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A GENERALIZATION OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. If μ is a finite measure on the unit disc and $k \geqslant 0$ is an integer, we study a generalization derived from Engliš's work, $T_\mu^{(k)}$, of the traditional Toeplitz operators on the Bergman space A^2, which are the case $k = 0$. Among other things, we prove that when $\mu \geqslant 0$, these operators are bounded if and only if μ is a Carleson measure, they are compact if and only if μ is a vanishing Carleson measure, and we obtain some estimates for their norms. Also, we use these operators to characterize the closure of the image of the Berezin transform applied to the whole operator algebra.

KEYWORDS: *Bergman space, Toeplitz operators, Berezin transform.*

MSC (2010): Primary 32A36; Secondary 47B35.

1. INTRODUCTION AND PRELIMINARIES

Let A^2 be the Bergman space of holomorphic function on the disc $\mathbb{D}$ with respect to the normalized area measure $\mathrm{d}A$, and $\mathfrak{L}(A^2)$ be the Banach space of bounded operators on A^2. If for $z \in \mathbb{D}$, $\varphi_z \in \mathrm{Aut}(\mathbb{D})$ denotes the involution that interchanges 0 and z, the change of variables operator $U_z f = (f \circ \varphi_z)\varphi_z'$ is unitary and self-adjoint. Here, $\varphi_z' = -K_z/\|K_z\|$, where K_z is the reproducing kernel for z, and $\|K_z\| = (1 - |z|^2)^{-1}$.

For $f, g, h \in A^2$, define the rank-one operator $(f \otimes g)h := \langle h, g\rangle f$. In particular, if $e_k = \sqrt{k+1}\, w^k$ ($k \geqslant 0$) is the standard basis of A^2, the operator $E_k := e_k \otimes e_k$ is the orthogonal projection onto the subspace generated by e_k. Hence, for every $z \in \mathbb{D}$ and $f, g \in A^2$ we have

$$\langle U_z E_0 U_z f, g\rangle = (1 - |z|^2)^2 f(z)\overline{g(z)}.$$

So, if $\mathrm{d}\widetilde{A}(z) = (1-|z|^2)^{-2}\,\mathrm{d}A(z)$ denotes the invariant area measure on $\mathbb{D}$ and $a \in L^\infty$, the traditional Toeplitz operator T_a can be written as

$$T_a = \int_D U_z E_0 U_z\, a(z)\, \mathrm{d}\widetilde{A}(z),$$

where the integral converges in the weak operator topology. This led Engliš in [5] to consider operators defined as above, where E_0 is replaced by more general operators R that are diagonal with respect to the standard basis (a radial operator). Among other results, he proved that if R is a radial operator in the trace class and $a \in L^\infty$, then

$$R_a := \int_D U_z R U_z\, a(z)\, \mathrm{d}\widetilde{A}(z) \in \mathfrak{L}(A^2) \quad \text{and} \quad \|R_a\| \leqslant \|R\|_{\mathrm{tr}}\, \|a\|_\infty.$$

Since such operator R is an ℓ^1-linear combination of the projections E_j, with the trace norm of R given by the corresponding ℓ_1-norm of its eigenvalues, the above result is equivalent to

$$T_a^{(j)} := \int_D U_z E_j U_z\, a(z)\, \mathrm{d}\widetilde{A}(z) \in \mathfrak{L}(A^2) \quad \text{and} \quad \|T_a^{(j)}\| \leqslant \|a\|_\infty$$

for every integer $j \geqslant 0$. We study this type of operators and a generalization $T_\mu^{(j)}$, where $a\,\mathrm{d}\widetilde{A}$ is replaced by the expression $(1-|z|^2)^{-2}\,\mathrm{d}\mu(z)$, for μ a measure whose variation $|\mathrm{d}\mu|$ is a Carleson measure. As in the well known case $j = 0$, these operators turned out to be bounded, and when μ is positive we find lower and upper bounds for their norms. We also characterize compactness and show that these operators are norm limits of traditional Toeplitz operators.

Useful tools for our study will be the n-Berezin transform and the invariant Laplacian. If $n \geqslant 0$ is an integer, the n-Berezin transform of $Q \in \mathfrak{L}(A^2)$ is

$$B_n(Q)(z) := (n+1)\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{(j+1)} \langle Q U_z e_j, U_z e_j \rangle. \tag{1.1}$$

In particular, if $Q = T_\mu$, where μ is a finite measure on $\mathbb{D}$, a straightforward calculation shows that

$$B_n(\mu)(z) := B_n(T_\mu)(z) = \int_D (n+1) \frac{(1-|\varphi_z(\zeta)|^2)^{n+2}}{(1-|\zeta|^2)^2}\, \mathrm{d}\mu(\zeta). \tag{1.2}$$

Observe that the last expression defines $B_n(\mu)$ for any measure μ of finite total variation, even if T_μ is not bounded. In particular, if $\mu = a\,\mathrm{d}A$ with $a \in L^1$, we write $B_n(a) := B_n(a\,\mathrm{d}A)$, which is also $B_n(T_a)$ if T_a is bounded. It is clear from the definition that $\|B_n(Q)\|_\infty \leqslant (n+1)2^n\|Q\|$. Also, it was shown in [10] that

$$B_n B_0(Q) = B_0 B_n(Q) \quad \text{and} \quad B_n(U_w Q U_w) = B_n(Q) \circ \varphi_w \tag{1.3}$$

for every $w \in \mathbb{D}$.

The Berezin transform B_0 of operators, which is given by (1.1) with $n = 0$, was introduced by Berezin in [2] as a tool to study spectral theory and to construct approximations of the exponential of an operator. It has being used extensively to study properties such as boundedness and compactness of Toeplitz, Hankel and other related operators.

The idea behind the transforms B_n of functions in L^1 goes back to Berezin (see [3]), and were explicitly used in [1] to prove a deep result about the eigenfunctions of B_0 in the context of the ball in $\mathbb{C}^n$. The extension of the definition of B_n to operators is quite natural and appears in [10], where it is used to prove approximation results in the same vein of Corollary 4.4 in the present paper.

The organization of the paper is as follows. In Section 2 we introduce the invariant Laplacian $\widetilde{\Delta}$ and prove some identities involving the interaction between $T_a^{(j)}$, B_n and $\widetilde{\Delta}$. This will establish the technical foundations for the remaining sections. In Section 3 we decompose $T_{B_n(S)}$ in terms of $T^{(j)}_{B_0(S)}$, and use it to give a characterization of the L^∞ closure of $B_0(\mathfrak{L}(A^2))$, which turns out to be an algebra. Section 4 contains the main results of the paper. We prove that if $\mu \geqslant 0$ and $k \geqslant 1$, the operator $T_\mu^{(k)}$ is bounded (compact) if and only if μ is a Carleson measure (respectively a vanishing Carleson measure), and estimate the norms. We also show that if μ is a complex measure whose variation $|\mu|$ is Carleson, then $T_\mu^{(k)}$ is the limit of traditional Toeplitz operators. All these results generalize known facts for $k = 0$. In the last section we construct an example to show that for any $k \geqslant 1$, $\|T_a^{(k)}\|$ is not majorized by $\sum\limits_{j=0}^{k-1} \|T_a^{(j)}\|$ independently of $a \in L^\infty$. In particular, the linear map $T_a \mapsto T_a^{(k)}$ is not bounded. We will write indistinctly $T_a^{(0)}$ or T_a for the traditional Toeplitz operator with symbol $a \in L^\infty$.

