International Journal of Bifurcation and Chaos, Vol. 20, No. 2 (2010) 331–339
© World Scientific Publishing Company
DOI: 10.1142/S0218127410025788

ABOUT THE EFFECTIVENESS OF DIFFERENT METHODS FOR THE ESTIMATION OF THE MULTIFRACTAL SPECTRUM OF NATURAL SERIES

ALEJANDRA FIGLIOLA, EDUARDO SERRANO*,
GUSTAVO PACCOSI and MARIEL ROSENBLATT
Instituto del Desarrollo Humano,
Universidad Nacional de General Sarmiento,
J. M. Gutiérrez 1150, Provincia de Buenos Aires, Argentina
**Escuela de Ciencia y Tecnología,*
Universidad Nacional de San Martín, Irigoyen 3100,
San Martín, Provincia de Buenos Aires, Argentina

Received November 12, 2008; Revised April 9, 2009

Complex natural systems present characteristics of *scalar invariance*. This behavior has been experimentally verified and a large related bibliography has been reported. *Multifractal Formalism* is a way to evaluate this kind of behavior. In the past years, different numerical methods to estimate the multifractal spectrum have been proposed. These methods could be classified into those that originated from the *wavelet analysis* and others from numerical approximations like the *Multifractal Detrended Fluctuation Analysis* (MFDFA), proposed by Kantelhardt and Stanley. Recently, S. Jaffard and co-workers proposed the *Wavelet Leaders* (WL) method that exploits the potential of wavelet analysis and the efficiency of the *Multiresolution Wavelet Schema*.

In a previous work, we checked that both methods are equivalent for estimating fractal properties in a series from singular measures. Now, we apply MFDFA and WL to natural signals with self-similar structures, but unknown multifractal spectrum. We observe that some differences appear in their respective estimations, particularly when the signals are corrupted with fractional Gaussian noise.

Keywords: Multifractal formalism; multifractal detrended fluctuation analysis; wavelet leaders, Hölder regularity.

1. Introduction

Frequently, complex natural systems present the characteristics of *scalar invariance*, behavior that has been experimentally verified in recent years, and exists a very rich related bibliography there reported from very diverse fields as physics, ecology, biology or economy. For more details, refer to [Stanley *et al.*, 2000; Ashkenazy *et al.*, 2001; Telesca *et al.*, 2005; Figliola *et al.*, 2007a, 2007b; Zunino *et al.*, 2008].

Multifractal Formalism, essentially, is based on the calculation of two sets of coefficients associated to the signals: the *Hölder exponents* H, that quantify local regularity of the signal or function f and the *multifractal spectrum* (MS) d_f that quantifies the multifractality of f. The MS associates each group of data with the same pointwise regularity (given by Hölder exponents) as the Hausdorff dimension of this set of points. In this way, a function is generated between the Hölder exponents and the Hausdorff dimension that is also known as *spectrum of singularities* [Falconer, 1997].

In 1995, Peng and co-workers presented the *Detrended Fluctuation Analysis* (DFA) [Peng *et al.*, 1995] developed to evaluate the scaling properties from a signal. The application of DFA had great success in the study of different fields such as the sequences of DNA or signals from physiological or economical phenomena [Stanley *et al.*, 2000; Zunino *et al.*, 2008]. A generalization of the DFA is the *Multifractal Detrended Fluctuation Analysis* (MFDFA), that was introduced by the pioneering work of Castro e Silva and Moreira and developed more recently by Kantelhardt and co-workers, who extended the DFA algorithm for the series with multifractal behavior or at least self-similarity behavior [Castro e Silva & Moreira, 1997; Kantelhardt *et al.*, 2002].

In recent years, Stéphane Jaffard has proposed another methodology for the characterization of Hölder exponents and their relationship with Hölder regularity and local oscillations. Jaffard named this method as *Wavelet Leaders* (WL) and he presented a new formulation in terms of the local suprema of the wavelet coefficients of the transformed signal, which he called the *wavelet leader* [Jaffard, 2004; Leshermes *et al.*, 2005].

The aim of this work is to compare the efficiency of WL and MFDFA methods for the estimation of the MS in natural series, as the case from an electroencephalogram (EEG) signal of a tonic–clonic epileptic seizure. In particular, we study the changes in the spectrum when the signal is corrupted with uncorrelated noise and with a signal of fractional Gaussian noise in two cases: anticorrelated and correlated monofractal series.

2. Multifractal Formalism

Let $\alpha \geq 0$ and $x_0 \in R$ and a locally bounded function $f : R^d \to R$. We say that $f \in C^\alpha(x_0)$ if there exists a constant $C > 0$ and a polynomial P with degree $\deg(P) < \alpha$ such as:

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \tag{1}$$

near the point x_0. The *pointwise Hölder exponent* of f in x_0 is:

$$H_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\} \tag{2}$$

These exponents describe the local regularity of f at x_0. If $H_f(x_0) < 1$, the function is not differentiable in x_0, and $P(x - x_0) = f(x_0)$. In this case, the exponent describes the nonsmoothness or the ruggedness of the function at the point and it expresses how spiky its graph is. The *multifractal analysis* becomes relevant when the set of points having the same Hölder exponents is dense inside the signal's domain. Following in this direction, we will denote:

$$E_f(H) = \{x \in \mathrm{dom}(f) : H_f(x) = H\} \tag{3}$$

the set of points having the same Hölder regularity. The MS, denoted by $d_f(H)$, is the *Hausdorff dimension* of any set $E_f(H)$, [Falconer, 1997; Mallat, 1999; Jaffard, 2004].

