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A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian

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Abstract

Our propose here is to provide a Hopf lemma and a strong minimum principle for weak supersolutions of

$$(-\Delta_p)^s u = c(x)|u|^{p-2}u \quad \text{in } \Omega$$

where Ω is an open set of $\mathbb{R}^N$, $s \in (0, 1)$, $p \in (1, +\infty)$, $c \in C(\overline{\Omega})$ and $(-\Delta_p)^s$ is the fractional p-Laplacian.

Keywords: Fractional p-Laplacian; Hopf's lemma; Strong minimum principle

1. Introduction

It is well known that the Hopf's lemma is one of the most useful and best known tool in the study of partial differential equations. Just to name a some of its applications, this lemma is crucial in the proofs of the strong maximum principle, and the anti-maximum principle and in

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the moving plane method. For a review on the topic in the local case, see for instance [27,28] and the references therein.

Our propose here is to provide a Hopf lemma and a strong minimum principle for the fractional p-Laplacian

$$(-\Delta_p)^s u(x) := 2\mathcal{K}(s,p,N) \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy \quad x\in\mathbb{R}^N$$

where $p\in(1,\infty)$, $s\in(0,1)$, and $\mathcal{K}(s,p,N)$ is a normalization factor. The fractional p-Laplacian is a nonlocal version of the p-Laplacian and is an extension of the fractional Laplacian ($p=2$).

In the last few years, the nonlocal operators have taken relevance because they arise in a number of applications in many fields, for instance, game theory, finance, image processing, Lévy processes, and optimization, see [7,9,16,3,14] and the references therein. From of the mathematical point of view, the fractional p-Laplacian has a great attractive since two phenomena are present in it: the nonlinearity of the operator and its nonlocal character. See for instance [10,5,6,12,21,23,24,31,19] and the references therein.

1.1. Statements of the main results

Before starting to state our results we need to introduce the theoretical framework for them.

Throughout this paper, Ω is an open set of $\mathbb{R}^N$, $s\in(0,1)$, $p\in(1,\infty)$ and to simplify notation, we omit the constant $\mathcal{K}(s,p,N)$. From now on, given a subset A of $\mathbb{R}^N$ we set $A^c=\mathbb{R}^N\setminus A$, and $A^2=A\times A$.

The fractional Sobolev spaces $W^{s,p}(\Omega)$ is defined to be the set of functions $u\in L^p(\Omega)$ such that

$$|u|^p_{W^{s,p}(\Omega)} := \int\limits_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy < \infty.$$

The fractional Sobolev spaces admit the following norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)}\right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\Omega)} := \int\limits_{\Omega} |u(x)|^p\,dx.$$

The space $W^{s,p}(\mathbb{R}^n)$ is defined similarly.

We will denote by $\widetilde{W}^{s,p}(\Omega)$ the space of all $u\in W^{s,p}(\Omega)$ such that $\tilde{u}\in W^{s,p}(\mathbb{R}^n)$, where $\tilde{u}$ is the extension by zero of u. The dual space of $\widetilde{W}^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ and the corresponding dual pairing is denoted by $\langle\cdot,\cdot\rangle$.

The space $\mathcal{W}^{s,p}(\Omega)$ is the space of all functions $u\in L^p_{loc}(\mathbb{R}^N)$ such that for any bounded $\Omega'\subseteq\Omega$ there is an open set $U\supset\supset\Omega'$ so that $u\in W^{s,p}(U)$, and

$$[u]_{s,p} := \int\limits_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} dx < \infty.$$

Remark 1.1. Suppose that Ω is bounded. Then $u \in \mathcal{W}^{s,p}(\Omega)$ if only if there is an open set $U \supset\supset \Omega$ such that $u \in W^{s,p}(U)$, and $[u]_{s,p} < \infty$. In addition, $\widetilde{W}^{s,p}(\Omega) \subset \mathcal{W}^{s,p}(\Omega)$.

Further informations on fractional Sobolev spaces and many references may be found in [1,11,13,18,21].

Now, let us introduce our notion of weak super(sub)-solution. Given $f \in W^{-s,p'}(\Omega)$, we say that $f \geq (\leq) 0$ if for any $\varphi \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \geq 0$ we have that $\langle f, \varphi \rangle \geq (\leq) 0$.

When Ω is bounded, we say that $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak super(sub)-solution of $(-\Delta_p)^s u = f$ in Ω if

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy \geq (\leq) \langle f, \varphi \rangle$$

for each $\varphi \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \geq 0$.

When Ω is unbounded, we say that $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak super(sub)-solution of $(-\Delta_p)^s u = f$ in Ω if for all bounded open set $\Omega' \subset \Omega$ we have that u is a weak super(sub)-solution of $(-\Delta_p)^s u = f$ in Ω'.

In both cases, u is a weak solution of $(-\Delta_p)^s u = f$ in Ω if u is a super and sub-solution of $(-\Delta_p)^s u = f$.