2. THE ROLE OF THE INVARIANT LAPLACIAN

If $\Delta = \partial\overline{\partial}$ denotes a quarter of the usual Laplacian, where ∂ and $\overline{\partial}$ are the traditional Cauchy–Riemann operators, the invariant Laplacian is $\widetilde{\Delta} := (1 - |z|^2)^2\Delta$. It is easy to check that $(\widetilde{\Delta} f) \circ \psi = \widetilde{\Delta}(f \circ \psi)$ for every $f \in C^2(\mathbb{D})$ and $\psi \in \mathrm{Aut}(\mathbb{D})$. If $a \in L^\infty$ is such that $\widetilde{\Delta} a \in L^1$, it is well known that $\widetilde{\Delta} B_0(a) = B_0(\widetilde{\Delta} a)$. When also $\widetilde{\Delta} a \in L^\infty$, this equality rewrites as $\widetilde{\Delta} B_0(T_a) = B_0(T_{\widetilde{\Delta} a})$. In accordance with this formula we give the following

DEFINITION 2.1. Let

$$\mathfrak{D} = \{S \in \mathfrak{L}(A^2) : \exists\, T \in \mathfrak{L}(A^2) \quad \text{such that } \widetilde{\Delta} B_0(S) = B_0(T)\},$$

and define $\widetilde{\Delta} : \mathfrak{D} \to \mathfrak{L}(A^2)$ by $\widetilde{\Delta} S = T$.

This definition says that $\widetilde{\Delta} B_0(S) = B_0(\widetilde{\Delta} S)$ for all $S \in \mathfrak{D}$. In [10] it is showed that if $S \in \mathfrak{L}(A^2)$ and $n \geqslant 1$ then

$$B_n(S) = \Big(1 - \frac{\widetilde{\Delta}}{n(n+1)}\Big) B_{n-1}(S). \tag{2.1}$$

Hence, a straightforward inductive argument shows that $\widetilde{\Delta} B_n(S) = B_n(\widetilde{\Delta} S)$ when $S \in \mathfrak{D}$ for $n \geqslant 0$. Also, the conformal invariance of $\widetilde{\Delta}$ and (1.3) immediately prove that if $S \in \mathfrak{D}$, then $U_w S U_w \in \mathfrak{D}$ and

$$\widetilde{\Delta}(U_w S U_w) = U_w(\widetilde{\Delta} S) U_w. \tag{2.2}$$

Observe also that (2.1) implies that $\widetilde{\Delta} B_n(S) \in L^\infty$ for every $S \in \mathfrak{L}(A^2)$.

LEMMA 2.2. *Let f, g, h, k be analytic on $\overline{\mathbb{D}}$. Then*
(i) $\widetilde{\Delta}(f \otimes g) = (f' \otimes g') + (z^2 f)' \otimes (z^2 g)' - 2\,(zf)' \otimes (zg)'$.
(ii) $\langle \widetilde{\Delta}(f \otimes g)h, k\rangle = \langle \widetilde{\Delta}(h \otimes k) f, g\rangle$.

Proof. (i) We have:

$$\begin{aligned} \widetilde{\Delta} B_0(f \otimes g) &= \widetilde{\Delta}(1 + |z|^4 - 2|z|^2) f\overline{g} \\ &= (1 - |z|^2)^2 [f'\overline{g'} + (z^2 f)'\overline{(z^2 g)'} - 2\,(zf)'\overline{(zg)'}] \\ &= B_0[(f' \otimes g') + (z^2 f)' \otimes (z^2 g)' - 2\,(zf)' \otimes (zg)']. \end{aligned}$$

(ii) By (i),

$$\begin{aligned} \widetilde{\Delta}(z^n \otimes z^m) = nm(z^{n-1} \otimes z^{m-1}) + (n+2)(m+2)(z^{n+1} \otimes z^{m+1}) \\ - 2(n+1)(m+1)(z^n \otimes z^m). \end{aligned} \tag{2.3}$$

Since $n\|z^{n-1}\|^2 = 1$ when $n > 0$, for any $j, k \geqslant 0$ we have

$$\langle \widetilde{\Delta}(z^n \otimes z^m) z^j, z^k\rangle = \begin{cases} 1 & \text{if } (j,k) = (m-1, n-1), \\ -2 & \text{if } (j,k) = (m,n), \\ 1 & \text{if } (j,k) = (m+1, n+1), \\ 0 & \text{otherwise.} \end{cases}$$

This clearly shows that $\langle \widetilde{\Delta}(z^n \otimes z^m) z^j, z^k\rangle = \langle \widetilde{\Delta}(z^j \otimes z^k) z^n, z^m\rangle$. The lemma follows by sesqui-linearity and an approximation argument. ∎

LEMMA 2.3. *Let μ be a measure of finite variation such that $T_\mu^{(k)}$ is bounded for all $k \geqslant 0$. Then $T_\mu^{(k)} \in \mathfrak{D}$ for all $k \geqslant 0$, and*

$$\frac{\widetilde{\Delta} T_\mu^{(k)}}{(k+1)} = k\, T_\mu^{(k-1)} + (k+2)\, T_\mu^{(k+1)} - 2(k+1)\, T_\mu^{(k)}, \tag{2.4}$$

or equivalently, $(k+1)(k+2)\,[T_\mu^{(k+1)} - T_\mu^{(k)}] = \widetilde{\Delta}[T_\mu^{(k)} + T_\mu^{(k-1)} + \cdots + T_\mu^{(0)}]$. *Formally, we set* $T_\mu^{(-1)} = 0$ *in* (2.4) *when* $k = 0$.

Proof. By (2.3) with $k = n = m$,

$$\frac{\widetilde{\Delta} E_k}{k+1} = kE_{k-1} + (k+2)E_{k+1} - 2(k+1)E_k, \tag{2.5}$$

where $E_{-1} := 0$. Since by (2.2), $\widetilde{\Delta}(U_w E_k U_w) = U_w(\widetilde{\Delta} E_k)U_w$, conjugating both members of the above equality with respect to U_w and integrating with respect to $(1-|w|^2)^{-2}\,\mathrm{d}\mu(w)$, we obtain (2.4), which is our claim. The second formula follows from (2.4) by induction on k. It is immediate for $k = 0$ and assuming that it holds for an integer $k-1 \geqslant 0$, we get

$$\begin{aligned}\widetilde{\Delta} T_\mu^{(k)} + \widetilde{\Delta}[T_\mu^{(k-1)} + \cdots + T_\mu^{(1)} + T_\mu^{(0)}] &= \widetilde{\Delta} T_\mu^{(k)} + k(k+1)[T_\mu^{(k)} - T_\mu^{(k-1)}] \\ &= (k+1)(k+2)[T_\mu^{(k+1)} - T_\mu^{(k)}].\end{aligned}$$

Finally, if the last formula holds, substracting the equality for $k-1$ from the equality for k, we obtain (2.4). ∎

LEMMA 2.4. *If b_n, $b \in L^\infty$ are such that $\|b_n\|_\infty \leqslant C$, a constant independent of n, and $b_n \to b$ pointwise, then $T_{b_n}^{(k)} \to T_b^{(k)}$ in the strong operator topology.*

Proof. We can assume that $b = 0$. For $f, g \in A^2$,

$$|\langle T_{b_n}^{(k)} f, g\rangle| \leqslant \langle T_{|b_n|}^{(k)} f, f\rangle^{1/2} \langle T_{|b_n|}^{(k)} g, g\rangle^{1/2} \leqslant \langle T_{|b_n|}^{(k)} f, f\rangle^{1/2} C^{1/2} \|g\|_2, \tag{2.6}$$

where the first inequality follows from Cauchy–Schwarz's inequality and the second because $\|T_{|b_n|}^{(k)}\| \leqslant \|b_n\|_\infty \leqslant C$. So, taking supremum in (2.6) over $\|g\|_2 = 1$ for any fixed value of n, we see that $\|T_{b_n}^{(k)} f\|_2 \leqslant C^{1/2} \langle T_{|b_n|}^{(k)} f, f\rangle^{1/2} \to 0$ as $n \to \infty$ by the dominated convergence theorem. ∎

