Finite cycle Gibbs measures on permutations of $\mathbb{Z}^d$

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Abstract

We consider Gibbs distributions on the set of permutations of $\mathbb{Z}^d$ associated to the Hamiltonian $H(\sigma) := \sum_x V(\sigma(x) - x)$, where σ is a permutation and $V : \mathbb{Z}^d \to \mathbb{R}$ is a strictly convex potential. Call finite-cycle those permutations composed by finite cycles only. We give conditions on V ensuring that for large enough temperature $\alpha > 0$ there exists a unique infinite volume ergodic Gibbs measure μ^α concentrating mass on finite-cycle permutations; this measure is equal to the thermodynamic limit of the specifications with identity boundary conditions. We construct μ^α as the unique invariant measure of a Markov process on the set of finite-cycle permutations that can be seen as a loss-network, a continuous-time birth and death process of cycles interacting by exclusion, an approach proposed by Fernández, Ferrari and Garcia. Define τ_v as the shift permutation $\tau_v(x) = x + v$. In the Gaussian case $V = \| \cdot \|^2$, we show that for each $v \in \mathbb{Z}^d$, μ_v^α given by $\mu_v^\alpha(f) = \mu^\alpha[f(\tau_v \cdot)]$ is an ergodic Gibbs measure equal to the thermodynamic limit of the specifications with τ_v boundary conditions. For a general potential V, we prove the existence of Gibbs measures μ_v^α when α is bigger than some v-dependent value.

1   Introduction

The Feynman-Kac representation of the Bose gas consists of trajectories of interacting Brownian motions in a fixed time interval, which start and finish at the points of a spatial point process [5] . In order to attempt a rigorous analysis of the model, several simplifications have been proposed over the years [5, 6, 12, 13]. In the resulting model, the starting and ending points belong to the d-dimensional lattice, and the interaction is reduced to an exclusion condition on the paths at the beginning and the end of the time interval. The state space is therefore the set of permutations or bijections $\sigma : \mathbb{Z}^d \to \mathbb{Z}^d$.

For a finite set $\Lambda \subset \mathbb{Z}^d$, denote by S_Λ the set of permutations σ that reduce to the identity outside Λ, i.e.,

$$S_\Lambda := \{\sigma \in S \ : \ \sigma(x) = x \text{ if } x \notin \Lambda\}. \tag{1.1}$$

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A function $V : \mathbb{Z}^d \to \mathbb{R}^+ \cup \{+\infty\}$ such that $V(\vec{0}) = 0$ is called a *potential*. We assume V is strictly convex and define the Hamiltonian

$$H_\Lambda(\sigma) := \sum_{x \in \Lambda} V(\sigma(x) - x), \quad \sigma \in S_\Lambda, \tag{1.2}$$

and associated measure G_Λ,

$$G_\Lambda(\sigma) := \frac{1}{Z_\Lambda} e^{-\alpha H_\Lambda(\sigma)}, \tag{1.3}$$

where Z_Λ is a normalizing constant. The nonnegative parameter α is called the temperature; we omit the dependence of G_Λ on α. We refer to the condition $\sigma(x) = x$ if $x \notin \Lambda$ as an *identity boundary condition*, and the finite volume measure G_Λ associated to a finite set $\Lambda \subset \mathbb{Z}^d$ is called a *specification*.

When the potential is Gaussian, $V(x) = \|x\|^2$, the value $e^{-\alpha\|\sigma(x)-x\|^2}$ is proportional to the density at the site $\sigma(x)$ of a Gaussian distribution with mean x and variance $1/(2\alpha)$. Hence, G_Λ is proportional to the joint density of the arrival points at time $1/(2\alpha)$ of a family of independent Brownian motions started at each point in Λ, which are conditioned to arrive at distinct points of Λ at that time. This is the case arising from the Feynmann-Kac representation of the Bose gas.

Given permutations τ, σ, define the composed permutation $\tau\sigma$ by $(\tau\sigma)(x) := \tau(\sigma(x))$ and let $\mu\tau$ be the law of $\tau\sigma$ when σ is distributed according to μ, that is

$$(\mu\tau)f = \int \mu(d\sigma)f(\tau\sigma), \tag{1.4}$$

for continuous real functions f. For any vector $v \in \mathbb{Z}^d$ denote by τ_v the shift permutation given by

$$\tau_v(x) := x + v\,. \tag{1.5}$$

A permutation τ is called a *ground state* if τ is a local minimum of the Hamiltonian $H_{\mathbb{Z}^d}$. Since V is strictly convex, the shift permutation τ_v is a ground state for any vector $v \in \mathbb{Z}^d$.

Results   Our main results are the following.

Identity boundary conditions.   In Theorem 2.1 we define a function $\alpha^*(V)$ such that when it is finite, for any $\alpha > \alpha^*(V)$, there exists an ergodic Gibbs measure μ equal to the thermodynamic limit of the specifications with identity boundary conditions at temperature α. The measure μ concentrates on finite-cycle permutations.

Shift boundary conditions.   In Theorem 2.3 we fix $v \in \mathbb{Z}^d$ and extend the results of Theorem 2.1 to τ_v-boundary conditions. That is, we define $\alpha_v^*(V)$, and assuming that it is finite, we show that for any temperature $\alpha > \alpha_v^*(V)$ there exists an ergodic Gibbs measure μ_v associated to τ_v boundary conditions such that $\mu_v\tau_{-v}$ concentrates on finite-cycle permutations.

Gaussian potential.   The physically relevant Gaussian potential $V(x) = \|x\|^2$ is covered by Theorem 2.1; in this case the results for τ_v-boundary conditions follow directly from the observation that the specifications $G_{\Lambda|\tau_v}$ matching τ_v at the boundary satisfy $G_{\Lambda|\tau_v} = G_\Lambda\tau_v$, a relation that extends to the limit $\mu_v = \mu\tau_v$. In particular, here $\alpha_v^*(V)$ is the same for all $v \in \mathbb{Z}^d$, $\alpha_v^*(V) = \alpha^*(V)$.

The statements of these results establish the existence of Gibbs measures μ as a weak limit of specifications. In fact, we obtain pointwise limits. For instance, in the proof of Theorem 2.1 we construct a coupled family of permutations $(\zeta_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$, each ζ_Λ distributed according to G_Λ ($G_{\mathbb{Z}^d} = \mu$), such that for $x \in \mathbb{Z}^d$ the random variables $\zeta_\Lambda(x)$ converge almost surely to $\zeta_{\mathbb{Z}^d}(x)$, as $\Lambda \nearrow \mathbb{Z}^d$.

In Section 5 we compute bounds for $\alpha^*(V)$. In the Gaussian case these computations yield explicit bounds, see (5.8).

Approach   The proofs follow the approach of Fernández, Ferrari and Garcia [4], relying on the fact that the Peierls-contour representation of the low temperature Gibbs measure for the Ising model is reversible for a loss network of contours interacting by exclusion. In the case of identity boundary conditions, instead of contours, we consider the finite cycles that compose a permutation. Let Γ be the set of finite cycles on $\mathbb{Z}^d$ with length larger than 1. A finite-cycle permutation is represented as a "gas" of finite cycles in Γ, and the Gibbs measure can be described as a product of independent Poisson random variables in the space $\{0, 1, \dots\}^\Gamma$, conditioned to non overlapping of cycles, that is, each site $x \in \mathbb{Z}^d$ belongs to at most one cycle. This is automatically well defined in finite volume. We explicitly construct an infinite volume random configuration $\eta \in \{0, 1\}^\Gamma$ with non overlapping cycles, $\eta(\gamma) = 1$ means that the cycle γ is present in the configuration η. This configuration is naturally associated to the permutation σ composed by the cycles indicated by η. We then show that σ is the almost sure limit as $\Lambda \nearrow \infty$ of permutations in S_Λ distributed according to the specifications G_Λ.

The loss network is a continuous-time Markov process $\eta_t \in \{0, 1\}^\Gamma$, having as unique invariant measure the target Gibbs measure. In this process, each cycle γ attempts to appear independently at a rate $w(\gamma)$ defined later in (2.4), and γ is allowed to join the existing configuration only if it does not overlap with the already present cycles. Cycles also die, independently, at rate 1. If α is sufficiently large this process is well defined in infinite volume, and a realization of the stationary process running for all $t \in \mathbb{R}$ can be constructed as a function of a family of space-time Poisson processes, the usually called Harris graphical construction. The condition for the existence of the stationary process is related to the absence of oriented percolation of cycles in the space–time realization of a *free process* in $\{0, 1, \dots\}^\Gamma$, where all cycles are allowed to be born, regardless whether they overlap with pre-existing cycles or not. The no-percolation condition follows from dominating the percolation cluster by a subcritical multitype branching process, a standard technique, see for instance [9]. The subcriticality condition for the branching process leads to the condition $\alpha > \alpha^*$.

Background and further prospects   The existence of a Gibbs measure concentrating on finite-cycle permutations of $\mathbb{Z}^d$ was first obtained by Gandolfo, Ruiz and Ueltschi [7] in the large temperature regime for the Gaussian potential. Recently, Betz [1] gave a condition yielding tightness of the specifications for a more general family of potentials, for any value of α, his results imply that thermodynamic limits of specifications with identity boundary conditions exist for any $\alpha > 0$ and dimension d. However, the problem of identifying these limits and their typical cycle length remains open.

