produces a parametrization of $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ for the corresponding complement $\mathcal{M} \in \mathcal{R}$.

4. A distinguished J-normal projection

Given a pseudo-regular subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ with fundamental symmetry J, Theorem 3.4 provides a parametrization of the set $\mathcal{Q}_{\mathcal{S}}$ of J-normal projections onto $\mathcal{S}$ in terms of three parameters r, A, B which can be arbitrarily chosen in $L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, $L(\mathcal{S}^\circ)$ and $L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, respectively. In particular, choosing r, A and B to be the null-operator in the corresponding spaces,

$$Q_0 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & b^{-1}c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

is a J-normal projection onto $\mathcal{S}$. Moreover, it is easy to see that $E_0 := Q_0^{\#}Q_0$ is the J-selfadjoint projection onto $\mathcal{S} \ominus \mathcal{S}^{\circ}$ (see Proposition 3.3) and $P_0 := Q_0 - E_0$ is the orthogonal projection onto $\mathcal{S}^{\circ}$, see Proposition 2.2.

The aim of this section is to show that Q_0 has minimal norm among the elements of $\mathcal{Q}_{\mathcal{S}}$, and to relate $\|Q_0\|$ with the Friedrichs angle between $\mathcal{S}$ and $\mathcal{S}^{[\perp]}$.

Given an arbitrary closed subspace $\mathcal{S}$ of $\mathcal{H}$, $Q \in L(\mathcal{H})$ is a projection onto $\mathcal{S}$ if and only if its block-operator matrix, according to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$, is given by

$$Q = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix},$$

for some $X \in L(\mathcal{S}^{\perp}, \mathcal{S})$. Then, it is easy to see that

$$\|Q\|^2 = 1 + \|X\|^2,$$

see e.g. [12]. By Theorem 3.4, $Q \in \mathcal{Q}_{\mathcal{S}}$ if and only if there exist r, A and B, such that

$$Q = \begin{pmatrix} I & 0 & B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, according to the above comment, to minimize the norm among the projections in $\mathcal{Q}_{\mathcal{S}}$ it suffices to minimize

$$\left\| \begin{pmatrix} B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ b^{-1}c & r \end{pmatrix} \right\|,$$

among all possible r, A and B.

Theorem 4.1. *Given a pseudo-regular subspace $\mathcal{S}$, the projection Q_0 has minimal norm among the projections in $\mathcal{Q}_{\mathcal{S}}$.*

Proof. Given $x = (x_1, x_2) \in (\mathcal{S}^{\perp} \ominus J(\mathcal{S}^{\circ})) \oplus J(\mathcal{S}^{\circ})$ with $\|x\| \leq 1$, observe that

$$\begin{aligned} \left\| \begin{pmatrix} 0 & 0 \\ b^{-1}c & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2 &= \|b^{-1}cx_1\|^2 \leq \|b^{-1}cx_1\|^2 + \|Bx_1\|^2 \\ &= \left\| \begin{pmatrix} B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ b^{-1}c & r \end{pmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right\|^2 \\ &\leq \left\| \begin{pmatrix} B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ b^{-1}c & r \end{pmatrix} \right\|^2, \end{aligned}$$

for every A, B and r. Therefore,

$$\left\| \begin{pmatrix} 0 & 0 \\ b^{-1}c & 0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ b^{-1}c & r \end{pmatrix} \right\|,$$

and the proof is completed. □

The following proposition shows that the minimum value for the norm of a projection in $\mathcal{Q}_\mathcal{S}$ can be calculated in terms of the norm of the uniform contraction c appearing in (3.2).

Proposition 4.2. *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$,*

$$\min\{\|Q\| : \ Q \in \mathcal{Q}_\mathcal{S}\} = \frac{1}{\sqrt{1 - \|c\|^2}}, \tag{4.2}$$

where c is the uniform contraction appearing in the block-operator matrix of J.

Proof. By Theorem 4.1, $\min\{\|Q\| : \ Q \in \mathcal{Q}_\mathcal{S}\} = \|Q_0\|$, where Q_0 is the distinguished projection given in (4.1). Furthermore, it was shown that

$$\|Q_0\|^2 = 1 + \|b^{-1}c\|^2,$$

where $b = u(I - cc^*)^{1/2}$ and $u \in L(\mathcal{S} \ominus \mathcal{S}^\circ)$ is a symmetry commuting with $|c^*|$. Observe that

$$\begin{aligned}\|b^{-1}c\|^2 &= \|u(I - cc^*)^{-1/2}c\|^2 = \|(I - cc^*)^{-1/2}c\|^2 = \\ &= \|c(I - c^*c)^{-1/2}\|^2 = \|(I - c^*c)^{-1/2}c^*c(I - c^*c)^{-1/2}\| = \\ &= \|c^*c(I - c^*c)^{-1}\| = \frac{\|c\|^2}{1 - \|c\|^2},\end{aligned}$$

and therefore $\|Q_0\|^2 = 1 + \dfrac{\|c\|^2}{1 - \|c\|^2} = \dfrac{1}{1 - \|c\|^2}$. □

On the other hand, the norm of a projection $Q \in L(\mathcal{H})$ can be calculated as:

$$\|Q\| = \frac{1}{\sin(\theta_1)} = \frac{1}{\sqrt{1 - \cos^2(\theta_1)}}, \tag{4.3}$$

where θ_1 is the minimal angle between the range and the nullspace of Q, see [19, Section VI.5.2] also [10,24]. Thus, in order to calculate $\|Q_0\|$ in terms of this minimal angle, it is necessary to calculate the nullspace of the distinguished projection.

