Decompositions and complexifications of homogeneous spaces

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Abstract

In this paper an extended CPR decomposition theorem for Finsler symmetric spaces of semi-negative curvature in the context of reductive structures is proven. This decomposition theorem is applied to give a geometric description of the complexification of some infinite dimensional homogeneous spaces.

Keywords: Banach-Lie group, coadjoint orbit, complexification, Corach-Porta-Recht decomposition, Finsler structure, flag manifold, homogeneous space, operator decomposition, reductive structure, Stiefel manifold.[1]

1 Introduction

In recent years, the geometrical study of operator algebras and their homogeneous spaces has become a central topic in the study of infinite dimensional geometry. It is a source of examples and counterexamples, and the operator algebra techniques (Banach algebras and C^* algebras, with their distinguished tools) are being used for obtaining results on abstracts infinite dimensional manifolds by studying their groups of automorphism, isometries, and their associated fiber bundles and G-bundles. See the recent book [3] by D. Beltita for a full account of these objects and a comprehensive list of references.

In particular, what we are interested in here, is the extension of certain results on the geometric description of complexifications of homogeneous spaces of Banach-Lie groups studied by Beltita and Galé in [1] and also the decompositions of the acting groups by means of a series of chained reductive structures.

In Section 2 the reader can find the basic facts about Finsler symmetric spaces; these are spaces of the form G/U endowed with a Finsler structure, where G is a Banach-Lie group and U is the fixed point set of an involution σ on G. A criteria that ensures that the spaces G/U have semi-negative curvature is recalled from the work of Neeb [11].

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In Section 3 we recall the definition of reductive structures, which can be interpreted as connection forms E on homogeneous spaces of the form G_A/G_B. Examples in the context of operator algebras are given: conditional expectations, their restrictions to Schatten ideals and projections to corners of operator algebras. The Corach-Porta-Recht splitting theorem by Conde and Larotonda [5] is used to prove an extended CPR-splitting theorem in the context of several reductive structures.
In Section 4 the CPR splitting theorem is used to give a geometric description of homogeneous spaces of the form G_A/G_B as associated principal bundles over U_A/U_B. Under additional hypothesis about the holomorphic character of G_A and the involution σ on G_A it is possible to interpret G_A/G_B as the complexification of U_A/U_B. Under these additional assumptions G_A/G_B is identified with the tangent bundle of U_A/U_B and it is shown that this identification has nice functorial properties related to the connection form E. Finally, we use the three examples of connection forms introduced in Section 3, to give a geometrical description of the complexifications of flag manifolds, coadjoint orbits in Schatten ideals and Stiefel manifolds respectively.

2 Finsler symmetric spaces

A connected Banach-Lie group G with an involutive automorphism σ is called a *symmetric Lie group*. Let $\mathfrak{g}$ be the Banach-Lie algebra of G, and let $U = \{g \in G : \sigma(g) = g\}$ be the subgroup of fixed points of σ. Then the Banach-Lie algebra $\mathfrak{u}$ of U is a closed and complemented subspace of $\mathfrak{g}$; a complement is given by the closed subspace

$$\mathfrak{p} = \{X \in \mathfrak{g} : \sigma_{*1}X = -X\},$$

where for a smooth map between manifolds $f : X \to Y$ we use the notation $f_* : T(X) \to T(Y)$ for the tangent map and $f_{*x} : T_x(X) \to T_{f(x)}(Y)$ for the tangent map at $x \in X$.
The Lie algebra $\mathfrak{u}$ is the eigenspace of σ_{*1} corresponding to the eigenvalue $+1$ and $\mathfrak{p}$ is the eigenspace corresponding to the eigenvalue -1. Since $\mathfrak{u}$ is complemented U is a Banach-Lie subgroup of G, and the quotient space $M = G/U$ has a Banach manifold structure. We denote by $q : G \to M$, $g \mapsto gU$ the quotient map which is a submersion, and by $Exp : \mathfrak{g} \to G$ the exponential map of G. We use the notation $e^X := Exp(X)$ for $X \in \mathfrak{g}$.
We also define $G^+ := \{g\sigma(g)^{-1} : g \in G\}$ which is a submanifold of G and note that there is a differential isomorphism $\phi : G/U \to G^+$, $gU \mapsto g\sigma(g)^{-1}$. See Section 5 in Chapter XIII of [8]. We use the notation $\sigma(g)^{-1} := g^*$ for $g \in G$.
Let L_g and R_g stand for the left and right translation diffeomorphisms on G. For $h \in G$, let $\mu_h : M \to M$, $\mu_h(q(g)) = q(hg) = q(L_h g)$. Then

$$(\mu_h)_{*q(g)} q_{*g} = q_{*hg}(L_h)_{*g}.$$

The map $q_{*1} : \mathfrak{p} \to T_o M$ is an isomorphism so that a generic vector in $T_{q(g)}M$ will be denoted by $(\mu_g)_{*o} q_{*1} X$ with $X \in \mathfrak{p}$. We use I_h to denote the interior automorphisms of G given by $I_h(g) = hgh^{-1}$, and Ad_h to denote the differential $(I_h)_{*1}$, which is an element of $\mathcal{B}(\mathfrak{g})$, the bounded linear maps that act on $\mathfrak{g}$. We note that $\sigma(I_u e^{tX}) = I_u e^{-tX}$ for every $X \in \mathfrak{p}$ and $u \in U$, so that $\sigma_{*1} Ad_u X = -Ad_u X$ and $\mathfrak{p}$ is Ad_U-invariant. Since σ is a group automorphism, σ_{*1} is an automorphism of Lie algebras and the following inclusions hold:

$$[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{u}, \quad [\mathfrak{u}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{u}.$$

In particular, $\mathfrak{p}$ is $ad_{\mathfrak{u}}$-invariant.
A way of giving M the structure of a Finsler manifold is establishing the following norm on the tangent space $T_{q(g)}(M)$ for each $g \in G$

$$\|(\mu_g)_{*o} q_{*1} X\|_{q(g)} := \|X\|_{\mathfrak{p}}$$

where $\|\cdot\|_{\mathfrak{p}}$ is any Ad_U-invariant norm $\mathfrak{p}$ compatible with any norm of $T_o(M)$ given by a local chart. To make the dependence of M with its underlying Banach-Lie group, involution and Finsler structure clear we shall write $M = G/U = Sym(G, \sigma, \|\cdot\|_{\mathfrak{p}})$ and we shall call M a *Finsler symmetric space.*
We say that $M = G/U$ has semi-negative curvature if for all $p \in M$ the operator between Banach spaces $(exp_p)_{*x} : T_p(M) \simeq T_x(T_p(M)) \to T_{exp_p(x)}(M)$ is expansive and surjective.
What follows is a criteria for semi-negative curvature for Finsler symmetric spaces due to K.-H. Neeb, [11, Prop. 3.15 and Th. 2.2]:

Theorem 2.1. *Let $M = G/U = Sym(G, \sigma, \|\cdot\|_{\mathfrak{p}})$ be a Finsler symmetric space. Then the following conditions are equivalent:*

1. *M has semi-negative curvature.*
2. *The operator $-(ad_X)^2|_{\mathfrak{p}}$ is dissipative for all $X \in \mathfrak{p}$.*
3. *The operator $1 + (ad_X)^2|_{\mathfrak{p}}$ is expansive and invertible for all $X \in \mathfrak{p}$.*
4. *The operator $X \in \mathfrak{p}$, $\frac{\sinh ad_X}{ad_X}|_{\mathfrak{p}}$ is expansive and invertible for all $X \in \mathfrak{p}$.*

Example 2.2. *If A is a unital C^*-algebra, G is the group of invertible elements of A endowed with the manifold structure given by the norm and $\sigma : G \to G$, $g \mapsto (g^{-1})^*$, then $U = \{g \in G : \sigma(g) = g\}$ is the group of unitary operators of A. In this case $\mathfrak{p} = A_s$ the set of self-adjoint elements of A and the uniform norm on A_s which we denote by $\|\cdot\|$ is Ad_U-invariant because it is unitarily invariant. We can identify the manifold G/U with the manifold of positive invertible elements G^+. It was proven in [6] that the manifold $M = G/U = Sym(G, \sigma, \|\cdot\|)$ has semi-negative curvature.*

Example 2.3. *Let $A = \mathcal{B}(\mathcal{H})$ stand for the set of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$, with the uniform norm denoted by $\|\cdot\|$. Let A_p be the ideal of p-Schatten operators with p-norm $\|\cdot\|_p$. Let G_p stand for the group of invertible operators in the unitized ideal, that is $G_p = \{g \in A^\times : g - 1 \in A_p\}$, then G_p is a Banach-Lie group (one of the so-called classical Banach-Lie groups [7]), and*

A_p identifies with its Banach-Lie algebra. Consider the involutive automorphism $\sigma : G_p \to G_p$ given by $g \mapsto (g^*)^{-1}$. Let $U_p \subseteq G_p$ stand for the unitary subgroup of fixed points of σ. In this case $\mathfrak{p}$ is the set of self-adjoint operators in A_p and the norm $\|\cdot\|_p$ on $\mathfrak{p}$ is Ad_{U_p}-invariant. We can identify the manifold G_p/U_p with the manifold of positive invertible operators in G_p. It was proven in Section 5 of [5] that the manifold $M_p = G_p/U_P = Sym(G_p, \sigma, \|\cdot\|_p)$ is simply connected and has semi-negative curvature.