PROPOSITION 2.5. *Let $a \in L^\infty \cap C^2(\mathbb{D})$ such that $\widetilde{\Delta} a \in L^\infty$. Then $\widetilde{\Delta} T_a^{(k)} = T_{\widetilde{\Delta} a}^{(k)}$.*

Proof. For $0 < r < 1$ consider the functions $a_r(z) = a(rz)$. It follows from the previous lemma that $T_{a_r}^{(j)} \to T_a^{(j)}$ in the strong operator topology when $r \to 1$ for all $j \geqslant 0$. Then (2.4) implies that $\widetilde{\Delta} T_{a_r}^{(k)} \overset{\text{sot}}{\to} \widetilde{\Delta} T_a^{(k)}$. Since $(\widetilde{\Delta} a_r)(z) = r^2(\widetilde{\Delta} a)(rz)$ is bounded by $\|\widetilde{\Delta} a\|_\infty$, the previous lemma says that $T_{\widetilde{\Delta} a_r}^{(k)} \overset{\text{sot}}{\to} T_{\widetilde{\Delta} a}^{(k)}$. Therefore it is enough to prove the lemma for a_r, meaning that we can assume that $a \in C^2(\overline{\mathbb{D}})$. First observe that

$$\widetilde{\Delta}_z B_0(U_w E_k U_w)(z) = \widetilde{\Delta} B_0(E_k)(\varphi_w(z)) = \widetilde{\Delta} B_0(E_k)(\varphi_z(w)) = \widetilde{\Delta}_w B_0(U_z E_k U_z)(w),$$

where the equality in the middle holds because $\widetilde{\Delta} B_0(E_k)$ is a radial function and $|\varphi_w(z)| = |\varphi_z(w)|$. Therefore

$$\begin{aligned}B_0(\widetilde{\Delta} T_a^{(k)})(w) = \widetilde{\Delta} B_0(T_a^{(k)})(w) &= \int \widetilde{\Delta}_w B_0(U_z E_k U_z)(w) a(z)\,\mathrm{d}\widetilde{A}(z) \\ &= \int \widetilde{\Delta}_z B_0(U_w E_k U_w)(z) a(z)\,\mathrm{d}\widetilde{A}(z) = \int \Delta_z B_0(U_w E_k U_w)(z) a(z)\,\mathrm{d}A(z),\end{aligned}$$

and since $B_0(U_zE_kU_z)(w) = B_0(U_wE_kU_w)(z)$ (because $B_0(E_k)$ is radial),

$$B_0(T^{(k)}_{\widetilde{\Delta}a})(w) = \int B_0(U_zE_kU_z)(w)(\widetilde{\Delta}a)(z)\,\mathrm{d}\widetilde{A}(z) = \int B_0(U_wE_kU_w)(z)(\Delta a)(z)\,\mathrm{d}A(z).$$

Since for every fixed $w \in \mathbb{D}$, the function

$$B_0(U_wE_kU_w)(z) = (1-|\varphi_w(z)|^2)^2(k+1)(|\varphi_w(z)|^2)^k \tag{2.7}$$

is defined for z in some neighborhood of $\overline{\mathbb{D}}$, the previous equalities and Green's theorem give

$$\begin{aligned} B_0(\widetilde{\Delta}T^{(k)}_a - T^{(k)}_{\widetilde{\Delta}a})(w) &= \int_{\mathbb{D}} [\Delta_z B_0(U_wE_kU_w)(z)a(z) - B_0(U_wE_kU_w)(z)(\Delta a)(z)]\,\mathrm{d}A(z) \\ &= \int_{\partial\mathbb{D}} \Big[a(z)\,\frac{\partial}{\partial n}B_0(U_wE_kU_w)(z) - B_0(U_wE_kU_w)(z)\frac{\partial a}{\partial n}(z)\Big]\,\frac{\mathrm{d}m(z)}{\pi}, \end{aligned}$$

where $\partial/\partial n$ is the derivative in the normal direction and $\mathrm{d}m(z)$ is the Lebesgue measure on $\partial\mathbb{D}$. A straightforward calculation from (2.7) shows that both

$$B_0(U_wE_kU_w)(z) \quad \text{and} \quad \frac{\partial}{\partial n}B_0(U_wE_kU_w)(z)$$

vanish when $|z| = 1$. The proposition follows because B_0 is one-to-one. ∎

COROLLARY 2.6. *If $a \in L^\infty$ is harmonic, $T^{(k)}_a = T_a$ for every integer $k \geqslant 0$.*

Proof. By Proposition 2.5, $\widetilde{\Delta}T^{(k)}_a = T^{(k)}_{\widetilde{\Delta}a} = 0$ for all $k \geqslant 1$. The corollary now follows from the second formula of Lemma 2.3. ∎

Taking $a \equiv 1$ in the Corollary, we see that $T^{(k)}_1$ is the identity for all $k \geqslant 0$. This also follows from the so called Schur orthogonality relations and it is the main ingredient in Engliš's proof of the result cited in the introduction. Indeed, the first inequality in (2.6) implies that if $a \in L^\infty$, then $\|T^{(k)}_a\| \leqslant \|a\|_\infty \|T^{(k)}_1\| = \|a\|_\infty$.

PROPOSITION 2.7. *Let μ be a finite measure such that $T^{(k)}_\mu$ is bounded for all $k \geqslant 0$. Then $T_{B_n(T^{(k)}_\mu)} = T^{(k)}_{B_n(\mu)}$.*

Proof. First we prove that $T_{B_0(T^{(k)}_\mu)} = T^{(k)}_{B_0(\mu)}$ by induction on k. For $k = 0$ there is nothing to prove. Suppose that the equality holds for $j = 0, \dots, k$. By Proposition 2.5, the commutativity of B_0 and $\widetilde{\Delta}$, and (2.4),

$$\begin{aligned} \widetilde{\Delta}T_{B_0(T^{(k)}_\mu)} &= T_{\widetilde{\Delta}B_0(T^{(k)}_\mu)} = T_{B_0(\widetilde{\Delta}T^{(k)}_\mu)} \\ &= (k+1)[k\,T_{B_0(T^{(k-1)}_\mu)} + (k+2)\,T_{B_0(T^{(k+1)}_\mu)} - 2(k+1)\,T_{B_0(T^{(k)}_\mu)}] \end{aligned}$$

and by (2.4),

$$\widetilde{\Delta} T^{(k)}_{B_0(\mu)} = (k+1)[k\, T^{(k-1)}_{B_0(\mu)} + (k+2)\, T^{(k+1)}_{B_0(\mu)} - 2(k+1)\, T^{(k)}_{B_0(\mu)}].$$

By inductive hypothesis the left members of these formulas are equal, implying that $T_{B_0(T^{(k+1)}_\mu)} = T^{(k+1)}_{B_0(\mu)}$. Now suppose that $k \geqslant 0$ is fixed and we prove the lemma by induction on n. So, suppose that the equality holds for $n-1 \geqslant 0$. Then

$$\begin{aligned} n(n+1)[T_{B_{n-1}(T^{(k)}_\mu)} - T_{B_n(T^{(k)}_\mu)}] = \widetilde{\Delta} T_{B_{n-1}(T^{(k)}_\mu)} &= \widetilde{\Delta} T^{(k)}_{B_{n-1}(\mu)} \\ &= n(n+1)[T^{(k)}_{B_{n-1}(\mu)} - T^{(k)}_{B_n(\mu)}], \end{aligned}$$

where the equality in middle holds by inductive hypothesis and the other two by Proposition 2.5 and (2.1). This proves our claim. ∎

3. T_{B_n} IN TERMS OF $T^{(j)}_{B_0}$ AND APPLICATIONS

It is clear that $B_0 : \mathfrak{L}(A^2) \to L^\infty$ is not multiplicative but less clear that its image is not a multiplicative set. We show this by constructing the following example.