If $E_f(H) \neq \emptyset$, we have $0 \leq d_f(H) \leq 1$. The spectrum is fractal if $0 < d_f(H) < 1$ for some H and it is multifractal if the same holds for different values of H.

In this way, the spectrum will give us a clear characterization of the global distribution of Hölder exponents, the fractal properties of underlying phenomena of the signal.

Let f be an unidimensional and locally bounded function. First, we define a numerable set of intervals:

$$I_{jk} = [r_j k, r_j(k+1)) \tag{4}$$

where r_j is a decreasing sequence and $\lim_{j\to+\infty} r_j = 0$. Note that $\mid I_{jk} \mid = r_j$. We will denote by $I_j(x_0)$ the unique interval of size r_j such that $x_0 \in I_{jk}$. We also denote by $MI_j(x_0)$ a M-dilation of the interval $I_j(x_0)$.

We suppose that there exists a non-negative set function:

$$\mu : \{I\} \to R_{\geq} 0 \tag{5}$$

related with Hölder exponents in this way:

$$\mu(MI_j(x_0)) \sim r_j^{H_f}(x_0) \tag{6}$$

for some dilation M. Then:

$$H_f(x_0) = \liminf_{r_j \to 0} \frac{\log(\mu(MI_j(x_0)))}{\log(r_j)} \tag{7}$$

In this case, it is possible to define the following *scaling function* (some authors named this function as *structure function*):

$$S(q, j) = r_j \sum_{k \in K_j} \mu(MI_{jk})^q \tag{8}$$

where K_j is the set of indexes such that $\mu(I_{jk}) > 0$.

The above hypothesis let us in Eq. (6) or (7), assert that the contribution to $S(q,j)$ of the intervals $I_j(x_0), (x_0 \in E_f(H))$ is given by $\mu(MI_{jk}) = r_j^H$.

Since we need $r_j^{-d_f(H)}$ intervals to cover $E_f(H)$, according to the Hausdorff dimension definition, it is possible to deduce:

$$S(q,j)(H) \sim r_j^{(1-d_f(H)+Hq)} \tag{9}$$

and considering when $r_j \to 0$ the principal contribution is from the smallest exponents, we can define the function:

$$\eta(q) = \inf_H (1 - d_f(H) + Hq) \tag{10}$$

Finally, if the function η is concave and supposing that the spectrum d_f is concave indeed, the duality relation for concave function gives us the formula:

$$d_f(H) = \inf_q (1 - \eta(q) + Hq) \tag{11}$$

and the formalism scheme is complete. We emphasize that this scheme has no empty domain. Because of the concavity, Hölder exponents and the parameters q could be related by the *Legendre Transform* of the function $\eta(q)$. More precisely, the exponent $H(q)$ is the unique value satisfying the equation:

$$H(q) = \frac{d\eta}{dq}(q) \tag{12}$$

for each $q \in R$.

In general, we can say that if μ is an increasing and nonnegative function formula, then $d_f(H)$ of Eq. (11) holds for dyadic intervals. Moreover, if μ is a Borel measure η is a concave function.

However, although we can choose the covering $\{I_{jk}\}$ and the function μ, we cannot ensure that $d_f(H)$ is a concave function and this is independent of our scheme design. Then, the multifractal formalism becomes a well intentioned tool for estimating the fractal properties of the signal. Moreover, in many cases the signal is given only by samples and the formalism must be discretized in some way. Actually, the problem is open to many choices, alternatives and variants. See [Jaffard, 2004] for comments and details.

In the next section we revise an alternative developed by Kantelhardt *et al.* [2002]: the MFDFA method.

3. Multifractal Detrended Fluctuation Analysis

The MFDFA method was presented as a generalization of DFA method and the advantage when compared with other methods has been proved [Kantelhardt *et al.*, 2002; Oświęcimka *et al.*, 2006]. The implementation of MFDFA does not involve more effort than the conventional DFA, just one additional step is required.

The MFDFA multifractal spectrum estimation of a one-dimensional series $\{x(i), i = 1, \ldots, N\}$, is based on the construction and the analysis of the *fluctuation function*, that is defined as:

$$F_s^2(\nu) = \frac{1}{s}\sum_{i=1}^{s}\{Y_s[(\nu-1)s+i]\}^2. \tag{13}$$

To obtain Eq. (13), we first calculate the *profile* of the series by the integration: $Y(k) = \sum_{i=1}^{k}[x(i) - \langle x \rangle]$, where $\langle x \rangle$ is the mean value of the series $\{x(i)\}$. The profile is cut into $N_s = N/s$ nonoverlapping segments of equal length s. The detrended time series for segment s, denoted by $Y_s(i)$, is calculated as the difference between the original time series and the fits,