3 Splitting of Finsler symmetric spaces

We recall some facts about the fundamental group of M and polar decompositions [11, Th. 3.14 and Th. 5.1]

Theorem 3.1. *Let $M = G/U = (G, \sigma, , \|\cdot\|_{\mathfrak{p}})$ be a Finsler symmetric space of semi-negative curvature, then*

1. *The exponential map $q \circ Exp : \mathfrak{p} \to M$ is a covering of Banach manifolds and*

$$\Gamma = \{X \in \mathfrak{p} : q(e^X) = q(1)\}$$

   is a discrete and additive subgroup of $\mathfrak{p} \cap Z(\mathfrak{g})$, with $\Gamma \simeq \pi_1(M)$ and $M \simeq \mathfrak{p}/\Gamma$. $Z(\mathfrak{g})$ denotes the center of the Banach-Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{p}$ and $q(e^X) = q(e^Y)$, then $X - Y \in \Gamma$.

2. *The polar map*

$$m : \mathfrak{p} \times U \to G, \quad (X, u) \mapsto e^X u$$

   is a surjective covering whose fibers are given by the sets $\{(X - Z, e^Z u) : Z \in \Gamma\}$, $u \in U$, $X \in \mathfrak{p}$. If M is simply connected the map m is a diffeomorphism.

In the context of C^*-algebras (Example 2.2), since G/U is simply connected and has semi-negative curvature we get the usual polar decomposition of invertible elements as a product of a positive invertible element and a unitary.

Corollary 3.2. *In the context of the previous theorem $G_A^+ = e^{\mathfrak{p}}$. Note that given $h \in G_A^+$ there is a $g \in G_A$ such that $h = g\sigma(g)^{-1}$. Using the polar decomposition in G_A there are $X \in \mathfrak{p}$ and $u \in U$ such that $g = e^X u$. Then $h = e^X u\sigma(e^X u)^{-1} = e^X uu^{-1}e^X = e^{2X} \in e^{\mathfrak{p}}$. We note also that $e^X = e^{\frac{1}{2}X}\sigma(e^{\frac{1}{2}X})^{-1} \in G_A^+$ for every $X \in \mathfrak{p}$.*

The following decomposition theorem in the context of Finsler symmetric spaces of semi-negative curvature was proven by Conde and Larotonda in [5].

Theorem 3.3. *Corach-Porta-Recht decomposition (CPR)*
Let $M = G/U = (G, \sigma, , \|\cdot\|_{\mathfrak{p}})$ be a simply connected Finsler symmetric space of semi-negative curvature. Let $p \in \mathcal{B}(\mathfrak{p})$ be an idempotent, $p^2 = p$. Let $\mathfrak{s} := Ran(p)$,

$\mathfrak{s}' := Ran(1-p) = Ker(p)$, so that $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{s}'$. If $ad^2_{\mathfrak{s}}(\mathfrak{s}) \subseteq \mathfrak{s}$, $ad^2_{\mathfrak{s}}(\mathfrak{s}') \subseteq \mathfrak{s}'$ and $\|p\| = 1$, then the maps

$$\Phi : U \times \mathfrak{s}' \times \mathfrak{s} \to G, \qquad (u, X, Y) \mapsto ue^X e^Y$$

$$\Psi : \mathfrak{s}' \times \mathfrak{s} \to G^+, \qquad (X, Y) \mapsto e^Y e^{2X} e^Y$$

are diffeomorphisms.

The following two definitions are from Beltita and Galé [2].

Definition 3.4. *A* ***reductive structure*** *is a triple $(G_A, G_B; E)$ where G_A is a real or complex connected Banach-Lie group with Banach-Lie algebra $\mathfrak{g}_A$, G_B is a connected Banach-Lie subgroup of G_A with Banach-Lie algebra $\mathfrak{g}_B$, and $E : \mathfrak{g}_A \to \mathfrak{g}_A$ is a $\mathbb{R}$-linear continuous transformation which satisfies the following properties: $E \circ E = E$; $RanE = \mathfrak{g}_B$, and for every $g \in G_B$ the diagram*

$$\begin{tikzcd}
\mathfrak{g}_A \arrow[r, "E"] \arrow[d, "Ad_g"'] & \mathfrak{g}_B \arrow[d, "Ad_g"] \\
\mathfrak{g}_A \arrow[r, "E"] & \mathfrak{g}_B
\end{tikzcd}$$

commutes.

Definition 3.5. *A* ***morphism of reductive structures*** *from $(G_A, G_B; E)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ is a homomorphism of Banach-Lie groups $\alpha : G_A \to \tilde{G_A}$ such that $\alpha(G_B) \subseteq \tilde{G_B}$ and such that the diagram*

$$\begin{tikzcd}
\mathfrak{g}_A \arrow[r, "E"] \arrow[d, "\alpha_{*1}"'] & \mathfrak{g}_B \arrow[d, "\alpha_{*1}"] \\
\tilde{\mathfrak{g}_A} \arrow[r, "\tilde{E}"] & \tilde{\mathfrak{g}_B}
\end{tikzcd}$$

commutes.
For example, a family of automorphisms of any reductive structure $(G_A, G_B; E)$ is given by $\alpha_g : x \mapsto gxg^{-1}$, $G_A \to G_A$, $(g \in G_B)$.

Now we introduce involutions in reductive structures:

Definition 3.6. *If $(G_A, G_B; E)$ is a reductive structure and σ is an involutive morphism of reductive structures we call $(G_A, G_B; E, \sigma)$ a* ***reductive structure with involution****. If $(G_A, G_B; E, \sigma)$ and $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$ are reductive structures with involution and α is a morphism of reductive structures from $(G_A, G_B; E)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E})$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$ then we call α a* ***morphism of reductive structures with involution*** *from $(G_A, G_B; E, \sigma)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$.*

Example 3.7. *Conditional expectations in C^*-algebras*
Let A and B be two unital C^*-algebras, such that B is a subalgebra of A and let $E : A \to B$ be a conditional expectation. This means that E is a linear projection on A with $RanE = B$, $E(1_A) = 1_B (= 1_A)$ and norm 1. By Tomiyama's theorem [15] the following holds

$$E(b_1 a b_2) = b_1 E(a) b_2 \qquad \textit{for all } a \in A; \quad b_1, b_2 \in B$$

$$E(a^*) = E(a)^* \qquad \textit{for all } a \in A.$$

Let G_Λ for $\Lambda \in \{A, B\}$ be the Banach-Lie group of invertible operators in Λ endowed with the topology given by the uniform norm. Then the Banach-Lie algebra of G_Λ is $\mathfrak{g}_\Lambda = \Lambda$. Since in this case we have $Ad_g(a) = gag^{-1}$ for each $g \in G_A$ and $a \in A$, the expectation E satisfies the conditions of Def. 3.4, so that $(G_A, G_B; E)$ is a reductive structure. In fact, this is a classical example that was the motivation of that definition in the paper [2].
If $(G_A, G_B; E)$ is a reductive structure that is derived from an inclusion of C^*-algebras and a conditional expectation as above then $\sigma : G_A \to G_A$, $a \mapsto (a^{-1})^*$ defines an involutive morphism of reductive structures since $\sigma_{*1} : A \to A$, $a \mapsto -a^*$ and

$$E(\sigma_{*1}(a)) = E(-a^*) = -E(a)^* = \sigma_{*1}(E(a)),$$

therefore $(G_A, G_B; E, \sigma)$ is a reductive structure with involution.
If for two triples $(A, B; E)$, $(\tilde{A}, \tilde{B}; \tilde{E})$ there is a bounded homomorphism $\phi : A \to \tilde{A}$ which satisfies $\phi \circ E = \tilde{E} \circ \phi$ then $\alpha := \phi|_{G_A}$ defines a morphism of reductive structures with involution from $(G_A, G_B; E, \sigma)$ to $(\tilde{G_A}, \tilde{G_B}; \tilde{E}, \tilde{\sigma})$.