Let $f, g \in A^2$ such that $T_f T_{\overline{g}}$ is bounded but $g \notin H^\infty$. To see that such functions exist, take for instance $f(z) = (1-z)^\alpha$ and $g(z) = (1-z)^{-\alpha}$, with $0 < \alpha < 1/2$. The elementary inequalities

$$|1-z|\Big(\frac{1-|w|}{1+|w|}\Big) \leqslant |1-\varphi_z(w)| \leqslant |1-z|\Big(\frac{1+|w|}{1-|w|}\Big)$$

yield

$$B_0(|f|^p)B_0(|g|^p)(z) = \int |1-\varphi_z|^{p\alpha} \mathrm{d}A \int \frac{\mathrm{d}A}{|1-\varphi_z|^{p\alpha}} \leqslant \Big[\int \Big(\frac{1+|w|}{1-|w|}\Big)^{p\alpha} \mathrm{d}A(w)\Big]^2 < \infty$$

if $0 < p < \alpha^{-1}$. Hence, there is some $p > 2$ such that $B_0(|f|^p)B_0(|g|^p)$ is bounded, which by Theorem 5.2 of [9] is a sufficient condition for the boundedness of $T_f T_{\overline{g}}$.

Since $g \notin H^\infty$, there is $h \in A^2$ such that $gh \notin A^2$, implying that the operator $(f \otimes gh)$ is not bounded. However, it is well defined on the reproducing kernels K_z and satisfies $(f \otimes gh)K_z = \overline{g}(z)\overline{h}(z)f \in A^2$ for all $z \in \mathbb{D}$. This holds because K_z also reproduces functions in the Bergman space A^1. In particular, its Berezin transform is defined, and

$$B_0(f \otimes gh)(z) = (1-|z|^2)^2 \overline{h}(z)\, f(z)\overline{g}(z) = B_0(1 \otimes h)(z)\, B_0(T_f T_{\overline{g}})(z).$$

So, if $B_0(\mathfrak{L}(A^2))$ is an algebra there must be $Q \in \mathfrak{L}(A^2)$ such that $B_0(Q) = B_0(f \otimes gh)(z)$. Consequently the function

$$F(z,w) := \langle Q K_{\overline{z}}, K_w \rangle - \langle (f \otimes gh) K_{\overline{z}}, K_w \rangle$$

is analytic on the bidisc $\mathbb{D}^2$ and vanishes on the points $(\bar{z}, z)$, implying that $F \equiv 0$. Since the span of the reproducing kernels is dense in A^2, we conclude that $\|f \otimes gh\| = \|Q\| < \infty$, a contradiction.

Despite the fact that $B_0(\mathfrak{L}(A^2))$ is not an algebra, we will see that its closure is a uniform algebra, in fact, the largest uniform algebra that previously known results allow. The key ingredient in the proof is the following decomposition of $T_{B_n(S)}$, for $S \in \mathfrak{L}(A^2)$.

LEMMA 3.1. *Let $S \in \mathfrak{L}(A^2)$ and $n \geqslant 0$ integer. Then*

$$T_{B_n(S)} = (n+1) \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j+1} \, T^{(j)}_{B_0(S)}. \tag{3.1}$$

Proof.

$$\begin{aligned} B_0\Big(\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j+1} T^{(j)}_{B_0(S)}\Big)(w) &= \int \sum_{j=0}^{n} \binom{n}{j} (-|\varphi_z(w)|^2)^j (1-|\varphi_z(w)|^2)^2 B_0(S)(z) \, \mathrm{d}\widetilde{A}(z) \\ &= \int \frac{(1-|\varphi_z(w)|^2)^{n+2}}{(1-|z|^2)^2} \, B_0(S)(z) \, \mathrm{d}A(z) \\ &= \frac{B_n(B_0(S))(w)}{(n+1)} = \frac{B_0(T_{B_n(S)})(w)}{(n+1)}, \end{aligned}$$

where the last equality holds because B_n and B_0 commute. The lemma follows because B_0 is one-to-one. ∎

For $z, w \in \mathbb{D}$, the expressions

$$\rho(z,w) = |\varphi_z(w)| \quad \text{and} \quad \beta(z,w) = \log \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

define the pseudo-hyperbolic and the hyperbolic metric, respectively. Consider the uniform algebra $\mathcal{A} \subset L^\infty(\mathbb{D})$ of functions that are uniformly continuous from the metric space $(\mathbb{D}, \beta)$ into the complex plane with the euclidean metric $(\mathbb{C}, |\cdot|)$. In [4] Coburn proved that $B_0(S)$ is a Lipschitz function between these metric spaces for every $S \in \mathfrak{L}(A^2)$. In particular, $B_0(\mathfrak{L}(A^2)) \subset \mathcal{A}$, a fact used in [10] to study some subalgebras of $\mathfrak{L}(A^2)$ in terms of their Berezin transforms. We see next that the inclusion is dense.

THEOREM 3.2. *The L^∞-closure of $B_0(\mathfrak{L}(A^2))$ is $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$. Replacing $B_0(S)$ by a in the chain of equalities of the previous proof (except for the last one), gives

$$B_0\Big((n+1) \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j+1} \, T^{(j)}_a\Big) = B_n(a).$$

Taking $\mathrm{d}\mu = a \mathrm{d}A$ in (1.2), a change of variables shows that

$$B_n(a)(z) = \int_D a(\varphi_z(\zeta))(n+1)(1-|\zeta|^2)^n \, \mathrm{d}A(\zeta) \to a(\varphi_z(0)) = a(z)$$

uniformly on z when $n \to \infty$, because since $a \in \mathcal{A}$, the functions $a \circ \varphi_z$ are equicontinuous at 0, and the probability measures $(n+1)(1-|\cdot|^2)^n \mathrm{d}A$ tend to accumulate all the mass at 0 when $n \to \infty$. Thus, $\mathcal{A} \subset \overline{B_0(\mathfrak{L}(A^2))}$. ∎

COROLLARY 3.3. *The set* $\{T_{B_0(S)} : S \in \mathfrak{L}(A^2)\}$ *is norm dense in* $\{T_a : a \in L^\infty\}$.

Proof. The last theorem implies that $\{T_{B_0(S)} : S \in \mathfrak{L}(A^2)\}$ is norm dense in $\{T_a : a \in \mathcal{A}\}$, which by Theorem 5.7 of [10] is norm dense in $\{T_a : a \in L^\infty\}$. ∎

The next result is an easy consequence of the identities in the previous section and Lemma 3.1. We need some notation first. Let $m \geqslant 0$ be an integer and $x = \{x_n\}_{n\geqslant 0}$ be a sequence of complex numbers. The m-difference of x, denoted $\Delta^m x$, is the sequence whose n-th term is

$$\Delta_n^m x := (-1)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j \, x_{n+j}, \quad \text{for } n \geqslant 0.$$

That is, Δ^m is the m-iteration of the difference operator $\Delta\{x_n\}_{n\geqslant 0} := \{x_{n+1} - x_n\}_{n\geqslant 0}$.