$$Y_s(i) = Y(i) - p_\nu(i), \tag{14}$$

where $p_\nu(i)$ is the fitting polynomial in the νth segment. Since, we use a polynomial fit of order 1, we denote the algorithm as 1-MFDFA, or for simplicity MFDFA. As the detrending of the time series is done by subtraction of the fits from the profile, these methods differ in their capability of eliminating trends in the data. For each of the N_s segments, the variance of the detrended time series $Y_s(i)$ is evaluated by averaging over all data points i in the νth segment. Then, averaging over all segments, it is possible to obtain the qth fluctuation function:

$$F_q(s) = \left\{\frac{1}{2N_s}\sum_{\nu=1}^{2N_s}[F_s^2(\nu)]^{q/2}\right\}^{1/q}, \tag{15}$$

where, in general, the index q can take any real value. For $q = 2$, the standard DFA procedure is retrieved. The scaling behavior of the fluctuation function is determined by analyzing log–log plots $F_q(s)$ versus s for each value of q. If the series $x(i)$ is long-range power-low correlated $F_q(s)$ increases, for large values of s, as a power-law:

$$F_q(s) \sim s^{h(q)}. \tag{16}$$

For more details, see [Peng *et al.*, 1995; Kantelhardt *et al.*, 2002].

For monofractal time series with compact support, $h(q)$ is independent of q, since the scaling behavior of the variance $F_s^2(\nu)$ is identical for all segments ν and the averaging procedure in Eq. (15) will give just this identical scaling behavior for all values of q. Only if small and large fluctuations scale differently, there will be a significant dependence of $h(q)$ on q. If we consider positive values of q, the segments ν with large variance $F_s^2(\nu)$ will dominate the average $F_q(s)$. Thus, for positive values of q, $h(q)$ describes the scaling behavior of the segments with large fluctuations. On the contrary, for negative values of q, the segments ν with small variance $F_s^2(\nu)$ will dominate the average $F_q(s)$. Hence, for negative values of q, $h(q)$ describes the scaling behavior of the segments with small fluctuations. When $q = 2$, $h(2)$ is the *Hurst exponent*. We will note the value of $h(2)$ as h.

Following from Eqs. (15) and (16) and assuming that the length N of the series is an integer multiple of the scale s,

$$\sum_{\nu=1}^{N/s} |Y(\nu s) - Y((\nu-1)s)|^q \sim s^{q\ h(q)-1} \tag{17}$$

Kantelhardt and co-workers showed that this multifractal formalism corresponds with the standard box counting theory and they related both formalisms. It is obvious that the term $|Y(\nu s) - Y((\nu - 1)s)|$ is identical to the sum of the numbers $x(i)$ within each segment ν of size s. This sum is the box probability $p_s(\nu)$ in the standard formalism for normalized series $x(i)$.

The scaling exponent $\eta(q)$ is usually defined via the partition function $Z_q(s)$,

$$Z_q(s) \equiv \sum_{\nu=1}^{N/s} |p_s(\nu)|^q \sim s^{\eta(q)} \tag{18}$$

where q is a real parameter.

Now, we can identify the partition function $Z_q(s)$ with the scaling function $S(q, j)$ defined by Eq. (8) taking the scale s as the j-interval r_j. So, using Eqs. (8) and (18), we conclude that they are identical and we obtain the relation between the two sets of multifractal scaling exponents:

$$\eta(q) = qh(q) - 1. \tag{19}$$

The Hölder exponent H and the multifractal spectrum $d_f(H)$ are related by $\eta(q)$ via a Legendre transform:

$$H = \eta'(q) \tag{20}$$

and

$$d_f(H) = qH - \eta(q), \tag{21}$$

From Eq. (16), we can conclude that

$$(F_q)^q(s) \cong s^{qh(q)} \tag{22}$$

for some function $h(q)$. If we identify

$$F_q^q(s) \sim S(q, j) \sim s_j^{1-d_f(H)+Hq} \tag{23}$$

it is possible to suppose that

$$qh(q) = \inf_H[1 - d_f(H) + Hq], \tag{24}$$

then

$$d_f(H) = \inf_q[1 - qh(q) + Hq] \tag{25}$$

Then, MFDFA can be framed into the multifractal formalism.

4. Wavelet Methods

The above described multifractal formalism is viewed to improve the original one proposed by Frisch and Parisi [1985] based on the scaling function:

$$S_{fp}(q, j) = 2^{-j} \sum_k \left| f\left(\frac{k+1}{2^j}\right) - f\left(\frac{k}{2^j}\right)\right|^q \tag{26}$$

Here, the associated measure function

$$\mu_{fp}(I_{jk}) = \left| f\left(\frac{k+1}{2^j}\right) - f\left(\frac{k}{2^j}\right)\right| \tag{27}$$

is defined for the dyadic intervals $I_{jk} = [(k/2^j), ((k+1)/2^j))$ for all integers $j > 0$ and k and it consists of first-order discrete differences.

We could replace the above measure by:

$$\mu'(I_{jk}) = 2^j \left| \int_{k/2^j}^{(1+k)/2^j} f(x) - f(x+1)dx \right| \tag{28}$$

or, more generally, by the *wavelet coefficients* of f.