Given a function $c \in L^1_{loc}(\Omega)$, we say that $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak super(sub)-solution of $(-\Delta_p)^s u = c(x)|u|^{p-1}u$ in Ω if $f = c(x)|u|^{p-2}u \in W^{-s,p'}(\Omega)$, and $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak super(sub)-solution of $(-\Delta_p)^s u = f$ in Ω. Finally, we say that u is a weak solution of $(-\Delta_p)^s u = c(x)|u|^{p-2}u$ in Ω if u is a super and sub-solution of $(-\Delta_p)^s u = c(x)|u|^{p-2}u$.

Our first result is the following minimum principle.

Theorem 1.2. *Let $c \in L^1_{loc}(\Omega)$ be a non-positive function and $u \in \mathcal{W}^{s,p}(\Omega)$ be a weak super-solution of*

$$(-\Delta_p)^s u = c(x)|u|^{p-2}u \quad \text{in } \Omega. \tag{1.1}$$

1. *If Ω is bounded, and $u \geq 0$ a.e. in Ω^c then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.*
2. *If $u \geq 0$ a.e. in $\mathbb{R}^N$ then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.*

Remark 1.3. Observe that $c \in L^1_{loc}(\Omega)$ then $c(x) \geq c^-(x) = \min\{0, c(x)\} \in L^1_{loc}(\Omega)$. Then, if u is a weak super-solution of (1.1) and $u \geq 0$ a.e. in $\mathbb{R}^N$ then u is also a weak super-solution of

$$(-\Delta_p)^s u = c^-(x)u^{p-1} \quad \text{in } \Omega.$$

Then, by Theorem 1.4, $u > 0$ a.e. in Ω or $u = 0$ a.e. in $\mathbb{R}^N$. That is, in the case $u \geq 0$ a.e. in $\mathbb{R}^n$, the non-positivity assumption on the function c is not necessary.

In fact, the previous result also holds for all measurable functions c for which there is $d \in L^1_{loc}(\Omega)$ such that $c \geq d$ a.e. in Ω.

Under the assumption that c and u also are continuous functions, by the properties that all continuous weak super-solutions are viscosity super-solutions and using a test function, we can remove “a.e.” in the statement of our previous theorem. For more details, see Section 2.

Theorem 1.4. *Let $c \in C(\overline{\Omega})$ be a non-positive function and $u \in \mathcal{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of (1.1).*

1. *If Ω is bounded, and $u \geq 0$ a.e. in Ω^c then either $u > 0$ in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.*
2. *If $u \geq 0$ a.e. in $\mathbb{R}^N$ then either $u > 0$ in Ω or $u = 0$ a.e. in $\mathbb{R}^N$.*

Lastly, we show our Hopf lemma.

Theorem 1.5. *Let Ω satisfy the interior ball condition in $x_0 \in \partial\Omega$, $c \in C(\overline{\Omega})$, and $u \in \mathcal{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a weak super-solution of (1.1).*

1. *If Ω is bounded, $c(x) \leq 0$ in Ω and $u \geq 0$ a.e. in Ω^c then either $u = 0$ a.e. in $\mathbb{R}^N$ or*
$$\liminf_{B_R \ni x \to x_0} \frac{u(x)}{\delta_R(x)^s} > 0 \tag{1.2}$$
where $B_R \subseteq \Omega$ and $x_0 \in \partial B_R$ and $\delta_R(x)$ is distance from x to B_R^c.
2. *If $u \geq 0$ a.e. in $\mathbb{R}^N$ then either $u = 0$ a.e. in $\mathbb{R}^N$ or (1.2) holds.*

Now, we give a brief resume about the Hopf's lemma and the strong minimum principle for the fractional Laplacian. In [8, Proposition 2.7] the authors show the strong minimum principle and a generalized Hopf lemma for fractional harmonic functions. Whereas, in [30], under the assumption Ω is a smooth bounded domain, it is proven a Hopf lemma for weak solutions of a Dirichlet problem. See also [15,29]. For a Hopf lemma with mixed boundary condition, see [2].

Finally, Theorems 1.4 and 1.5 are known for the fractional Laplacian only for pointwise solutions, see [17]. See also [20] for $p = 2$ and [26] for $p \neq 2$. Thus, our results generalize the results of [17] in two way: for nonlinear operators and weak solutions.

To complete the introduction, we want to make a little remark related to our result and the optimal regularity of the Dirichlet problem. Given $f \in L^\infty(\Omega)$, if $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and u is a weak solution of

$$(-\Delta_p)^s u = f(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \Omega^c,$$

then, by [21, Theorem 1.1], there is $\alpha = \alpha(N, s, p) \in (0, 1)$ such that $u \in C^\alpha(\overline{\Omega})$. In fact that, we cannot expect more than s-Hölder continuity, see [21, Section 3].

Also, by Theorem 1.5, we can deduce that $\alpha \leq s$. Suppose that there exists a function $c \in C(\overline{\Omega})$ such that $c \leq 0$ in Ω and $c(x)|u(x)|^{p-2}u(x) \leq f(x)$ (for instance, if $f \geq 0$ we can take $c \equiv 0$). Then u is a weak super-solution of

$$(-\Delta_p)^s u = c(x)|u(x)|^{p-2}u(x) \quad \text{in } \Omega.$$

Thus, by Theorem 1.5, $\alpha \leq s$.