Example 3.8. *We use the notation of Example 2.3. Let $B \subseteq A = \mathcal{B}(\mathcal{H})$ be a C^*-subalgebra, and let $E : A \to B$ be a conditional expectation with range A such that E sends trace-class operators to trace-class operators and E is compatible with the trace, that is $Tr(E(x)) = Tr(x)$ for any trace-class operator $x \in A$. Let $p \geq 1$, $B_p = B \cap A_p$,*

$$G_{A,p} = \{g \in A^\times : g - 1 \in A_p\} \;\; \textit{and} \;\; G_{B,p} = \{g \in A^\times : g - 1 \in B_p\}.$$

Then $\mathfrak{g}_{A,p} := A_p$ and $\mathfrak{g}_{B,p} = B_p$ are the Banach-Lie algebras of $G_{A,p}$ and $G_{B,p}$ respectively. It was proven in Section 5 of [5] that $E_p = E|_{A_p} : A_p \to B_p$ and that $\|E_p\| = 1$. It easy to see that $(G_{A,p}, G_{B,p}; E_p, \sigma)$ is a reductive structure with involution.

Example 3.9. *Corners*
Let $\mathcal{H}$ be a Hilbert space, $n \geq 1$ and p_i, $i = 1, \ldots, n+1$ be pairwise orthogonal non-zero projections with range $\mathcal{H}_i$ and $\sum_{i=1}^{n-1} p_i = 1$. Let G_A be the group of invertible

elements of $\mathcal{B}(\mathcal{H})$ and let

$$G_B = \left\{ \begin{pmatrix} g_1 & 0 & \dots & 0 & 0 \\ 0 & g_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & g_n & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : g_i \text{ invertible in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \dots, n \right\};$$

where we write operators in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_{n+1})$ as $(n+1) \times (n+1)$ matrices with the corresponding operator entries.
In this case $\mathfrak{g}_A = \mathcal{B}(\mathcal{H})$ and

$$\mathfrak{g}_B = \left\{ \begin{pmatrix} X_1 & 0 & \dots & 0 & 0 \\ 0 & X_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & X_n & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} : X_i \text{ in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \dots, n \right\}.$$

If we consider the map $E : \mathfrak{g}_A \to \mathfrak{g}_B$, $X \mapsto \sum_{i=1}^{n} p_i X p_i$ and $\sigma = (\cdot)^{*-1}$ it is easily verified that $(G_A, G_B; E, \sigma)$ is a reductive structure with involution. Note that $\|E\| = 1$.

Definition 3.10. *If (G_A, σ) is a symmetric Banach-Lie group we say that a connected subgroup $G_B \subseteq G_A$ is **involutive** if $\sigma(G_B) = G_B$.*

Remark 3.11. *If $G_B \subseteq G_A$ is an involutive Banach-Lie subgroup with Banach-Lie algebra $\mathfrak{g}_B \subseteq \mathfrak{g}_A$ and $\mathfrak{g}_A = \mathfrak{p} \oplus \mathfrak{u}$ is the eigenspace decomposition of σ_{*1}, we can write $\mathfrak{g}_B = \mathfrak{p}_B \oplus \mathfrak{u}_B$, where $\mathfrak{p}_B := \mathfrak{p} \cap \mathfrak{g}_B$ and $\mathfrak{u}_B := \mathfrak{u} \cap \mathfrak{g}_B$.*

Proposition 3.12. *Given a Finsler symmetric space*

$$M_A = G_A/U_A = Sym(G_A, \sigma, \| \cdot \|_{\mathfrak{p}})$$

of semi-negative curvature, if G_B is an involutive subgroup, then

$$M_B = G_B/U_B = Sym(G_B, \sigma|_{G_B}, \| \cdot \|_{\mathfrak{p}_B})$$

is a Finsler symmetric space of semi-negative curvature. Also, the inclusion $\Gamma_B \subseteq \Gamma_A \cap \mathfrak{p}_B$ holds. In particular, if M_A is simply connected then M_B is also simply connected.

Proof. We can restrict the Ad_{U_A}-invariant norm of $M_A = G_A/U_A$ to $\mathfrak{p}_B$ to give $M_B = G_B/U_B$ a Ad_{U_B}-invariant norm. Since for each $X \in \mathfrak{p}$ the operator $-(ad_X)^2|_{\mathfrak{p}}$ is dissipative and $-(ad_X)^2|_{\mathfrak{p}}(\mathfrak{p}_B) \subseteq \mathfrak{p}_B$ for all $X \in \mathfrak{p}_B$, we conclude that the operator $-(ad_X)^2|_{\mathfrak{p}_B}$ is dissipative for all $X \in \mathfrak{p}_B$. Therefore $M_B = G_B/U_B = Sym(G_B, \sigma|_{G_B}, \| \cdot \|_{\mathfrak{p}_B})$ has semi-negative curvature.
If $X \in \Gamma_B$ then $q_B \circ Exp_B(X) = o_B$ so that $Exp_A(X) = Exp_B(X) \in U_B \subseteq U_A$ and $q_A \circ Exp_A = o_A$. We conclude that $\Gamma_B \subseteq \Gamma_A \cap \mathfrak{p}_B$. ∎

Remark 3.13. *If $(G_A, G_B; E)$ is a reductive structure, since $Ad_g \circ E = E \circ Ad_g$ for each $g \in G_B$ we see that $Ad_g(KerE) \subseteq KerE$ for every $g \in G_B$. If σ is an involutive automorphism of reductive structures and $\mathfrak{g}_A = \mathfrak{u} \oplus \mathfrak{p}$ is the decomposition into eigenspaces of σ_{*1} then $Ad_{U_A}(\mathfrak{p}) \subseteq \mathfrak{p}$ and $Ad_{U_A}(\mathfrak{u}) \subseteq \mathfrak{u}$, so that the actions $Ad : U_B \to \mathcal{B}(\mathfrak{p}_E)$ and $Ad : U_B \to \mathcal{B}(\mathfrak{u}_E)$ are well defined, where we denote $\mathfrak{p}_E := KerE \cap \mathfrak{p}$ and $\mathfrak{u}_E := KerE \cap \mathfrak{u}$.*

Theorem 3.14. *Extended CPR splitting*
If for $n \geq 2$ we have the following inclusions of connected Banach-Lie groups, the following maps between its Banach-Lie algebras

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n$$

$$\mathfrak{g}_1 \xleftarrow{E_2} \mathfrak{g}_2 \xleftarrow{E_3} \ldots \xleftarrow{E_n} \mathfrak{g}_n$$

and a morphism $\sigma : G_n \to G_n$ such that:

- *$(G_n, G_{n-1}; E_n, \sigma)$,$(G_{n-1}, G_{n-2}; E_n, \sigma|_{G_{n-1}})$,..., $(G_2, G_1; E_2, \sigma|_{G_2})$ are reductive structures with involution.*
- *$M_n = G_n/U_n = Sym(G_n, \sigma, \|\cdot\|)$ is a simply connected Finsler symmetric space of semi-negative curvature.*
- *$\|E_k|_{\mathfrak{p}_k}\| = 1$ for $k = 2, \ldots, n$, where the norm is the norm of the previous item restricted to $\mathfrak{p}_k := \mathfrak{p} \cap \mathfrak{g}_k$.*

Then the maps

$$\Phi_n : U_n \times \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n$$
$$(u_n, X_n, \ldots, X_2, Y_1) \mapsto u_n e^{X_n} \ldots e^{X_2} e^{Y_1}$$

$$\Psi_n : \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_n^+$$
$$(X_n, \ldots, X_2, Y_1) \mapsto e^{Y_1} e^{X_2} \ldots e^{X_{n-1}} e^{2X_n} e^{X_{n-1}} \ldots e^{X_2} e^{Y_1}$$

are diffeomorphisms, where $\mathfrak{p}_{E_k} := KerE_k \cap \mathfrak{p}_k$ for $k = 2, \ldots, n$.