PROPOSITION 3.4. *Let* $f, g, h, k \in A^2$ *and integers* $n, j \geqslant 0$. *Then*

$$\langle T_{B_n(f\otimes g)} h, k\rangle = \langle T_{B_n(h\otimes k)} f, g\rangle \quad \textit{and}$$

$$\int \langle U_w e_j, h\rangle \overline{\langle U_w e_j, k\rangle} \, f(w)\overline{g(w)} \, \mathrm{d}A(w) = \int \langle U_w e_j, f\rangle \overline{\langle U_w e_j, g\rangle} \, h(w)\overline{k(w)} \, \mathrm{d}A(w).$$

In particular,

$$\int |\langle U_w e_j, h\rangle|^2 \, |f(w)|^2 \, \mathrm{d}A(w) = \int |\langle U_w e_j, f\rangle|^2 \, |h(w)|^2 \, \mathrm{d}A(w). \tag{3.2}$$

Proof. Since $\|T_{B_n(f\otimes g)}\| \leqslant C_n \|f\|_2 \|g\|_2$, it is enough to assume that all the functions are polynomials. Since $B_0(f\otimes g) = (1-|z|^2)^2 f\overline{g}$, the first assertion is clear for $n = 0$. So, assuming that the result holds up to n, by (2.1) we need to prove the equality for $\widetilde{\Delta} B_n$ instead of B_n.

$$\begin{aligned}
\langle \widetilde{\Delta} T_{B_n(f\otimes g)} h, k\rangle &= \langle B_n(f\otimes g)h, k\rangle + \langle B_n((z^2 f)'\otimes(z^2 g)')h, k\rangle - 2\langle B_n((zf)'\otimes(zg)')h, k\rangle \\
&= \langle B_n(h\otimes k)f, g\rangle + \langle B_n(h\otimes k)(z^2 f)', (z^2 g)'\rangle - 2\langle B_n(h\otimes k)(zf)', (zg)'\rangle \\
&= \langle B_n(h\otimes k), \Delta(1-|z|^2)^2 \overline{f} g\rangle = \langle \widetilde{\Delta} B_n(h\otimes k), \overline{f} g\rangle = \langle \widetilde{\Delta} T_{B_n(h\otimes k)} f, g\rangle,
\end{aligned}$$

where the first equality follows from Proposition 2.5, the commutativity of B_n and $\widetilde{\Delta}$, and Lemma 2.2, the second equality holds by inductive hypothesis, the fourth one by Green's theorem, and the last one by Proposition 2.5 again. Writing $\sigma_j(z) = z^j$, (3.1) says that

$$\frac{T_{B_n(f\otimes g)}}{n+1} = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j+1} T^{(j)}_{B_0(f\otimes g)} = (-1)^n \int \Delta_0^n (U_w\sigma_j \otimes U_w\sigma_j)\, f(w)\overline{g(w)}\, \mathrm{d}A(w).$$

Therefore the equality $\langle T_{B_n(f\otimes g)}h, k\rangle = \langle T_{B_n(h\otimes k)}f, g\rangle$ rewrites as

$$\Delta_0^n \int \langle U_w\sigma_j, h\rangle \overline{\langle U_w\sigma_j, k\rangle} f(w)\overline{g}(w)\, \mathrm{d}A(w) = \Delta_0^n \int \langle U_w\sigma_j, f\rangle \overline{\langle U_w\sigma_j, g\rangle} h(w)\overline{k}(w)\, \mathrm{d}A(w),$$

and the second claim follows by induction on n. ∎

4. CARLESON MEASURES AS SYMBOLS

A positive measure μ on $\mathbb{D}$ is called a *Carleson measure* if $A^2 \subset L^2(\mathrm{d}\mu)$. If in addition the inclusion is compact, μ is called a *vanishing Carleson measure*. Among the many known characterizations of Carleson measures (see p. 123 of [13] for comments and references), a positive measure μ is Carleson if and only if $\|B_0(\mu)\|_\infty < \infty$, a quantity that is equivalent to the operator norm of the inclusion of A^2 in $L^2(\mu)$. Another characterization comes from replacing the kernel of the Berezin integral by a box kernel. Indeed, if $0 < r < 1$ and $v \in \mathbb{D}$, consider the pseudo-hyperbolic disk

$$D(v,r) := \{z \in \mathbb{D} : |\varphi_v(z)| \leqslant r\} \quad \text{and its area} \quad |D(v,r)| := \int_{D(v,r)} \mathrm{d}A.$$

If μ is a positive measure on $\mathbb{D}$ and $0 < r < 1$, there is a constant $C(r) > 0$ depending only on r such that

$$\frac{1}{C(r)} \sup_{v\in\mathbb{D}} \frac{\mu(D(v,r))}{|D(v,r)|} \leqslant \|B_0(\mu)\|_\infty \leqslant C(r) \sup_{v\in\mathbb{D}} \frac{\mu(D(v,r))}{|D(v,r)|}. \tag{4.1}$$

Clearly, if the above supremum is finite for some r then it is finite for all $0 < r < 1$. Finally, a positive measure μ is Carleson if and only if T_μ is bounded (see pp. 111–112 of [13]). We shall see that the same holds for $T_\mu^{(k)}$ when $k \geqslant 1$. For a positive measure μ write $\mathrm{d}\widetilde{\mu} := (1-|z|^2)^{-2}\, \mathrm{d}\mu$.

LEMMA 4.1. *Let μ be a positive finite measure on $\mathbb{D}$. Then*

$$\frac{\mu(D(v,r))}{|D(v,r)|} \Big[\frac{r(1-r^2)}{4}\Big]^2 \leqslant \widetilde{\mu}(D(v,r)) \leqslant \frac{\mu(D(v,r))}{|D(v,r)|} \Big[\frac{4r}{(1-r^2)^2}\Big]^2$$

for every $v \in \mathbb{D}$ and $0 < r < 1$.

Proof. Since by p. 60 of [13], $|D(v,r)| = [(r(1-|v|^2))/(1-|v|^2r^2)]^2$,

$$\widetilde{\mu}(D(v,r)) = \int_{D(v,r)} \frac{\mathrm{d}\mu(\xi)}{(1-|\xi|^2)^2} = \frac{1}{|D(v,r)|} \int_{D(v,r)} \Big[\frac{r(1-|v|^2)}{(1-|\xi|^2)(1-|v|^2r^2)}\Big]^2 \mathrm{d}\mu(\xi).$$

The lemma follows immediately from the easy inequalities, valid for $\xi \in D(v,r)$:

$$\frac{(1-r^2)}{4} \leqslant \frac{(1-|v|^2)}{(1-|\xi|^2)} \leqslant \frac{4}{(1-r^2)}. \qquad \blacksquare$$

THEOREM 4.2. *Let μ be a positive finite measure on $\mathbb{D}$. Then $T_\mu^{(k)}$ is bounded if and only if μ is a Carleson measure, in which case,*

$$\frac{C}{(k+2)}\, \|B_0(\mu)\|_\infty \leqslant \|T_\mu^{(k)}\| \leqslant 4(k+2)\|B_0(\mu)\|_\infty, \tag{4.2}$$

where $C > 0$ is an absolute constant.