Proposition 4.3. *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$,*

$$Q_0 = P_{\mathcal{S}//(\mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ) \dot{+} J(\mathcal{S}^\circ)},$$

i.e. the nullspace of Q_0 coincides with $(\mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ) \dot{+} J(\mathcal{S}^\circ)$.

Proof. Given a vector $x \in \mathcal{H}$, decompose it as $x = (x_1, x_2, x_3, x_4)$, where $x_1 \in \mathcal{S}^\circ$, $x_2 \in \mathcal{S} \ominus \mathcal{S}^\circ$, $x_3 \in \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)$, $x_4 \in J(\mathcal{S}^\circ)$. Then, note that $Q_0 x = 0$ if and only if

$$x_1 = 0 \quad \text{and} \quad x_2 = (-b^{-1}c)\, x_3.$$

By (3.2), if $x_3 \in \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)$ it follows that

$$J\begin{pmatrix} 0 \\ (-b^{-1}c)\, x_3 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (d - c^* b^{-1} c)\, x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d^{-1}\, x_3 \\ 0 \end{pmatrix},$$

because $-b^{-1}c = cd^{-1}$ and therefore

$$d - c^* b^{-1} c = d + c^* c d^{-1} = (d^2 + c^* c) d^{-1} = d^{-1}.$$

Hence, $J(\{(0, (-b^{-1}c)x_3, x_3, 0) : \ x_3 \in \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)\}) = R(d^{-1}) = \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)$ and

$$\{(0, (-b^{-1}c)x_3, x_3, 0) : \ x_3 \in \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)\} = \mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ.$$

Thus, the nullspace of Q_0 is $N(Q_0) = \mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ \dotplus J(\mathcal{S}^\circ)$. $\square$

Finally, observe that

$$c_0(\mathcal{S}, N(Q_0)) = c(\mathcal{S}, \mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ \dotplus J(\mathcal{S}^\circ)) = c(\mathcal{S}, \mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ) = c(\mathcal{S}, \mathcal{S}^{[\perp]}),$$

where the second equality is a consequence of $J(\mathcal{S}^\circ) \subseteq \mathcal{S}^\perp$. Thus,

$$\|Q_0\| = \frac{1}{\sqrt{1 - c_0^2(\mathcal{S}, N(Q_0))}} = \frac{1}{\sqrt{1 - c^2(\mathcal{S}, \mathcal{S}^{[\perp]})}}. \qquad \square$$

An immediate consequence of (4.2) is that the norm of the uniform contraction c has the following geometrical interpretation:

Corollary 4.4. *If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$ and c is the uniform contraction appearing in the block-operator matrix (3.2) of J, then*

$$\|c\| = c(\mathcal{S}, \mathcal{S}^{[\perp]}),$$

where $c(\mathcal{S}, \mathcal{S}^{[\perp]})$ is the cosine of the Friedrichs angle between $\mathcal{S}$ and $\mathcal{S}^{[\perp]}$.

Before ending this section, note that Q_0 is not the unique projection with minimal norm in $\mathcal{Q}_\mathcal{S}$. In fact, given $A \in L(\mathcal{S}^\circ)$ with $A^* = -A$ and $\|A\| \le \|b^{-1}c\|$, consider

$$Q = \begin{pmatrix} I & 0 & 0 & Aa \\ 0 & I & b^{-1}c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 3.4, Q is a J-normal projection onto $\mathcal{S}$. Furthermore,

$$\|Q\|^2 = 1 + \left\| \begin{pmatrix} 0 & Aa \\ b^{-1}c & 0 \end{pmatrix} \right\|^2 = 1 + \max\{\|b^{-1}c\|^2, \|Aa\|^2\} =$$
$$= 1 + \max\{\|b^{-1}c\|^2, \|A\|^2\} = 1 + \|b^{-1}c\|^2 = \|Q_0\|^2.$$

5. The optimal J-normal projection onto a subspace of $\mathbb{C}^n$

In the previous section, given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, it was shown that the distinguished projection Q_0 has minimal operator norm among the elements of $\mathcal{Q}_\mathcal{S}$. However this condition does not characterize Q_0 because there are infinitely many projections in $\mathcal{Q}_\mathcal{S}$ with the same operator norm as Q_0.