Proof. Note that Prop. 3.12 implies that $M_k := G_k/U_k$ are simply connected Finsler symmetric spaces of semi-negative curvature for $k = 2, \ldots, n$. We prove the statement about the map Φ for the case $n = 2$ and then prove the statement for $n > 2$ by induction.
Since $E_2 \circ \sigma_{*1} = \sigma_{*1} \circ E_2$, $E_2(\mathfrak{p}_2) \subseteq \mathfrak{p}_2$, we can consider $p := E_2|_{\mathfrak{p}_2} : \mathfrak{p}_2 \to \mathfrak{p}_2$. We see that $\|p\| = 1$ and $Ker(p) = Ran(1-p) = \mathfrak{p}_{E_2}$. Also, since $E_2^2 = E_2$ and $Ran(E_2) = \mathfrak{g}_1$, $Ran(p) = \mathfrak{p}_1$. The condition $ad^2_{\mathfrak{p}_1}(\mathfrak{p}_1) \subseteq \mathfrak{p}_1$ of the statement of the CPR splitting 3.3 is trivial. Also note that for every $g \in G_1$ and for every $X \in \mathfrak{g}_2$, $Ad_g(E_2(X)) = E_2(Ad_g(X))$. If $Y \in \mathfrak{g}_1$ and we differentiate $Ad_{e^{tY}}(E_2(X)) = E_2(Ad_{e^{tY}}(X))$ at $t = 0$ we get $ad_Y(E_2(X)) = E_2(ad_Y(X))$ and therefore $ad_{\mathfrak{g}_1}(KerE_2) \subseteq KerE_2$.

Since $ad^2_{\mathfrak{p}_2}(\mathfrak{p}_2) \subseteq \mathfrak{p}_2$ we conclude that $ad^2_{\mathfrak{p}_1}(\mathfrak{p}_{E_2}) \subseteq \mathfrak{p}_{E_2}$. The CPR splitting (Th. 3.3) implies the existence of a diffeomorphism

$$\Phi_2 : U_2 \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1 \to G_2$$

$$(u_2, X_2, Y_1) \mapsto u_2 e^{X_2} e^{Y_1}.$$

Assume now that $n > 2$ and that the theorem is true for $k = n-1$ and $k = 2$. We prove that Φ_n is surjective. If $g_n \in G_n$ then the splitting theorem applied to the reductive structure $(G_n, G_{n-1}; E_n)$ implies the existence of $u_n \in U_n$, $X_n \in \mathfrak{p}_{E_n}$ and Y_{n-1} such that $g_n = u_n e^{X_n} e^{Y_{n-1}}$. Since $e^{Y_{n-1}} \in G_{n-1}$ applying the splitting theorem in the case $k = n-1$ we get $u_{n-1} \in U_{n-1}$, $X_{n-1} \in \mathfrak{p}_{E_{n-1}}, \dots,$ $X_2 \in \mathfrak{p}_{E_2}$ and $Y_1 \in \mathfrak{p}_1$ such that $e^{Y_{n-1}} = u_{n-1} e^{X_{n-1}} \dots e^{X_2} e^{Y_1}$. Then

$$g_n = u_n e^{X_n} e^{Y_{n-1}} = u_n e^{X_n} u_{n-1} e^{X_{n-1}} \dots e^{X_2} e^{Y_1} = u_n u_{n-1} e^{Ad_{u_{n-1}^{-1}} X_n} e^{X_{n-1}} \dots e^{X_2} e^{Y_1}$$

is in the image of Φ_n because $Ad_{u_{n-1}^{-1}} X_n \in \mathfrak{p}_{E_n}$.
We prove that Φ_n is injective. Assume that

$$u_n e^{X_n} e^{X_{n-1}} \dots e^{X_2} e^{Y_1} = u'_n e^{X'_n} e^{X'_{n-1}} \dots e^{X'_2} e^{Y'_1}.$$

Since $e^{X_{n-1}} \dots e^{X_2} e^{Y_1} \in G_{n-1}$ there are $u_{n-1} \in U_{n-1}$ and $Y_{n-1} \in \mathfrak{p}_{n-1}$ such that

$$u_{n-1} e^{Y_{n-1}} = e^{X_{n-1}} \dots e^{X_2} e^{Y_1}.$$

Also there are $u'_{n-1} \in U_{n-1}$ and $Y'_{n-1} \in \mathfrak{p}_{n-1}$ such that

$$u'_{n-1} e^{Y'_{n-1}} = e^{X'_{n-1}} \dots e^{X'_2} e^{Y'_1}.$$

Then

$$u_n u_{n-1} e^{Ad^{-1}_{u_{n-1}} X_n} e^{Y_{n-1}} = u'_n u'_{n-1} e^{Ad_{u'^{-1}_{n-1}} X'_n} e^{Y'_{n-1}}$$

and because of the uniqueness of the splitting theorem for $k = 2$ we conclude that

$$\begin{array}{rcl} u_n u_{n-1} & = & u'_n u'_{n-1} \\ Ad_{u_{n-1}^{-1}} X_n & = & Ad_{u'^{-1}_{n-1}} X'_n \\ Y_{n-1} & = & Y'_{n-1}. \end{array} \qquad (1)$$

Since $u_{n-1} e^{Y_{n-1}} = e^{X_{n-1}} \dots e^{X_2} e^{Y_1}$ and $u'_{n-1} e^{Y'_{n-1}} = e^{X'_{n-1}} \dots e^{X'_2} e^{Y'_1}$

$$u_{n-1}^{-1} e^{X_{n-1}} \dots e^{X_2} e^{Y_1} = e^{Y_{n-1}} = e^{Y'_{n-1}} = u'^{-1}_{n-1} e^{X'_{n-1}} \dots e^{X'_2} e^{Y'_1}$$

the uniqueness of the splitting theorem for $k = n-1$ implies that $u_{n-1} = u'_{n-1}$, $X_{n-1} = X'_{n-1}, \dots,$ $X_2 = X'_2$ and $Y_1 = Y'_1$. The equalities in (1) say that $u_n = u'_n$ and $X_n = X'_n$ also hold.
We prove that Ψ_n is bijective based on the fact that Φ_n is bijective. If $p_n \in G_A^+$ then $p_n = g_n g_n^*$ for some $g_n \in G_n$. Because Φ_n is surjective there are $u_n \in U_n$,

$X_n \in \mathfrak{p}_{E_n}, \ldots,\ X_2 \in \mathfrak{p}_{E_2}$ and $Y_1 \in \mathfrak{p}_1$ such that $g_n^* = u_n e^{X_n} \ldots e^{X_2} e^{Y_1}$. Then $p_n = g_n g_n^* = e^{Y_1} e^{X_2} \ldots e^{2X_n} \ldots e^{X_2} e^{Y_1}$ and we conclude that Ψ_n is surjective. To see that Ψ_n is injective let assume that $e^{Y_1} e^{X_2} \ldots e^{2X_n} \ldots e^{X_2} e^{Y_1} = e^{Y_1'} e^{X_2'} \ldots e^{2X_n'} \ldots e^{X_2'} e^{Y_1'}$. If $g_n := e^{Y_1} e^{X_2} \ldots e^{X_n}$ and $g'_n := e^{Y_1'} e^{X_2'} \ldots e^{X_n'}$ then $g_n g_n^* = g'_n {g'}_n^*$ and therefore there is an $u_n \in U_n$ such that $g_n u_n = g_n'$. Then $u_n e^{X_n} \ldots e^{X_2} e^{Y_1} = e^{X_n'} \ldots e^{X_2'} e^{Y_1'}$ and we conclude that $(X_n, \ldots, X_2, Y_1) = (X_n', \ldots, X_2', Y_1')$.
We prove that Φ_n is a diffeomorphism by induction. The CPR splitting states that Φ_2 is a diffeomorphism. Assume that $n > 2$ and that Φ_{n-1} is a diffeomorphism. If $g_n \in G_n$ then $g_n = u_n(g_n) e^{X_n(g_n)} e^{Y_{n-1}(g_n)}$, where $(u_n, X_n, Y_{n-1}) : G_n \to U_n \times \mathfrak{p}_{E_n} \times \mathfrak{p}_{n-1}$ is smooth because the inverse of the CPR splitting is smooth in the case $n = 2$. If we denote $f(g_n) := e^{Yn-1(g_n)}$ then f is a smooth map and

$$f(g_n) = u_{n-1}(f(g_n)) e^{X_{n-1}(f(g_n))} \ldots e^{X_2(f(g_n))} e^{Y_1(f(g_n))}$$

where

$$(u_{n-1}, X_{n-1}, \ldots, X_2, Y_1) : G_{n-1} \to U_{n-1} \times \mathfrak{p}_{E_{n-1}} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1$$

is a smooth map. Since

$$g_n = u_n(g_n) e^{X_n(g_n)} u_{n-1}(f(g_n)) e^{X_{n-1}(f(g_n))} \ldots e^{X_2(f(g_n))} e^{Y_1(f(g_n))} =$$

$$u_n(g_n) u_{n-1}(f(g_n)) e^{Ad_{u_{n-1}^{-1}(f(g_n))} X_n(g_n)} e^{X_{n-1}(f(g_n))} \ldots e^{X_2(f(g_n))} e^{Y_1(f(g_n))}$$

we get that $\Phi_n^{-1} : G_n \to U_n \times \mathfrak{p}_{E_n} \times \cdots \times \mathfrak{p}_{E_2} \times \mathfrak{p}_1$

$$g_n \mapsto (u_n(g_n) u_{n-1}(f(g_n)), Ad_{u_{n-1}^{-1}(f(g_n))} X_n(g_n), \ldots, X_2(f(g_n)), Y_1(f(g_n)))$$

is smooth.
We prove next that $\Psi^{-1} = (\overline{X_n}, \ldots, \overline{X_2}, \overline{Y_1})$ is smooth. Let $g_n \in G_n$, then if $p_n = g_n^* g_n$,

$$p_n = e^{(\overline{Y}_1(p_n))} e^{(\overline{X}_2(p_n))} \ldots e^{(\overline{X}_{n-1}(p_n))} e^{(2\overline{X}_n(p_n))} e^{(\overline{X}_{n-1}(p_n))} \ldots e^{(\overline{X}_2(p_n))} e^{(\overline{Y}_1(p_n))}.$$

