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Subresultants, Sylvester sums and the rational interpolation problem ☆

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ABSTRACT

We present a solution for the classical univariate rational interpolation problem by means of (univariate) subresultants. In the case of Cauchy interpolation (interpolation without multiplicities), we give explicit formulas for the solution in terms of symmetric functions of the input data, generalizing the well-known formulas for Lagrange interpolation. In the case of the osculatory rational interpolation (interpolation with multiplicities), we give determinantal expressions in terms of the input data, making explicit some matrix formulations that can independently be derived from previous results by Beckermann and Labahn.

1. Introduction

The *Cauchy interpolation problem* or rational interpolation problem, considered already in Cauchy (1841), Rosenhain (1845), Predonzan (1953), is the following:

Let K be a field, $a, b \in \mathbb{Z}_{\geq 0}$, and set $\ell = a + b$. Given a set $\{x_0, \ldots, x_\ell\}$ of $\ell + 1$ distinct points in K, and $y_0, \ldots, y_\ell \in K$, determine—if possible—polynomials $A, B \in K[x]$ such that $\deg(A) \leq a$, $\deg(B) \leq b$

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and

$$\frac{A}{B}(x_i) = y_i, \quad 0 \le i \le \ell. \tag{1}$$

This might be considered as a generalization of the classical Lagrange interpolation problem for polynomials, where $b = 0$ and $a = \ell$. In contrast with that case, there is not always a solution to this problem, since for instance by setting $y_0 = \cdots = y_a = 0$, the numerator A is forced to be identically zero, and therefore the remaining y_{a+k}, $1 \le k \le \ell - a$, have to be zero as well. However, when there is a solution, then the rational function A/B is unique as shown below.

The obvious generalization of the Cauchy interpolation problem receives the name *osculatory rational interpolation problem* or rational Hermite interpolation problem:

Let K be a field, $a, b \in \mathbb{Z}_{\ge 0}$, and set $\ell = a + b$. Given a set $\{x_0, \dots, x_k\}$ of $k+1$ distinct points in K, $a_0, \dots, a_k \in \mathbb{Z}_{\ge 0}$ such that $a_0 + \cdots + a_k = \ell + 1$, and $y_{i,j} \in K$, $0 \le i \le k$, $0 \le j < a_i$, determine—if possible—polynomials $A, B \in K[x]$ such that $\deg(A) \le a$, $\deg(B) \le b$ and

$$\left(\frac{A}{B}\right)^{(j)}(x_i) = j!\,y_{i,j}, \quad 0 \le i \le k,\ 0 \le j < a_i. \tag{2}$$

This problem has also been extensively studied from both an algorithmic and theoretical point of view, see for instance Salzer (1962), Kahng (1969), Wuytack (1975), Beckermann and Labahn (2000), Tan and Fang (2000) and the references therein. A unified framework, which relates the rational interpolation problem with the Euclidean algorithm, is presented in Antoulas (1988), and also in the book von zur Gathen and Gerhard (2003, Section 5.7), where it is called *rational function reconstruction*. In Theorem 2.2 below, we translate these results to the subresultants context, which enables us to obtain some explicit expressions in terms of the input data for both problems.

For the Cauchy interpolation problem, there exists an explicit closed formula in terms of the input data that can be derived from the results on symmetric operators in a suitable ring of polynomials presented in Lascoux (2003), as shown in Lascoux (2013). Theorem 3.1 recovers this expression from the relationship between subresultants and the Sylvester sums introduced in Sylvester (1853), see also Lascoux and Pragacz (2003), D'Andrea et al. (2007, 2009), Roy and Szpirglas (2011), Krick and Szanto (2012).

We also present in Theorem 4.2 an explicit determinantal expression for the solution of the osculatory rational interpolation problem in terms of the input data, giving it as a quotient of determinants of generalized Vandermonde-type (and Wronskian-type) matrices. This generalizes straight-forwardly the corresponding known determinantal expression for the classical Hermite interpolation problem, setting another unified framework for all these interpolation problems. As mentioned in Remark 4.4 below, this determinantal expression can actually also be derived following the work of Beckermann and Labahn (2000), as we concluded from a recent useful discussion with George Labahn.

Since no closed formula for subresultants in terms of roots with multiplicities is known yet—except for very few exceptions, see D'Andrea et al. (2013)—a generalization of Theorem 3.1 to the osculatory rational interpolation problem is still missing, and some more work on the subject must be done in order to shed light to the problem.

2. Subresultants and the rational interpolation problem

Let us start by showing that a solution A/B for the rational interpolation problem, when it exists, is unique.