If $\mathcal{H}$ is a finite-dimensional Krein space, every subspace of $\mathcal{H}$ is pseudo-regular. Thus, there are J-normal projections onto every subspace $\mathcal{S}$ of $\mathcal{H}$. On the other hand, it is an immediate consequence of the block-operator matrix representation (4.1) that Q_0 is the unique projection in $\mathcal{Q}_\mathcal{S}$ with minimal Frobenius norm. Thus, in the finite-dimensional case, there is a minimal-norm criterion to determine Q_0.

This section is devoted to explore in detail this particular case, with the aim to obtain alternative optimality criteria to characterize the distinguished projection Q_0. Hereafter, the underlaying Krein space is $\mathcal{H} = \mathbb{C}^n$, for some fixed $n \in \mathbb{N}$. Therefore, given subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{C}^n$ with $\dim \mathcal{S} = m$ and $\dim \mathcal{T} = k$, $L(\mathcal{S}, \mathcal{T})$ is identified with $\mathbb{C}^{k\times m}$, the vector space of $k \times m$ matrices with complex entries.

5.1. Eigenvalues, singular values and unitarily invariant norms

These paragraphs are devoted to recalling some well-known aspects of matrix analysis that are required in the following.

Given a selfadjoint matrix $A \in \mathbb{C}^{n\times n}$, $\lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ denotes the vector of eigenvalues of A (counted with multiplicities) arranged in non-increasing order. The following are two well-known variational principles for eigenvalues of selfadjoint matrices, see e.g. [8].

Theorem 5.1 *(Cauchy's Interlacing Theorem). Let $A \in \mathbb{C}^{n\times n}$ be selfadjoint, and let $B \in \mathbb{C}^{k\times k}$ be its compression to a k-dimensional subspace. Then, for $j = 1, \ldots, k$,*

$$\lambda_j(A) \geq \lambda_j(B) \geq \lambda_{n-k+j}(A). \tag{5.1}$$

Theorem 5.2 *(Weyl). Let $A \in \mathbb{C}^{n\times n}$ be selfadjoint and $H \in \mathbb{C}^{n\times n}$ be positive with rank k. Then, for $j = 1, \dots, n-k$,*

$$\lambda_j(A+H) \geq \lambda_j(A) \geq \lambda_{j+k}(A+H).$$

Furthermore, if $\lambda_j(A+H) = \lambda_j(A)$ for $j = 1, \dots, n$ then $H = 0$.

A norm $|||\cdot|||$ on $\mathbb{C}^{n\times n}$ is unitarily invariant if

$$|||A||| = |||UAV|||,$$

for any pair of unitary matrices $U, V \in \mathbb{C}^{n\times n}$. For instance, the Frobenius norm defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

is unitarily invariant.

For an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, $s_j(A) = \lambda_j(|A|)$ for $j = 1, \dots, n$ stands for the singular values of A (counted with multiplicities, arranged in non-increasing order). Singular values can be used to define different unitarily invariant norms on $\mathbb{C}^{n\times n}$ via symmetric gauge functions, see [8, Chapter IV]. A specially important family of such norms are the Ky Fan norms. Given $k = 1, \dots, n$, the k-th Ky Fan norm is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A).$$

As the reader may have noticed, the operator norm is the same as the first Ky Fan norm $\|A\|_{(1)}$. The main result on Ky Fan norms is Fan's Dominance Theorem [14]:

Theorem 5.3. *Given $A, B \in \mathbb{C}^{n\times n}$, if*

$$\|A\|_{(k)} \leq \|B\|_{(k)} \quad \textit{for } k = 1, \dots, n,$$

then $|||A||| \leq |||B|||$ for every unitarily invariant norm on $\mathbb{C}^{n\times n}$.

5.2. The optimal J-normal projection

Suppose that $\mathbb{C}^n$ is a Krein space with fundamental symmetry $J \in \mathbb{C}^{n\times n}$. Given a J-degenerated subspace $\mathcal{S}$ of $\mathbb{C}^n$ with $\dim \mathcal{S} = m$, the aim of this section is to present different optimal characterizations of the distinguished projection Q_0 among the infinitely many J-normal projections onto $\mathcal{S}$.

Assuming that $\dim \mathcal{S} \ominus \mathcal{S}^\circ = k$, J is decomposed again according to the decomposition $\mathbb{C}^n = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus (\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)) \oplus J(\mathcal{S}^\circ)$ as

$$J = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ a^* & 0 & 0 & 0 \end{pmatrix},$$

where $a \in \mathbb{C}^{(m-k)\times(m-k)}$ is unitary, $b = b^* \in GL(\mathbb{C}^{k\times k})$, $c \in \mathbb{C}^{k\times(n+k-2m)}$ is a uniform contraction and $d = d^* \in GL(\mathbb{C}^{(n+k-2m)\times(n+k-2m)})$ satisfying the equation

$$bc = -cd.$$

Then, by Theorem 3.4, $Q \in \mathcal{Q}_\mathcal{S}$ if and only if there exist $r \in \mathbb{C}^{k\times(m-k)}$, $A \in \mathbb{C}^{(m-k)\times(m-k)}$ and $B \in \mathbb{C}^{(n+k-2m)\times k}$ such that

$$Q = \begin{pmatrix} I & 0 & B & C \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $C = \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \in \mathbb{C}^{(m-k)\times(m-k)}$.