Since $g_n = u_n(g_n) e^{X_n(g_n)} \ldots e^{X_2(g_n)} e^{Y_1(g_n)}$ where $\Phi^{-1} = (u_n, X_n, \ldots, X_2, Y_1)$, we get

$$p_n := g_n^* g_n = e^{Y_1(g_n)} e^{X_2(g_n)} \ldots e^{X_{n-1}(g_n)} e^{2X_n(g_n)} e^{X_{n-1}(g_n)} \ldots e^{X_2(g_n)} e^{Y_1(g_n)}$$

so that

$$(\overline{X_n}, \ldots, \overline{X_2}, \overline{Y_1}) = (X_n, \ldots, X_2, Y_1) \circ \pi$$

where $\pi : G_n \to G_n^+, g_n \to g_n^* g_n$. Since π is a submersion we conclude that $\Psi^{-1} = (\overline{X}_n, \ldots, \overline{X}_2, \overline{Y}_1)$ is smooth.

$\square$

Remark 3.15. *We note that in the context of the previous theorem, if* $F_{k,j} := E_{j+1} \circ \cdots \circ E_k$*, then* $(G_k, G_j; F_{k,j})$ *is a reductive structure and* $\|F_{k,j}|_{\mathfrak{p}_k}\| = 1$.

Remark 3.16. *The splitting theorem of Porta and Recht [14] asserts that if we have a unital inclusion of C^*-algebras $B \subseteq A$ and a conditional expectation $E : A \to B$ then the map*

$$\Phi : U_A \times \mathfrak{p}_E \times \mathfrak{p}_B \to G_A$$
$$(u, X, Y) \mapsto ue^X e^Y$$

is a diffeomorphism, where $\mathfrak{p}_E$ are the self-adjoint elements of $KerE$ and $\mathfrak{p}_B$ are the self-adjoint elements of B.
Theorem 3.14 in the case $n = 2$ is a formulation of the CPR splitting (Theorem 3.3) in the context of reductive structures. The Porta-Recht splitting theorem is a special case of the previous theorem if we consider $(G_A, G_B; E, \sigma)$ derived from the triple $(A, B; E)$ as in Example 3.7 and verify that the conditions of the theorem are satisfied because of what was stated in Example 2.2. The CPR theorem covers the case where the inclusion of algebras and the map E are not unital, as in Example 3.9 of reductive structures. It also covers the case where the symmetric space and reductive structure are derived from unitized ideals of operators as in Example 2.3 and Example 3.8, see [5].
The CPR theorem in the context of several reductive structures (Theorem 3.14) covers for example the case of multiple unital inclusions of C^*-algebras and conditional expectations between them

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$$
$$A_1 \xleftarrow{E_2} A_2 \xleftarrow{E_3} \dots \xleftarrow{E_n} A_n.$$

4 Complexifications of homogeneous spaces

Proposition 4.5 to Remark 4.13 here are extensions of Section 5 of [1], from the context of C^*-algebras to the context of Finsler symmetric spaces of semi-negative curvature with reductive structures.

Definition 4.1. *Let X be a Banach manifold. A **complexification** of X is a complex Banach manifold Y endowed with an anti-holomorphic involutive diffeomorphism σ such that the fixed point submanifold $Y_0 = \{y \in Y : \sigma(y) = y\}$ is a strong deformation retract of Y and is diffeomorphic to X.*

Example 4.2. *Let $M = G/U = Sym(G, \sigma, \|\cdot\|)$ be a simply connected Finsler symmetric space of semi-negative curvature. Theorem 3.1 guaranties that U is a strong deformation retract of G. If G is a complex analytic group and σ is anti-holomorphic, then G is a complexification of U. In the context of C^*-algebras the group of invertible elements G is a complexification of the group of unitary elements U with $\sigma = (\cdot)^{-1*}$. Note that U is not a complex analytic manifold.*

Definition 4.3. *Let (G_A, σ) be a symmetric Banach-Lie group with involutive subgroup G_B. We define $\sigma_G : G_A/G_B \to G_A/G_B$, $uG_B \mapsto \sigma(u)G_B$ and $\lambda : U_A/U_B \hookrightarrow G_A/G_B$, $uU_B \mapsto uG_B$.*

We now give a criteria which implies that U_A/U_B is diffeomorphic to the fixed point set of the involution σ_G.

Proposition 4.4. *If $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ is a Finsler symmetric space of semi-negative curvature, G_B is an involutive subgroup of G_A, and $\Gamma \subseteq \mathfrak{p}_B$, then $G_A^+ \cap G_B = G_B^+$.*

Proof. Since $G_B^+ \subseteq G_A^+ \cap G_B$ always holds, it is enough to prove that $G_A^+ \cap G_B \subseteq G_B^+$. By Cor. 3.2 $G_A^+ = e^{\mathfrak{p}}$ and $G_B^+ = e^{\mathfrak{p}_B}$. If $g \in G_A^+ \cap G_B$ then there is an $X \in \mathfrak{p}$ such that $g = e^X$. Since G_B is an involutive subgroup G_B/U_B has semi-negative curvature and using the polar decomposition of Th. 3.1 in G_B guaranties the existence of $u \in U_B$ and $Y \in \mathfrak{p}_B$ such that $g = ue^Y$. Then, Theorem 3.1 applied to G_A tells us that for certain $Z \in \Gamma$, $u = e^Z$ and $Y = X - Z$. Since $\Gamma \subseteq \mathfrak{g}_B$ we conclude that $X \in \mathfrak{g}_B$ and therefore $g \in G_B^+$. $\square$

Proposition 4.5. *If $G_B^+ = G_A^+ \cap G_B$, then $\lambda(U_A/U_B) = \{s \in G_A/G_B : \sigma_G(s) = s\}$.*

Proof. The inclusion $\subseteq$ is obvious. Given $s = uG_B$ such that $\sigma_G(s) = s$, $u^{-1}\sigma(u) \in G_B$. Since $u^{-1}\sigma(u) \in G_A^+$ the hypothesis $G_B^+ = G_A^+ \cap G_B$ implies that $u^{-1}\sigma(u) \in G_B^+$, and therefore there exists $w \in G_B$ such that $u^{-1}\sigma(u) = ww^*$. Then $uw = \sigma(u)w^{*-1} = \sigma(u)\sigma(w) = \sigma(uw)$, so that $uw \in U_A$ and $s = uG_B = uwG_B = \lambda(uwU_B)$. $\square$

We give a geometric description of the complexification G_A/G_B of U_A/U_B in the context of reductive structures. This can be seen as an infinite dimensional version of Mostow fibration, see [9, 10] and Section 3 of [4].