2. Preliminaries

Let's start by introducing the notations and definitions that we will use in this work. We also gather some preliminaries properties which will be useful in the forthcoming sections.

If $t \in \mathbb{R}$ and $q > 0$, we will denote $|t|^{q-1}t$ by t^q. For all functions $u\colon \Omega \to \mathbb{R}$ we define

$$u_+(x) := \max\{u(x), 0\} \quad \text{and} \quad u_-(x) := \max\{-u(x), 0\},$$
$$\Omega_+ := \{x \in \Omega\colon u(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega\colon u(x) < 0\}.$$

Our next remark shows that u_+ and u_- belong to the same space as u.

Remark 2.1. If $\mathcal{X} = W^{s,p}(\Omega)$ or $\widetilde{W}^{s,p}(\Omega)$ or $\mathcal{W}^{s,p}(\Omega)$, and $u \in \mathcal{X}$ then $u_+, u_- \in \mathcal{X}$ owing to

$$|u_-(x) - u_-(y)| \le |u(x) - u(y)| \quad \text{and} \quad |u_+(x) - u_+(y)| \le |u(x) - u(y)|$$

for all $x, y \in \Omega$.

The proof of the following results can be found in [21].

Lemma 2.2. *[21, Lemma 2.8] Suppose $f \in L^1_{loc}(\Omega)$ and $u \in \mathcal{W}^{s,p}(\Omega)$ is a weak solution of $(-\Delta_p)^s u = f$ in Ω. Let $v \in L^1_{loc}(\mathbb{R}^N)$ be such that*

$$\operatorname{dist}(\operatorname{supp}(v), \Omega) > 0 \text{ and } \int\limits_{\Omega^c} \frac{|v(x)|^{p-1}}{(1+|x|)^{N+sp}} dx < \infty,$$

and define for a.e. Lebesgue point $x \in \Omega$ of u

$$h(x) = 2 \int\limits_{\operatorname{supp} v} \frac{(u(x) - u(y) - v(y))^{p-1} - (u(x) - u(y))^{p-1}}{|x-y|^{N+sp}} dy.$$

Then $u + v \in \mathcal{W}^{s,p}(\Omega)$ and is a weak solution of $(-\Delta_p)^s(u+v) = f + h$ in Ω.

Theorem 2.3. *[21, Theorem 3.6] Let Ω be a bounded domain such that $\partial\Omega$ is $C^{1,1}$, and $\delta_\Omega(x) = \operatorname{dist}(x, \Omega^c)$. There exists $\rho = \rho(N, p, s, \Omega)$ such that δ_Ω^s is a weak solution of $(-\Delta_p)^s \delta_\Omega^s = f$ in $\Omega_\rho = \{x \in \Omega\colon \delta_\Omega(x) < \rho\}$ for some $f \in L^\infty(\Omega_\rho)$.*

Proposition 2.4. *[21, Proposition 2.10] Let Ω be bounded, $u, v \in \mathcal{W}^{s,p}(\Omega)$ satisfy $u \ge v$ in Ω^c and for all $\varphi \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \ge 0$*

$$\int\limits_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy \ge$$
$$\int\limits_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^{p-1}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy.$$

Then $u \ge v$ a.e. in Ω.

We also have a comparison principle for sub-solutions and super-solutions of (1.1).

Proposition 2.5. *Let Ω be bounded, $u, v \in \mathcal{W}^{s,p}(\Omega)$ be nonnegative super-solution and sub-solution of (1.1) in Ω respectively. If $c(x) \le 0$ in Ω and $u \ge v$ a.e. in Ω^c then $u \ge v$ a.e. in Ω.*

Proof. We first observe that since $u, v \in \mathcal{W}^{s,p}(\Omega)$ we have that $(v-u)_+ \in \widetilde{W}^{s,p}(\Omega)$. Then, using that u, v are super-solution and sub-solution of (1.1) in Ω respectively, we get

$$\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^{p-1}-(u(x)-u(y))^{p-1}}{|x-y|^{N+sp}}((v(x)-u(x))_+ - (v(y)-u(y))_+)dxdy$$
$$\le \int_{\Omega} c(x)(v(x)^{p-1}-u(x)^{p-1})(v(x)-u(x))_+dx \le 0.$$

The proof follows by the argument of [23, Lemma 9]. □

Our next result is referred to the regularity of the weak solutions.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^N$ be smooth bounded domain and $c \in L^\infty(\Omega)$. If $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (1.1) then there is $\alpha \in (0,1)$ such that $u \in C^\alpha(\overline{\Omega})$.*

Proof. By [25, Lemma 2.3] and bootstrap argument, we have that $u \in L^\infty(\Omega)$. Therefore, by [21, Theorem 1.1], $u \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$. □

2.1. Viscosity solution

In he rest of this section, Ω is bounded open set with smooth boundary and $c \in C(\overline{\Omega})$.