Since wavelets are natural tools in multifractal analysis, because the concept of self-similarity is implicit in the wavelet analysis, the wavelet coefficients provide a time-scale decomposition of the signal f reflecting the scaling properties.

Now, we can define the multifractal formalism based on the wavelet coefficients: the *Wavelet Continous Multifractal Formalism*, from the scaling function:

$$S_{wc}(q, j) = 2^{-j} \sum_{k \in K_j} |c_{jk}|^q \tag{29}$$

where K_j is the set of indexes such that $|c_{jk}| > 0$ are the wavelet coefficients of the discrete wavelet transform of the function f.

The $\mu_{wc}(I_{jk}) = |c_{jk}|$ is a non-negative function and we stress that it may be non-increasing.

On the other hand, there is a direct correlation between wavelet coefficients and pointwise Hölder regularity [Jaffard, 2004]. If $f \in C^\alpha(x_0)$ then, for all $j > 0$,

$$|c_{jk}| \leq C2^{-j\alpha}(1 + |2^j x_0 - k|)^\alpha \tag{30}$$

for some constant $C > 0$. When f has a nonoscillating singularity in x_0, like a *cusp point*, the significant coefficients lie near the point:

$$|c_{jk}| \sim C2^{-j\alpha} \tag{31}$$

and the local measure is increasing around the point. But this is not the case when the function has oscillating singularities, like a *chirp*. Then, the maximum coefficients may be placed far from the singular point and the last property fails. These circumstances are obstacles for the multifractal formalism. Particularly, the scaling function becomes unstable for negative values of the parameter q, (see [Jaffard, 2004] for more details).

5. The Wavelet Leader Multifractal Formalism

In recent years, Stéphane Jaffard and co-workers developed a new wavelet method for the characterization of pointwise Hölder exponent and the relationship between Hölder regularity and local oscillation. They give the formulation of the criterion in terms of the local suprema of the wavelet coefficients, called *Wavelet Leaders*.

Let ψ be an orthonormal wavelet, smooth, having fast decay and several null moments. It is centered in $x_c = 1/2$ and it is essentially localized on the interval $[0, 1]$. We suppose that each wavelet coefficient c_{jk} — corresponding to the wavelet transform of the signal f or of the series $\{x(i)\}$ — is localized on the dyadic interval $I_{jk} = [(k/2^j), (k + 1/2^j))$ [Meyer, 1990; Mallat, 1999].

We will denote $3I_{jk} = [((k-1)/2^j), ((k+2)/2^j))$ the dilated intervals, recalling that we can choose another scaling factor $M \geq 3$. Then, *wavelet leaders* d_{jk} are defined as follows:

$$d_{jk} = \sup\{|c_{lh}| : I_{lh} \subset 3I_{jk}\} \tag{32}$$

The last definition indicates that to compute d_{jk} we consider the indexes $2^{l-j}(k-1) \leq h \leq 2^{l-j}(k+2)$, for each $l \geq j - 1$.

It defines the set function:

$$\mu_{wl}(3I_{jk}) = d_{jk} \tag{33}$$

This is a non-negative function and it is increasing, that is, if $3I_{j'k'} \subset 3I_{jk}$ then $d_{j'k'} \leq d_{jk}$ and it follows that $\mu_{wl}(3I_{j'k'}) \leq \mu_{wl}(3I_{jk})$. Moreover, this is proof of the suitable correlation between the wavelet leaders and the pointwise Hölder regularity [Jaffard, 2004].

Let $I_j(x_0)$ be the unique dyadic interval containing the point x_0 and $d_j(x_0)$ the corresponding wavelet leader. If $f \in C^\alpha(x_0)$ then, for all $j > 0$,

$$|d_j(x_0)| \leq C2^{-j\alpha} \tag{34}$$

for some constant $C > 0$.

It is possible to consider that the wavelet leaders can compute a suitable scaling formula for the full parameter range and also holds when the method is applied to signal embodying oscillating singularities [Jaffard, 2004; Leshermes *et al.*, 2005].

Then, it defines the scaling function:

$$S_{wl}(q, j) = 2^{-j} \sum_{k \in K_j} d_{jk}^q \tag{35}$$

where K_j is the set of the indexes such that $|c_{jh}| > 0$ for some $I_{jh} \in I_{jk}$ [Jaffard, 2004].