Remark 4.6. *Since the actions $Ad : U_B \to \mathcal{B}(\mathfrak{p}_E)$ and $Ad : U_B \to \mathcal{B}(\mathfrak{u}_E)$ are well defined we get the homogeneous vector bundles $U_A \times_{U_B} \mathfrak{p}_E \to U_A/U_B$ and $U_A \times_{U_B} \mathfrak{u}_E \to U_A/U_B$, $[(u, X)] \mapsto uU_B$, where the actions of U_B on $U_A \times_{U_B} \mathfrak{p}_E$ and $U_A \times_{U_B} \mathfrak{u}_E$ are given by $v \cdot (u, X) = (uv^{-1}, Ad_v X)$.*

Theorem 4.7. *Let $M_A = G_A/U_A = Sym(G_A, \sigma, \|\cdot\|)$ be a simply connected Finsler symmetric space of semi-negative curvature and $(G_A, G_B; E, \sigma)$ a reductive structure with involution such that $\|E|_{\mathfrak{p}}\| = 1$. Consider $\Psi_0^E : U_A \times \mathfrak{p}_E \to G_A$, $(u, X) \mapsto ue^X$ and $\kappa : (u, X) \mapsto [(u, X)]$ the quotient map. Then there is a unique real analytic, U_A-equivariant diffeomorphism $\Psi^E : U_A \times_{U_B} \mathfrak{p}_E \to G_A/G_B$ such that the diagram*

$$\begin{tikzcd} U_A \times \mathfrak{p}_E \arrow[r, "\Psi_0^E"] \arrow[d, "\kappa"'] & G_A \arrow[d, "q"] \\ U_A \times_{U_B} \mathfrak{p}_E \arrow[r, "\Psi^E"] & G_A/G_B \end{tikzcd}$$

commutes.
Therefore the homogeneous space G_A/G_B has the structure of an U_A-equivariant fiber bundle over U_A/U_B with the projection given by the composition

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Xi} U_A/U_B$$

$$ue^X G_B \mapsto [(u,X)] \mapsto uU_B \qquad \textit{for } u \in U_A \textit{ and } X \in \mathfrak{p}_E$$

and typical fiber $\mathfrak{p}_E$*.*

Proof. To prove that Ψ^E is well defined we show that for $u \in U_A$, $v \in U_B$ and $X \in \mathfrak{p}_E$

$$\begin{aligned} q(\Psi_0^E(u,X)) &= ue^X G_B = uv^{-1}e^{Ad_v X} vG_B = uv^{-1}e^{Ad_v X} G_B \\ &= q(\Psi_0^E(uv^{-1}, Ad_v X)) = q(\Psi_0^E(v \cdot (u,X))) \end{aligned}$$

The uniqueness of Ψ^E is a consequence of the surjectivity of κ.
Theorem 3.14 for the case $n = 2$ implies the existence of a diffeomorphism

$$\Phi : U_A \times \mathfrak{p}_E \times \mathfrak{p}_B \to G_A$$

$$(u, X, Y) \mapsto ue^X e^Y.$$

If $gG_B \in G_A/G_B$ there is $(u,X,Y) \in U_A \times \mathfrak{p}_E \times \mathfrak{p}_B$ such that $g = ue^X e^Y$ and we get $gG_B = ue^X e^Y G_B = ue^X G_B$, proving the surjectivity of Φ.
To see that Ψ^E is also injective assume that $u_1 e^{X_1} G_B = u_2 e^{X_2} G_B$. Then there is a $b \in G_B$ such that $u_1 e^{X_1} b = u_2 e^{X_2}$. Since G_B is an involutive connected subgroup of G_A and G_A/U_A has semi-negative curvature, Proposition 3.12 states that G_B/U_B has also semi-negative curvature and we can apply the polar decomposition (Proposition 3.1) in G_B: there are unique $v \in U_B$ and $Y \in \mathfrak{p}_B$ such that $b = ve^Y$.
Then

$$(u_1 v) e^{Ad_{v^{-1}} X_1} e^Y = u_1 e^{X_1} v e^Y = u_1 e^{X_1} b = u_2 e^{X_2}$$

and applying $(\Phi)^{-1}$ to this equality we get $(u_1 v, Ad_{v^{-1}} X_1, Y) = (u_2, X_2, 0)$, which implies that $v^{-1} \cdot (u_1, X_1) = (u_2, X_2)$.
Finally, we prove that Ψ^E is an analytic diffeomorphism. Since κ is a submersion and $\Psi^E \circ \kappa$ $(= q \circ \Psi_0^E)$ is a real analytic map Ψ^E is real analytic. Since the map $\Phi^{-1} : g \mapsto (u(g), X(g), Y(g))$ is analytic, the map $\sigma : g \mapsto [(u(g), X(g))]$, $G_A \to U_A \times_{U_B} \mathfrak{p}_E$ is also analytic. Since q is a submersion and $\sigma = (\Psi^E)^{-1} \circ q$ we see that $(\Psi^E)^{-1}$ is analytic. □

Corollary 4.8. *If we analyse the diagram of the previous theorem in the tangent spaces using the following identifications* $T_{(1,0)}(U_A \times \mathfrak{p}_E) \simeq \mathfrak{u}_A \times \mathfrak{p}_E$, $T_{[(1,0)]}(U_A \times_{U_B} \mathfrak{p}_E) \simeq \mathfrak{u}_E \times \mathfrak{p}_E$ *and* $T_o(G_A/G_B) \simeq KerE$ *then*

$$(\Phi_0^E)_{*(1,0)} : \mathfrak{u}_A \times \mathfrak{p}_E \to \mathfrak{g}_A, \qquad (Y,Z) \mapsto Y + Z$$

$$\kappa_{*(1,0)} : \mathfrak{u}_A \times \mathfrak{p}_E \to \mathfrak{u}_E \times \mathfrak{p}_E, \qquad (Y,Z) \mapsto ((1-E)Y, Z)$$

$$q_{*1} : \mathfrak{g}_A \mapsto KerE, \qquad W \mapsto (1-E)W$$

and therefore

$$(\Phi^E)_{*[(1,0)]} : \mathfrak{u}_E \times \mathfrak{p}_E \to KerE, \qquad ((1-E)Y, Z) \mapsto (1-E)(Y+Z) = (1-E)Y + Z.$$

We conclude that

$$(\Phi^E)_{*[(1,0)]} : \mathfrak{u}_E \times \mathfrak{p}_E \to KerE, \qquad (X, Z) \mapsto X + Z$$

is an isomorphism.

Corollary 4.9. *If we assume the conditions of the previous theorem, the fixed point set of of the involution σ_G on $G_A/G_B \simeq U_A \times_{U_B} \mathfrak{p}_E$ is diffeomorphic to U_A/U_B and U_A/U_B is a strong deformation retract of G_A/G_B. If G_A is a complex analytic group and σ is anti-holomorphic then G_A/G_B is a complexification of U_A/U_B.*
If we define $\tau_G : U_A \times_{U_B} \mathfrak{p}_E \to U_A \times_{U_B} \mathfrak{p}_E$, $[(u, X)] \mapsto [(u, -X)]$, then the following diagram

$$\begin{tikzcd}
U_A \times_{U_B} \mathfrak{p}_E \arrow[r, "\tau_G"] \arrow[d, "\Psi^E"'] & U_A \times_{U_B} \mathfrak{p}_E \arrow[d, "\Psi^E"] \\
G_A/G_B \arrow[r, "\sigma_G"] & G_A/G_B
\end{tikzcd}$$

commutes.

Proof. Note that $\Gamma = \{0\}$ so that Prop. 4.4 implies $G_B^+ = G_B \cap G_A^+$ and Prop. 4.5 states that U_A/U_B is diffeomorphic to the set of fixed points on σ_G.
Alternatively, the diagram tells us that the set of fixed points of the involution σ_G is $\Psi^E(\{[(u, X)] \in U_A \times_{U_B} \mathfrak{p}_E : \tau_G([(u, X)]) = [(u, X)]\}) = \Psi^E(\{[(u, 0)] : u \in U_A\}) = \{uG_B : u \in U_A\} = \lambda(U_A/U_B)$.
If we define $F : (U_A \times_{U_B} \mathfrak{p}_E) \times [0, 1] \to U_A \times_{U_B} \mathfrak{p}_E, ([(u, X)], t) \mapsto [(u, tX)]$ we see that $\{[(u, 0)] : u \in U_A\}$ is a strong deformation retract of $U_A \times_{U_B} \mathfrak{p}_E$ and $\{[(u, 0)] : u \in U_A\}$ is diffeomorphic to U_A/U_B.
If σ is anti-holomorphic then σ_G is anti-holomorphic (see [12]) and it follows from Definition 4.1 that G_A/G_B is a complexification of U_A/U_B. $\square$

Theorem 4.10. *If we assume that the conditions of Theorem 4.7 are satisfied then there is a U_A-equivariant diffeomorphic vector bundle map from the associated vector bundle $U_A \times_{U_B} \mathfrak{u}_E \to U_A/U_B$ to the tangent bundle $T(U_A/U_B) \to U_A/U_B$ given by $\alpha^E : U_A \times_{U_B} \mathfrak{u}_E \to T(U_A/U_B)$, $[(u, X)] \mapsto (\mu_u)_{*o} q_{*1} X$, where the action of U_A on $T(U_A/U_B)$ is given by $u \cdot - = (\mu_u)_* -$ for every $u \in U_A$.*

Proof. Let $\alpha : U_A \times U_A/U_B \to U_A/U_B$ be given by $(u, vU_B) \mapsto uvU_B$, then $\partial_2 \alpha : U_A \times T(U_A/U_B) \to T(U_A/U_B)$, $(u, V) \mapsto (\mu_u)_* V$. Since $E \circ \sigma_{*1} = \sigma_{*1} \circ E$ $E(\mathfrak{u}) \subseteq \mathfrak{u}$, and since $E(\mathfrak{g}_A) = \mathfrak{g}_B$ we get the decomposition $\mathfrak{u} = \mathfrak{u}_B \oplus \mathfrak{u}_E$. Then $\mathfrak{u}_E \simeq T_o(U_A/U_B)$, $X \mapsto q_{*1} X$ and restricting $\partial_2 \alpha$ to $U_A \times T_o(U_A/U_B)$ we get a map $\alpha_0^E : U_A \times \mathfrak{u}_E \to T(U_A/U_B)$, $(u, X) \mapsto (\mu_u)_{*o} q_{*1} X$.
We assert that there is a unique U_A-equivariant diffeomorphism $\alpha^E : U_A \times_{U_B} \mathfrak{u}_E \to T(U_A/U_B)$ such that $\alpha^E \circ \kappa = \alpha_0^E$, where κ is the quotient map $(u, X) \mapsto [(u, X)]$.