Biskup and Richthammer [3] consider the one dimensional case and strictly convex potentials satisfying some additional growth conditions. They prove that the set of all ground states associated to H in (1.2) is $\{\tau_v : v \in \mathbb{Z}\}$, $\tau_v(x) = x + v$ as in (1.5), and that for each ground
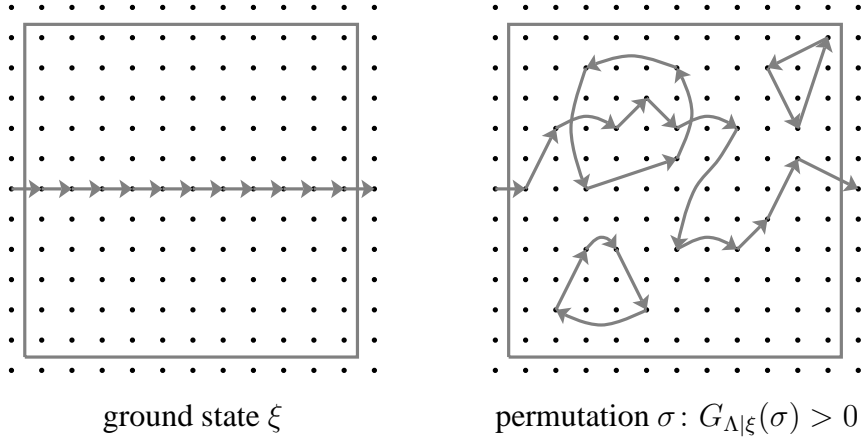
| ground state ξ | permutation $\sigma\colon G_{\Lambda\mid\xi}(\sigma) > 0$ |

Figure 1: A dot at site x means that $\xi(x) = x$ while an arrow from x to y means that $\xi(x) = y$. The left picture represents the ground state ξ described in (1.6). On the right we see a permutation with positive probability according to $G_{\Lambda\mid\xi}$ defined in (2.2). The square represents the box Λ.

state τ_v and temperature $\alpha > 0$ there is a Gibbs measure μ_v^α. Furthermore, they show that the set of extremal Gibbs measures is $\mathcal{G}_{\alpha,e} = \{\mu_v^\alpha, \ v \in \mathbb{Z}\}$, that is, each extremal Gibbs measure is associated to a ground state. The measure μ_v^α is translation invariant and supported on configurations having exactly n infinite cycles. They also prove that for any $\alpha > 0$, the measure μ_v^α has a regeneration property, which in the case $v = 0$ entails the convergence as $\Lambda \nearrow \mathbb{Z}$ of the specifications G_Λ with identity boundary conditions to μ_0^α. In particular, this implies that for $d = 1$, identity boundary conditions lead to finite cycles, for all temperatures.

Infinite cycles. In d–dimensions our results say that under identity boundary conditions, for α large enough, the Gibbs measures concentrate on finite-cycle permutations. On the other hand, for the Gaussian potential and small α, Gandolfo, Ruiz and Ueltschi [7] performed numerical simulations of the 3-dimension specification associated to a box Λ yielding cycles with *macroscopic* length, i.e., length that grows proportionally to the size of Λ. More recent numerical results by Grosskinsky, Lovisolo and Ueltschi [10] suggest that the scaled down size of these macroscopic cycles converges to a Poisson-Dirichlet distribution. See also Goldschmidt, Ueltschi and Windridge [8] for a discussion relating cycle representations and fragmentation-coagulation models, where the Poisson-Dirichlet distributions appear naturally. The authors in [10] argue that the situation should be similar in higher dimensions, in contrast to the case $d = 2$. In 2–dimensions it is expected that the size of the cycles grows as $\Lambda \nearrow \mathbb{Z}^2$, but in this case the length would not be macroscopic, a conjecture that is supported by numerical simulations in [1, 7]. The question remains whether a positive fraction of sites belongs to these mesoscopic cycles. Betz [1] provides numerical evidence that for $d = 2$ long cycles are fractals in the thermodynamic limit, and conjectures a connection to Schramm-Loewner evolution.

Domain of attraction of Gibbs measures. Let $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ and $e_1 = (1, 0, \dots, 0)$ denote the first vector in the canonical basis. In a forthcoming paper, Yuhjtman considers the Gaussian potential with ground state ξ defined by

$$\xi(x) = \begin{cases} x + e_1 & \text{if } x_2 = \cdots = x_d = 0, \\ x & \text{otherwise,} \end{cases} \tag{1.6}$$
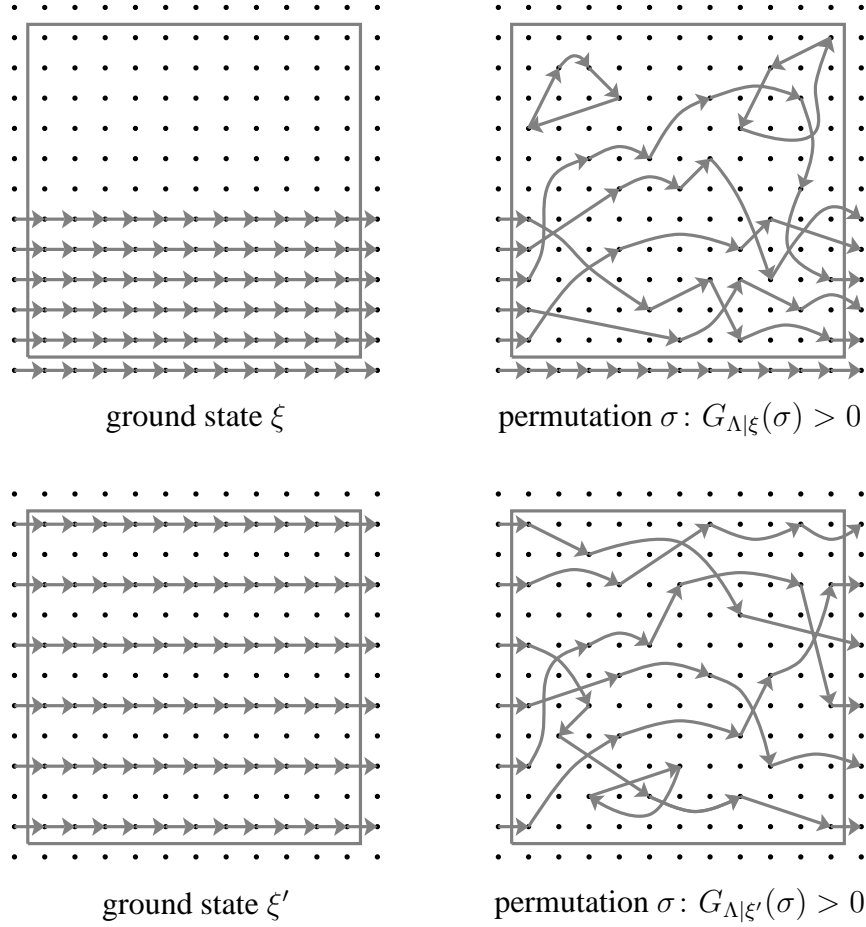
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ground state ξ

permutation $\sigma : G_{\Lambda|\xi}(\sigma) > 0$

ground state ξ'

permutation $\sigma : G_{\Lambda|\xi'}(\sigma) > 0$

Figure 2: The ground states ξ and ξ' and permutations with positive probability for the specifications with ξ and ξ' boundary conditions respectively.

(see Figure 1) and shows that the thermodynamic limit of $G_{\Lambda|\xi}$, the specifications with boundary conditions given by ξ, is equal to the measure μ associated to identity boundary conditions. In particular, in dimensions higher than 1, the one-to-one correspondence between ground states and extremal Gibbs measures fails to hold. It would be interesting to find the domain of attraction of each Gibbs state. That is, if μ is a Gibbs measure, one would like to characterize the set $\left\{\xi \,:\, \lim_{\Lambda \nearrow \mathbb{Z}^d} G_{\Lambda|\xi} = \mu \right\}$.

Further translation invariant Gibbs measures. Set $d \geq 2$ and consider the ground states $\xi, \xi' : \mathbb{Z}^d \to \mathbb{Z}^d$ given by

$$\xi(x) = \begin{cases} x & \text{if } x_d \geq 0, \\ x + e_1 & \text{if } x_d < 0 \end{cases}, \qquad \xi'(x) = \begin{cases} x & \text{if } x_d \text{ is even}, \\ x + e_1 & \text{if } x_d \text{ is odd}. \end{cases} \qquad (1.7)$$

Our approach requires translation invariance of the boundary conditions, which are satisfied neither by ξ nor by ξ' (see Figure 2). The conjecture is that the thermodynamic limit arising from any of these boundary conditions should lead to a Gibbs measure with $\frac{1}{2}$ - density of paths crossing the hyperplane $x_1 = 0$ from left to right. In connection to these ground states, it would be interesting to describe the macroscopic shape determined by these left-right crossing paths.

Permutations of point processes. When the points are distributed according to a point process there are two possibilities. In the quenched case one studies the random permutation of a fixed point configuration. In this case we expect that our approach would be useful to show that for almost all point configuration there is a unique Gibbs measure when the temperature is high enough in relation to the point density ρ. The 1–dimensional quenched case is studied by Biskup and Richthammer [3], who prove that there are no infinite cycles for any value of the temperature. Süto [14, 15] investigates the annealed case, where one jointly averages point positions and permutations. By integrating over the former, it is then possible to explicitly identify the temperature α_0 below which infinite cycles appear, Süto points out that this is equivalent to Bose-Einstein condensation in the Bose gas. These results are generalized by Betz and Ueltschi in [2].

Organization of the article We introduce notation and describe rigorously the results in Section 2, we then sketch the techniques in Subsection 2.1. We construct the loss network approach of [4] in Section 3, and prove the main results in Section 4. Section 5 contains bounds for $\alpha^*(V)$.

2 Notation and Results

Denote by S the set of permutations of $\mathbb{Z}^d$, that is

$$S := \{\sigma : \mathbb{Z}^d \to \mathbb{Z}^d, \sigma \text{ bijective}\},$$

equipped with the product topology generated by the sets $\{\sigma \in S : \sigma(x) = y\}$, $x, y \in \mathbb{Z}^d$, and the associated Borel sigma-algebra $\mathcal{B}$. Given a permutation $\xi \in S$ and a finite set $\Lambda \subseteq \mathbb{Z}^d$, let

$$S_{\Lambda|\xi} := \{\sigma \in S : \sigma(x) = \xi(x), \ x \in \Lambda^c\}, \tag{2.1}$$

be the set of permutations that match ξ outside of Λ. Let I be the identity permutation, $I(x) = x$ for all $x \in \mathbb{Z}^d$, and denote $S_\Lambda := S_{\Lambda|I}$. Let $V : \mathbb{Z}^d \to \mathbb{R}^+ \cup \{+\infty\}$ be a strictly convex potential with $V(\vec{0}) = 0$ and recall the definition (1.2) of the Hamiltonian $H_\Lambda : S \to \mathbb{R}$.

Fix $\alpha > 0$. The Hamiltonian determines a family of probability measures called specifications, indexed by the set of finite $\Lambda \subset \mathbb{Z}^d$ and permutations ξ, defined by

$$G_{\Lambda|\xi}(\sigma) := \frac{1}{Z_{\Lambda|\xi}} \exp(-\alpha H_\Lambda(\sigma)), \quad \sigma \in S_{\Lambda|\xi}, \tag{2.2}$$

where $Z_{\Lambda|\xi}$ is the normalizing constant $Z_{\Lambda|\xi} := \sum_{\sigma \in S_{\Lambda|\xi}} \exp(-\alpha H_\Lambda(\sigma))$. Denote $G_\Lambda := G_{\Lambda|I}$.