The following result shows that Q_0 is the unique projection in $\mathcal{Q}_\mathcal{S}$ with minimal singular values.

Theorem 5.4. *If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$ and $Q \in \mathcal{Q}_\mathcal{S}$, then*

$$s_j(Q_0) \le s_j(Q), \quad \text{for } j = 1, \dots, n.$$

Furthermore, Q_0 is the unique projection in $\mathcal{Q}_\mathcal{S}$ with this property.

Proof. Assume that $\dim \mathcal{S} = m$. If $X = \begin{pmatrix} B & C \\ b^{-1}c & r \end{pmatrix} \in \mathbb{C}^{m\times(n-m)}$, note that

$$QQ^* = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ X^* & 0 \end{pmatrix} = \begin{pmatrix} I + XX^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the singular values of Q are

$$s_j(Q) = \begin{cases} \sqrt{1 + \lambda_j}, & \text{if } 1 \le j \le m, \\ 0 & \text{if } m+1 \le j \le n, \end{cases}$$

where $\lambda_j = \lambda_j(XX^*)$. Also, observe that

$$XX^* = \begin{pmatrix} B & C \\ b^{-1}c & r \end{pmatrix} \begin{pmatrix} B^* & c^*b^{-1} \\ C^* & r^* \end{pmatrix} = \begin{pmatrix} BB^* + CC^* & D \\ D^* & b^{-1}cc^*b^{-1} + rr^* \end{pmatrix},$$

where $D = Bc^*b^{-1} + Cr^*$. Thus, by Cauchy's Interlacing Theorem, it follows that

$$\lambda_j \geq \lambda_j(b^{-1}cc^*b^{-1} + rr^*), \quad j = 1, \ldots, k,$$

with $k = \dim \mathcal{S} \ominus \mathcal{S}^\circ$. Moreover, by Weyl's theorem, it follows that

$$\lambda_j(b^{-1}cc^*b^{-1} + rr^*) \geq \lambda_j(b^{-1}cc^*b^{-1}), \quad j = 1, \ldots, k. \tag{5.2}$$

In the particular case of Q_0, it is easy to see that

$$X_0X_0^* = \begin{pmatrix} 0 & 0 \\ 0 & b^{-1}cc^*b^{-1} \end{pmatrix}.$$

So, if $1 \leq j \leq k$ then

$$s_j(Q_0) = \sqrt{1 + \lambda_j(X_0X_0^*)} = \sqrt{1 + \lambda_j(b^{-1}cc^*b^{-1})} \leq \sqrt{1 + \lambda_j(XX^*)} = s_j(Q).$$

On the other hand, if $k + 1 \leq j \leq m$, it is obvious that $s_j(Q_0) = 1 \leq s_j(Q)$. Finally, if $m + 1 \leq j \leq n$ then $s_j(Q_0) = s_j(Q) = 0$.

Now, suppose that there exists $Q \in \mathcal{Q}_\mathcal{S}$ such that $s_j(Q_0) = s_j(Q)$ for every $j = 1, \ldots, n$. Equivalently, $\lambda_j(XX^*) = \lambda_j(X_0X_0^*)$ for $j = 1, \ldots, m$, where

$$Q = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_0 = \begin{pmatrix} I & X_0 \\ 0 & 0 \end{pmatrix}.$$

Hence, it is assumed that equality holds in (5.2) for every $j = 1, \ldots, k$. Thus, by Weyl's theorem, $r = 0$. So,

$$X = \begin{pmatrix} B & C \\ b^{-1}c & 0 \end{pmatrix} \quad \text{and} \quad XX^* = \begin{pmatrix} BB^* & Bc^*b^{-1} \\ b^{-1}cB^* & b^{-1}cc^*b^{-1} \end{pmatrix} + \begin{pmatrix} CC^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, applying one more time interlacing and Weyl's theorems,

$$\lambda_j(XX^*) \geq \lambda_j\left(\begin{pmatrix} BB^* & Bc^*b^{-1} \\ b^{-1}cB^* & b^{-1}cc^*b^{-1} \end{pmatrix}\right) \geq \lambda_j(b^{-1}cc^*b^{-1}) = \lambda_j(X_0X_0^*),$$

for $j = 1, \ldots, m$. Then, $\lambda_j(XX^*) = \lambda_j(X_0X_0^*)$ for $j = 1, \ldots, m$ implies that $C = 0$, or equivalently, $X = \begin{pmatrix} B & 0 \\ b^{-1}c & 0 \end{pmatrix}$.

Finally, it is necessary to use that the non-zero eigenvalues of XX^* and X^*X coincide. Then, note that

$$X^*X = \begin{pmatrix} B^*B + c^*b^{-2}c & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X_0^*X_0 = \begin{pmatrix} c^*b^{-2}c & 0 \\ 0 & 0 \end{pmatrix},$$

and by a similar argument it follows that $B = 0$. Thus, $X = X_0$, or equivalently, $Q = Q_0$. □

The following is a direct consequence of Theorem 5.4 and Fan's Dominance Theorem.