Proof. First let us assume that $T_\mu^{(k)}$ is a bounded operator. For $k \geqslant 1$ consider the function $f(x) = (k+1)x^k(1-x)^2$ defined in $[0,1]$. This function reaches its maximum at $x = k/(k+2)$. If $(k-1/2)/(k+2) \leqslant x \leqslant (k+1)/(k+2)$ (that is, $x = (k+y)/(k+2)$ with $-1/2 \leqslant y \leqslant 1$), then

$$f(x) = f\Big(\frac{k+y}{k+2}\Big) = (k+1)\Big[\frac{k+y}{k+2}\Big]^k\Big[\frac{2-y}{k+2}\Big]^2 \geqslant \frac{(k+1)}{(k+2)^2}\Big[1-\frac{5/2}{k+2}\Big]^k \geqslant \frac{c_1}{(k+2)},$$

where $c_1 > 0$ is a constant independent of k. This means that there is an absolute constant $c_1 > 0$ such that for all $k \geqslant 1$,

$$(k+1)|z|^{2k}(1-|z|^2)^2 \geqslant \frac{c_1}{(k+2)} \quad \text{if } \frac{k-1/2}{k+2} \leqslant |z|^2 \leqslant \frac{k+1}{k+2}. \tag{4.3}$$

Now, let $0 < r \leqslant z_k := \sqrt{k/(k+2)}$. By the geometric arguments in p. 3 of [6], $D(z_k,r)$ is contained in the annulus

$$\frac{z_k - r}{1-rz_k} \leqslant |w| \leqslant \frac{z_k+r}{1+rz_k}.$$

Thus, if we choose $r \leqslant \sqrt{k/(k+2)}$ small enough so that

$$\sqrt{\frac{k-1/2}{k+2}} \leqslant \frac{\sqrt{k/(k+2)}-r}{1-r\sqrt{k/(k+2)}} \quad \text{and} \quad \frac{\sqrt{k/(k+2)}+r}{1+r\sqrt{k/(k+2)}} \leqslant \sqrt{\frac{k+1}{k+2}} \tag{4.4}$$

for all $k \geqslant 1$, then $D(z_k,r)$ is contained in the annulus $(k-1/2)/(k+2) \leqslant |z|^2 \leqslant (k+1)/(k+2)$, implying that the inequalities in (4.3) hold for $z \in D(z_k,r)$. We see next that $0 < r \leqslant 1/10$ does the trick. Clearing r from (4.4) we get the equivalent inequalities

$$r \leqslant \frac{\sqrt{k/(k+2)}-\sqrt{(k-1/2)/(k+2)}}{[1-\sqrt{k/(k+2)}\sqrt{(k-1/2)/(k+2)}]} \quad \text{and} \quad r \leqslant \frac{\sqrt{(k+1)/(k+2)}-\sqrt{k/(k+2)}}{[1-\sqrt{k/(k+2)}\sqrt{(k+1)/(k+2)}]},$$

meaning that r must be bounded by

$$\min\Big\{\frac{\sqrt{k+2}}{[\sqrt{k}+\sqrt{k-1/2}]}\,\frac{1/2}{[k+2-\sqrt{k^2-k/2}]},\ \frac{\sqrt{k+2}}{[\sqrt{k}+\sqrt{k+1}]}\,\frac{1}{[k+2-\sqrt{k^2+k}]}\Big\}.$$

The claim follows because this minimum is bounded below by

$$\begin{aligned}\frac{\sqrt{k+2}}{[\sqrt{k}+\sqrt{k+1}]}\,\frac{1/2}{[k+2-\sqrt{k^2-k/2}]} &\geqslant \frac{1/4}{[k+2-\sqrt{k^2-k/2}]}\\ &= \frac{k+2+\sqrt{k^2-k/2}}{[18k+16]} \geqslant \frac{2k+3/2}{[18k+16]} > \frac{1}{10}.\end{aligned}$$

Therefore, if $r \leqslant 1/10$,

$$\begin{aligned}B_0(T_\mu^{(k)})(w) &= \int (k+1)|\varphi_w(z)|^{2k}(1-|\varphi_w(z)|^2)^2\frac{\mathrm{d}\mu(z)}{(1-|z|^2)^2}\\ &\geqslant \int_{D(\varphi_w(z_k),r)} (k+1)|\varphi_w(z)|^{2k}(1-|\varphi_w(z)|^2)^2\frac{\mathrm{d}\mu(z)}{(1-|z|^2)^2}\\ &\overset{\text{by (4.3)}}{\geqslant} \frac{c_1}{(k+2)}\int_{D(\varphi_w(z_k),r)} \frac{\mathrm{d}\mu(z)}{(1-|z|^2)^2}\\ &= \frac{c_1}{(k+2)}\widetilde{\mu}(D(\varphi_w(z_k),r)).\end{aligned}\tag{4.5}$$

Taking the supremum for $w \in \mathbb{D}$ and using that $\{\varphi_w(z_k) :\ w \in \mathbb{D}\} = \mathbb{D}$ for any fixed $z_k \in \mathbb{D}$, we get

$$\|T_\mu^{(k)}\| \geqslant \|B_0(T_\mu^{(k)})\|_\infty \geqslant \frac{c_1}{(k+2)}\sup_v \widetilde{\mu}(D(v,r))\tag{4.6}$$

for any $r \leqslant 1/10$. By (4.1), Lemma 4.1 and (4.6), there are absolute constants C_0, C_1 and C_2, such that

$$\|B_0(\mu)\|_\infty \leqslant C_0\sup_v\frac{\mu(D(v,\frac{1}{10}))}{|D(v,\frac{1}{10})|} \leqslant C_1\sup_v\widetilde{\mu}(D(v,\tfrac{1}{10})) \leqslant C_2(k+2)\|T_\mu^{(k)}\|.$$

This proves the first inequality in (4.2).

Now suppose that μ is a Carleson measure and let $F(z) = \sum a_j e_j(z) \in A^2$. For $0 \leqslant t < 2\pi$ and $0 \leqslant r < 1$ we have

$$\begin{aligned}|\langle F(\mathrm{e}^{\mathrm{i}t}z),(U_re_k)(z)\rangle|^2 &= \sum_{j,l} a_j\overline{a}_l\,\langle e_j(\mathrm{e}^{\mathrm{i}t}z),(U_re_k)(z)\rangle\,\overline{\langle e_l(\mathrm{e}^{\mathrm{i}t}z),(U_re_k)(z)\rangle}\\ &= \sum_{j,l} a_j\overline{a}_l\,\mathrm{e}^{\mathrm{i}(j-l)t}\,\langle e_j,U_re_k\rangle\,\overline{\langle e_l,U_re_k\rangle},\end{aligned}$$

and since $|\langle F, U_{re^{it}} e_k\rangle| = |\langle F(z), (U_r e_k)(e^{-it}z)\rangle| = |\langle F(e^{it}z), (U_r e_k)(z)\rangle|$, then

$$\int_0^{2\pi} |\langle F, U_{re^{it}} e_k\rangle|^2 \frac{dt}{2\pi} = \sum_j |a_j|^2 |\langle e_j, U_r e_k\rangle|^2 \geqslant |a_k|^2 |\langle e_k, U_r e_k\rangle|^2$$

$$= |\langle F, e_k\rangle|^2 |\langle e_k, U_r e_k\rangle|^2 = |\langle F, e_k\rangle|^2 \int_0^{2\pi} |\langle e_k, U_{re^{it}} e_k\rangle|^2 \frac{dt}{2\pi}.$$

Multiplying by $2r\,dr$ and integrating yields

$$\int |\langle F, U_z e_k\rangle|^2 \, dA(z) \geqslant |\langle F, e_k\rangle|^2 \int |\langle e_k, U_z e_k\rangle|^2 \, dA(z).$$