Proposition 2.1. *If the osculatory rational interpolation problem (2) has a solution, then there exists a unique pair (A, B) with $\gcd(A, B) = 1$ and A monic such that A/B is a solution.*

Proof. If there is a solution, then, cleaning common factors and dividing by the leading coefficient of A, there is a solution satisfying the same degree bounds with $\gcd(A, B) = 1$ and A monic. Assume

A_1/B_1 and A_2/B_2 are both solutions of the same type. Then, $(A_1/B_1)^{(j)}(x_i) = (A_2/B_2)^{(j)}(x_i)$ implies

$$\left(\frac{A_1B_2 - A_2B_1}{B_1B_2}\right)^{(j)}(x_i) = 0 \quad \text{for } 0 \le i \le k,\ 0 \le j < a_i,$$

which inductively implies that $(A_1B_2 - A_2B_1)^{(j)}(x_i) = 0$ for the $\ell + 1$ conditions. But $A_1B_2 - A_2B_1$ is a polynomial of degree at most ℓ, and therefore $A_1B_2 = A_2B_1$. Therefore, $A_1 = cA_2$ and $B_1 = cB_2$ with $c \in K \setminus \{0\}$. Both A_1 and A_2 are monic, so $c = 1$ and the claim follows. $\square$

Our results are consequences of interpreting the rational interpolation problem in terms of conditions of subresultants of the following two polynomials:

- $f := \prod_{j=0}^{k}(x - x_j)^{a_j}$, which we write $f = \sum_{i=0}^{\ell+1} f_i x^i$. Note that $f_{\ell+1} = 1$.
- $g = \sum_{i=0}^{\ell} g_i x^i \in K[x]$, the Hermite interpolation polynomial satisfying $g^{(j)}(x_i) = y_{i,j}$ for $0 \le i \le k$, $0 \le j < a_i$ (where we assume $g_i = 0$ for $\deg(g) < i \le \ell$ in case $\deg(g) < \ell$).

We can assume in what follows that at least one of the $y_{i,j}$ is non-zero, as otherwise the solution of the rational interpolation problem is the 0 function.

For $d \le \ell$, consider the d-th subresultant polynomial $\mathrm{Sres}_d(f, g)$ of f and g, defined as

$$\mathrm{Sres}_d(f,g) := \det \begin{array}{cl} 2\ell+1-2d & \\ \left|\begin{array}{ccccc} f_{\ell+1} & \cdots & \cdots & f_{d+1-(\ell-d-1)} & x^{\ell-d-1}f(x) \\ & \ddots & & \vdots & \vdots \\ & & f_{\ell+1}\ \cdots & f_{d+1} & x^0 f(x) \\ \hline g_\ell & \cdots & \cdots & g_{d+1-(\ell-d)} & x^{\ell-d}g(x) \\ & \ddots & & \vdots & \vdots \\ & & g_\ell\ \cdots & g_{d+1} & x^0 g(x) \end{array}\right| & \begin{array}{l} \ell-d \\ \\ \\ \ell+1-d \end{array} \end{array} . \tag{3}$$

Note that the previous definition makes sense even if $\deg(g) = m < \ell$, and agrees for $d \le m$ with the usual definition of subresultant of f and g given by the matrix of the right size $\ell + 1 + m - 2d$, since f is monic. For $m < d < \ell$ we have, according to the definition above, that $\mathrm{Sres}_d(f, g) = 0$, and for $d = \ell$, $\mathrm{Sres}_\ell(f, g) = g = \mathrm{Sres}_m(f, g)$.

We have the universal subresultant Bézout identity

$$\mathrm{Sres}_d(f,g) = F_d f + G_d g, \tag{4}$$

where

$$F_d := \det \begin{array}{cl} 2\ell+1-2d & \\ \left|\begin{array}{ccccc} f_{\ell+1} & \cdots & \cdots & f_{d+1-(\ell-d-1)} & x^{\ell-d-1} \\ & \ddots & & \vdots & \vdots \\ & & f_{\ell+1}\ \cdots & f_{d+1} & x^0 \\ \hline g_\ell & \cdots & \cdots & g_{d+1-(\ell-d)} & 0 \\ & \ddots & & \vdots & \vdots \\ & & g_\ell\ \cdots & g_{d+1} & 0 \end{array}\right| & \begin{array}{l} \ell-d \\ \\ \\ \ell+1-d \end{array} \end{array} , \tag{5}$$

and

$$G_d := \det \begin{array}{cl} 2\ell+1-2d & \\ \left|\begin{array}{cccccc} f_{\ell+1} & \cdots & & \cdots & f_{d+1-(\ell-d-1)} & 0 \\ & \ddots & & & \vdots & \vdots \\ & & f_{\ell+1} & \cdots & f_{d+1} & 0 \\ \hline g_\ell & \cdots & & \cdots & g_{d+1-(\ell-d)} & x^{\ell-d} \\ & \ddots & & & \vdots & \vdots \\ & & g_\ell & \cdots & g_{d+1} & x^0 \end{array}\right| & \begin{array}{l} \ell-d \\ \\ \\ \ell+1-d \end{array} \end{array} . \qquad (6)$$

Observe that $\deg(G_d) \le \ell - d$, if $G_d \ne 0$.