Following [22], we define our notion of viscosity super-solution of (1.1). We start to introduce some notation

$$L_{s,p}(\mathbb{R}^N) = \left\{ u \in L^{p-1}_{loc}(\mathbb{R}^N) \colon [u]_{s,p} < \infty \right\}.$$

The set of critical points of a differential function u and the distance from the critical points are denoted by

$$N_u := \{x \in \Omega \colon \nabla u(x) = 0\}, \quad d_u(x) := \operatorname{dist}(x, N_u),$$

respectively. Let $D \subset \Omega$ be an open set. We denote the class of C^2-functions whose gradient and Hessian are controlled by d_u as

$$C^2_\beta(D) := \left\{ u \in C^2(\Omega) \colon \sup_{x \in D} \left(\frac{\min\{d_u(x), 1\}^{\beta-1}}{|\nabla u(x)|} + \frac{|D^2u(x)|}{d_u(x)^{\beta-2}} \right) < \infty \right\}.$$

We are now in condition to introduce our definition. We say that a function $u \colon \mathbb{R}^N \to [-\infty, \infty]$ is a viscosity super-solution of (1.1) if it satisfies the following four assumptions:

(VS1) $u < \infty$ a.e. in $\mathbb{R}^N$ and $u > -\infty$ everywhere in Ω;
(VS2) u is lower semicontinuous in Ω;

(VS3) If $\phi \in C^2(B_r(x_0))$ for some $B_r(x_0) \subset \Omega$ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in $B_r(x_0)$, and one of the following holds
(a) $p > 2/(2-s)$ or $\nabla\phi(x_0) \neq 0$;
(b) $1 < p \leq 2/(2-s)$; $\nabla\phi(x_0) = 0$ such that x_0 is an isolate critical point of ϕ, and $\phi \in C^2_\beta(B_r(x_0))$ for some $\beta > sp/p-1$;
then $(-\Delta_p)^s\phi_r(x_0) \geq c(x_0)u(x_0)^{p-1}$, where

$$\phi_r(x) = \begin{cases} \phi & \text{if } x \in B_r(x_0), \\ u(x) & \text{otherwise;} \end{cases} \tag{2.3}$$

(VS4) $u_- \in L_{s,p}(\mathbb{R}^N)$.

A function u is a viscosity sub-solution of (1.1) if $-u$ is a viscosity super-solution. Finally, u is a viscosity solution if it is both a viscosity super-solution and sub-solutions.

To prove the following results, we borrow ideas and techniques of [23, Proposition 11].

Lemma 2.7. *Let $c \in C(\overline{\Omega})$. If $u \in \mathcal{W}^{s,p}(\Omega) \cap C(\overline{\Omega})$ is a weak super-solution of (1.1) such that $u \geq 0$ in Ω^c then u is a viscosity super-solution of (1.1).*

Proof. Let's observe that, by our assumptions, u satisfies (VS1), (VS2) and (VS4). Thus, we only need verify property (VS3). We prove it by contradiction. Suppose the conclusion in the lemma is false. Then there exist $x_0 \in \Omega$, and $\phi \in C^2(B_r(x_0))$ such that

- $\phi(x_0) = u(x_0)$, and $\phi \leq u$ in $B_r(x_0) \subset \Omega$;
- Either (a) or (b) in (VS3) holds;
- $(-\Delta_p)^s\phi_r(x_0) < c(x_0)|\phi_r(x_0)|^{p-2}\phi_r(x_0) = c(x_0)|u(x_0)|^{p-2}u(x_0)$.

Then, by continuity (see [22, Lemma 3.8]), there is $\delta \in (0, r)$ such that

$$(-\Delta_p)^s\phi_r(x) < c(x)u(x)^{p-1}$$

for all $x \in B_\delta(x_0)$.

By [22, Lemma 3.9], there exist $\theta > 0$, $\rho \in (0, \delta/2)$ and $\mu \in C^2_0(B_{\rho/2}(x_0))$ with $0 \leq \mu \leq 1$ and $\mu(x_0) = 1$ such that $v = \phi_r + \theta\mu$ satisfies

$$\sup_{B_\rho(x_0)} |(-\Delta_p)^s\phi_r(x) - (-\Delta_p)^s v(x)| < \inf_{B_{\delta/2}(x_0)} c(x)u(x)^{p-1} - (-\Delta_p)^s\phi_r(x).$$

Then

$$(-\Delta_p)^s v(x) < c(x)u(x)^{p-1}$$

for all $x \in B_\rho(x_0)$. Then, $v = \phi_r \leq u$ in $B_\rho(x_0)^c$ and by [23, Lemma 10], for any $\varphi \in \widetilde{W}^{s,p}(B_\rho(x_0))$, $\varphi \geq 0$

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy \geq \int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy.$$

Therefore, by Proposition 2.4, $u \geq v$ in B_ρ. Thus $u(x_0) = \phi_r(x_0) > \phi_r(x_0) + \theta = v(x_0)$ which is a contradiction. □

3. Strong minimum principle

Let us now prove a strong minimum principle for weak super-solutions of (1.1). To this end, we follow the ideas in [4] and prove first the next logarithmic lemma (see [12, Lemma 1.3]).