So, taking $F = U_w f$ we get

$$\int |\langle U_w f, U_z e_k\rangle|^2 \, dA(z) \geqslant |\langle U_w f, e_k\rangle|^2 \int |\langle e_k, U_z e_k\rangle|^2 \, dA(z).$$

Writing $\lambda = (z\overline{w} - 1)/(1 - w\overline{z})$, we have $U_w U_z = U_{\varphi_w(z)} V_\lambda$, where $(V_\lambda h)(\omega) = \lambda h(\lambda\omega)$ for $h \in A^2$. Consequently,

$$|\langle U_w f, U_z e_k\rangle| = |\langle f, U_w U_z e_k\rangle| = |\langle f, U_{\varphi_w(z)} e_k\rangle|,$$

and the change of variables $v = \varphi_w(z)$ in the first integral above yields

$$\int |\langle f, U_v e_k\rangle|^2 \, |\varphi_w'(v)|^2 \, dA(v) \geqslant |\langle U_w f, e_k\rangle|^2 \int |\langle e_k, U_z e_k\rangle|^2 \, dA(z).$$

Integrating with respect to $d\widetilde{\mu}(w)$,

$$\int_{\mathbb{D}} \Big[\int \frac{(1-|v|^2)^2}{|1-\overline{w}v|^4} \, d\mu(w) \Big] \, |\langle f, U_v e_k\rangle|^2 \, d\widetilde{A}(v) \geqslant c_k \int_{\mathbb{D}} |\langle U_w f, e_k\rangle|^2 \, d\widetilde{\mu}(w), \tag{4.7}$$

where

$$c_k = \int |\langle e_k, U_z e_k\rangle|^2 \, dA(z) \overset{\text{by (3.2)}}{=} \int |\langle U_z e_k, 1\rangle|^2 \, |e_k(z)|^2 \, dA(z)$$

$$= \int (1-|z|^2)^2 \, |e_k(z)|^4 \, dA(z) = (k+1)^2 \int_0^1 (1-x)^2 \, x^{2k} \, dx \geqslant \frac{1}{4(k+2)}.$$

Thus, going back to (4.7),

$$\|B_0(\mu)\|_\infty \, \|f\|^2 \geqslant \langle T^{(k)}_{B_0(\mu)} f, f\rangle \geqslant c_k \langle T^{(k)}_\mu f, f\rangle \geqslant \frac{1}{4(k+2)} \, \langle T^{(k)}_\mu f, f\rangle.$$

This proves the second inequality in (4.2). ∎

It would be interesting to know how sharp are the bounds in (4.2) except for absolute multiplicative constants when k tends to infinity, especially the upper bound.

REMARK 4.3. Observe that by (4.6) and the subsequent inequality, we also showed that

$$\frac{C}{(k+2)}\|B_0(\mu)\|_\infty \leqslant \|B_0(T_\mu^{(k)})\|_\infty \leqslant \|T_\mu^{(k)}\|,$$

and that the last formula of the proof says that $4(k+2)T^{(k)}_{B_0(\mu)} \geqslant T_\mu^{(k)}$ as positive operators.

Suppose that μ is a complex measure on $\mathbb{D}$ such that its variation $|\mu|$ is Carleson. By (2.6) with measures instead of functions, we see that $\|T_\mu^{(k)}\| \leqslant \|T_{|\mu|}^{(k)}\|$ for all $k \geqslant 0$, so $T_\mu^{(k)} \in \mathfrak{L}(A^2)$. It is worth noticing that the converse does not hold, since there are finite measures μ such that T_μ is bounded but $|\mu|$ is not Carleson. The next result was proved in Corollary 2.5 of [11] for $k = 0$. In particular, it shows that when $a \in L^\infty$, $T_a^{(k)}$ is a limit of classical Toeplitz operators.

COROLLARY 4.4. *Let μ be a finite measure on $\mathbb{D}$ such that $|\mu|$ is a Carleson measure and $k \geqslant 0$ be an integer. Then*

$$T_{B_n(T_\mu^{(k)})} \to T_\mu^{(k)} \quad \textit{when } n\to\infty.$$

Proof. Decomposing $\mu = \mu_1 + \mathrm{i}\mu_2$, where each μ_j is a real measure, and using Jordan decomposition with both μ_1 and μ_2, we can assume without loss of generality that $\mu \geqslant 0$. By Lemma 4.1 of [12], if $Q \in \mathfrak{L}(A^2)$ satisfies $\|T_{\widetilde{\Delta} B_n(Q)}\| \leqslant C$, where C is independent of n, then $T_{B_n(Q)} \to Q$. So, we need to prove the above inequality for $Q = T_\mu^{(k)}$. By Propositions 2.5 and 2.7, and (2.4),

$$T_{\widetilde{\Delta} B_n(T_\mu^{(k)})} = \widetilde{\Delta} T_{B_n(T_\mu^{(k)})} = \widetilde{\Delta} T^{(k)}_{B_n(\mu)} = (k+1)[kT^{(k-1)}_{B_n(\mu)} + (k+2)T^{(k+1)}_{B_n(\mu)} - 2(k+1)T^{(k)}_{B_n(\mu)}].$$

Since $B_n(\mu)\,\mathrm{d}A$ is a Carleson measure satisfying

$$\|B_0 B_n(\mu)\|_\infty = \|B_n B_0(\mu)\|_\infty \leqslant \|B_0(\mu)\|_\infty,$$

using (4.2) in the above equality gives $\|T_{\widetilde{\Delta} B_n(T_\mu^{(k)})}\| \leqslant 4^2(k+3)^3\|B_0(\mu)\|_\infty$, which does not depend on n. ∎

It is well known that for a positive measure μ on $\mathbb{D}$, the condition of being a vanishing Carleson measure is equivalent to $B_0(\mu)(z) \to 0$ when $|z| \to 1$, and also to the compactness of T_μ (see pp. 112–115 of [13], also Proposition 3 of [7]). We aim to prove the same result for $T_\mu^{(k)}$ when k is any nonnegative integer.

LEMMA 4.5. *If $f_n \in A^2$ is a sequence that tends weakly to 0 then $\langle f_n, U_w e_k\rangle \to 0$ uniformly for w in compact sets of $\mathbb{D}$.*

Proof. By the Banach–Steinhaus theorem (see p. 44 of [8]) the norms $\|f_n\|$ are uniformly bounded and by Lemma 4.3 of [10] the function $w \mapsto U_w e_k$ is uniformly continuous on compact sets. Thus, the Cauchy–Schwarz inequality shows that the scalar functions $F_n(w) = \langle f_n, U_w e_k\rangle$ are equicontinuous on compact sets.