So, following the development in (9) and (10) and from (23) and (24), it is possible to deduce:

$$\eta_{wl}(q) = \liminf_{j \to +\infty} \frac{\log(S_{wl}(q, j))}{\log(2^j)} \tag{36}$$

Due to above properties of μ_{wl}, the function η is concave. Suppose that the spectrum d_f is concave indeed, we can complete the multifractal formalism, in the following way:

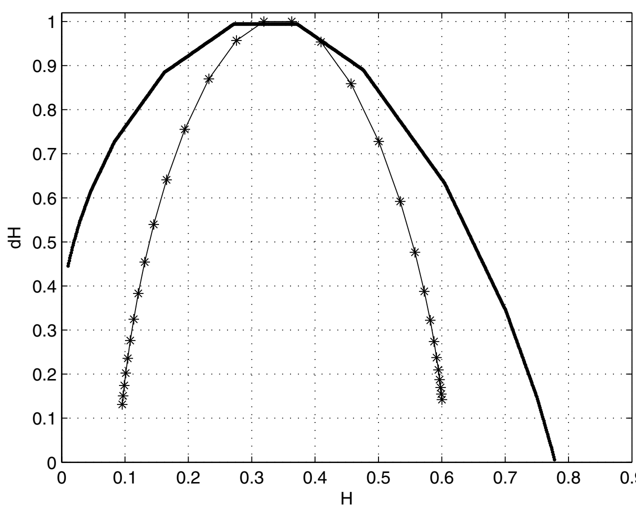
$$d_f(H) = \inf_q (1 - \eta_{wl}(q) + Hq) \tag{37}$$

and MFDFA and WL methods are framed into the multifractal formalism. (See [Serrano & Figliola, 2009] for more details.)

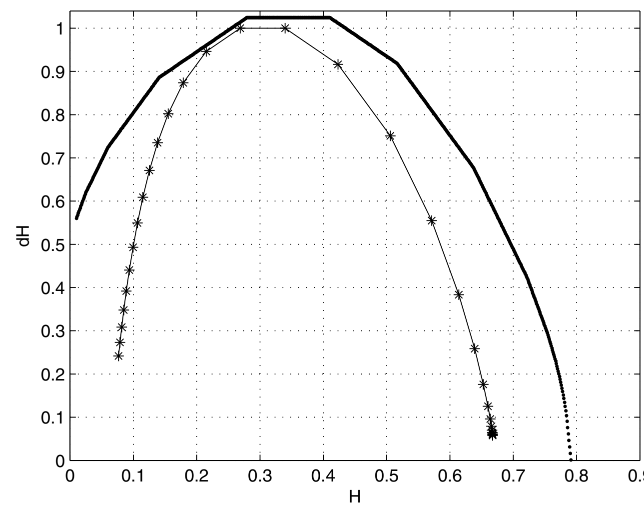
6. Applications

We apply both estimators, MFDFA and WL, to the multifractal analysis of three natural series corresponding to an EEG from an epileptic seizure for the same subject. For the three cases, we use the first period of the seizures corresponding to the tonic stages. The data is collected from a secondary generalized *tonic-clonic epilepsy seizure* from a female epileptic patient. The diagnosis of the patient is pharmaco-resistant epilepsy and has no other accompanying disorders. Scalp electrodes with bimastoideal reference were applied following the 10–20 international system. Each signal was digitized at 409.6 Hz through a 12 bit A/D converter and filtered with an "anti-aliasing" 8 pole low pass Bessel filter, with a cutoff frequency of 50 Hz. Then, the signal was digitally filtered with a 1–50 Hz bandpass filter (medical frequency range of interest for diagnosis). The series were analyzed at the right central region, $C4$ derivation. This electrode has been chosen after visual inspection of the EEG, by the physician's team, as the one with the least number of artifacts. The time intervals of the pre-ictal stage which present artifacts (ocular and other movements, etc.) were marked by the physician's team and were excluded in the subsequent analysis.

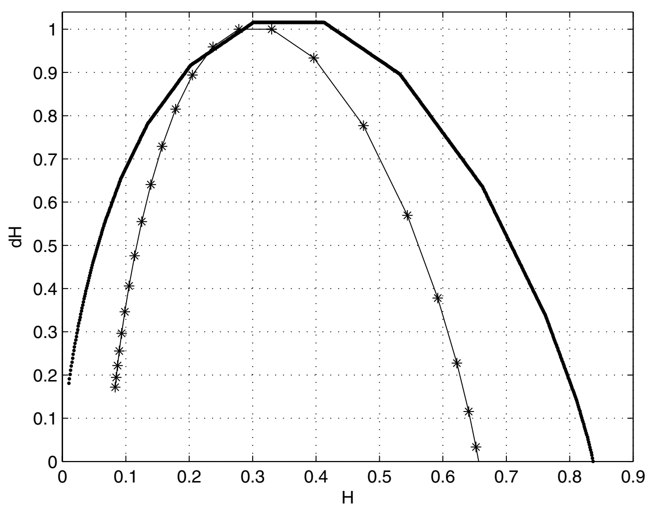
We have more than enough evidence of the multifractal behavior of the EEG signals from an epileptic seizures and also in EEG from NO REM sleep [Pan *et al.*, 2004; Figliola *et al.*, 2007a, 2007b; Rosenblatt & Figliola, 2007]. For this work, we confirm the experimental evidence by using three different series corresponding to tonic stages. Figures 1(a)–1(c) show the MS estimated from MFDFA and WL methods for the three series, with 4096 data points each one. We enumerate them as 1, 2 and 3, respectively.



(a)



(b)



(c)

Fig. 1. Multifractal spectrum estimated from MFDFA method (fine solid line with asterisks) and from WL method (thick solid line) for the EEG tonic-clonic epileptic seizures series corresponds to the series (a) 1, (b) 2 and (c) 3.