Lemma 3.1. *Let $c \in L^1_{loc}(\Omega)$, and $u \in \mathcal{W}^{s,p}(\Omega)$ be a weak super-solution of (1.1). If $u \geq 0$ a.e. in $B_R(x_0) \subset\subset \Omega$ then for any $B_r = B_r(x_0) \subset B_{R/2}(x_0)$ and $0 < h < 1$ we have that*

$$\int_{B_r^2} \frac{1}{|x-y|^{N+sp}} \left| \log\left(\frac{u(x)+h}{u(y)+h}\right)\right|^p dxdy \leq$$

$$Cr^{N-sp}\left\{ h^{1-p} r^{sp} \int_{B_{2r}^c} \frac{u_-(y)^{p-1}}{|y-x_0|^{N+sp}} dy + 1 \right\} + \|c\|_{L^1(B_{2r})},$$

where C depends only on N, s, and p.

Proof. Let $0 < r < R/2$, $0 < h < 1$ and $\phi \in C_0^\infty(B_{3r/2})$ be such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\phi| < Cr^{-1} \text{ in } B_{3r/2} \subset B_R.$$

Since $v = (u+h)^{1-p}\phi^p \in \widetilde{W}^{s,p}(\Omega)$ and u is a super-solution of (1.1), we have that

$$\begin{aligned} &\int_{B_{3r/2}} c(x) \frac{u^{p-1}(x)\phi^p(x)}{(u(x)+h)^{p-1}} dx \\ &\leq \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}} \left(\frac{\phi^p(x)}{(u(x)+h)^{p-1}} - \frac{\phi^p(y)}{(u(y)+h)^{p-1}} \right) dxdy. \end{aligned} \tag{3.4}$$

In the proof of Lemma 1.3 in [12], it is showed that right side of the above inequality is bounded by

$$Cr^{N-sp}\left\{ h^{1-p} r^{sp} \int_{(B_{2r})^c} \frac{(u_-(y))^{p-1}}{|y-x_0|^{N+sp}} dy + 1 \right\}$$
$$- \int_{(B_r)^2} \frac{1}{|x-y|^{N+sp}} \left| \log\left(\frac{u(x)+h}{u(y)+h}\right)\right|^p dxdy,$$

where C depends only on N, s, and p. Then, by (3.4) and using that $0 \le u^{p-1}(u+h)^{1-p}\phi^p \le 1$ in $B_{3r/2}$, the lemma holds. □

Lemma 3.2. *Let $c \in L^1_{loc}(\Omega)$ be a non-positive function and u be a weak super-solution of (1.1). Then,*

(a) If $u \ge 0$ a.e. in Ω^c and $u = 0$ a.e. in Ω then $u = 0$ a.e. in $\mathbb{R}^N$.
(b) If Ω is bounded, and $u \ge 0$ a.e. in Ω^c then $u \ge 0$ a.e. in Ω.

Proof. First, we prove (a). Let $\varphi \in C_0^\infty(\Omega)$ be non-negative function, then

$$\begin{aligned} 0 &\le \int\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}} dxdy \\ &= -2 \int\limits_{\Omega^c\times\Omega} \frac{u(y)^{p-1}\varphi(x)}{|x-y|^{N+sp}} dxdy \end{aligned}$$

due to $u = 0$ a.e. in Ω. Thus, since $u \ge 0$ a.e. in Ω^c then $u = 0$ a.e. in Ω^c. Hence $u = 0$ a.e. in $\mathbb{R}^N$.

Now we prove (b). Since $u \in \mathcal{W}^{s,p}(\Omega)$ and $u \ge 0$ in Ω^c we have that $u_- \in \widetilde{W}^{s,p}(\Omega)$. Then

$$\begin{aligned} 0 \le -\int\limits_{\Omega} c(x)(u_-(x))^p dx &= \int\limits_{\Omega} c(x)(u(x))^{p-1}u_-(x)dx \\ &\le \int\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(u_-(x)-u_-(y))}{|x-y|^{N+sp}} dx \end{aligned}$$

owing to $c(x) \le 0$ in Ω and u is a weak super-solution of (1.1). Observe that

$$\begin{aligned} &(u(x)-u(y))^{p-1}(u_-(x)-u_-(y)) \le \\ &\qquad \le \begin{cases} -(u_-(x)-u_-(y))^p & \text{if } x, y \in \Omega_-, \\ -(u_-(x)+u_+(y))^{p-1}u_-(x) & \text{if } x \in \Omega_-, \ y \in \Omega_-^c \end{cases} \end{aligned}$$

consequently

$$\begin{aligned} 0 &\le \int\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(u_-(x)-u_-(y))}{|x-y|^{N+sp}} dx \\ &\le -\int\limits_{\Omega_-^2} \frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}} dxdy - 2 \int\limits_{\Omega_-\times\Omega_-^c} \frac{(u_-(x)+u_+(y))^{p-1}u_-(x)}{|x-y|^{N+sp}} dxdy \\ &\le 0. \end{aligned}$$

Therefore $u_- \equiv 0$ a.e. in $\mathbb{R}^N$. Then in both cases we have that $u \ge 0$ a.e. in $\mathbb{R}^N$. □

Now, we prove our strong minimum principle under the assumption that Ω is connected.