Corollary 5.5. *If $\mathcal{S}$ is a subspace of $\mathbb{C}^n$ then*

$$|||Q_0||| \leq |||Q||| \quad \text{for every } Q \in \mathcal{Q}_{\mathcal{S}},$$

where $|||\cdot|||$ stands for an arbitrary unitarily invariant norm on $\mathbb{C}^{n\times n}$.

Moreover, Q_0 is the unique minimal norm element in $\mathcal{Q}_{\mathcal{S}}$ not only with respect to the Frobenius norm but also with respect to infinitely many unitarily invariant norms. Given $1 \leq p < \infty$, the Schatten p-norm of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j(A)^p\right)^{1/p}.$$

Corollary 5.6. *Given a subspace $\mathcal{S}$ of $\mathbb{C}^n$ and a fixed p ($1 \leq p < \infty$) Q_0 is the unique minimal norm element in $\mathcal{Q}_{\mathcal{S}}$ with respect to the Schatten p-norm, i.e. if $Q \in \mathcal{Q}_{\mathcal{S}}$ is such that $\|Q\|_p = \|Q_0\|_p$ then $Q = Q_0$.*

Proof. By Corollary 5.5, Q_0 minimizes every unitarily invariant norm $|||\cdot|||$, in particular, $\|Q_0\|_p \leq \|Q\|_p$ for every $Q \in \mathcal{Q}_{\mathcal{S}}$. On the other hand, if $Q \in \mathcal{Q}_{\mathcal{S}}$ satisfies $\|Q\|_p = \|Q_0\|_p$ then

$$\sum_{j=1}^{n} s_j(Q_0)^p = \sum_{j=1}^{n} s_j(Q)^p.$$

Since the singular values of Q and Q_0 are disposed in non-increasing order and, by Theorem 5.4, $s_j(Q_0) \leq s_j(Q)$ for $j = 1, \dots, n$, it follows that the above equality holds if and only if

$$s_j(Q_0) = s_j(Q) \quad \text{for } j = 1, \dots, n.$$

Hence, by Theorem 5.4, $Q = Q_0$. □

Final remarks. Theorem 5.4 can also be geometrically interpreted in terms of the principal angles between two subspaces.

If $\mathcal{M}$ and $\mathcal{W}$ are subspaces of $\mathbb{C}^n$ whose dimensions are k and l, respectively, assume that $k \leq l$. Then, note that the principal angles between $\mathcal{M}$ and $\mathcal{W}$ are those angles $0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \frac{\pi}{2}$ whose cosines are the non-zero singular values of $P_{\mathcal{W}} P_{\mathcal{M}}$, see Remark 2.1.

Furthermore, if $\mathcal{M}$ and $\mathcal{W}$ are such that $\mathbb{C}^n = \mathcal{M} \dotplus \mathcal{W}^\perp$, then $k = l$ and

$$P_{\mathcal{M}//\mathcal{W}^\perp} = (P_\mathcal{W} P_\mathcal{M})^\dagger.$$

Therefore, the non-zero singular values of $P_{\mathcal{M}//\mathcal{W}^\perp}$ are the cosecant of the principal angles between $\mathcal{M}$ and $\mathcal{W}$,

$$s_i(P_{\mathcal{M}//\mathcal{W}^\perp}) = \frac{1}{\sin(\theta_i(\mathcal{M}, \mathcal{W}))}, \quad i = 1, \dots, k, \tag{5.3}$$

see e.g. [3, Remark 3.3] or [18].

Using this ideas, Theorem 5.4 can be reinterpreted in the following manner:

Corollary 5.7. *Let $\mathcal{S}$ be an m-dimensional subspace of $\mathbb{C}^n$ and consider the projection $Q_0 = P_{\mathcal{S}//(\mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ) \oplus J(\mathcal{S}^\circ)}$. If $Q \in L(\mathcal{H})$ is any J-normal projection onto $\mathcal{S}$, then*

$$\theta_i(\mathcal{S}, N(Q)^\perp) \le \theta_i(\mathcal{S}, N(Q_0)^\perp) \quad \textit{for } i = 1, \dots, m. \tag{5.4}$$

Proof. First, observe that for any J-normal projection Q onto $\mathcal{S}$, $\dim(N(Q)^\perp) = \dim(\mathcal{S})$ because both subspaces are complementary to $N(Q)$ in $\mathbb{C}^n$. Applying (5.3) to Q_0 and Q, it follows that for $i = 1, \dots, m$,

$$\frac{1}{\sin(\theta_i(\mathcal{S}, N(Q_0)^\perp))} = s_i(Q_0) \le s_i(Q) = \frac{1}{\sin(\theta_i(\mathcal{S}, N(Q)^\perp))}.$$