Table 1. Parameters estimated with WL and MFDFA methods for the series. The subindex points out each series.

	WL_1	$MFDFA_1$	WL_2	$MFDFA_2$	WL_3	$MFDFA_3$
$H_{\min}$	0.010	0.096	0.010	0.076	0.010	0.083
$H_{\max}$	0.778	0.600	0.791	0.668	0.837	0.672
ΔH	0.768	0.505	0.781	0.591	0.872	0.590
H^*	0.272	0.341	0.280	0.304	0.301	0.304

We choose four parameters to characterize the spectrum. They are: $H_{\min}$ and $H_{\max}$ that correspond to the minimum and maximum of the Hölder set: H; the range $\Delta H = H_{\max} - H_{\min}$ and the H^* that correspond to the Hölder of the maximum Hausdorff dimension (i.e. $d_f(H^*) = \max(d_f)$).

Table 1 synthesizes the differences between both methods for the estimation of the multifractal spectrum of the three series above mentioned.

Note that in all cases, the WL method estimates larger Hölder exponents range (ΔH) than the MFDFA method, but the H^*s are similar.

We analyze the effects when we add *fractional Gaussian noise* in the natural series. The *fractional Brownian motion* (fBm) is the only family of processes which is Gaussian, self-similar, and endowed with stationary increments. The normalized family of these Gaussian processes $\{B^h(t), t > 0\}$ is the one with $B^h(0) = 0$ zero mean, and covariance given by:

$$E[(B^h(t_1), B^h(t_2)] = \frac{1}{2}(t_1^{2h} + t_2^{2h} - |t_1 - t_2|^{2h})$$

$E[\cdot]$ refers to the average computed with a Gaussian probability density and $t_1, t_2 \in R$. The exponent h — mentioned in Sec. 3 — is the *Hurst exponent* and $0 \leq h \leq 1$. The fGn is introduced as the process $\{W^h(t), t > 0\}$, obtained from the fBm increments for discrete time [Palma, 2007; Rosso *et al.*, 2007].

$$W^h(t) = B^h(t+1) - B^h(t)$$

This is a stationary Gaussian process with zero mean and covariance given by:

$$\begin{aligned} Cov(k) &= [W^h(t), W^h(t+1)] \\ &= \frac{1}{2}[(k+1)^{2h} - 2k^{2h} + |k-1|^{2h}], \quad k > 0 \end{aligned}$$

The power spectrum associated to the fGn is given by $\Phi_{W^h}(\omega) \propto (1/|\omega|^\beta)$, with $\beta = 2h - 1$. (Note that $\beta = \eta(2)$ from Eq. (19).)

By using $h < 0.5$ we can generate a long-range anti-correlated, $h = 0.5$ uncorrelated, or with $h > 0.5$ positively long-range correlated monofractal series. When we add uncorrelated noise, no significant changes are observed at the spectra, also when the amplitude of the noise is about the 50% of the maximum amplitude of the signal. Nevertheless when $h \neq 0.5$, these appear unimportant differences.

Figure 2 shows the multifractal spectra estimated by WL and MFDFA methods when we add to the natural series number 1 correlated fGn with $h = 0.8$. The maximum amplitude of the fGn is 50% of the maximum amplitude of the natural series. Figure 3 shows the same MS for the same series but with a Hurst exponent $h = 0.3$ for the additional fGn process and with the same proportion between the amplitude of the series and the fGn.

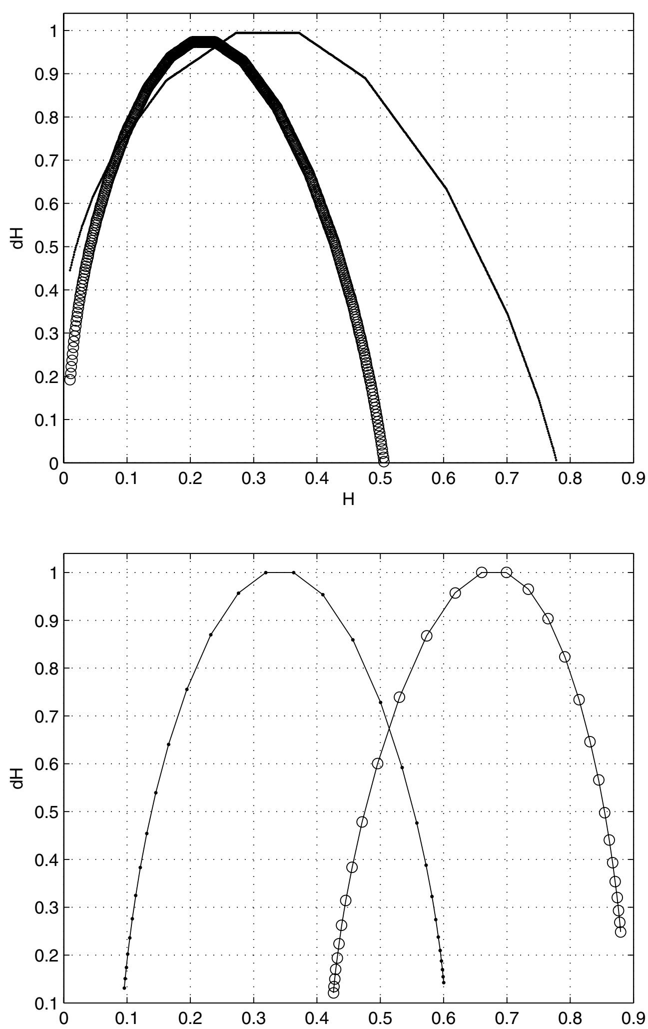
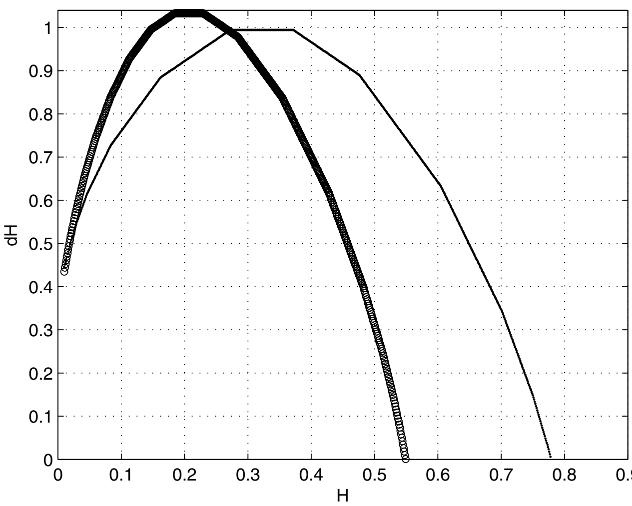