A measure μ on S is said to be Gibbs at temperature α for the family of specifications $(G_{\Lambda|\xi})$ if the conditional distribution of μ on Λ given ξ outside Λ coincides with the specification $G_{\Lambda|\xi}$. That is, for finite $\Lambda \subseteq \mathbb{Z}^d$ and $\xi \in S$,

$$\mu(\cdot \mid \sigma(x) = \xi(x), x \in \Lambda^c) = G_{\Lambda|\xi} \quad \text{for } \mu\text{-almost all } \xi \in S.$$

We denote the set of Gibbs measures at temperature α by $\mathcal{G}^\alpha$, and let $\mathcal{G} = \cup_{\alpha > 0} \mathcal{G}^\alpha$.

Take $n \geq 2$. A *finite cycle* γ of length $|\gamma| = n$ associated to the set of distinct sites $x_1, \ldots, x_n$ is a permutation $\gamma \in S$ such that $\gamma(x) = x$ for all $x \notin \{x_1, \ldots, x_n\}$, $x_{i+1} = \gamma(x_i)$ for all

$i \in \{1, \dots, n\}$, with the convention $x_{n+1} = x_1$. An *infinite cycle* γ associated to a doubly infinite sequence of distinct sites $\dots, x_{-1}, x_0, x_1, \dots$ is a permutation such that $\gamma(x) = x$ if $x \neq x_i$ for any i and $x_{i+1} = \gamma(x_i)$ for all i. The support of a cycle γ associated to $x_1, \dots, x_n$ is $\{\gamma\} = \{x_1, \dots, x_n\}$. Denote the set of finite cycles by

$$\Gamma := \{\gamma \in S : \gamma \text{ is a cycle with } \{\gamma\} \text{ finite}\} \quad \text{and} \quad \Gamma_\Lambda := \{\gamma \in \Gamma : \{\gamma\} \subset \Lambda\}, \quad (2.3)$$

the set of cycles with support contained in Λ. We say that two permutations are *disjoint* if their supports are so.

Denote $\sigma\sigma'$ the composition of the permutations σ, σ':

$$(\sigma\sigma')(x) := \sigma(\sigma'(x)).$$

Any permutation $\sigma \neq I$ can be written as a finite or countable composition of disjoint cycles:

$$\sigma = \dots \gamma_2 \gamma_1, \qquad \{\gamma_i\} \cap \{\gamma_j\} = \emptyset, \ \text{ for all } i \neq j,$$

note that the order of the cycles in this composition does not matter. The identity has no cycle decomposition. A permutation σ is called *finite-cycle* if all cycles in its decomposition are finite. In this case we identify $\sigma \neq I$ with the "gas of cycles" $\{\gamma_1, \dots, \gamma_k\}$, $k = k(\sigma) \in \mathbb{N} \cup \{+\infty\}$, while the identity I is identified with the empty set. We denote $\gamma \in \sigma$ when γ is one of the cycles in the decomposition of σ.

For a finite cycle $\gamma \in \Gamma$, define the *weight* of γ by

$$w(\gamma) := \exp\left\{-\alpha \sum_{x \in \{\gamma\}} V\big(\gamma(x) - x\big)\right\}. \quad (2.4)$$

Since γ is a cycle and V is strictly convex, the sum in (2.4) is strictly positive, which in turn implies $w(\gamma) \in (0, 1)$ for all $\alpha > 0$. Define

$$\beta(V, \alpha) := \sum_{\gamma \in \Gamma, \{\gamma\} \ni \vec{0}} |\gamma| \, w(\gamma). \quad (2.5)$$

If $\beta(V, \alpha)$ is finite for some α, then $\beta(V, \alpha)$ is decreasing in α and $\beta(V, \alpha) < 1$ for all $\alpha > \alpha^*$ defined by

$$\alpha^*(V) := \inf\{\alpha : \beta(V, \alpha) < 1\}. \quad (2.6)$$

If $\beta(V, \alpha) = \infty$ for all α we set $\alpha^* = \infty$.

In our first theorem we give sufficient conditions on α for the existence of a Gibbs measure as limit of specifications with identity boundary conditions. The proof follows the lines proposed in [4] to construct the infinite volume limit of the contour representation for the Ising model at low temperature. We include the proof for the convenience of the reader.

Theorem 2.1. *Identity boundary conditions.*

Fix a strictly convex potential $V : \mathbb{Z}^d \to \mathbb{R}^+ \cup \{+\infty\}$ satisfying $V(\vec{0}) = 0$. Assume $\alpha^*(V) < \infty$. Then, for each $\alpha > \alpha^*(V)$ there exists a random process $(\zeta_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ on $(S_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ such that
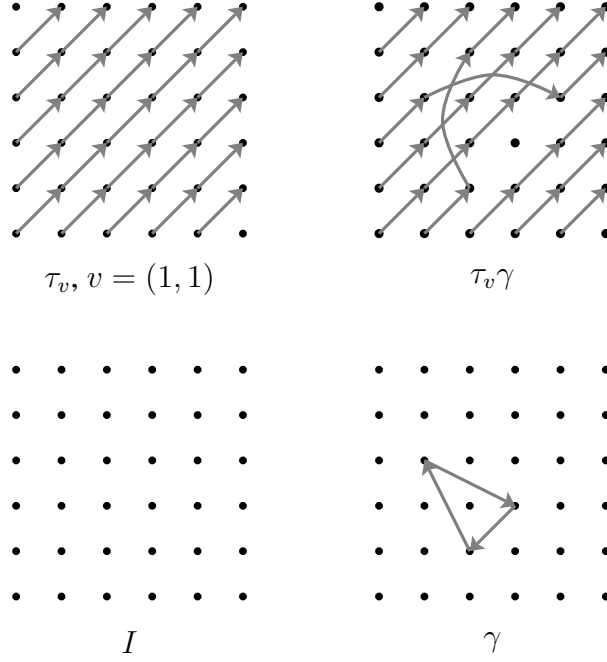
Figure 3: Local perturbation $\tau_v\gamma$ of τ_v, $v = (1, 1)$, introduced by the cycle γ.

(i) for finite Λ, ζ_Λ is distributed according to G_Λ, the specification with identity boundary conditions,

(ii) $\lim_{\Lambda \nearrow \mathbb{Z}^d} \zeta_\Lambda(x) = \zeta_{\mathbb{Z}^d}(x)$ almost surely, for each $x \in \mathbb{Z}^d$. Call μ the distribution of $\zeta_{\mathbb{Z}^d}$. Then, $\lim_{\Lambda \nearrow \mathbb{Z}^d} G_\Lambda = \mu$ weakly.

(iii) μ is an ergodic Gibbs measure at temperature α with mean jump $\vec{0}$.

(iv) μ is the unique Gibbs measure for the specifications G_Λ, supported on the set of finite-cycle permutations of $\mathbb{Z}^d$.

We next consider more general boundary conditions.

We will say that the permutation σ' is a *local perturbation* of σ if the set $\{x \in \mathbb{Z} : \sigma'(x) \neq \sigma(x)\}$ is finite; in this case, the energy difference between σ' and σ is defined by

$$H(\sigma') - H(\sigma) := \sum_{x:\sigma(x)\neq\sigma'(x)} \big(V(\sigma'(x) - x) - V(\sigma(x) - x)\big).$$

A *ground state* is a permutation $\xi \in S$ such that for any local perturbation ξ' of ξ, $H(\xi') - H(\xi) \geq 0$. For $v \in \mathbb{Z}^d$, the shift permutation $\tau_v \in S$ defined in (1.5) is a ground state: given a finite cycle γ, the permutation $\tau_v\gamma$ is a local perturbation of τ_v with energy difference

$$H(\tau_v\gamma) - H(\tau_v) = \sum_{x \in \{\gamma\}} \big[V(\gamma(x) + v - x) - V(v)\big] > 0, \tag{2.7}$$

by the strict convexity of V.

The next lemma says that a local perturbation of τ_v is a composition of a finite number of finite cycles with τ_v. We leave the proof to the reader. See Figure 3.

Lemma 2.2. *If τ'_v is a local perturbation of τ_v, then there exist disjoint finite cycles $\gamma_1, \ldots, \gamma_n$ in Γ such that $\tau'_v = \tau_v \gamma_1 \ldots \gamma_n$. If V is strictly convex, then $H(\tau'_v) - H(\tau_v) > 0$.*

In the following theorem we establish conditions on α that allow to extend the results of Theorem 2.1 to τ_v-boundary conditions. For a finite cycle $\gamma \in \Gamma$, denote the *v-weight* of γ by

$$w_v(\gamma) := \exp\Big\{-\alpha \sum_{x \in \{\gamma\}} \big(V(\gamma(x) + v - x) - V(v)\big)\Big\}. \tag{2.8}$$

Given a measure μ and a permutation ξ recall the definition of the shifted measure $\mu\xi$ from (1.4). In order to obtain the result, we first consider the composition of a configuration with τ_v boundary conditions with the permutation τ_{-v} to produce a finite cycle permutation with cycles weighted by w_v. We then apply Theorem 2.1 to this random permutation, take limits in $\Lambda \nearrow \mathbb{Z}^d$, and as a last step compose the resulting measure with the permutation τ_v to recover the initial boundary conditions.

Let

$$\alpha_v^*(V) := \inf\{\alpha : \beta_v(V, \alpha) < 1\}, \quad \text{where} \quad \beta_v(V, \alpha) := \sum_{\gamma \in \Gamma, \{\gamma\} \ni \vec{0}} |\gamma| \, w_v(\gamma). \tag{2.9}$$

Theorem 2.3. τ_v *boundary conditions.*

Fix a vector $v \in \mathbb{Z}^d$ and a strictly convex potential $V : \mathbb{Z}^d \to \mathbb{R}^+ \cup \{+\infty\}$ such that $V(\vec{0}) = 0$. If $\alpha_v^*(V)$ is finite, then for any $\alpha > \alpha_v^*(V)$ there exists a random process $(\zeta_{\Lambda,v}, \Lambda \subseteq \mathbb{Z}^d)$ on $(S_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ such that

(i) For finite Λ, $\tau_v \zeta_{\Lambda,v}$ is distributed according to $G_{\Lambda|\tau_v}$, the specification with τ_v boundary conditions.