The result below expresses the existence and uniqueness of the solution of the osculatory rational interpolation problem in terms of the subresultant sequence of f and g.

Theorem 2.2. *With notation as above, let $0 \le d \le a$ be the maximal index such that $\mathrm{Sres}_d(f,g) \ne 0$. Then $\deg(G_d) \le b$ and the osculatory rational interpolation problem (2) has a solution if and only if $G_d(x_i) \ne 0$ for $1 \le i \le k$. In that case the solution is given by*

$$\frac{A}{B} = \frac{\mathrm{Sres}_d(f,g)}{G_d},$$

where moreover $\gcd(\mathrm{Sres}_d(f,g), G_d) = 1$.

This result is strongly related to Theorem 5.16 from von zur Gathen and Gerhard (2003), which expresses the existence and uniqueness of the solution of the osculatory rational interpolation problem in terms of the Extended Euclidean Scheme for f and g. We recall its statement below, as well as Lemma 5.15 and a consequence of Lemma 3.15(v) of the same reference.

Theorem 2.3 *(von zur Gathen and Gerhard (2003, Theorem 5.16, Lemmas 5.15 and 3.15(v))). With notation as above, let $r_i = s_i f + t_i g$, $i \ge 0$, be the successive remainders in the Extended Euclidean Scheme for f and g, and s_i, t_i the corresponding Bézout coefficients.*

(1) *The osculatory rational interpolation problem (2) has a solution A/B if and only if the minimal row $r_j = s_j f + t_j g$ such that $d_j := \deg(r_j) \le a$ satisfies $\gcd(r_j, t_j) = 1$. If this is the case, $A/B = r_j/t_j$ is the solution (and in particular $\deg(t_j) \le b$).*
(2) *Let $r = sf + tg \ne 0$ be such that $\deg(r_j) \le \deg(r) < \deg(r_{j-1})$ and $\deg(r) + \deg(t) < \ell + 1 = \deg(f)$. Then there exists $c \in K$ such that $r = cr_j$, $s = cs_j$, $t = ct_j$. Moreover, $\gcd(s,t) = 1$.*

Proof of Theorem 2.2. We consider the minimal j in the Extended Euclidean Scheme such that $d_j := \deg(r_j) \le a$: by Theorem 2.3(1), there is a solution $A/B = r_j/t_j$ to our problem if and only if $\gcd(r_j, t_j) = 1$. Observe that for $d_{j-1} := \deg(r_{j-1})$ we have $a < d_{j-1}$, i.e. $d_j \le a < d_{j-1}$.

Let $d \le a$ be the largest such that $\mathrm{Sres}_d(f,g) \ne 0$. One has $\mathrm{Sres}_d(f,g) = F_d f + G_d g$ with $\deg(\mathrm{Sres}_d(f,g)) + \deg(G_d) \le \ell < \ell + 1 = \deg(f)$. Moreover, by the Fundamental Theorem of Polynomial Remainder Sequences, Collins (1967), Brown and Traub (1971) or Geddes et al. (1996, Theorem 7.4), $\mathrm{Sres}_{d_j}(f,g)$ and $\mathrm{Sres}_{d_{j-1}-1}(f,g)$ are (non-zero) constant multiples of r_j (and $\mathrm{Sres}_{d'}(f,g) = 0$ for $d_j < d' < d_{j-1} - 1$). This implies that $d_j \le d < d_{j-1}$. Therefore, applying Theorem 2.3(2), there exists $c \in K^\times$ such that

$$\mathrm{Sres}_d(f,g) = cr_j, \qquad F_d = cs_j \quad \text{and} \quad G_d = ct_j$$

with $\gcd(F_d, G_d) = 1$. This implies, by the definition of f,

$$\gcd\big(\mathrm{Sres}_d(f,g), G_d\big) = 1 \quad \Longleftrightarrow \quad G_d(x_i) \ne 0 \quad \text{for } 0 \le i \le k.$$

This concludes the proof. $\square$

Remark 2.4. In the statement of Theorem 2.2, one can replace the hypothesis *"let $0 \le d \le a$ be the maximal index such that* $\mathrm{Sres}_d(f, g) \neq 0$*"* by *"let $a \le d \le \ell$ be the minimal index such that* $\mathrm{Sres}_d(f, g) \neq 0$*"*. This is due to the Fundamental Theorem of Polynomial Remainder Sequences mentioned in the previous proof, since if $\mathrm{Sres}_a(f, g) = 0$, then one has that for $\mathrm{Sres}_k(f, g)$ and $\mathrm{Sres}_j(f, g)$ coincide up to a non-zero constant, for the maximal $k < a$ such that $\mathrm{Sres}_k(f, g) \neq 0$ and the minimal $j > a$ such that $\mathrm{Sres}_j(f, g) \neq 0$. Accordingly, one can replace the corresponding hypothesis in Theorems 3.1 and 4.2 below.