Fig. 2. Top: Multifractal spectrum estimated from WL. The fine line corresponds to the natural series of an EEG tonic-clonic epileptic seizures and the thick line is the natural series with fGn with $h = 0.8$. Bottom: Multifractal spectrum estimated from MFDFA. The fine solid line with asterisks corresponds to the natural series and the line with empty circles the natural series with fGn with $h = 0.8$.



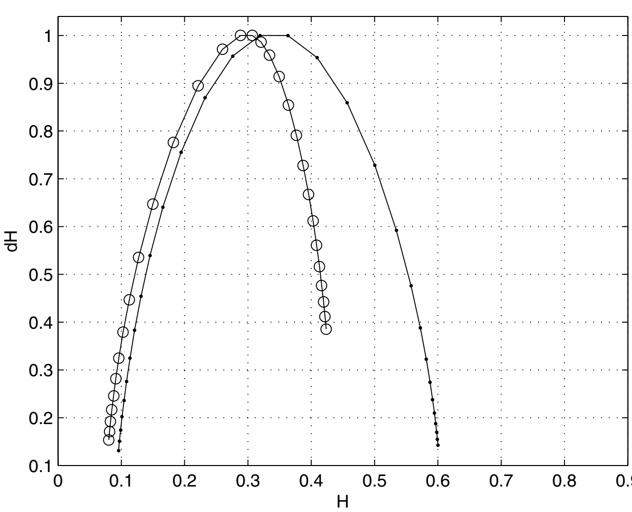


Fig. 3. Top: Multifractal spectrum estimated from WL. The fine line corresponds to the natural series of an EEG tonic-clonic epileptic seizures and the thick line is the natural series with fGn with $h = 0.3$. Bottom: Multifractal spectrum estimated from MFDFA. The fine solid line with little asterisks corresponds to the natural series and the line with empty circles corresponds to the natural series with fGn with $h = 0.3$.

It is very interesting to note that the spectrum of the series with fGn estimated by MFDFA method is translated approximately by keeping its shape. H^* (that corresponds to the Hölder of the maximum Hausdorff dimension) is close to Hurst exponent: in the case of correlated fGn series $h = 0.8$ and $H^* = 0.699$ and in the case of anti-correlated fGn series $h = 0.3$ and $H^* = 0.343$. Also, we remark that in this case, the shape of the spectrum with noise is more narrow than the spectrum without noise (seems that it tends to a monofractal spectrum). On the other hand, the multifractal spectrum of the series plus fGn is narrowed by estimating it using the WL method, translating it into a range of minor Hölder exponents, for the uncorrelated case as well for the correlated one.

7. Conclusions

The aim of this work is to analyze the effectiveness of the two presented methods for estimating the multifractal spectrum of natural series. In previous works, we found evidence of agreement between both methods using numerical simulations of series computed from Cantor measures and binomial cascades. In these cases, it was possible to compare the estimations with the theoretical spectra, but this is not possible for the natural series provided by systems whose properties are unknown. For the series without fGn, both MS are in agreement, in spite of the range of MS from WL are larger than the MS from MFDFA.

We also investigate the effect in the estimations when a monofractal series is added to the natural ones. For these purposes, we use a series of fGn with different Hurst exponents: uncorrelated, anticorrelated and correlated monofractal series.

For this last situation, we conclude that uncorrelated noise does not make a difference but adding correlated fGb may produce significant differences. While the MS of the series with noise is more narrow than the MS of the series estimated by WL, the same MS estimated by MFDFA are translated, by aligning its maximum with the values of the Hurst exponents. We conjecture that this behavior from the MFDFA is a generalization of DFA and this methodology strongly adapts to detect the fractality of fBm [Peng *et al.*, 1995; Kantelhard *et al.*, 2002].