(ii) $\lim_{\Lambda \nearrow \mathbb{Z}^d} \tau_v \zeta_{\Lambda,v}(x) = \tau_v \zeta_{\mathbb{Z}^d,v}(x)$ almost surely, for all $x \in \mathbb{Z}^d$. Calling μ_v the law of $\tau_v \zeta_{\mathbb{Z}^d,v}$, we get $\lim_{\Lambda \nearrow \mathbb{Z}^d} G_{\Lambda|\tau_v} = \mu_v$ weakly.

(iii) μ_v is an ergodic Gibbs measure at temperature α with mean jump v.

(iv) $\mu_v \tau_{-v}$ is the only measure with cycle weights w_v supported on the set of finite-cycle permutations of $\mathbb{Z}^d$.

We finally consider separately the Gaussian potential. Although it is in principle covered by the previous results, it is worth pointing out that in this case the associated v-weights do not actually depend on v, $w_v(\gamma) = w(\gamma)$ for all $\gamma \in \Gamma$, with the consequence that the shift boundary condition measures are just the composition of the identity boundary conditions Gibbs measure μ with τ_v, $\mu_v = \mu\tau_v$, and the value of α^* is the same for all $v \in \mathbb{Z}^d$, $\alpha_v^* = \alpha_{\vec{0}}^*$. We also compute an explicit bound on α^*.

Theorem 2.4. *The Gaussian case.*

Let $V(x) = \|x\|^2$, then $\alpha^*(V) \leq \big(1.44504^{1/d} - 1\big)^{-2}$.

Fix $\alpha > \alpha^*(V)$, let $(\zeta_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ be the process constructed in Theorem 2.1, and let μ be the distribution of $\zeta_{\mathbb{Z}^d}$. Then, for each $v \in \mathbb{Z}^d$,

(i) for finite Λ, $G_\Lambda \tau_v = G_{\Lambda|\tau_v}$, the specification with τ_v boundary conditions. In particular, $\tau_v \zeta_\Lambda$ has distribution $G_{\Lambda|\tau_v}$.

(ii) $\lim_{\Lambda \nearrow \mathbb{Z}^d} \tau_v \zeta_\Lambda(x) = \tau_v \zeta_{\mathbb{Z}^d}(x)$ almost surely, for all $x \in \mathbb{Z}^d$. As a consequence $\lim_{\Lambda \nearrow \mathbb{Z}^d} G_{\Lambda|\tau_v} = \mu\tau_v$ weakly.

(iii) $\mu\tau_v$ is an ergodic Gibbs measure at temperature α and mean jump v.

2.1 Sketch of the proofs

Identity boundary conditions. Consider a finite $\Lambda \subset \mathbb{Z}^d$ and recall $S_\Lambda = S_{\Lambda|I}$ is the set of permutations that equal the identity outside of Λ.

A finite-cycle permutation $\sigma \in S$ can be identified with the configuration $\eta \in \{0,1\}^\Gamma$ defined by $\eta(\gamma) = \mathbf{1}\{\gamma \in \sigma\}$. Thus S_Λ can be described as a subset of $\{0,1\}^{\Gamma_\Lambda}$:

$$S_\Lambda = \left\{ \eta \in \{0,1\}^{\Gamma_\Lambda} : \eta(\gamma)\eta(\gamma') = 0 \text{ if } \{\gamma\} \cap \{\gamma'\} \neq \emptyset, \text{ for all } \gamma, \gamma' \in \Gamma_\Lambda \right\}. \tag{2.10}$$

Recall the definition (2.4) of weight of a cycle γ. The specification in Λ with identity boundary conditions (2.2) can now be written as

$$G_\Lambda(\eta) = \frac{1}{Z_\Lambda} \prod_{\gamma \in \Gamma_\Lambda} w(\gamma)^{\eta(\gamma)}, \qquad \eta \in S_\Lambda. \tag{2.11}$$

We interpret the measure G_Λ as the distribution of the gas of cycles with weights w and interacting by exclusion. This is the setup proposed in [4] to study the contour representation of the low temperature Ising model.

Let now $S^o = \{0, 1, \dots\}^\Gamma$. Note that in S^o cycles may have intersecting support; indeed, the same cycle may have multiplicity larger than 1. Given a configuration $\eta \in S^o$, $\eta(\gamma)$ counts the number of times the cycle γ is present in η. Let μ^o be the product measure on S^o with marginal Poisson$(w(\gamma))$ for each $\gamma \in \Gamma$. If η^o has law μ^o, then the random variable $\eta^o(\gamma)$ is Poisson with mean $w(\gamma)$, and the random variables $\eta^o(\gamma)$, $\gamma \in \Gamma$ are independent. For finite Λ, G_Λ is just the law μ^o conditioned to S_Λ:

$$G_\Lambda = \mu^o(\cdot \,|S_\Lambda). \tag{2.12}$$

We claim that for large enough α we can construct a Poisson measure on S^o conditioned to the event that each cycle is present at most once, and present cycle supports are disjoint. That is, the measure is supported on the set of configurations associated to finite-cycle permutations of $\mathbb{Z}^d$. Since this set has zero μ^o-probability, an argument is required to give a proper sense to this notion. For α large we construct μ as the invariant measure for a continuous-time birth and death process of cycles interacting by exclusion, and show that it concentrates on finite-cycle permutations. We also prove that μ is the limit as $\Lambda \to \infty$ of G_Λ given by (2.12).

Given a cycle $\gamma \in \Gamma$, consider the rates of a continuous-time birth and death process on $\{0, 1, \dots\}$ defined by

$$\text{birth rates: } q_\gamma(k, k+1) := w(\gamma), \quad \text{death rates: } q_\gamma(k+1, k) := k+1, \ \ k \geq 0. \tag{2.13}$$

We construct birth and death processes with the above rates as a function of a Poisson process. Let $\mathcal{N}$ be a Poisson process on $\Gamma \times \mathbb{R} \times \mathbb{R}^+$ with rate measure $w(\gamma) \times dt \times e^{-s}ds$.

If the point $(\gamma, t', s') \in \mathcal{N}$, we say that a cycle γ is born at time t' and lives until $t' + s'$. Define η^o as the number of cycles γ *alive* at time t. By construction $(\eta_t^o(\gamma), t \in \mathbb{R})$ is a time-stationary continuous-time birth and death process with rates q_γ given in (2.13); that is, at any time a new copy of a cycle γ is born at rate $w(\gamma)$, whereas existing copies die independently at rate 1. The marginal distribution of $\eta_t^o(\gamma)$ is Poisson with mean $w(\gamma)$, for each $t \in \mathbb{R}$. Letting $\eta_t^o := (\eta_t^o(\gamma) : \gamma \in \Gamma)$, the process $(\eta_t^o, t \in \mathbb{R})$ is a family of stationary independent birth and death processes with marginal distribution μ^o at any time t.

Our goal is to perform such a graphical construction for a birth and death process with the same rates, subject to an exclusion rule as follows. Now the point $(\gamma, t, s) \in \mathcal{N}$ represents a birth attempt of a cycle γ at time t (see Figure 4), but the cycle will be effectively born only if its support $\{\gamma\}$ does not intersect the support of any of the cycles already present at that time t. When the process is restricted to a finite set Λ, the points in $\{(\gamma, t, s) \in \mathcal{N}, \gamma \in \Gamma_\Lambda\}$ can be ordered by their birth time t. Since the free process is empty infinitely often: $\eta_t^o(\gamma) = 0$ for all $\gamma \in \Gamma_\Lambda$ for infinitely many positive and negative times, it is possible to iteratively decide for each (γ, t, s) if it actually produces a birth of γ in the model with exclusion, or not. We so construct a stationary birth and death process $(\eta_t^\Lambda, t \in \mathbb{R})$ on Γ^Λ with rates $(q_\gamma, \gamma \in \Gamma^\Lambda)$ subjected to the exclusion condition on cycles in Λ. The marginal distribution of η_t^Λ is G_Λ.

Figure 4: The representation of the point $\varphi = (\gamma, t, s)$. Time is going down.

In infinite volume the above argument does not work because the configuration is never empty. Instead, for each point $(\gamma, t, s) \in \mathcal{N}$ one can look for the points of $\mathcal{N}$ born prior to t that could interfere with the birth of the cycle γ at time t. This set is called the *clan of ancestors* of $(\gamma, t, s) \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$. If the clan of ancestors of any point is finite with probability one, then it is possible to construct the stationary loss network of finite cycles in $\mathbb{Z}^d$. We call $(\eta_t, t \in \mathbb{R})$ the resulting Markov process, obtained as a deterministic function of $\mathcal{N}$. Let us suggestively denote by μ, the notation previously used to name the Gibbs measure, the distribution of the permutation with cycles η_t for a given time t. Since the construction is time-stationary, the measure μ does not depend on t: it is an invariant measure for the process. In fact one can check that μ is reversible for the process. We show that μ is the thermodynamic limit of G_Λ and the unique invariant measure for the process (η_t).

In order to prove that μ is the thermodynamic limit of G_Λ, we construct a stationary family of processes $(\eta_t^\Lambda, t \in \mathbb{R})$ for any $\Lambda \subset \mathbb{Z}^d$ as a function of a unique realization $\mathcal{N}$ of the Poisson process; a coupling. For finite Λ, the marginal distribution of η_t^Λ is G_Λ. We use the finiteness of the clan of ancestors to show that for each finite-cycle γ, $\eta_t^\Lambda(\gamma)$ converges to $\eta_t(\gamma)$ as $\Lambda \nearrow \mathbb{Z}^d$, for almost all realizations of the point process $\mathcal{N}$. In particular, this proves that G_Λ converges weakly to μ and yields several properties of the limit.

To show that the clan of ancestors of a point (γ, t, s) is finite we dominate it by a multitype branching process and then show that the condition $\beta(V, \alpha) < 1$ is a sufficient condition for the branching process to die out. We give more details of these processes in Section 3.