Since by hypothesis $F_n \to 0$ pointwise, Ascoli's theorem (see p. 394 of [8]) implies that $F_n \to 0$ uniformly on compact sets. ∎

LEMMA 4.6. *If $a \in L^\infty$ has compact support then $T_a^{(k)}$ is compact.*

Proof. Let $f_n, g_n \in A^2$ be sequences such that f_n tends weakly to 0 and $\|g_n\| \leqslant 1$. Then

$$|\langle T_a^{(k)} f_n, g_n\rangle| \leqslant \|a\|_\infty\, \widetilde{A}(\operatorname{supp} a) \sup_{w \in \operatorname{supp} a} |\langle f_n, U_w e_k\rangle \langle U_w e_k, g_n\rangle|,$$

where the last factor tends to 0 by the previous lemma, since $|\langle U_w e_k, g_n\rangle| \leqslant 1$. ∎

THEOREM 4.7. *Let μ be a positive finite measure on $\mathbb{D}$. Then $T_\mu^{(k)}$ is compact if and only if μ is a vanishing Carleson measure.*

Proof. Suppose that μ is a vanishing Carleson measure and let $0 < r < 1$. By Remark 4.3,

$$0 \leqslant T_\mu^{(k)} \leqslant 4(k+2) T^{(k)}_{B_0(\mu)} = 4(k+2)[T^{(k)}_{\chi_{rD} B_0(\mu)} + T^{(k)}_{\chi_{D \setminus rD} B_0(\mu)}].$$

By Lemma 4.6 the first operator in the sum is compact, and by Engliš's result ([5], Theorem 1),

$$\|T^{(k)}_{\chi_{D\setminus rD} B_0(\mu)}\| \leqslant \|\chi_{D\setminus rD} B_0(\mu)\|_\infty \to 0 \quad \text{when } r \to 1.$$

Thus, $T_\mu^{(k)}$ is compact. Conversely, suppose now that $T_\mu^{(k)}$ is compact. Then $B_0(T_\mu^{(k)})(w) \to 0$ when $|w| \to 1$, which together with (4.5) says that there are $z_k \in \mathbb{D}$ and $0 < r < 1$ such that

$$\widetilde{\mu}(D(\varphi_w(z_k), r)) \to 1 \quad \text{when } |w| \to 1.$$

If $V \subset \mathbb{D}$ is such that $\mathbb{D} \setminus V$ is compact, the same holds for the set $\{\varphi_w(z_k) : w \in V\}$, for any fixed $z_k \in \mathbb{D}$. Therefore $\widetilde{\mu}(D(v,r)) \to 1$ when $|v| \to 1$, which together with Lemma 4.1 gives

$$\frac{\mu(D(v,r))}{|D(v,r)|} \to 0 \quad \text{as } |v| \to 1.$$

Then μ is a vanishing Carleson measure by pp. 111–114 of [13] . ∎

5. EXAMPLE OF BAD BEHAVIOUR

As far as I know there is no accurate estimate for $\|T_a\|$ when $a \in L^\infty$ is arbitrary, which obviously remains true for $\|T_a^{(k)}\|$ when $k \geqslant 1$. It would be interesting to know if at least $\|T_a^{(k)}\|$ is majorized by $\|T_a\|$, or more generally, if for some given $k \geqslant 1$, there exists a positive constant C_k depending only on k such that

$$\|T_a^{(k)}\| \leqslant C_k(\|T_a^{(0)}\| + \cdots + \|T_a^{(k-1)}\|) \quad \text{for all } a \in L^\infty. \tag{5.1}$$

By Theorem 4.2 this is certainly the case when $a \geqslant 0$ or when $a\,\mathrm{d}A$ is replaced by any Carleson measure. Unfortunately (5.1) does not hold for any $k \geqslant 1$, as the example that we construct next will show.

LEMMA 5.1. *For $a \in L^\infty$ and $\ell \geqslant 0$ there are constants $c_0, \dots, c_\ell$ depending only on ℓ such that*

$$T_a^{(\ell)} = c_0 \widetilde{\Delta}^0 T_a + \cdots + c_\ell \widetilde{\Delta}^\ell T_a.$$

Proof. By the second formula of Lemma 2.3,

$$T_a^{(\ell)} = T_a^{(0)} + \widetilde{\Delta} \sum_{m=0}^{\ell-1} \frac{1}{(m+1)(m+2)} [T_a^{(m)} + T_a^{(m-1)} + \cdots + T_a^{(0)}].$$

This proves the lemma for $\ell = 1$ and assuming inductively that it holds for $T_a^{(m)}$ with $m = 1, \dots, \ell - 1$, it also shows that it holds for $T_a^{(\ell)}$. ∎

COROLLARY 5.2. *For all $k \geqslant 0$ and $a \in L^\infty$ there is $C_k > 0$ such that*

$$\sum_{\ell=0}^{k} \|T_a^{(\ell)}\| \leqslant C_k \sum_{\ell=0}^{k} \|\widetilde{\Delta}^\ell T_a\|.$$

The proof of Lemma 5.1 clearly shows that both the lemma and its corollary hold if $a\,\mathrm{d}A$ is replaced by any finite measure μ such that $T_\mu^{(k)}$ is bounded for every $k \geqslant 0$. In particular, they hold when $|\mu|$ is a Carleson measure.

Let $k \geqslant 1$ and suppose that (5.1) holds. This, together with (2.4) imply the first of the following inequalities

$$\|\widetilde{\Delta}^k T_a\| \leqslant C_1(k) \sum_{\ell=0}^{k-1} \|T_a^{(\ell)}\| \leqslant C_2(k) \sum_{\ell=0}^{k-1} \|\widetilde{\Delta}^\ell T_a\| \quad \text{for all } a \in L^\infty,$$

for some $C_1(k) > 0$, where the second inequality comes from the corollary. Thus, the next example disproves (5.1).

EXAMPLE 5.3. We claim that if $k \geqslant 1$ there is no positive constant C_k such that

$$\|\widetilde{\Delta}^k T_a\| \leqslant C_k \sum_{\ell=0}^{k-1} \|\widetilde{\Delta}^\ell T_a\| \quad \text{for all } a \in L^\infty.$$

For $j \geqslant 0$ recall that $E_j = e_j \otimes e_j$, and we write $E_j = 0$ if $j < 0$. An iteration of (2.5) shows that $\widetilde{\Delta}^\ell E_j$ is a linear combination of $E_{j-\ell}, \dots, E_{j+\ell}$ in such a way that there are positive constants c_ℓ and C_ℓ independent of j with $c_\ell (j+1)^{2\ell} \leqslant \|\widetilde{\Delta}^\ell E_j\| \leqslant$

$C_\ell (j+1)^{2\ell}$ for all $\ell \geqslant 0$. In particular, if $0 \leqslant \ell \leqslant k$, there are constants c and C depending only on k such that

$$c(j+1)^{2\ell} \leqslant \|\widetilde{\Delta}^\ell E_j\| \leqslant C(j+1)^{2\ell} \quad \forall \ell = 0,\dots,k \text{ and } j \geqslant 0.$$

By Theorem 4.3 of [12], $T_{B_n(E_j)} \to E_j$ when $n \to \infty$. Hence, Proposition 2.5, the commutativity of B_n and $\widetilde{\Delta}$, and the previous comments yield

$$\widetilde{\Delta}^\ell T_{B_n(E_j)} = T_{\widetilde{\Delta}^\ell B_n(E_j)} = T_{B_n(\widetilde{\Delta}^\ell E_j)} \to \widetilde{\Delta}^\ell E_j \quad \text{as } n \to \infty.$$

Therefore for each pair of integers $k, j \geqslant 0$ we can choose $n = n(k,j)$ large enough so that

$$\frac{c}{2}(j+1)^{2\ell} \leqslant \|\widetilde{\Delta}^\ell T_{B_n(E_j)}\| \leqslant 2C(j+1)^{2\ell} \quad \forall \ell = 0,\dots,k.$$

Taking $a_j := (j+1)^{-2k} B_n(E_j) \in L^\infty$, the above inequalities show that,

$$\sum_{\ell=0}^{k-1} \|\widetilde{\Delta}^\ell T_{a_j}\| \leqslant 2C \sum_{p=1}^{k} \frac{1}{(j+1)^{2p}} \leqslant \frac{2C}{(j+1)^2 - 1} \quad \text{while } \frac{c}{2} \leqslant \|\widetilde{\Delta}^k T_{a_j}\|$$

for all $j \geqslant 1$. Taking $j \to \infty$ shows our claim.

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