Since $\sin(x)$ is an increasing function for $x \in [0, \frac{\pi}{2}]$, the above inequalities are equivalent to $\theta_i(\mathcal{S}, N(Q)^\perp) \le \theta_i(\mathcal{S}, N(Q_0)^\perp)$, for $i = 1, \dots, m$. $\square$

There is something else to say about Q_0 as the unique minimal norm element of $\mathcal{Q}_\mathcal{S}$ with respect to a unitarily invariant norm. In fact, the main idea behind the proof of Corollary 5.6 is that the function $\Phi_p : \mathbb{R}^n \to [0, +\infty)$ given by

$$\Phi_p(x_1, x_2, \dots, x_n) = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p},$$

is strictly convex, i.e. for every $x, y \in \mathbb{R}^n$, $x \neq y$ and for every $t \in (0, 1)$,

$$\Phi_p(tx + (1-t)y) < t\Phi_p(x) + (1-t)\Phi_p(y).$$

Moreover, the function Φ_p is a symmetric gauge function on $\mathbb{R}^n$ and the Schatten p-norm on $\mathbb{C}^{n \times n}$ is defined by

$$\|A\|_p = \Phi_p(s(A)),$$

where $s(A)$ is the vector of singular values of $A \in \mathbb{C}^{n\times n}$. As it was mentioned before, there is an intimate relation between unitarily invariant norms and symmetric gauge functions. By [8, Thm. IV.2.1], if Φ is a symmetric gauge function on $\mathbb{R}^n$ then $|||A|||_\Phi := \Phi(s(A))$ is a unitarily invariant norm on $\mathbb{C}^{n\times n}$ and, conversely, if $|||\cdot|||$ is a unitarily invariant norm on $\mathbb{C}^{n\times n}$ then

$$\Phi_{|||\ |||}(x) = |||\operatorname{diag}(x)|||, \quad x \in \mathbb{R}^n,$$

defines a symmetric gauge function on $\mathbb{R}^n$. Furthermore, by [4, Cor. 2.5], $|||\cdot|||_\Phi$ is strictly convex if and only if Φ has the same property. Therefore, it is possible to extend Corollary 5.6 to every strictly convex unitarily invariant norm on $\mathbb{C}^{n\times n}$.

Corollary 5.8. *If $|||\cdot|||$ is a strictly convex unitarily invariant norm on $\mathbb{C}^{n\times n}$, then Q_0 is the unique minimal norm element in $\mathcal{Q}_\mathcal{S}$, i.e. if $Q \in \mathcal{Q}_\mathcal{S}$ satisfies $|||Q||| = |||Q_0|||$ then $Q = Q_0$.*

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Appendix A. Proofs

Along this appendix operators are treated as block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus (\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)) \oplus J(\mathcal{S}^\circ), \tag{A.1}$$

i.e. every $T \in L(\mathcal{H})$ is identified with the block-operator matrix

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \begin{matrix} \mathcal{S}^\circ \\ \mathcal{S} \ominus \mathcal{S}^\circ \\ \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ) \\ J(\mathcal{S}^\circ) \end{matrix}, \tag{A.2}$$

where $t_{ij} = P_i T|_{\mathcal{T}_j}$ if P_i denotes the orthogonal projection onto $\mathcal{T}_i$ and $\mathcal{T}_1 = \mathcal{S}^\circ$, $\mathcal{T}_2 = \mathcal{S} \ominus \mathcal{S}^\circ$, $T_3 = \mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)$ and $\mathcal{T}_4 = J(\mathcal{S}^\circ)$.

Proof of Proposition 3.1. Assume that J is the fundamental symmetry. Hence, $J = J^* = J^2$. Notice that $P_1 J|_{\mathcal{T}_i} = 0$ for $i = 1, 2, 3$ because $\mathcal{T}_1$ is J-orthogonal to $\mathcal{T}_i$ for $i = 1, 2, 3$. Also, $P_4 J|_{\mathcal{T}_i} = 0$ for $i = 2, 3, 4$ because $\mathcal{T}_4$ is J-orthogonal to $\mathcal{T}_i$ for $i = 2, 3, 4$. Then, since J is selfadjoint it follows that

$$J = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ a^* & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \end{matrix}. \tag{A.3}$$

On the other hand, the equation $J^2 = I$ can be rewritten as

$$\begin{cases} aa^* = I_{\mathcal{S}^\circ} \\ b^2 + cc^* = I_{\mathcal{S} \ominus \mathcal{S}^\circ} \\ c^* c + d^2 = I_{\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ)} \\ a^* a = I_{J(\mathcal{S}^\circ)} \\ bc + cd = 0 \end{cases}. \tag{A.4}$$

Then, $a \in L(\mathcal{T}_4, \mathcal{T}_1)$ is an isometric isomorphism. Furthermore, $\mathcal{S}$ is pseudo-regular if and only if $\mathcal{S} \ominus \mathcal{S}^\circ$ is regular, and the regularity of $\mathcal{S} \ominus \mathcal{S}^\circ$ is equivalent to the range inclusion

$$R(c) \subseteq R(b),$$

see [12, Prop. 3.3]. Hence, the second equation in (A.4) implies that $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq R(b)$. Hence, b is an invertible selfadjoint operator in $L(\mathcal{S} \ominus \mathcal{S}^\circ)$. Analogously, $d \in GL(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ))$.