For any fixed $t \in \mathbb{R}$, the process $(\eta_t^\Lambda : \Lambda \subseteq \mathbb{Z}^d)$ satisfies the properties attributed to the process $(\zeta_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ in Theorem 2.1.
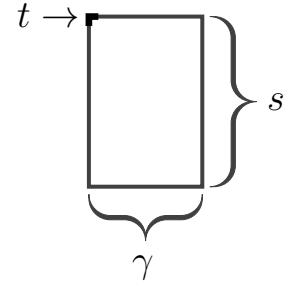
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v-jump boundary conditions. The specifications associated to the potential

$$V_v(y) := V(y + v) - V(y) \tag{2.14}$$

with identity boundary conditions are given by $G_{\Lambda|\tau_v}\tau_{-v}$ on S_Λ, for any finite $\Lambda \subseteq \mathbb{Z}^d$. We prove that for $\alpha_v^*(V) < \infty$, V_v satisfies the conditions of Theorem 2.1 to obtain a process $(\zeta_{\Lambda,v}, \Lambda \subset \mathbb{Z}^d)$ on $(S_\Lambda, \Lambda \subseteq \mathbb{Z}^d)$ with the properties stated in that theorem. We then use the fact that $\tau_v\zeta_{\Lambda,v}$ has law $G_{\Lambda|\tau_v}$ to obtain Theorem 2.3.

v-jump boundary conditions for the Gaussian potential. When the potential is Gaussian, $V = \|\cdot\|^2$, we have $G_{\Lambda|\tau_v} = G_\Lambda\tau_v$, a fact proven in Section 4. This is the key to the proof of Theorem 2.4.

3 Loss networks of finite cycles

We here construct the invariant measure of a loss network of cycles and show that it is the Gibbs measure related to the specifications G_Λ with identity boundary conditions. The section consists of a review of [4] described in terms of cycles instead of contours.

Loss network Take a potential V and a set $\Lambda \subset \mathbb{Z}^d$. Recall the definition (2.10) of $S_\Lambda \subset \{0,1\}^{\Gamma_\Lambda}$. We introduce a continuous-time Markov process in S_Λ called *loss network* of finite cycles. We say that two cycles are *compatible* if their supports are disjoint. Given a configuration $\eta \in S_\Lambda$ of the process, we add a new cycle γ at rate $w(\gamma)$, if it is compatible with η, that is, if γ is compatible with all cycles γ' with $\eta(\gamma') = 1$. If γ and η are not compatible, then the cycle is not added and the attempt is lost, hence the name loss network. Finally, any cycle in η is deleted at rate one. Loss networks were introduced as stochastic models of a telecommunication network in which calls are routed between nodes around a network. In our case the nodes are the d-dimensional integers and a call uses the nodes in the support of a non-identity cycle. Each node has capacity to support at most one call and hence arriving calls that would occupy an already busy node are lost. An account of the properties of loss networks can be found in Kelly [11].

Denote $\gamma \sim \eta$ if γ is compatible with η; in particular $\gamma \sim \eta$ implies $\eta(\gamma) = 0$. The loss network process on $S_\Lambda \subset \{0,1\}^{\Gamma_\Lambda}$ has formal generator

$$\mathcal{L}^\Lambda f(\eta) = \sum_{\gamma \in \Gamma_\Lambda} w(\gamma)\mathbf{1}_{\{\gamma \sim \eta\}}\left[f(\eta + \delta_\gamma) - f(\eta)\right] + \sum_{\gamma \in \Gamma_\Lambda}\left[f(\eta - \delta_\gamma) - f(\eta)\right], \tag{3.1}$$

where f is a test function, and $\delta_\gamma(\gamma') = 1$ if and only if $\gamma' = \gamma$. When Λ is finite, the loss network is a well defined, irreducible Markov process on a finite state space, with a unique invariant measure.

The next lemma shows that G_Λ defined in (2.11) is reversible for the loss network η_t^Λ; the proof is left to the reader.

Lemma 3.1. *Let Λ be finite. The measure G_Λ is reversible for the dynamics (3.1). In particular, this is the unique invariant measure, and the weak limit of the distribution of the process starting from any initial permutation as $t \to \infty$.*

In the following we show that when $\alpha > \alpha^*(V)$ given in (2.6) there exists a stationary process with generator (3.1) for any $\Lambda \subseteq \mathbb{Z}^d$. The proof relies on a coupling argument applying the Harris graphical construction of the process: to each configuration of an appropriate Poisson process $\mathcal{N}$ we associate a realization of the loss network, $\mathcal{N} \mapsto (\eta_t^\Lambda)$, for any $\Lambda \subseteq \mathbb{Z}^d$. We now introduce the basic elements of the argument.

The Poisson process. Let $\mathcal{N}$ be a Poisson process on $\Gamma \times \mathbb{R} \times \mathbb{R}^+$ with intensity measure

$$d(\gamma, t, s) = w(\gamma) \, dt \, e^{-s} ds \, .$$

This process can be thought of as a product of independent Poisson processes on $\mathbb{R} \times \mathbb{R}^+$, indexed by $\gamma \in \Gamma$.

The free process. Given the Poisson process $\mathcal{N}$, define the *free process* $(\eta_t^o, t \in \mathbb{R})$ on $\{0, 1, \dots\}^\Gamma$ by

$$\eta_t^o(\gamma) := \sum_{(\gamma, t', s') \in \mathcal{N}} \mathbf{1}\{t' \leq t < t' + s'\} \, . \tag{3.2}$$

If a point $(\gamma, t, s) \in \mathcal{N}$, we say that a cycle γ is born at time t and lives s time units. We represent it as a cylinder with base γ, height s with its higher point located at t. See Figure 4 where the basis is represented by a segment.

The construction implies that cycles of type γ are born independently at rate $w(\gamma)$, and each of them lives for an exponential time of parameter 1; there may be more than one cycle of type γ present at any given time. The process η_t^o is thus obtained as the product of independent birth and death processes $\left(\eta_t^o(\gamma) : \gamma \in \Gamma\right)$, with birth rates $w(\gamma)$ and death rate 1. The generator of η_t^o is given by

$$\mathcal{L}^o f(\eta) = \sum_{\gamma \in \Gamma} w(\gamma) \left[f(\eta + \delta_\gamma) - f(\eta) \right] + \sum_{\gamma \in \Gamma} \eta(\gamma) \left[f(\eta - \delta_\gamma) - f(\eta) \right],$$

where $f : \{0, 1, \dots\}^\Gamma \to \mathbb{R}$ is any local test function in the domain of $\mathcal{L}^o$. It is easy to see that the product measure μ^o on $\{0, 1, \dots\}^\Gamma$ with Poisson marginals

$$\mu^o(\eta : \eta(\gamma) = k) = \frac{e^{-w(\gamma)}(w(\gamma))^k}{k!}$$

is reversible for the free process. Indeed, this is the law of the configuration η_t^o defined in (3.2), for any fixed $t \in \mathbb{R}$.

The clan of ancestors. We will construct a stationary version of the loss network in infinite volume starting from the stationary free process, by simply erasing those cycles that violate the exclusion condition at birth. In order to make sense of this construction we need to consider the clan of ancestors of each point $(\gamma, t, s) \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$, as follows.

The first generation of ancestors of $\varphi = (\gamma, t, s)$ is the subset of $\mathcal{N}$ defined by

$$\mathcal{A}_1^\varphi := \{(\gamma', t', s') \in \mathcal{N} : \gamma' \not\sim \gamma, \, t' < t < t' + s'\}.$$

where, as before, two cycles γ and γ' are incompatible, $\gamma \not\sim \gamma'$, if their supports have non empty intersection; in particular, a cycle is incompatible with itself: $\gamma \not\sim \gamma$. Iteratively, the
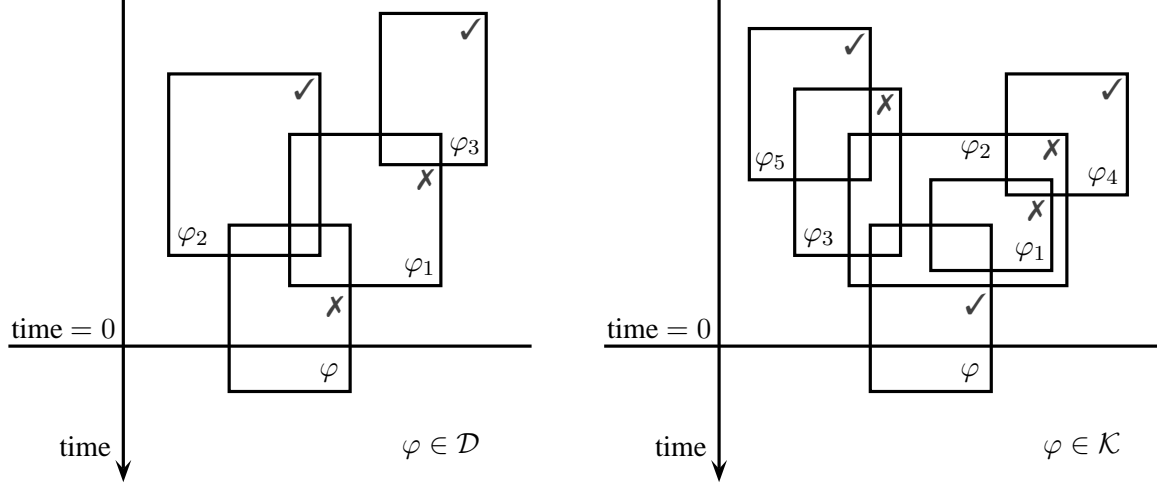
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Figure 5: The clan of ancestors of the point $\varphi = (\gamma, t, s)$ in two scenarios. On the left φ is deleted, while on the right it is kept.

$(n + 1)$-th generation of ancestors of φ is the union of the first generation of ancestors of the points belonging to the n-th generation of ancestors of φ, that is,

$$\mathcal{A}_{n+1}^{\varphi} := \bigcup_{\varphi' \in \mathcal{A}_n^{\varphi}} \mathcal{A}_1^{\varphi'}.$$

The *clan of ancestors* of φ is the union of all generations of ancestors:

$$\mathcal{A}^{\varphi} := \cup_{n \geq 1} \mathcal{A}_n^{\varphi}. \tag{3.3}$$

See Figure 5 with two scenarios. On the left $\mathcal{A}_1^{\varphi} = \{\varphi_1, \varphi_2\}$ and $\mathcal{A}_2^{\varphi} = \{\varphi_3\}$. On the right $\mathcal{A}_1^{\varphi} = \{\varphi_1, \varphi_2, \varphi_3\}$, $\mathcal{A}_2^{\varphi} = \{\varphi_2, \varphi_3, \varphi_4, \varphi_5\}$, $\mathcal{A}_3^{\varphi} = \{\varphi_3, \varphi_4, \varphi_5\}$ and $\mathcal{A}_4^{\varphi} = \{\varphi_5\}$.