To prove that α^E is well defined we see that for every $u \in U_A$, $v \in U_B$ and $X \in \mathfrak{u}_E$

$$\begin{aligned}
\alpha_0^E(v \cdot (u, X)) &= \alpha_0^E(uv^{-1}, Ad_v X) = (\mu_{uv^{-1}})_{*o} q_{*1} Ad_v X \\
&= (\mu_{uv^{-1}})_{*o} q_{*1} (I_v)_{*1} X = (\mu_{uv^{-1}} q I_v)_{*1} X \\
&= (\mu_u \mu_{v^{-1}} q L_v R_{v^{-1}})_{*1} X = (\mu_u q L_{v^{-1}} L_v R_{v^{-1}})_{*1} X \\
&= (\mu_u q R_{v^{-1}})_{*1} X = (\mu_u q)_{*1} = (\mu_u)_{*o} q_{*1} X = \alpha_0^E(u, X)
\end{aligned}$$

The uniqueness of α^E is a consequence of the surjectivity of κ. α^E is surjective because $(\mu_u)_{*o} : T_o(U_A/U_B) \to T_{q(u)}(U_A/U_B)$ is bijective for every $u \in U_A$. To see that α^E is injective assume that $(\mu_{u_1})_{*o} q_{*1} X_1 = (\mu_{u_2})_{*o} q_{*1} X_2$. Then $q(u_1) = q(u_2)$ and therefore there is a $v \in U_B$ such that $u_1 v = u_2$. Then

$$\begin{aligned}
(\mu_{u_1})_{*o} q_{*1} X_1 &= (\mu_{u_2})_{*o} q_{*1} X_2 = (\mu_{u_1 v} q)_{*1} X_2 = (\mu_{u_1} \mu_v q)_{*1} X_2 \\
&= (\mu_{u_1} \mu_v q R_{v^{-1}})_{*1} X_2 = (\mu_{u_1} q L_v R_{v^{-1}})_{*1} X_2 \\
&= (\mu_{u_1} q I_v)_{*1} X_2 = (\mu_{u_1})_{*o} q_{*1} Ad_v X_2
\end{aligned}$$

so that $Ad_v X_2 = X_1$ and we conclude that $v \cdot (u_2, X_2) = (u_1, X_1)$. □

Lemma 4.11. *If σ is a anti-holomorphic involutive automorphism of a complex Banach-Lie group G_A then $i\mathfrak{u} = \mathfrak{p}$.*

Proof. If $X \in \mathfrak{u}$, $\sigma_{*1} X = X$ and $\sigma_{*1}(iX) = -i\sigma_{*1} X = -iX$ so that $iX \in \mathfrak{p}$. The other inclusion is proved in a similar way. □

Example 4.12. *If G_A is the group of invertible elements of a C^*-algebra A then the previous lemma applies and we get $\mathfrak{p} = A_s$ the set of self-adjoint elements of A and $\mathfrak{u} = i\mathfrak{p} = iA_s = A_{as}$ the set of skew-adjoint elements of A.*

Remark 4.13. *Assume the conditions of Theorem 4.7 are satisfied and that G_A is a complex analytic group, $\mathfrak{u} = i\mathfrak{p}$, and E is $\mathbb{C}$-linear. Since $Ad_g(iX) = iAd(X)$ for every $g \in G_A$ and $X \in \mathfrak{g}_A$ we conclude that $\Theta : U_A \times_{U_B} \mathfrak{p}_E \to U_A \times_{U_B} \mathfrak{u}_E$, given by $[(u, X)] \mapsto [(u, iX)]$ is well defined. Theorem 4.7 and Theorem 4.10 imply that the composition*

$$G_A/G_B \xrightarrow{(\Psi^E)^{-1}} U_A \times_{U_B} \mathfrak{p}_E \xrightarrow{\Theta} U_A \times_{U_B} \mathfrak{u}_E \xrightarrow{\alpha^E} T(U_A/U_B)$$

is a U_A-equivariant diffeomorphism between the complexification G_A/G_B and the tangent bundle $T(U_A/U_B)$ of the homogeneous space U_A/U_B. Under the above identification the involution σ_G is the map $V \mapsto -V$, $T(U_A/U_B) \to T(U_A/U_B)$.

The isomorphism in the last remark gives the tangent bundle of U_A/U_B a complex manifold structure which depends on the map E.

The following proposition shows that the diffeomorphism between G_A/G_B and $T(U_A/U_B)$ respects the natural morphisms that can be defined between homogeneous spaces of the form G_A/G_B and tangent bundles of homogeneous spaces given by $T(U_A/U_B)$.

Proposition 4.14. *Let $(G_A, G_B; E; \sigma)$ and $(\tilde{G_A}, \tilde{G_B}; \tilde{E}; \tilde{\sigma})$ be reductive structures with involution that satisfy the conditions of the previous remark and let $\alpha : G_A \to \tilde{G_A}$ be a holomorphic morphism of reductive structures with involution. If we define $\alpha_G : G_A/G_B \to G_A/G_B$, $gG_B \mapsto \alpha(g)\tilde{G_B}$ and $\alpha_U : U_A/U_B \to U_A/U_B$, $uU_B \mapsto \alpha(u)\tilde{U_B}$ then the diagram*

$$\begin{tikzcd}
G_A/G_B \arrow[d, "\alpha_G"'] & U_A \times_{U_B} \mathfrak{u}_E \arrow[l, "\sim"'] \arrow[r, "\sim"] & T(U_A/U_B) \arrow[d, "\alpha_{U*}"] \\
\tilde{G_A}/\tilde{G_B} & \tilde{U_A} \times_{\tilde{U_B}} \tilde{\mathfrak{u}_E} \arrow[l, "\sim"'] \arrow[r, "\sim"] & T(\tilde{U_A}/\tilde{U_B})
\end{tikzcd}$$

commutes, where the horizontal arrows correspond to the morphisms of Rem. 4.13.

Proof. Since $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$, $\alpha(U_B) \subseteq \tilde{U_B}$ and α_U is well defined. Since $\alpha_{*1} \circ \sigma_{*1} = \tilde{\sigma}_{*1} \circ \alpha_{*1}$, $\alpha_{*1}(\mathfrak{u}) \subseteq \tilde{\mathfrak{u}}$. Also $E \circ \alpha_{*1} = \alpha_{*1} \circ E$ implies $\alpha_{*1}(KerE) \subseteq Ker\tilde{E}$ so that $\alpha_{*1}(\mathfrak{u}_E) \subseteq \tilde{\mathfrak{u}_E}$. Given $u \in U_A$ and $X \in \mathfrak{u}_E$, $\alpha(u) \in \tilde{U_A}$ and $\alpha_{*1}X \in \tilde{u_E}$ and we have the following diagram

$$\begin{tikzcd}
ue^{iX}G_B \arrow[d, mapsto, "\alpha_G"'] & {[(u,X)]} \arrow[l, mapsto] \arrow[r, mapsto] & (\mu_u)_{*o}q_{*1}X \arrow[d, mapsto, "\alpha_{U*}"] \\
\alpha(u)e^{i\alpha_{*1}(X)}\tilde{G_B} & {[(\alpha(u), \alpha_{*1}(X))]} \arrow[l, mapsto] \arrow[r, mapsto] & (\tilde{\mu}_{\alpha(u)})_{*o}\tilde{q}_{*1}\alpha_{*1}X
\end{tikzcd}$$

It is enough to verify that the values in the vertical arrows correspond to the stated morphisms. $\alpha_G(ue^{iX}G_B) = \alpha(u)e^{\alpha_{*1}(iX)}\tilde{G_B} = \alpha(u)e^{i\alpha_{*1}(X)}\tilde{G_B}$ since $\alpha_{*1}(iX) = i\alpha_{*1}(X)$ because α is holomorphic. Since $\alpha_U \circ \mu_u = \tilde{\mu}_{\alpha(u)} \circ \alpha_U$ and $\tilde{q} \circ \alpha = \alpha_U \circ q$ we get $\alpha_{U*q(u)}(\mu_u)_{*o}q_{*1}X = (\tilde{\mu}_{\alpha(u)})_{*o}\alpha_{U*o}q_{*1}X = (\tilde{\mu}_{\alpha(u)})_{*o}\tilde{q}_{*1}\alpha_{*1}X$. ☐

There are two basic examples of homogeneous spaces U_A/U_B in the infinite dimensional context, the flag manifolds and the Stiefel manifolds. Coadjoint orbits are examples of flag manifolds.