Theorem 2.2 has the advantage that it can be applied to produce explicit formulae for the Cauchy and the osculatory rational interpolation problems in terms of the input data, as we show in the next sections.

3. The Cauchy interpolation problem formula

We now present the closed expression in terms of the data for the Cauchy interpolation problem. For $U, V \subset K$, we set $R(U, V) := \prod_{u\in U, v\in V}(u - v)$.

Theorem 3.1. *Given (a, b), $X := \{x_0, \dots, x_\ell\}$ and $y_0, \dots, y_\ell$ as in Problem (1), let d be maximal such that $0 \le d \le a$ and*

$$A_0 := \sum_{X'\subset X, |X'|=d} R\big(x, X'\big)\Big(\prod_{x_j\notin X'} y_j\Big)/R\big(X\setminus X', X'\big) \in K[x]$$

is not identically zero. Set

$$B_0 := \sum_{X''\subset X, |X''|=\ell-d} R\big(X'', x\big)\Big(\prod_{x_j\in X''} y_j\Big)/R\big(X'', X\setminus X''\big) \in K[x].$$

Then $\deg(B_0) \le b$ and a solution $\frac{A}{B}$ for the Cauchy interpolation problem (1) exists if and only if $B_0(x_i) \neq 0$ for $0 \le i \le \ell$. In that case the solution is given by

$$\frac{A}{B} = \frac{A_0}{B_0}.$$

Proof. Let as before $f = \prod_{i=0}^{\ell}(x - x_i)$, and g be the unique polynomial of degree bounded by ℓ which satisfies $g(x_i) = y_i$ for $0 \le i \le \ell$. Denote by Z the set of roots of g in $\overline{K}$, the algebraic closure of K.

Let d be maximal such that $0 \le d \le a$ and $\mathrm{Sres}_d(f, g) \neq 0$. We apply Theorem 2.2 and Sylvester's single-sum formula in roots for $\mathrm{Sres}_d(f, g)$ (see for instance the original paper of Sylvester (1853, Art. 21) or the many other references on the topic) and for G_d (Sylvester (1853, Art. 29), or Krick and Szanto (2012, remark after Lemma 6)):

$$\begin{aligned}
\mathrm{Sres}_d(f, g) &= \sum_{|X'|=d} R\big(x, X'\big)\frac{R(X\setminus X', Z)}{R(X\setminus X', X')} = \sum_{|X'|=d} R\big(x, X'\big)\frac{\prod_{x_j\notin X'} g(x_j)}{R(X\setminus X', X')} \\
&= \sum_{|X'|=d} R\big(x, X'\big)\frac{\prod_{x_j\notin X'} y_j}{R(X\setminus X', X')} = A_0, \\
G_d &= (-1)^{\ell-d} \sum_{|X''|=\ell-d} R\big(x, X''\big)\frac{R(X'', Z)}{R(X'', X\setminus X'')} \\
&= \sum_{|X''|=\ell-d} R\big(X'', x\big)\frac{R(X'', Z)}{R(X'', X\setminus X'')} = \sum_{|X''|=\ell-d} R\big(X'', x\big)\frac{\prod_{x_j\in X''} g(x_j)}{R(X'', X\setminus X'')}
\end{aligned}$$

$$= \sum_{|X''|=\ell-d} R\left(X'', x\right) \frac{\prod_{x_j \in X''} y_j}{R(X'', X \setminus X'')} = B_0,$$

where both $X', X'' \subset X$. The claim follows from Theorem 2.2. □

Remark 3.2. Observe that when $a = \ell$ then $\mathrm{Sres}_\ell(f, g) = g \neq 0$ and Theorem 3.1 specializes to the well-known *Lagrange interpolation polynomial* associated to the data $\{(x_i, y_i)\}_{0 \le i \le \ell}$, that is

$$\frac{A_0}{B_0} = \sum_{0 \le i \le \ell} y_i \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} = \sum_{0 \le i \le \ell} y_i \frac{R(x, X \setminus \{x_i\})}{R(x_i, X \setminus \{x_i\})}.$$

The gap $d < a$ in Theorem 2.2 may appear, as the following example shows.