Observe that $c^* c < I$ because $d^2 = I - c^* c \in GL(\mathcal{T}_3)^+$. Therefore, $\|c\| < 1$. Also, if $b = u|b|$ is the polar decomposition of b, it follows that $|b|^2 = b^2 = I - cc^*$. Hence, $|b| = (I - cc^*)^{1/2}$ and u is a symmetry commuting with $(I - cc^*)^{1/2}$. Thus, u also commutes with $|c^*|$.

Analogously, if $d = v|d|$ is the polar decomposition of d, it is easy to see that $|d| = (I - c^* c)^{1/2}$ and v is a symmetry commuting with $|c|$.

Finally, notice that the condition $bc + cd = 0$ can be rephrased as

$$uc = -cv,$$

because $(I - cc^*)^{1/2}$ is invertible and $(I - cc^*)^{1/2} c = c(I - c^* c)^{1/2}$. □

Proof of Proposition 3.3. Given $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, consider the block-operator matrix

$$E = \begin{pmatrix} 0 & ar^*b & ar^*c & ar^*br \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $E^2 = E$ and, recalling that $a^*a = I_{J(\mathcal{S}^\circ)}$, it follows that

$$JE = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & c & br \\ 0 & c^* & c^*b^{-1}c & c^*r \\ 0 & r^*b & r^*c & r^*br \end{pmatrix},$$

which is clearly selfadjoint. Therefore, $E \in L(\mathcal{H})$ is a J-selfadjoint projection. The assertion on the range of E is immediate.

Conversely, if $\mathcal{M}$ is a regular complement of $\mathcal{S}^\circ$ in $\mathcal{S}$, denote by $E_\mathcal{M}$ the J-selfadjoint projection onto $\mathcal{M}$. Since $R(E_\mathcal{M}) = \mathcal{M} \subseteq \mathcal{S}$ it follows that $P_{\mathcal{S}^\perp}E_\mathcal{M} = 0$, so that the third and fourth row in the matrix representation of $E_\mathcal{M}$ are zero. Also, since $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(E_\mathcal{M})$, it follows that $E_\mathcal{M}P_{\mathcal{S}^\circ} = 0$. So that the first column is also zero. Therefore,

$$E_\mathcal{M} = \begin{pmatrix} 0 & u & v & w \\ 0 & p & q & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $u \in L(\mathcal{S} \ominus \mathcal{S}^\circ, \mathcal{S}^\circ)$, $v \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, $w \in L(J(\mathcal{S}^\circ), \mathcal{S}^\circ)$ and $p \in L(\mathcal{S} \ominus \mathcal{S}^\circ)$, $q \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$, $r \in L(J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ satisfy

$$\begin{cases} up = u \\ uq = v \\ ur = w \end{cases} \quad \text{and} \quad \begin{cases} p^2 = p \\ pq = q \\ pr = r \end{cases}.$$

Thus, $p = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_\mathcal{M}|_{\mathcal{S}\ominus\mathcal{S}^\circ}$ is a projection with

$$R(p) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_\mathcal{M}(\mathcal{S} \ominus \mathcal{S}^\circ) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_\mathcal{M}(\mathcal{S}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{M}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{S}) = \mathcal{S} \ominus \mathcal{S}^\circ,$$

because $\mathcal{S}^\circ \subseteq N(P_{\mathcal{S}\ominus\mathcal{S}^\circ}) \cap N(E_\mathcal{M})$. Hence, $p = I_{\mathcal{S}\ominus\mathcal{S}^\circ}$.

Furthermore, $E_\mathcal{M}$ is J-selfadjoint if and only if

$$JE_\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & bq & br \\ 0 & c^* & c^*q & c^*r \\ 0 & a^*u & a^*v & a^*w \end{pmatrix}$$

is selfadjoint, or equivalently, if

$$\begin{cases} bq = c \\ br = u^*a \end{cases} \quad \text{and} \quad \begin{cases} a^*v = r^*c \\ a^*w = w^*a \end{cases}. \tag{A.5}$$

First, notice that $q = b^{-1}c$. Then, by (A.4), $aa^* = I_{\mathcal{S}^\circ}$ and it follows that $u = ar^*b$ and $v = ar^*c$. Finally, $w = ur$ implies that

$$w = ar^*br.$$

Therefore,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & ar^*b & ar^*c & ar^*br \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.6}$$

and the proof is completed. $\square$

Finally, a block-matrix representation of a projection $Q \in L(\mathcal{H})$ onto $\mathcal{S}$ is needed. Since Q is a projection onto $\mathcal{S}$, Q is given by

$$Q = \begin{pmatrix} I & 0 & x_1 & x_2 \\ 0 & I & y_1 & y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.7}$$

where $x_1 \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, $x_2 \in L(J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, $y_1 \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$ and $y_2 \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S} \ominus \mathcal{S}^\circ)$.