Lemma 3.3. *Let $c \in L^1_{loc}(\Omega)$ be a non-positive function and $u \in \mathcal{W}^{s,p}(\Omega)$ be a weak super-solution of (1.1).*

1. *If Ω is bounded and connected, and $u \geq 0$ a.e. in Ω^c then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in Ω.*
2. *If Ω is connected, $u \geq 0$ a.e. in $\mathbb{R}^N$ then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in Ω.*

Proof. By Lemma 3.2, $u \geq 0$ a.e. in $\mathbb{R}^N$.

Proceeding as in the proof of Theorem A.1 in [4] and using Lemma 3.1, we have that If Ω is bounded and connected $u \neq 0$ a.e. in Ω, then $u > 0$ a.e. in Ω.

If Ω is unbounded and connected, then there is a sequence of bounded connected open sets $\{\Omega_n\}_{n\in\mathbb{N}}$ such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$. If $u \neq 0$ a.e. in Ω then there is $n_0 \in \mathbb{N}$ such that $u \neq 0$ a.e. in Ω_n for all $n \geq n_0$. Thus $u > 0$ a.e. in Ω_n for all $n \geq n_0$, since for all $n \geq n_0$ we have that Ω_n is a bounded connected open set, u is be a nonnegative weak super-solution of $(-\Delta)^s_p u = c(x)u^{p-1}$ in Ω_n and $u \neq 0$ a.e. in Ω_n. Therefore $u > 0$ a.e. in Ω. □

In fact, as our operator is non-local, we do not need to assume that the domain is connected.

Lemma 3.4. *Let $c \in L^1_{loc}(\Omega)$ be a non-positive function and $u \in \mathcal{W}^{s,p}(\Omega)$ be a weak super-solution of (1.1).*

1. *If Ω is bounded, and $u \geq 0$ a.e. in Ω^c then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in Ω.*
2. *If $u \geq 0$ a.e. in $\mathbb{R}^N$ then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in Ω.*

Proof. By Lemma 3.3, we only need to show that $u \neq 0$ a.e. in Ω if only if $u \neq 0$ a.e. in all connected components of Ω. That is, we only need to show that if $u \not\equiv 0$ in Ω then $u \not\equiv 0$ in all connected components of Ω.

Suppose, on the contrary, that there is a connected component U of Ω such that $u = 0$ a.e. in U. Since u is a weak super-solution of (1.1), it follows from Lemma 3.2 that $u \geq 0$ in $\mathbb{R}^N$. Moreover, for any nonnegative function $\varphi \in \widetilde{W}^{s,p}(\Omega)$ we get

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dxdy \\
&= -2\int_U \int_{U^c} \frac{(u(x))^{p-1}\varphi(y)}{|x-y|^{N+sp}}\,dxdy
\end{aligned}
$$

due to $u = 0$ a.e. in U. Then $u = 0$ a.e. in U^c, that is $u = 0$ a.e. in $\mathbb{R}^N$, which is a contradiction to the fact that $u \neq 0$ a.e. in Ω. □

Then, by Lemmas 3.2 and 3.4, we get Theorem 1.2.

To conclude this section, we prove Theorem 1.4. The key of the proof is Theorem 1.2 and the next result.

Lemma 3.5. *Let $c \in C(\overline{B_R(x_0)})$ and $u \in \mathcal{W}^{s,p}(B_R(x_0)) \cap C(\overline{B_R(x_0)})$ be a weak super-solution of*

$$(-\Delta_p)^s u = c(x)u^{p-1} \quad \text{in } B_R(x_0). \tag{3.5}$$

If $u \geq 0$ in $B_R(x_0)^c$ then either $u > 0$ in $B_R(x_0)$ or $u = 0$ a.e. in $\mathbb{R}^N$.

Proof. We will show that if there is $x_\star \in B_R = B_R(x_0)$ such that $u(x_\star) = 0$ then $u = 0$ a.e. in $\mathbb{R}^N$.

We start observing that, by Lemma 3.2, $u \geq 0$ a.e. in B_R. Moreover, by Theorem 1.2, either $u > 0$ a.e. in B_R or $u = 0$ a.e. in $\mathbb{R}^N$.

On the other hand, by Lemma 2.7, u is a viscosity super-solution of (3.5). Then, since $u \geq 0$ in B_R, for any $\varepsilon > 0$ and $\beta > \max\{2, 2/2-s\}$ the function

$$\phi^\varepsilon = -\varepsilon|x - x_\star|^\beta$$

is an admissible test function. Therefore

$$(-\Delta_p)^s \phi_r^\varepsilon(x_\star) \geq c(x_\star)\phi_r^\varepsilon(x_\star)^{p-1} = 0$$

for some $r \in (0, R - |x_\star - x_0|)$. See (2.3) for the definition of ϕ_r^ε.