Example 3.3. We consider the Cauchy interpolation problem with $a = 3$, $b = 2$, and the associated input data

$$X = (x_0, \ldots, x_5) \quad \text{where } x_0, \ldots, x_5 \text{ are the 6 different roots of } x^6 - 1 \text{ in } \overline{K},$$
$$Y = (y_0, \ldots, y_5) \quad \text{with } y_i = x_i^5 + 2 \text{ for } 0 \le i \le 5,$$

for a field K of characteristic $\neq 2, 3$. In this case we have

$$f = x^6 - 1 \quad \text{and} \quad g = x^5 + 2.$$

An explicit computation shows that $\mathrm{Sres}_3(f, g) = \mathrm{Sres}_2(f, g) = 0$. However,

$$\mathrm{Sres}_1(f, g) = 8 + 16x, \qquad F_1 = -8, \qquad G_1 = 8x.$$

We easily verify that $G_1(x_i) \neq 0$ for $0 \le i \le 5$. Hence by Theorem 2.2, $d = 1$ and

$$\frac{A}{B} = \frac{8 + 16x}{8x} = \frac{1 + 2x}{x}$$

is the solution to this Cauchy interpolation problem, which can be checked straightforwardly since

$$\frac{1 + 2x_i}{x_i} = \frac{1}{x_i} + 2 = x_i^5 + 2 = y_i, \quad i = 0, \ldots, 5.$$

4. The osculatory rational interpolation formula

Before stating our main result for the osculatory rational interpolation problem, we need to set a notation.

Notation 4.1. *Set $a_0, \ldots, a_k \in \mathbb{N}$ such that $a_0 + \cdots + a_k = \ell + 1$, as in Problem (2). We define*

- *$\overline{X} := ((x_0, a_0); \ldots; (x_k, a_k))$ an array of pairs in $K \times \mathbb{N}$ and $Y := (\boldsymbol{y}_0, \ldots, \boldsymbol{y}_k)$ where $\boldsymbol{y}_i = (y_{i,0}, \ldots, y_{i,a_i-1})$. We call $(\overline{X}, Y)$ the* input data *for the osculatory rational interpolation problem.*
- *Set $u \in \mathbb{N}$. The* generalized Vandermonde *or* confluent matrix *(e.g. Kalman (1984)) of size $u + 1$ associated to $\overline{X}$ is the (non-necessarily square) matrix $V_{u+1}(\overline{X}) \in K^{(u+1)\times(\ell+1)}$ defined by*

$$V_{u+1}(\overline{X}) := \begin{array}{c} \ell+1 \\ \boxed{\begin{array}{c|c|c} V_{u+1}(x_0, a_0) & \ldots & V_{u+1}(x_k, a_k) \end{array}} \end{array} u+1\, ,$$

where for any t, $V_{u+1}(x_i, t+1) \in K^{(u+1)\times(t+1)}$ is defined by

$$V_{u+1}(x_i,t+1) := \begin{array}{c} t+1 \\ \begin{array}{|ccccc|} \hline 1 & 0 & 0 & \dots & 0 \\ x_i & 1 & 0 & \dots & 0 \\ x_i^2 & 2x_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_i^u & ux_i^{u-1} & \binom{u}{2}x_i^{u-2} & \dots & \binom{u}{t}x_i^{u-t} \\ \hline \end{array} \end{array} u+1$$

- *We define the matrix $U_{u+1}(\overline{X}, Y) \in K^{(u+1)\times(\ell+1)}$ associated to $\overline{X}$ and Y as:*

$$U_{u+1}(\overline{X},Y) := \begin{array}{c} \ell+1 \\ \begin{array}{|c|c|c|} \hline U_{u+1}(x_0;\boldsymbol{y}_0) & \dots & U_{u+1}(x_k;\boldsymbol{y}_k) \\ \hline \end{array} \end{array} u+1,$$

where for any t, $U_{u+1}(x_i, \boldsymbol{y}_i) \in K^{(u+1)\times(t+1)}$ is defined by

$$U_{u+1}(x_i,\boldsymbol{y}_i) = \begin{array}{c} t+1 \\ \begin{array}{|cccc|} \hline y_{i,0} & y_{i,1} & \dots & y_{i,t} \\ y_{i,0}x_i & y_{i,1}x_i + y_{i,0} & \dots & y_{i,t}x_i + y_{i,t-1} \\ \vdots & \vdots & & \vdots \\ y_{i,0}x_i^u & y_{i,1}x_i^u + uy_{i,0}x_i^{u-1} & \dots & \sum_{j=0}^{t}\binom{u}{j}y_{i,t-j}x_i^{u-j} \\ \hline \end{array} \end{array} u+1,$$

where

$$\big(U_{u+1}(x_i,\boldsymbol{y}_i)\big)_{k+1,l+1} = \sum_{j=0}^{l}\binom{k}{j} y_{i,l-j}x_i^{k-j},$$

with the convention that when $u < j$, $\binom{u}{j} = 0$.