Kept and deleted points. Assume $\mathcal{A}^{\varphi}$ finite for all $\varphi \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$, for almost all realizations of $\mathcal{N}$. Fix $\mathcal{D}_0 = \emptyset$, and for $n \geq 1$ let

$$\mathcal{K}_n := \{\varphi \in \mathcal{N} : \mathcal{A}_1^{\varphi} \setminus \mathcal{D}_{n-1} = \emptyset\}, \qquad \mathcal{D}_n := \{\varphi \in \mathcal{N} : \mathcal{A}_1^{\varphi} \cap \mathcal{K}_n \neq \emptyset\}.$$

Let $\mathcal{K} := \cup_n \mathcal{K}_n \subseteq \mathcal{N}$ be the set of *kept* points, and $\mathcal{D} := \cup_n \mathcal{D}_n \subseteq \mathcal{N}$ be the set of *deleted* points. As a consequence of the finiteness of the clans of ancestors, every point is either kept or deleted. Indeed, to determine whether a point φ is in $\mathcal{K}$ or $\mathcal{D}$, it suffices to inspect its clan of ancestors $\mathcal{A}^{\varphi}$. In Figure 5 we have checked the kept points and crossed the deleted ones.

Stationary loss network. Assume $\mathcal{A}^{\varphi}$ finite for all $\varphi \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$, for almost all realizations of $\mathcal{N}$. Define the *stationary loss network* $(\eta_t, t \in \mathbb{R})$ by

$$\eta_t(\gamma) := \sum_{(t', s'):(\gamma, t', s') \in \mathcal{N}} \mathbf{1}\{t' \leq t < t' + s'\} \, \mathbf{1}\{(\gamma, t', s') \in \mathcal{K}\}. \tag{3.4}$$

This is the set of cycles associated to kept points alive at time t. Note that $\eta_t(\gamma) \in \{0, 1\}$. The process $(\eta_t, t \in \mathbb{R})$ is stationary by construction, let us call μ its stationary distribution,

$$\mu := \text{ law of } \eta_t, \text{ for any } t \in \mathbb{R}. \tag{3.5}$$

The reader can prove the following result.

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Proposition 3.2. *Assume* $\mathcal{A}^\varphi$ *finite for all* $\varphi \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$, *for almost all realizations of* $\mathcal{N}$. *Then, the process* $(\eta_t,\, t \in \mathbb{R})$ *defined in* (3.4) *is Markov with generator* (3.1) *and invariant measure* μ *as in* (3.5).

Thermodynamic limit. The set of kept points is a deterministic function of $\mathcal{N}$: $\mathcal{K} = \mathcal{K}(\mathcal{N})$. Since the process $(\eta_t,\, t \in \mathbb{R})$ is a function of the kept points, it is also a function of $\mathcal{N}$: $(\eta_t) = (\eta_t)(\mathcal{N})$. Given $\Lambda \subset \mathbb{Z}^d$ define the Poisson process associated to the cycles in Γ_Λ,

$$\mathcal{N}^\Lambda := \{(\gamma, t, s) \in \mathcal{N} \,:\, \{\gamma\} \subset \Lambda\},$$

the corresponding set of kept points $\mathcal{K}^\Lambda := \mathcal{K}(\mathcal{N}^\Lambda)$, and the loss network of cycles in Λ

$$(\eta_t^\Lambda) := (\eta_t)(\mathcal{N}^\Lambda). \tag{3.6}$$

Clearly $\mathcal{N}^\Lambda$ is a function of $\mathcal{N}$. When the clan of ancestors of any point is finite for almost all realizations of $\mathcal{N}$, we have managed to define all processes $(\eta_t^\Lambda,\, t \in \mathbb{R})$, $\Lambda \subseteq \mathbb{Z}^d$, as a function of the same realization $\mathcal{N}$ of the point process. In particular notice that $\eta_t^{\mathbb{Z}^d} = \eta_t$.

When Λ is finite, the finiteness of the clan of ancestors is guaranteed and in this case (η_t^Λ) is an irreducible Markov process in the finite state space $S_\Lambda \subset \{0,1\}^{\Gamma_\Lambda}$ with generator $\mathcal{L}^\Lambda$ given by (3.1). By Lemma 3.1, the distribution of η_t^Λ is the measure G_Λ, which is reversible for the process, for any $t \in \mathbb{R}$.

We now state and prove the thermodynamic limit.

Theorem 3.3. *Existence of almost sure thermodynamic limit.*

Assume $\mathcal{A}^\varphi$, *the clan of ancestors of* φ, *is finite for all* φ *for almost all realizations of* $\mathcal{N}$. *Then for any fixed* $t \in \mathbb{R}$ *and* $\gamma \in \Gamma$, $\lim_{\Lambda \nearrow \mathbb{Z}^d} \eta_t^\Lambda(\gamma) = \eta_t(\gamma)$ *almost surely. In particular, as* $\Lambda \nearrow \mathbb{Z}^d$, G_Λ *converges weakly to* μ, *the stationary law of* η_t *in* (3.5).

Proof. Take a realization $\mathcal{N}$ such that $\mathcal{A}^\varphi$ is finite for all $\varphi \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$. It suffices to show that for any $\gamma \in \Gamma$ and $t \in \mathbb{R}$, there exists a set $\Lambda_t(\mathcal{N}, \gamma)$ such that if Λ contains $\Lambda_t(\mathcal{N}, \gamma)$, then $\eta_t^\Lambda(\gamma) = \eta_t(\gamma)$. Take the point (γ, t, s) (s is irrelevant here) and define

$$\Lambda_t(\mathcal{N}, \gamma) := \bigcup_{(\gamma', t', s') \in \mathcal{A}^{(\gamma, t, s)}} \{\gamma'\}, \tag{3.7}$$

the union of the supports of the cycles γ' present in the clan of ancestors of (γ, t, s). Now if Λ contains $\Lambda_t(\mathcal{N}, \gamma)$, then the clan of ancestors restricted to Λ is the same as the non-restricted clan: $\mathcal{A}^{(\gamma, t, s)}(\mathcal{N}^\Lambda) = \mathcal{A}^{(\gamma, t, s)}(\mathcal{N})$. This implies $\eta_t^\Lambda(\gamma) = \eta_t(\gamma)$ for all $\Lambda \supset \Lambda_t(\mathcal{N}, \gamma)$. $\qquad\square$

Theorem 3.4. *Uniqueness.*

Assume $\mathcal{A}^\varphi$ *is finite for any point* $\varphi = (\gamma, t, s)$, *for almost all realizations of* $\mathcal{N}$. *Let* ν *be an invariant measure for the loss network dynamics defined by* (3.1) *supported on finite-cycle configurations. Then* $\nu = \mu$, *the law of the stationary process* η_t *at any fixed time* t.

Proof. Let $(\eta_t)_{t \in \mathbb{R}}$ denote the stationary loss network. Consider a family $(s(\theta) : \theta \in \Gamma)$ of iid random variables with exponential distribution of rate 1. For any $\eta' \in \{0,1\}^\Gamma$ define

$$\mathcal{I}_{\eta'}(u) := \{(\theta, u, s(\theta)) : \eta'(\theta) = 1\}. \tag{3.8}$$

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and the set of $\mathcal{N}$-points born after u,

$$\mathcal{N}_{[u,\infty)} = \{(\gamma, t', s) \in \mathcal{N} : t' \geq u\}. \tag{3.9}$$

Let $(\eta'_{[u,t]} : t \geq u)$ be the coupled loss network with initial configuration $\eta'_u = \eta'$ that updates using the points in $\mathcal{N}_{[u,\infty)}$, and such that each initial cycle $\theta \in \eta'$ dies at time $u + s(\theta)$. Then

$$(\eta'_{[u,t]}, t \geq u) := (\eta_t, t \geq u)(\mathcal{I}_{\eta'}(u) \cup \mathcal{N}_{[u,\infty)}). \tag{3.10}$$

Note that the distribution of $\eta'_{[-t,0]}$ is the same as $\eta'_{[0,t]}$.

We can compare the latter process with the stationary process at time 0: for any $\gamma \in \Gamma$

$$\left|\eta_0(\gamma) - \eta'_{[-t,0]}(\gamma)\right| \leq \mathbf{1}\{\mathcal{A}^{(\gamma,0,s)} \not\subset \mathcal{N}_{[-t,\infty)}\} + \mathbf{1}\{\mathcal{A}^{(\gamma,0,s)} \not\sim \mathcal{I}_{\eta'}(-t)\}, \tag{3.11}$$

where the death time $s \in (\gamma, 0, s)$ is in fact irrelevant to the computation. Equation (3.11) says that if each point in the clan of ancestors of $(\gamma, 0, s)$ is born after time $-t$, and it is compatible with all points associated to the cycles θ in the initial configuration η', then the cycle γ belongs to both configurations or to none of them. Since the clan of ancestors is finite, for any cycle γ:

$$\lim_{t \to \infty} \left|\eta_0(\gamma) - \eta'_{[-t,0]}(\gamma)\right| = 0 \qquad a.s.. \tag{3.12}$$

Sample a random η' distributed according to the invariant measure ν, then $\eta'_{[-t,0]}$ has law ν for all t and $\eta'_{[-t,0]}(\gamma) \to \eta_0(\gamma)$ almost surely for all finite cycle γ. If $f : \{0,1\}^\Gamma \to \mathbb{R}$ is a bounded cylindrical function this implies $f(\eta'_{[-t,0]}) \to_{t\to\infty} f(\eta_0)$ almost surely and

$$\nu f = \int \nu(d\eta') f(\eta'_{[-t,0]}) \xrightarrow[t\to\infty]{} E f(\eta_0) = \mu f, \tag{3.13}$$

i.e., $\nu = \mu$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Conditions for the clan of ancestors to be finite   The results of this section depend crucially on the hypothesis that the clan of ancestors of any point $\varphi \in \Gamma \times \mathbb{R} \times \mathbb{R}^+$ be finite for almost all realizations of $\mathcal{N}$.