Furthermore, if $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ then $P = Q - E_{\mathcal{M}}$ is a projection onto $\mathcal{S}^\circ$ such that $PP^\# = P^\# P = 0$. Moreover, by (A.6), P has the form

$$P = Q - E_{\mathcal{M}} = \begin{pmatrix} I & -ar^*b & x_1 - ar^*c & x_2 - ar^*br \\ 0 & 0 & y_1 - b^{-1}c & y_2 - r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

But, $R(P) = \mathcal{S}^\circ$ if and only if

$$y_1 = b^{-1}c \quad \text{and} \quad y_2 = r.$$

Also, $PP^\# = 0$ if and only if $PJP^* = 0$, or equivalently,

$$\begin{pmatrix} I & -ar^*b & z_1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ a^* & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ -bra^* & 0 & 0 & 0 \\ z_1^* & 0 & 0 & 0 \\ z_2^* & 0 & 0 & 0 \end{pmatrix} = 0,$$

where $z_1 = x_1 - ar^*c$ and $z_2 = x_2 - ar^*br$. But the above equation is equivalent to

$$2\,\mathrm{Re}(z_2a^*) - 2\,\mathrm{Re}(z_1c^*bra^*) + z_1dz_1^* + ar^*b^3ra^* = 0. \tag{A.8}$$

Since $z_2 = x_2 - ar^*br$,

$$2\,\mathrm{Re}(z_2a^*) = 2\,\mathrm{Re}(x_2a^* - ar^*bra^*) = 2\,\mathrm{Re}(x_2a^*) - 2ar^*bra^*. \tag{A.9}$$

Analogously, since $z_1 = x_1 - ar^*c$,

$$2\,\mathrm{Re}(z_1c^*bra^*) = 2\,\mathrm{Re}(x_1c^*bra^* - ar^*cc^*bra^*) = 2\,\mathrm{Re}(x_1c^*bra^*) - 2ar^*cc^*bra^*, \tag{A.10}$$

because $ar^*cc^*bra^* = -ar^*cdc^*ra^*$ is selfadjoint. Also,

$$z_1dz_1^* = (x_1 - ar^*c)d(x_1^* - c^*ra^*) = x_1dx_1^* - 2\,\mathrm{Re}(x_1dc^*ra^*) + ar^*cdc^*ra^*. \tag{A.11}$$

Using the identities of (A.9), (A.10) and (A.11) in (A.8), it follows that

$$\begin{aligned}&2\,\mathrm{Re}(x_2a^*) - 2\,\mathrm{Re}(x_1c^*bra^*) + x_1dx_1^* - 2\,\mathrm{Re}(x_1dc^*ra^*) + ar^*b^3ra^* = \\ &\quad = 2ar^*bra^* - 2ar^*cc^*bra^* - ar^*cdc^*ra^*.\end{aligned}$$

Hence, recalling that $c^*b = -dc^*$:

$$\begin{aligned}2\,\mathrm{Re}(x_2a^*) + x_1dx_1^* &= -ar^*b^3ra^* + 2ar^*bra^* - ar^*bcc^*ra^* \\ &= 2ar^*bra^* - ar^*b(b^2 + cc^*)ra^* \\ &= 2ar^*bra^* - ar^*bra^* = ar^*bra^*.\end{aligned}$$

Therefore,

$$\mathrm{Re}(x_2a^*) = \tfrac{1}{2}(ar^*bra^* - x_1dx_1^*). \tag{A.12}$$

Then, considering the operators $A = x_2a^* \in L(\mathcal{S}^\circ)$ and $B = x_1 \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, it follows that

$$x_2 = (x_2a^*)a = (\tfrac{1}{2}(A - A^*) + \tfrac{1}{2}(ar^*bra^* - BdB^*))a.$$

Thus,

$$Q = \begin{pmatrix} I & 0 & B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.13}$$

and it is possible to parametrize the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ as follows:

Proof of Theorem 3.4. Suppose that $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$, i.e. $Q \in L(\mathcal{H})$ is a J-normal projection onto $\mathcal{S}$ satisfying $QQ^\# = Q^\# Q = E_\mathcal{M}$. Then, $P = Q - E_\mathcal{M}$ is a projection onto $\mathcal{S}^\circ$ such that $PP^\# = P^\# P = 0$. Hence, by the discussion above, there exist $A \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$ such that Q can be written as in (A.13).

Conversely, given $A \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp \ominus J(\mathcal{S}^\circ), \mathcal{S}^\circ)$, consider

$$Q = \begin{pmatrix} I & 0 & B & \frac{1}{2}(A - A^* + ar^*bra^* - BdB^*)a \\ 0 & I & b^{-1}c & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows immediately that $P = Q - E_\mathcal{M}$ satisfies $PP^\# = P^\# P = 0$, i.e. $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$. □

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