Then

$$0 \leq \varepsilon^{p-1} \int\limits_{B_r(x_\star)} |y - x_\star|^{\beta(p-1)-N-ps} dy - \int\limits_{B_r(x_0)^c} \frac{u(y)^{p-1}}{|y - x_\star|^{N+ps}} dy. \tag{3.6}$$

Since $\beta > \max\{2, 2/2-s\}$, we get

$$\varepsilon^{p-1} \int\limits_{B_r(x_\star)} |y - x_\star|^{\beta(p-1)-N-ps} dy = \frac{\varepsilon^{p-1}\omega_N}{\beta(p-1) - ps} r^{\beta(p-1)-ps} \to 0 \quad \text{as } \varepsilon \to 0,$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Then, by (3.6), we get $u = 0$ a.e. in $B_r(x_\star)^c$. Therefore $u = 0$ a.e. $\mathbb{R}^N$. □

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.2, we have that $u > 0$ a.e. in Ω or $u = 0$ *a.e.* in $\mathbb{R}^N$. Suppose that there is $x_0 \in \Omega$ such that $u(x_0) = 0$. Since Ω is open, $c, u \in C(\Omega)$, there is $R > 0$ such that $B_R(x_0) \subset \Omega$ and $c, u \in C(\overline{B_R(x_0)})$. Moreover, since u is a weak super-solution of (1.1), we have that u is a weak super-solution of

$$(-\Delta_p)^s u = c(x)u^{p-1} \quad \text{in } B_R(x_0).$$

Then, by Lemma 3.5, $u = 0$ a.e. in $\mathbb{R}^N$ since $u(x_0) = 0$. Therefore $u > 0$ in Ω or $u = 0$ in Ω. □

Remark 3.6. In the case $u \geq 0$ a.e. in $\mathbb{R}^n$, the non-positivity assumption over the function c is not necessary, see Remark 1.3.

4. A Hopf lemma

First, we can show the Hopf's lemma in a ball.

Lemma 4.1. *Let B be a ball in $\mathbb{R}^N$ of radius $R > 0$, $c, u \in C(\overline{B})$, u be a weak super-solution of*

$$(-\Delta_p)^s u = c(x)u^{p-1} \quad \text{in } B$$

and $\delta(x) = \operatorname{dist}(x, B^c)$. If $u \geq 0$ a.e. in $\mathbb{R}^N$ or $c(x) \leq 0$ in B and $u \geq 0$ a.e. in B^c then either $u \equiv 0$ a.e. in $\mathbb{R}^N$ or

$$\liminf_{B \ni x \to x_0} \frac{u(x)}{\delta^s(x)} > 0 \tag{4.7}$$

for all $x_0 \in \partial B$.

Proof. By Theorem 1.4 and Remark 3.6, we have that either $u = 0$ a.e. in $\mathbb{R}^N$ or $u > 0$ in B. Suppose $u \not\equiv 0$ in B, then we want to show that (4.7) holds true for all $x_0 \in \partial B$.

By Theorem 2.3, there exists $\rho = \rho(N, p, s, B) > 0$ such that δ^s is a weak solution of $(-\Delta_p)^s \delta^s = f$ in $B_\rho = \{x \in B : \delta(x) < \rho\}$ for some $f \in L^\infty(B_\rho)$. Let $K \subset\subset B_\rho^c \cap B$ be a closed ball and $\alpha > 0$ be a constant (to be determined later). Owing to Lemma 2.2, $w = \delta^s + \alpha\chi_K$ is a weak solution of $(-\Delta_p)^s w = f + h_\alpha$ in B_ρ where

$$h_\alpha(x) = 2\int\limits_K \frac{(\delta^s(x) - \delta^s(y) - \alpha)^{p-1} - (\delta^s(x) - \delta^s(y))^{p-1}}{|x-y|^{N+sp}} dxdy$$

for a.e. $x \in B_\rho$. Since $u \in L^\infty(\overline{B})$ and $\operatorname{dist}(K, B_\rho) > 0$, it is clear that

$$h_\alpha(x) \to -\infty \quad \text{uniformly in } \overline{B_\rho} \quad \text{as } \alpha \to \infty.$$

Then, we choose α large enough such that

$$\sup\{f(x) + h_\alpha(x) : x \in B_\rho\} \leq \inf\{c(x)(u(x))^{p-1} : x \in B_\rho\}. \tag{4.8}$$

Let $\varepsilon \in (0, 1)$ be such that

$$\varepsilon(R^s + \alpha) < \inf\{u(x) : x \in B, \delta(x) \geq \rho\}.$$

Thus $v = \varepsilon w \leq u$ in B_ρ^c and using (4.8) we have that for all $\varphi \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \geq 0$

$$\int\limits_{\mathbb{R}^{2N}} \frac{(v(x) - v(y))^{p-1}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy$$

$$\leq \int\limits_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^{p-1}(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} dxdy.$$

Then, by Proposition 2.4, $v \le u$ in B_ρ. Therefore

$$\varepsilon \le \frac{u(x)}{\delta^s(x)} \quad \forall x \in B_\rho.$$

Thus (4.7) holds true for all $x_0 \in \partial B$. $\square$

To conclude this section, we show our Hopf Lemma.