Fix $\varphi = (\gamma, t, s)$. For each $\theta \in \Gamma$ and $n \geq 1$, define

$$A_n(\gamma, \theta) := \left|\{(t', s') : (\theta, t', s') \in \mathcal{A}_n^{(\gamma,t,s)}\}\right|,$$

the number of θ-points $\in \mathcal{N}$ that belong to the n-th generation of ancestors of (γ, t, s), and denote the number of θ-points in the clan of ancestors of (γ, t, s) by

$$A(\gamma, \theta) := \sum_{n \geq 0} A_n(\gamma, \theta). \tag{3.14}$$

The total size of the clan of ancestors is $|\mathcal{A}^{(\gamma,t,s)}| = \sum_{\theta \in \Lambda} A(\gamma, \theta)$. We set conditions on V and α that ensure this sum is finite.

Subcritical multitype branching process.   We dominate the number of points in the clan of ancestors by a branching process. Ancestors in the clan become descendants for the branching process, hence time runs backwards for the branching process.

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Let B_n be a discrete time multitype branching process with type-space Γ and offspring distribution $A_1(\gamma, \theta)$. The number of children of type θ in the n-th generation is defined by $B_0(\gamma, \theta) = 1\{\theta = \gamma\}$, and for $n \geq 0$,

$$B_{n+1}(\gamma, \theta) = \sum_{\gamma' \in \Gamma} \sum_{i=1}^{B_n(\gamma, \gamma')} A_{1,n+1,i}(\gamma', \theta)$$

where $A_{1,n,i}(\gamma, \theta)$ are independent random variables with the same distribution as $A_1(\gamma, \theta)$. Let $B(\gamma, \theta) := \sum_n B_n(\gamma, \theta)$ be the total number of descendants of type θ of a cycle γ.

Lemma 3.5. *$A(\gamma, \cdot)$ is stochastically dominated by $B(\gamma, \cdot)$.*

Proof. The branching process B_{n+1} counts twice or more times those cycles θ in the $(n + 1)$-th generation that intersect more than one γ' on the n-th generation, while A_{n+1} counts them only once. For details see [9] and [4]. $\quad\square$

We conclude that if the branching process is subcritical then the clan of ancestors is finite almost surely.

Mean number of ancestors. Let

$$m(\gamma, \theta) := E\big[A_1(\gamma, \theta)\big].$$

By stationarity, the law of $A_1(\gamma, \theta)$ does not depend on t. Also, the property of being an ancestor of (γ, t, s) is determined by the type γ and its birth time t: $A_1(\gamma, \theta)$ does not depend on s. The random variable $A_1(\gamma, \theta)$ has Poisson distribution with mean $m(\gamma, \theta)$.

A point (θ, t', s') in the first generation of ancestors of (γ, t, s) must satisfy $\theta \not\sim \gamma$, $t' < t$ and $s' \geq t - t'$. Hence,

$$m(\gamma, \theta) = w(\theta)\mathbf{1}\{\gamma \not\sim \theta\} \int_{-\infty}^{t} dt' \int_{t-t'}^{\infty} ds' \, e^{-s'} = w(\theta)\mathbf{1}\{\gamma \not\sim \theta\}. \tag{3.15}$$

Since $B_1(\gamma, \theta)$ has the same law as $A_1(\gamma, \theta)$, the mean matrix of the branching process is given by m and the mean number of descendants of type θ from an individual of type γ after n branchings is given by

$$E\big[B_n(\gamma, \theta)\big] = m^n(\gamma, \theta). \tag{3.16}$$

Lemma 3.6. *For fixed $\gamma \in \Gamma$ and $n \geq 1$ the following inequality holds*

$$\sum_{n \geq 1} \sum_{\theta \in \Gamma} m^n(\gamma, \theta) \leq \sum_{n \geq 1} |\gamma| \beta^n, \tag{3.17}$$

where $\beta = \beta(V, \alpha)$ was defined in (2.5). In particular $\beta(V, \alpha) < 1$ implies that the expected number of descendants of a finite cycle γ is finite.

Proof. Recall $m(\gamma, \theta) = \mathbf{1}\{\theta \not\sim \gamma\}\, w(\theta)$, and bound

$$
\begin{aligned}
\sum_{\theta \in \Gamma} m^n(\gamma, \theta) &\leq \sum_{\theta \in \Gamma} |\theta| m^n(\gamma, \theta) \\
&= |\gamma| \sum_{\gamma_1 \not\sim \gamma} \frac{|\gamma_1|}{|\gamma|} w(\gamma_1) \sum_{\gamma_2 \not\sim \gamma_1} \frac{|\gamma_2|}{|\gamma_1|} w(\gamma_2) \cdots \sum_{\theta \not\sim \gamma_{n-1}} \frac{|\theta|}{|\gamma_{n-1}|} w(\theta) \\
&\leq |\gamma| \Big(\sum_{\theta \ni \vec{0}} |\theta| w(\theta)\Big)^n = |\gamma| \beta^n, \tag{3.18}
\end{aligned}
$$

where the inequality in (3.18) follows from

$$\sum_{\gamma':\gamma'\not\sim\gamma} |\gamma'| w(\gamma') \le |\gamma| \sum_{\gamma'\ni\vec{0}} |\gamma'| w(\gamma'). \qquad \square$$

Corollary 3.7. *If $\alpha > \alpha^*(V)$, as defined in (2.6), then the clan of ancestors of any point (γ, t, s) is finite for almost all realizations of $\mathcal{N}$.*

Proof. By Lemma 3.5 the size of the clan of ancestors is dominated by the total population of the branching process. This population is finite if $\beta(\alpha) < 1$ by Lemma 3.6. $\qquad \square$

4 Proofs of the main theorems

Proof of Theorem 2.1.

Let η_t^Λ and $\eta_t = \eta_t^{\mathbb{Z}^d}$ be the stationary processes defined in (3.6) and (3.4). Fix an arbitrary time $t \in \mathbb{R}$ and consider the t marginal of the process η_t^Λ. Define $\zeta_\Lambda \in S_\Lambda$ as the permutation with cycles indicated by $\eta_t^\Lambda \in \{0,1\}^{\Gamma_\Lambda}$, for each $\Lambda \subseteq \mathbb{Z}^d$. That is, $\gamma \in \zeta_\Lambda$ if and only if $\eta_t^\Lambda(\gamma) = 1$.

(i) For finite Λ the marginal distribution of the process η_t^Λ at each fixed $t \in \mathbb{R}$ is G_Λ, as discussed following (3.6).

(ii) Thermodynamic limit: Under the condition $\alpha > \alpha^*(V)$, Corollary 3.7 implies that the clan of ancestors of any point in $\mathcal{N}$ is finite with probability one. Then Theorem 3.3 implies the almost sure thermodynamic limit $\zeta_\Lambda(x) \to \zeta_{\mathbb{Z}^d}(x)$, as $\Lambda \nearrow \mathbb{Z}^d$, and hence $G_\Lambda \to \mu$, the law of $\eta_t^{\mathbb{Z}^d}$.

(iii) By item (ii) μ is a weak limit of specifications, hence a Gibbs measure. Since $\eta_t^{\mathbb{Z}^d}$ is a space-time translation invariant function of the Poisson process $\mathcal{N}$, the ergodicity of $\mathcal{N}$ implies the spatial ergodicity of the law of $\eta_t^{\mathbb{Z}^d}$.

(iv) Uniqueness follows from Theorem 3.4. $\qquad \square$

τ_v **boundary conditions. Proof of Theorem 2.3.**   Fix $v \in \mathbb{Z}^d$ and $\alpha > \alpha_v^*(V)$ and define the potential V_v as in (2.14). Then $V_v(0) = 0$ and V_v is strictly convex because V is. Also, $\alpha^*(V_v) = \alpha_v^*(V) < \infty$ by hypothesis. Hence V_v satisfies the hypothesis of Theorem 2.1 and there exist a process $(\zeta_{\Lambda,v}, \Lambda \subseteq \mathbb{Z}^d)$, $\zeta_{\Lambda,v} \in S_\Lambda$, such that (i) to (iv) of that theorem hold.

(i) If Λ is finite and $\sigma \in S_\Lambda$,

$$\begin{aligned}
P(\tau_v \zeta_{\Lambda,v} = \tau_v \sigma) &= P(\zeta_{\Lambda,v} = \sigma) \\
&= \frac{1}{Z_{\Lambda,v}} \prod_{\gamma \in \sigma} \exp\Big\{ \sum_{x \in \{\gamma\}} V_v(\gamma(x) - x) \Big\} \\
&= \frac{1}{Z_{\Lambda,v}} \prod_{\gamma \in \sigma} \exp\Big\{ \sum_{x \in \{\gamma\}} \big[V(\gamma(x) + v - x) - V(v) \big] \Big\} \\
&= G_{\Lambda|\tau_v}(\tau_v \sigma).
\end{aligned} \qquad (4.1)$$

The remaining items follow from (i) and the statements (ii) to (iv) of Theorem 2.1. $\qquad \square$

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The Gaussian potential. Proof of Theorem 2.4. Assume V is the Gaussian potential $V(x) = \|x\|^2$ in $\mathbb{Z}^d$. Let ρ_0 as in (5.5) and $\alpha > 0$ such that $\sum_{x \in \mathbb{Z}^d \setminus \{\vec{0}\}} \exp(-\alpha \|x\|^2) < \rho_0$. By Lemma 5.1 below this implies $\alpha^*(\|\cdot\|^2) < \infty$. We compute the explicit bound for $\alpha^*(\|\cdot\|^2)$ later in (5.8).

Fix $v \in \mathbb{Z}^d$.

(i) Any permutation $\xi \in S_{\Lambda|\tau_v}$ is a finite perturbation of τ_v and by Lemma 2.2, $\tau_{-v}\xi = \gamma_1 \ldots \gamma_n$, a composition of disjoint finite cycles in S_Λ. We can then write

$$G_{\Lambda|\tau_v}(\xi) = \frac{1}{Z_{v,\Lambda}} \exp\left\{-\alpha \sum_{\gamma \in \tau_{-v}\xi} \sum_{x \in \{\gamma\}} (\|\gamma(x) + v - x\|^2 - \|v\|^2) - \alpha \sum_{x \in \Lambda} \|v\|^2\right\} \quad (4.2)$$

$$= \frac{1}{Z'_{v,\Lambda}} \prod_{\gamma \in \tau_{-v}\xi} w_v(\gamma), \quad (4.3)$$

where $Z'_{v,\Lambda} = Z_{v,\Lambda} \exp\{\alpha \sum_{x \in \Lambda} \|v\|^2\}$ and w_v is defined in (2.4). If γ is a cycle with support in Λ

$$\sum_{x \in \{\gamma\}} \left(\|\gamma(x) + v - x\|^2 - \|v\|^2\right) = \sum_{x \in \{\gamma\}} \|x - \gamma(x)\|^2 + 2v \cdot \sum_{x \in \{\gamma\}} (x - \gamma(x))$$

$$= \sum_{x \in \{\gamma\}} \|x - \gamma(x)\|^2 \quad (4.4)$$

as $\sum_{x \in \{\gamma\}} x - \gamma(x) = 0$. This implies $w_v(\gamma) = w(\gamma)$ defined in (2.4), and $G_{\Lambda|\tau_v}(\xi) = G_\Lambda(\tau_{-v}\xi)$ for $\xi \in S_{\Lambda|\tau_v}$. But this is equivalent to $G_{\Lambda|\tau_v} = G_\Lambda \tau_v$.