The next determinantal expression presents the solution of the osculatory rational interpolation problem in terms of the input data as follows:

Theorem 4.2. *Under the notation above, let d be maximal such that $0 \le d \le a$ and*

$$A_0 := -\det \begin{array}{c} \begin{array}{cc} \ell+1 & 1 \end{array} \\ \begin{array}{|c|c|} \hline V_{d+1}(\overline{X}) & \begin{array}{c} 1 \\ \vdots \\ x^d \end{array} \\ \hline U_{\ell-d+1}(\overline{X},Y) & \mathbf{0} \\ \hline \end{array} \end{array} \begin{array}{l} d+1 \\ \\ \ell-d+1 \end{array} \in K[x] \tag{7}$$

is not identically zero. Set

$$B_0 := \det \begin{array}{c} \begin{array}{cc} \ell+1 & 1 \end{array} \\ \begin{array}{|c|c|} \hline V_{d+1}(\overline{X}) & \mathbf{0} \\ \hline U_{\ell-d+1}(\overline{X},Y) & \begin{array}{c} 1 \\ \vdots \\ x^{\ell-d} \end{array} \\ \hline \end{array} \end{array} \begin{array}{l} d+1 \\ \\ \ell-d+1 \end{array} \in K[x]. \tag{8}$$

Then $\deg(B_0) \le b$, *and a solution* $\frac{A}{B}$ *for the osculatory rational interpolation problem* (2) *exists if and only if* $B_0(x_i) \neq 0$ *for* $0 \le i \le k$. *In that case the solution is given by*

$$\frac{A}{B} = \frac{A_0}{B_0}.$$

To prove this result we need the following lemma, that we prove at the end of the section.

Lemma 4.3. *Let* $d \le \ell$. *Then*

$$\mathrm{Sres}_d(f,g) = (-1)^{\ell+1-d}\det\big(V_{\ell+1}(\overline{X})\big)^{-1}\det \begin{array}{cc} \begin{array}{cc} \ell+1 & \quad 1 \end{array} & \\ \left|\begin{array}{c|c} V_{d+1}(\overline{X}) & \begin{matrix} 1 \\ \vdots \\ x^d \end{matrix} \\ \hline U_{\ell-d+1}(\overline{X},Y) & \mathbf{0} \end{array}\right| & \begin{array}{c} d+1 \\ \\ \ell-d+1 \end{array} \end{array}$$

and

$$G_d = (-1)^{\ell-d}\det\big(V_{\ell+1}(\overline{X})\big)^{-1}\det \begin{array}{cc} \begin{array}{cc} \ell+1 & \quad 1 \end{array} & \\ \left|\begin{array}{c|c} V_{d+1}(\overline{X}) & \mathbf{0} \\ \hline U_{\ell-d+1}(\overline{X},Y) & \begin{matrix} 1 \\ \vdots \\ x^{\ell-d} \end{matrix} \end{array}\right| & \begin{array}{c} d+1 \\ \\ \ell+1-d \end{array} \end{array}.$$

Proof of Theorem 4.2. The maximal index $d \le a$ such that $A_0 \neq 0$ clearly coincides with the maximal index $d \le a$ such that $\mathrm{Sres}_d(f,g) \neq 0$, since these two quantities only differ by a non-zero constant. Analogously, $G_d(x_i) \neq 0 \Leftrightarrow B_0(x_i) \neq 0$. Finally, $\frac{A_0}{B_0} = \frac{\mathrm{Sres}_d(f,g)}{G_d}$. $\square$

Remark 4.4. As we checked after a useful discussion with George Labahn at the 2013 SIAM Conference on Applied Algebraic Geometry, the matrix formulations for A_0 and B_0 in Theorem 4.2 can actually be derived from the *Mahler systems* introduced by Beckermann and Labahn (2000) to solve a more general class of problems. Indeed, by translating our situation into their general framework (see Beckermann and Labahn (2000, Example 2.3)), and using the standard Hermite dual basis to produce their matrices, it can be seen that the determinants appearing in the right hand side of (7) and (8) coincide with those defining $p^{(\ell)}(\vec{n}, z)$ in Beckermann and Labahn (2000, Section 5).