Example 4.15. *Flag manifolds*
Let $\mathcal{H}$ be a Hilbert space and let p_i, $i = 1, \ldots, n$ be pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$ each with range $\mathcal{H}_i$ such that $\sum_{i=1}^n p_i = 1$. If we consider the action of the unitary group U_A of $\mathcal{B}(\mathcal{H})$ on the set of n-tuples of pairwise orthogonal projections with sum 1 given by $u \cdot (q_1, \ldots, q_n) = (uq_1u^*, \ldots, uq_nu^*)$ then the orbit of $(p_1, \ldots, p_n)$ can be considered as an infinite dimensional version of a flag manifold. This orbit is isomorphic to U_A/U_B where

$$U_B = \left\{ \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_n \end{pmatrix} : u_i \textit{ unitary in } \mathcal{B}(\mathcal{H}_i) \textit{ for } i = 1, \ldots, n \right\};$$

and we write the operators in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$ as $n \times n$-matrices with corresponding operator entries. If we consider the group G_A of invertible operators in $\mathcal{B}(\mathcal{H})$ with the usual involution σ, the involutive subgroup

$$G_B = \left\{ \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_n \end{pmatrix} : g_i \text{ invertible in } \mathcal{B}(\mathcal{H}_i) \text{ for } i = 1, \dots, n \right\};$$

and the conditional expectation $E : \mathfrak{g}_A \to \mathfrak{g}_B$, $X \mapsto \sum_{i=1}^n p_i X p_i$ then we are in the context of Example 3.7 and Th. 4.7, Th. 4.10 and Rem. 4.13 give a geometric description of the complexification of the flag manifold.

Other examples of flag manifolds in the infinite dimensional context are coadjoint orbits in operator ideals, which now can be described geometrically.

Example 4.16. *Coadjoint orbits*
In the setting of Example 3.8 let $1 \leq p < \infty$ and q such that $1/p + 1/q = 1$. The Banach-Lie algebra of the Banach-Lie group $G_{A,p}$ is $\mathfrak{g}_{A,p} = A_p$, the ideal of p-Schatten operators (A_∞ is the ideal of compact operators). The Banach-Lie algebra of the real Banach-Lie group $U_{A,p}$ is $\mathfrak{u}_{A,p}$, the skew-adjoint p-Schatten operators. The trace provides strong duality pairings $\mathfrak{g}_{A,p}^* \simeq \mathfrak{g}_{A,q}$ and $\mathfrak{u}_{A,p}^* \simeq \mathfrak{u}_{A,q}$.
We denote by $Ad^* : G_{A,p} \mapsto \mathcal{B}(\mathfrak{g}_{A,p})$, $Ad_g^*(X) = (Ad_{g^{-1}})^*(X) = gXg^{-1}$ for $g \in G_{A,p}$ and $X \in \mathfrak{g}_{A,p}^* \simeq \mathfrak{g}_{A,q}$, the coadjoint action of $G_{A,p}$ and by $Ad^* : U_{A,p} \mapsto \mathcal{B}(\mathfrak{u}_{A,p})$, $Ad_u^*(X) = (Ad_{u^{-1}})^*(X) = uXu^{-1}$ for $u \in U_{A,p}$ and $X \in \mathfrak{u}_{A,p}^* \simeq \mathfrak{u}_{A,q}$, the coadjoint action of $U_{A,p}$.
For a fixed $X \in \mathfrak{u}_{A,q} \subseteq \mathfrak{g}_{A,q}$ let $\mathcal{O}_G(X) = \{Ad_g^*(X) : g \in G_{A,p}\}$ be the coadjoint orbit of X under the action of $G_{A,p}$ and $\mathcal{O}_U(X) = \{Ad_u^*(X) : g \in U_{A,p}\}$ be the coadjoint orbit of X under the action of $U_{A,p}$. Since X is a compact skew-adjoint operator it is diagonalizable, i.e. there is a finite or countable sequence of pairwise orthogonal projections $(p_i)_{i=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ such that $\sum_{i=1}^N p_i = 1$ and $X = \sum_{i=1}^N \lambda_i p_i$, where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $(\lambda_i)_{i=1}^N \subseteq i\mathbb{R}$. The map $E : Y \mapsto \sum_{i=1}^N p_i Y p_i$ is a conditional expectation from A onto the C^*-subalgebra $B = \{Y \in A : p_i Y = Y p_i \text{ for all } i \geq 1\}$. This conditional expectation sends trace-class operators to trace-class operators and preserves the trace, so the conditions on E in Example 3.8 are satisfied. The coadjoint isotropy group of X for the action of $G_{A,p}$ is $\{g \in G_{A,p} : gXg^{-1} = X\} = G_{B,p}$ and the coadjoint isotropy group of X for the action of $U_{A,p}$ is $\{u \in U_{A,p} : uXu^{-1} = X\} = U_{B,p}$. This follows from the fact that an operator commutes with a diagonalizable operator if and only if it leaves all the eigenspaces of the diagonalizable operator invariant. Thus, making the identifications $\mathcal{O}_G(X) \simeq G_{A,p}/G_{B,p}$ and $\mathcal{O}_U(X) \simeq U_{A,p}/U_{B,p}$, Th. 4.7, Th. 4.10 and Rem. 4.13 give a geometric description of the complexification of the flag manifold; there is a $U_{A,p}$-equivariant diffeomorphic fiber bundle map between $\mathcal{O}_G(X)$ and $T(\mathcal{O}_U(X))$ covering the identity map of $\mathcal{O}_U(X)$.

For further reading on the coadjoint orbits in the infinite dimensional setting, see Section 7 in [13].

Likewise, it is now possible to give a geometric description of the complexification of the Stiefel manifolds.

Example 4.17. *Stiefel manifolds*
Let $\mathcal{H}$ be a Hilbert space and let p_i, $i = 1, 2$ be pairwise orthogonal projections in $\mathcal{B}(\mathcal{H})$ each with range $\mathcal{H}_i$ such that $p_1 + p_2 = 1$. If we consider the action of the unitary group U_A of $\mathcal{B}(\mathcal{H})$ on the set of partial isometries given by by $u \cdot v = uv$ then the orbit of p_1 can be considered as an infinite dimensional version of a Stiefel manifold. This orbit is isomorphic to U_A/U_B where

$$U_B = \left\{ \begin{pmatrix} 1 & 0 \\ & \\ 0 & u \end{pmatrix} : u \text{ is unitary in } \mathcal{B}(\mathcal{H}_2) \right\}.$$

and we write the operator in $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ as 2×2-matrices with corresponding operator entries. If we consider the group G_A of invertible operators in $\mathcal{B}(\mathcal{H})$ with the usual involution σ, the involutive subgroup

$$G_B = \left\{ \begin{pmatrix} 1 & 0 \\ & \\ 0 & g \end{pmatrix} : g \text{ is invertible in } \mathcal{B}(\mathcal{H}_2) \right\}.$$

and the map $E : \mathfrak{g}_A \to \mathfrak{g}_B$, $X \mapsto (1-p)X(1-p)$ then we are in the context of Example 3.9 and Th. 4.7, Th. 4.10 and Rem. 4.13 give a geometric description of the complexification of the Stiefel manifold.

Remark 4.18. *The case of the flag manifold with two projections is the infinite dimensional Grassmannian. The case of the Grassmannian where the decomposition of $\mathcal{H}$ is $\mathcal{H} = \mathbb{C}\eta \oplus (\mathbb{C}\eta)^{\perp}$ for a non-zero vector $\eta \in \mathcal{H}$ is the projective space $\mathbb{P}(\mathcal{H})$. The special case of the Stiefel manifold where the decomposition of $\mathcal{H}$ is $\mathcal{H} = \mathbb{C}\eta \oplus (\mathbb{C}\eta)^{\perp}$ for a non-zero vector $\eta \in \mathcal{H}$ is the unit sphere in the Hilbert space $\mathcal{H}$.*

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