(ii) Take $v \in \mathbb{Z}^d$ and $\alpha > \alpha^*(V)$ and let ζ_Λ be as constructed in Theorem 2.1. Since ζ_Λ has law G_Λ, by (i), $\tau_v \zeta_\Lambda$ has distribution $G_{\Lambda|\tau_v}$. The almost sure thermodynamic limit of item (ii), Theorem 2.1 implies

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \tau_v \zeta_\Lambda(x) = \tau_v \zeta_{\mathbb{Z}^d}(x) \quad \text{a.s.,} \quad \text{for all } x \in \mathbb{Z}^d.$$

Now $\zeta_{\mathbb{Z}^d}$ is distributed according to μ, then

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} G_{\Lambda|\tau_v} = \mu\tau_v \quad \text{weakly.}$$

(iii) The ergodicity of $\mu\tau_v$ follows from the ergodicity of μ proved in Theorem 2.1 (iii). $\qquad \square$

5 Bounds on $\alpha^*(V)$

5.1 A general bound

We start with a general bound. Following [7], define

$$\rho(V, \alpha) := \sum_{x \in \mathbb{Z}^d \setminus \{\vec{0}\}} e^{-\alpha V(x)}. \quad (5.1)$$

The proof of the following lemma is taken from the proof of Theorem 2.1 in [7].

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Lemma 5.1. *Call $\beta = \beta(V, \alpha)$ and let $\rho = \rho(V, \alpha)$. Then,*

$$\beta \le \frac{\rho}{\left[1 - \rho\right]^2} - \rho. \tag{5.2}$$

Proof. Compute

$$\beta = \sum_{\theta \ni \vec{0}} |\theta| w(\theta) = \sum_{n \ge 2} n \sum_{\theta \ni \vec{0}; \, |\theta| = n} w(\theta). \tag{5.3}$$

The second sum, indexed by θ, on the right of (5.3) can be re-written as

$$\sum_{x_1, \ldots, x_{n+1} \in \mathbb{Z}^d} \mathbf{1}\{x_1 = x_{n+1} = 0; \, x_i \ne x_j, \, i, j \in \{1, \ldots, n\}\} \prod_{i=1}^{n} e^{-\alpha V(x_{i+1} - x_i)}. \tag{5.4}$$

Dominate the indicator function in (5.4) by $\mathbf{1}\{x_1 = 0; \, x_i \ne x_{i+1}, \, i \in \{1, \ldots, n\}\}$, to dominate (5.4) by

$$\sum_{y_1, \ldots, y_n \in \mathbb{Z}^d \setminus \{\vec{0}\}} \prod_{i=1}^{n} e^{-\alpha V(y_i)} = \left(\sum_{x \in \mathbb{Z}^d \setminus \{\vec{0}\}} e^{-\alpha V(x)} \right)^n = \rho^n.$$

We conclude that $\beta \le \sum_{n \ge 2} n \rho^n$ which is equivalent to (5.2). $\qquad\square$

Let

$$\rho_0 \text{ be the unique solution } r \in [0, 1] \text{ to } \frac{r}{(1 - r)^2} - r = 1. \tag{5.5}$$

Solving the equation one gets $\rho_0 \approx 0.44504$.

Corollary 5.2. *If $\rho(V, \alpha) < \rho_0$ then $\beta(V, \alpha) < 1$. In particular,*

$$\alpha^*(V) \le \inf\{\alpha > 0 : \rho(V, \alpha) \le \rho_0\}. \tag{5.6}$$

5.2 Examples

The Gaussian potential In this case

$$\rho(\|\cdot\|^2, \alpha) = \sum_{z \in \mathbb{Z}^d \setminus \{\vec{0}\}} e^{-\alpha \|z\|^2} = \left(\sum_{k \in \mathbb{Z}} e^{-\alpha k^2} \right)^d - 1$$

$$\le \left(1 + \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \right)^d - 1 = \left(1 + \sqrt{\pi/\alpha} \right)^d - 1, \tag{5.7}$$

which implies the following explicit bound for α^*:

$$\alpha^*(\|\cdot\|^2) \le \pi\left((\rho_0 + 1)^{1/d} - 1\right)^{-2} \approx \pi\left(1.44504^{1/d} - 1\right)^{-2}. \tag{5.8}$$

For $d = 2$ this gives $\alpha^* \le 76.9176$; for $d = 3$, $\alpha^* \le 184.305$.

Differentiable, strictly convex potentials Let $V : \mathbb{R}^d \to \mathbb{R}^+ \cup \{+\infty\}$ be a potential such that for each $v \in \mathbb{Z}^d$ there exists a constant $m(v) > 0$ satisfying

$$V(y) \geq V(x) + \nabla V(x)^T \cdot (y - x) + m(v)\|y - x\|^2 \qquad \text{for any } x, y \in \mathbb{R}^d. \tag{5.9}$$

Then, for any cycle $\gamma \in \Gamma$,

$$\sum_{x \in \gamma} V(\gamma(x) + v - x) - V(v) \geq m(v) \sum_{x \in \gamma} \|\gamma(x) - x\|^2, \tag{5.10}$$

and $\alpha_v^*(V) \leq \frac{1}{m(v)} \alpha^*(\| \cdot \|^2)$.

In particular, if V is a strongly convex potential then (5.9) holds with a constant m uniformly in $v \in \mathbb{Z}^d$, and $\alpha_v^*(V) \leq \frac{1}{m} \alpha^*(\| \cdot \|^2)$. For instance, in 1-dimension, x^2 and e^{x^2} are strongly convex potentials.

Polynomial potentials Let $V : \mathbb{R}^d \to \mathbb{R}^+$ be a strictly convex polynomial, $V(\vec{0}) = 0$, with a positive definite Hessian at all points. Given $v \in \mathbb{Z}^d$, there exists $b(v) > 0$ such that

$$\left[V(v + y) - V(v) - \nabla V(v) \cdot y \right] \mathbf{1}_{\|y\| \geq b(v)} \geq \frac{1}{2} V(y) \, \mathbf{1}_{\|y\| \geq b(v)} \tag{5.11}$$

Let now $\gamma \in \Gamma$, and write

$$\sum_{x \in \{\gamma\}} V(\gamma(x) - x + v) - V(v)$$

$$= \sum_{x \in \{\gamma\}} V(\gamma(x) - x + v) - V(v) - \nabla V(v) \cdot (\gamma(x) - x) = I_1 + I_2$$

with

$$I_1 = \sum_{x \in \{\gamma\}, \, \|x - \gamma(x)\| < b(v)} V(\gamma(x) - x + v) - V(v) - \nabla V(v) \cdot (\gamma(x) - x)$$

$$I_2 = \sum_{x \in \{\gamma\}, \, \|x - \gamma(x)\| \geq b(v)} V(\gamma(x) - x + v) - V(v) - \nabla V(v) \cdot (\gamma(x) - x)$$

By (5.11)

$$I_2 \geq \frac{1}{2} \sum_{x \in \{\gamma\}, \, \|x - \gamma(x)\| \geq b(v)} V(\gamma(x) - x). \tag{5.12}$$

On the other hand, since the set $\{y \in \mathbb{Z}^d, \|y - v\| < b(v)\}$ is finite and the Hessian $HV = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq d}$ is positive definite at all points, there exists $m > 0$ such that

$$V(\gamma(x) - x + v) - V(v) - \nabla V(v) \cdot (\gamma(x) - x) \geq m \|\gamma(x) - x\|^2$$

for all $\|\gamma(x) - x\| < b(v)$. As a result,

$$I_1 \geq m \sum_{x \in \{\gamma\}, \, \|x - \gamma(x)\| < b(v)} \|\gamma(x) - x\|^2. \tag{5.13}$$

Finally, for any integer neighborhood of the origin that excludes the origin itself there exists another constant m' such that $m'V(x) \leq \|x\|^2$. Together with (5.12, 5.13), we obtain

$$\sum_{x \in \{\gamma\}} V(\gamma(x) - x + v) - V(v) \geq C(v) \sum_{x \in \{\gamma\}} V(\gamma(x) - x)$$

for some constant $C(v) > 0$, and $\alpha_v^*(V) \geq \frac{1}{C(v)} \alpha^*(V)$, where $\alpha^*(V)$ can be bounded as in Corollary 5.2.

Finally, note that some naturally arising polynomial potentials such as $V(x) = x^4$ fail to have a positive definite Hessian at all points. In this case the above argument still applies, provided the set of points where the Hessian is not positive definite does not affect the computation leading to (5.13). In other words, one just needs to check that the Hessian appearing in the remainder term of the 1st degree Taylor expansion of $V(z)$ around v is positive definite, for all (finitely many) integer points z in the neighborhood $\|z - v\| < b(v)$. For instance, in the case of $V(x) = x^4$, or its d-dimensional version $V(x) = \|x\|^4$, the Hessian fails to be positive definite only at the origin, and the argument works fine.

Finite-range potentials We say that a potential V is *finite-range* if $V(x) = \infty$ for all but a finite number of x's. Consider for instance the nearest neighbor potential $V : \mathbb{Z}^2 \to \{0, 1, \infty\}$ defined by $V(0) = 0$, $V(x) = 1$ if $\|x\| = 1$ and $V(x) = \infty$ if $\|x\| > 1$. In $d = 2$ the specifications for the permutations are very similar to the ones for the Peierls contours of the Ising model. There are some differences between these models: (a) while contours may have self intersections, cycles are not allowed to; (b) two-point cycles do not determine a contour, and (c) given the set of at least 3 sites in the support of a cycle, there are two possible ways of going through them, clockwise and counter clockwise. Other than these observations, the approach works exactly as in the contour case studied in [4].

Acknowledgments

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