Remark 4.5. Let us note that in particular, Problem (2) for $a = \ell$ corresponds to the ordinary *Hermite interpolation problem*, i.e. the determination of the Hermite interpolation polynomial g associated to the input data

$$\overline{X} = \big((x_0, a_0), \ldots, (x_k, a_k)\big), \qquad Y = (\boldsymbol{y}_0, \ldots, \boldsymbol{y}_k) \quad \text{where } \boldsymbol{y}_i = (y_{i,j})_{0 \le j < a_i},$$

which is the unique polynomial of degree less than or equal to ℓ such that

$$g^{(j)}(x_i) = j!\,y_{i,j}, \quad 0 \le i \le k,\ 0 \le j < a_i.$$

In this case, Theorem 4.2 specializes to the well-known determinantal expression for the polynomial g, that is

$$g=-\frac{\det\left[\begin{array}{c|c} V_{\ell+1}(\overline{X}) & \begin{matrix}1\\ \vdots\\ x^{\ell}\end{matrix} \\ \hline Y & 0\end{array}\right]}{\det(V_{\ell+1}(\overline{X}))},$$

setting in this way a unified determinantal framework for polynomial and rational interpolation problems.

Remark 4.6. We remark that for the Cauchy interpolation problem (1), the solution described by Theorem 4.2 gives

$$\frac{A}{B}=-\frac{\begin{array}{cc} \begin{array}{ccc|c} & \ell+1 & & 1 \end{array} & \\ \det\left[\begin{array}{ccc|c} 1 & \cdots & 1 & 1\\ \vdots & & \vdots & \vdots\\ x_0^d & \cdots & x_\ell^d & x^d\\ \hline y_0 & \cdots & y_\ell & 0\\ \vdots & & \vdots & \vdots\\ y_0x_0^{\ell-d} & \cdots & y_\ell x_\ell^{\ell-d} & 0 \end{array}\right] & \begin{array}{l} d+1 \\ \\ \ell-d+1 \end{array} \end{array}}{\begin{array}{cc} \begin{array}{ccc|c} & \ell+1 & & 1 \end{array} & \\ \det\left[\begin{array}{ccc|c} 1 & \cdots & 1 & 0\\ \vdots & & \vdots & \vdots\\ x_0^d & \cdots & x_\ell^d & 0\\ \hline y_0 & \cdots & y_\ell & 1\\ \vdots & & \vdots & \vdots\\ y_0x_0^{\ell-d} & \cdots & y_\ell x_\ell^{\ell-d} & x^{\ell-d} \end{array}\right] & \begin{array}{l} d+1 \\ \\ \ell-d+1 \end{array} \end{array}}.$$

Example 4.7. We consider the osculatory rational interpolation problem with $a=b=2$, $k=2$, and the associated input data

$$\overline{X}=\big((x_0,a_0),(x_1,a_1)\big)\quad\text{with } (x_0,a_0)=(1,2),\ (x_1,a_1)=(2,3)$$
$$Y=(\boldsymbol{y}_0,\boldsymbol{y}_1)\quad\text{with } \boldsymbol{y}_0=(y_{0,0},y_{0,1})=(2,3),\ \boldsymbol{y}_1=(y_{1,0},y_{1,1},y_{1,2})=(6,7,8).$$

We have

$$f=(x-1)^2(x-2)^3\quad\text{and}\quad g=-8+23x-20x^2+8x^3-x^4.$$

By explicit computation, we get

$$\mathrm{Sres}_2(f,g)=35x-25x^2,\qquad F_2=-25+5x,\qquad G_2=25-25x+5x^2.$$

We easily verify that $G_2(x_i)\neq 0$ for $i=1,2$ if $5\neq 0$ in K. Hence by Theorem 2.2, $d=a=2$, and

$$\frac{A}{B}=\frac{35x-25x^2}{25-25x+5x^2}=\frac{7x-5x^2}{5-5x+x^2}$$

is the solution to the rational interpolation problem, which can be checked straightforwardly.

Proof of Lemma 4.3. By D'Andrea et al. (2013, Theorem 2.5),

$$\mathrm{Sres}_d(f,g)=(-1)^{\ell+1-d}\det\big(V_{\ell+1}(\overline{X})\big)^{-1}\det\begin{array}{cc} \begin{array}{cc}\quad\ell+1\quad & 1\end{array} & \\ \left[\begin{array}{c|c} V_{d+1}(\overline{X}) & \begin{matrix}1\\ \vdots\\ x^d\end{matrix} \\ \hline W_{g,\ell-d+1}(\overline{X}) & \mathbf{0}\end{array}\right] & \begin{array}{c} d+1 \\[2ex] \ell-d+1\end{array}\end{array},$$

where $W_{g,\ell-d+1}(\overline{X})$ is the *generalized Wronskian* of size $\ell-d+1$ associated to $\overline{X}$, i.e. the matrix

$$W_{g,\ell-d+1}(\overline{X}):=\begin{array}{cc} \ell+1 & \\ \left[\begin{array}{c|c|c} W_{g,\ell-d+1}(x_0,a_0) & \dots & W_{g,\ell-d+1}(x_k,a_k)\end{array}\right] & \ell-d+1\end{array}\in K^{(\ell-d+1)\times(\ell+1)}$$