Proof of Theorem 1.5. By Theorem 1.4 and Remark 3.6, we have that either $u = 0$ a.e. in $\mathbb{R}^N$ or $u > 0$ in Ω. Suppose $u \not\equiv 0$ in Ω. Since Ω satisfies the interior ball condition in $x_0 \in \partial\Omega$, there is a ball $B \subset \Omega$ such that $x_0 \in \partial B$. Then, by Lemma 4.1, (1.2) holds true. $\square$

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References

[1] R.A. Adams, Sobolev Spaces, Pure and Applied Mathematics, vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, London, 1975.
[2] B. Barrios, M. Medina, Strong maximum principles for fractional elliptic and parabolic problems with mixed boundary conditions, ArXiv e-prints, 2016.
[3] J. Bertoin, Lévy Processes, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996.
[4] L. Brasco, G. Franzina, Convexity properties of Dirichlet integrals and Picone-type inequalities, Kodai Math. J. 37 (2014) 769–799.
[5] L. Brasco, E. Parini, The second eigenvalue of the fractional p-Laplacian, Adv. Calc. Var. 9 (2016) 323–355.
[6] L. Brasco, E. Parini, M. Squassina, Stability of variational eigenvalues for the fractional p-Laplacian, Discrete Contin. Dyn. Syst. 36 (2016) 1813–1845.
[7] L. Caffarelli, Non-Local Diffusions, Drifts and Games, Springer, Berlin, Heidelberg, 2012, pp. 37–52.
[8] L.A. Caffarelli, J.-M. Roquejoffre, Y. Sire, Variational problems for free boundaries for the fractional Laplacian, J. Eur. Math. Soc. (JEMS) 12 (2010) 1151–1179.
[9] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[10] L.M. Del Pezzo, A. Quaas, Global bifurcation for fractional p-Laplacian and an application, Z. Anal. Anwend. 35 (2016) 411–447.
[11] F. Demengel, G. Demengel, Functional Spaces for the Theory of Elliptic Partial Differential Equations, Universitext, Springer, London, 2012, translated from the 2007 French original by Reinie Erné.
[12] A. Di Castro, T. Kuusi, G. Palatucci, Local behavior of fractional p-minimizers, Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016) 1279–1299.
[13] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[14] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Grundlehren der Mathematischen Wissenschaften, vol. 219, Springer-Verlag, Berlin, New York, 1976, translated from the French by C.W. John.
[15] M.M. Fall, S. Jarohs, Overdetermined problems with fractional Laplacian, ESAIM Control Optim. Calc. Var. 21 (2015) 924–938.
[16] G. Gilboa, S. Osher, Nonlocal operators with applications to image processing, Multiscale Model. Simul. 7 (2008) 1005–1028.
[17] A. Greco, R. Servadei, Hopf's lemma and constrained radial symmetry for the fractional Laplacian, Math. Res. Lett. 23 (2016) 863–885.

[18] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[19] A. Iannizzotto, S. Liu, K. Perera, M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, Adv. Calc. Var. 9 (2016) 101–125.
[20] A. Iannizzotto, S. Mosconi, M. Squassina, h^s versus c^0-weighted minimizers, NoDEA Nonlinear Differential Equations Appl. 22 (2015) 477–497.
[21] A. Iannizzotto, S. Mosconi, M. Squassina, Global Hölder regularity for the fractional p-Laplacian, Rev. Mat. Iberoam. 32 (2016) 1353–1392.
[22] J. Korvenpää, T. Kuusi, E. Lindgren, Equivalence of solutions to fractional p-Laplace type equations, J. Math. Pures Appl. (2017), in press.
[23] E. Lindgren, P. Lindqvist, Fractional eigenvalues, Calc. Var. Partial Differential Equations 49 (2014) 795–826.
[24] J.M. Mazón, J.D. Rossi, J. Toledo, Fractional p-Laplacian evolution equations, J. Math. Pures Appl. 9 (105) (2016) 810–844.
[25] S. Mosconi, K. Perera, M. Squassina, Y. Yang, The Brezis–Nirenberg problem for the fractional p-Laplacian, Calc. Var. Partial Differential Equations 55 (55) (2016) 105.
[26] S. Mosconi, M. Squassina, Nonlocal problems at nearly critical growth, Nonlinear Anal. 136 (2016) 84–101.
[27] P. Pucci, J. Serrin, The strong maximum principle revisited, J. Differential Equations 196 (2004) 1–66.
[28] P. Pucci, J. Serrin, The Maximum Principle, Progress in Nonlinear Differential Equations and their Applications, vol. 73, Birkhäuser Verlag, Basel, 2007.
[29] X. Ros-Oton, Nonlocal elliptic equations in bounded domains: a survey, Publ. Mat. 60 (2016) 3–26.
[30] X. Ros-Oton, J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. 9 (101) (2014) 275–302.
[31] J.L. Vázquez, The Dirichlet problem for the fractional p-Laplacian evolution equation, J. Differential Equations 260 (2016) 6038–6056.