On the other hand, Jaffard's method is based on Hölder regularity and wavelet transform and it is biased to characterize the Hausdorff dimension of the sets of points having the same exponents. Hence, it can detect in the perturbed series different Hölder regularity structures and hence more concentrated spectra. Currently, we cannot give a categorical explanation of these differences, but we hope that future works will provide more results about these conjectures.

Acknowledgments

The authors wish to thank the support of the Agencia para la Promoción Científica y Técnica, CONICET, the Universidad Nacional de General Sarmiento and the Universidad de San Martín, Argentina.

References

Ashkenazy, Y., Ivanov, P. Ch., Havlin, S., Peng, Ch-K., Goldberger, A. L. & Stanley, H. E. [2001] "Magnitude and sign correlations in heartbeat fluctuations," *Phys. Rev. Lett.* **86**, 1900–1909.

Castro e Silva, A. & Moreira, J. G. [1997] "Roughness exponents to calculate multi/affine fractal exponents," *Phys. A* **235**, 327–333.

Falconer, K. [1997] *Techniques in Fractal Geometry* (John Wiley, NY).

Figliola, A., Serrano, E., Rostas, J. A. P., Hunter, M. & Rosso, O. A. [2007a] "Study of EEG brain maturation signals with multifractal detrended fluctuation analysis," *Proc. Nonequilibrium Statistical Mechanics and Nonlinear Physics, XV Conf.*, eds. Descalzi, O., Rosso, O. A. & Larrondo, H. A. (American Institute of Physics), pp. 190–195.

Figliola, A., Serrano, E. & Rosso, O. A. [2007b] "Dynamic behavior of tonic-clonic epileptic seizures," *Eur. Phys. J.* **143**, 117–123.

Frisch, U. & Parisi, G. [1985] *Fully Developed Turbulence and Intermittency, Proc. Inter. Sch. Phys. Enrico Fermi* (North Holland), pp. 84–88.

Jaffard, S. [2004] "Wavelet techniques in multifractal analysis, fractal geometry and applications: A jubilee of Benoit Mandelbrot," *Proc. Symp. Pure Mathematics, Providence*, eds. Lapidus, M. & van Frankenhuijsen, M., AMS, **72**, Part 2, pp. 91–151.

Kantelhardt, J. W., Zschiegner, S. A., Koscielny-Bunde, E., Havlin, S., Bunde, A. & Stanley, H. E. [2002] "Multifractal detrended fluctuation analysis of nonstationary time series," *Phys. A* **316**, 87–114.

Lashermes, B., Jaffard, S. & Abry, P. [2005] "Wavelet leaders in multifractal analysis," *Proc. IEEE Int. Conf. Acoust., Sp., and Sig.* **4**, 161–164.

Mallat, S. [1999] *A Wavelet Tour of Signal Processing* (Ac. Press, San Diego).

Oświęcimka, P., Kwapień, J. & Drożdż, S. [2006] "Wavelet modulo maxima versus detrended fluctuation analysis of multifractal structures," *Phys. Rev. E* **74**, 8–17.

Palma, W. [2007] *Long Memory Time Series* (Wiley-Interscience, John Wiley, NJ).

Pan, C. P., Zheng, B., Wu, Y. Z. & Tang, X. W. [2004] "Detrended fluctuation analysis of human brain electroencephalogram," *Phys. Lett. A* **329**, 130–135.

Peng, C. K., Buldyrev, S. V., Goldberger, A. L., Havlin, S., Mantegna, R. N., Simons, M. & Stanley, H. E. [1995] "Statistical properties of DNA sequences," *Phys. A* **221**, 180-192.

Rosenblatt, M. & Figliola, A. [2007] "Estudio del espectro multifractal de una señal no estacionaria con," *Multifractal Detrended Fluctuation Analysis*, AMCA Serie Mec. Comp., Vol. XXVI, pp. 2162–2179.

Rosso, O. A., Zunino, L., Perez, D. G., Figliola, A., Larrondo, H., Caravaglia, M., Martin, M. T. & Plastino, A. [2007] "Extracting features of Gaussian self-similar stochastic processes via the Bandt–Pompe approach," *Phys. Rev. E* **76**, 061114, 1–6.

Serrano, E. & Figliola, A. [2009] "Wavelet leaders: A new method to estimate the multifractal singularity spectra," *Physica A* **388**, 2793–2805.

Stanley, H. E., Amaral, L. A., Gopikrishnan, P., Ivanov, P. Ch., Keitt, T. H. & Plerou, V. [2000] "Scale invariance and universality: Organizing principles in complex systems," *Phys. A* **281**, 60–68.

Telesca, L., Lapenna, V. & Macchiato, M. [2005] "Multifractal fluctuations in seismic interspike series," *Phys. A* **354**, 629–640.

Zunino, L., Tabak, B., Figliola, A., Perez, D. G., Garavaglia, M. & Rosso, O. A. [2008] "A multifractal approach for the stock market inefficiency," *Phys. A* **387**, 26–37.