where

$$W_{g,\ell-d+1}(x_i,a_i):=\begin{array}{cc} a_i & \\ \left[\begin{array}{cccc} g(x_i) & g'(x_i) & \dots & \frac{g^{(a_i-1)}(x_i)}{(a_i-1)!} \\ (xg)(x_i) & (\xi g)'(x_i) & \dots & \frac{(xg)^{(a_i-1)}(x_i)}{(a_i-1)!} \\ \vdots & \vdots & & \vdots \\ (x^{\ell-d}g)(x_i) & (x^{\ell-d}g)'(x_i) & \dots & \frac{(x^{\ell-d}g)^{(a_i-1)}(x_i)}{(a_i-1)!}\end{array}\right] & \ell-d+1\end{array}\in K^{(\ell-d+1)\times a_i}.$$

In the same way, we prove now that

$$G_d=(-1)^{\ell-d}\det\big(V_{\ell+1}(\overline{X})\big)^{-1}\det\begin{array}{cc} \begin{array}{cc}\quad\ell+1\quad & 1\end{array} & \\ \left[\begin{array}{c|c} V_{d+1}(\overline{X}) & \mathbf{0} \\ \hline W_{g,\ell-d+1}(\overline{X}) & \begin{matrix}1\\ \vdots\\ x^{\ell-d}\end{matrix}\end{array}\right] & \begin{array}{c} d+1 \\[2ex] \ell+1-d\end{array}\end{array}.$$

For that, we consider the following matrices:

$$M_f:=\begin{array}{cc} 2\ell-d+2 & \\ \left[\begin{array}{ccccc|c} f_0 & \dots & f_{\ell+1} & & & 0 \\ & \ddots & & \ddots & & \vdots \\ & & f_0 & \dots & f_{\ell+1} & 0\end{array}\right] & \ell-d\end{array},$$

$$M_g:=\begin{array}{cc} 2\ell-d+2 & \\ \left[\begin{array}{ccccc|c} g_0 & \dots & g_\ell & & & x^0 \\ & \ddots & & \ddots & & \vdots \\ & & g_0 & \dots & g_\ell & x^{\ell-d}\end{array}\right] & \ell+1-d\end{array}$$

and

$$U_d := \begin{array}{cl} 2\ell-d+2 & \\ \left[\begin{array}{c} I_{d+1} \\ \hline M_f \\ \hline M_g \end{array}\right] & \begin{array}{l} d+1 \\ \ell-d \\ \ell+1-d \end{array} \end{array},$$

where I_{d+1} is the $(d+1)\times(2\ell-d+2)$ matrix with the identity matrix on the left and zero otherwise. Then from the definition of G_d we have that

$$G_d = \det(U_d).$$

Also, similarly as in the proof of D'Andrea et al. (2013, Theorem 2.5), we have

$$\begin{array}{rc} & 2\ell-d+2 \\ \begin{array}{r} d+1 \\ \ell-d \\ \ell+1-d \end{array} & \left[\begin{array}{c} I_{d+1} \\ \hline M_f \\ \hline M_g \end{array}\right] \end{array} \begin{array}{cl} \begin{array}{cc} \ell+1 & \ell-d+1 \end{array} & \\ \left[\begin{array}{c|c} V_{2\ell-d+1}(\overline{X}) & \mathbf{0} \\ \hline 0 & \mathrm{Id}_{\ell-d+1} \end{array}\right] & \begin{array}{l} \ell+1 \\ \ell-d+1 \end{array} \end{array}$$

$$= \begin{array}{cl} \begin{array}{ccc} \ell+1 & \ell-d & 1 \end{array} & \\ \left[\begin{array}{c|c|c} V_{d+1}(\overline{X}) & * & 0 \\ \hline \mathbf{0} & M'_f & 0 \\ \hline W_{g,\ell+1-d}(\overline{X}) & * & \begin{array}{c} 1 \\ \vdots \\ x^{\ell-d} \end{array} \end{array}\right] & \begin{array}{l} d+1 \\ \ell-d \\ \ell+1-d \end{array} \end{array},$$

where M'_f is a triangular matrix with $f_{\ell+1} = 1$ in its diagonal. This shows the formula for G_d.

Finally, we simply show that $W_{g,\ell-d+1}(\overline{X}) = U_{\ell-d+1}(\overline{X}, Y)$ by computing the entries of $W_{g,\ell-d+1}(\overline{X})$: we apply Leibniz rule and the fact that $g^{(t-j)}(x_i) = (t-j)!y_{i,j}$ for $0 \le i \le k$ and $0 \le j < a_i$:

$$\frac{(x^u g)^{(t)}(x_i)}{t!} = \sum_{j=0}^{t} \binom{u}{j} x_i^{u-j} y_{i,t-j}. \qquad \square$$

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