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An explicit description of the second cohomology group of a quandle

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Abstract We use the inflation-restriction sequence and a result of Etingof and Graña on the rack cohomology to give a explicit description of 2-cocycles of finite indecomposable quandles with values in an abelian group. Several applications are given.

1 Introduction and main results

1.1. Quandles are non-associative algebraic structures introduced independently by Joyce [17] and Matveev [20] in connection with knot theory. They produce powerful invariants similar to those obtained by coloring [6,22]. Quandles turned out to be useful in different branches of algebra, topology and geometry since they have connections to several different topics such as permutation groups [16], quasigroups [24], symmetric spaces [25], Hopf algebras [2], etc.

Quandles have a very interesting cohomology theory that first appeared in [4] and independently in [12]. This theory is somewhat based on the rack cohomology introduced in Fenn et al. [11]. As in the case of groups, 2nd quandle cohomology groups can be used to produce new quandles by means of extensions.

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The explicit computation of quandle cohomology groups is an important problem relevant to different areas of current research. The 2nd quandle cohomology group is particularly important since it has many applications going from knot theory to Hopf algebras.

In Carter et al. [4], used quandle cohomology classes to produce powerful invariants of classical links and their higher dimensional analogs. The invariants based on quandle 2-cocycles improve the effectiveness of the quandle-coloring invariants since, for example, they distinguish knots from their mirror images. These invariants require an explicit description of 2-cocycles.

In the Hopf algebra context, quandles and their cohomology parametrize Yetter–Drinfeld modules. In turn these modules are crucial ingredients in the classification problem of finite-dimensional Hopf algebras with non-abelian coradical. Indeed, an important step of the lifting method proposed by Andruskiewitsch and Schneider to solve this classification problem is the explicit computation of the 2nd cohomology of finite quandles, see [1].

1.2. In this work we give an explicit description of the second cohomology group of a finite indecomposable quandle. Our presentation is made by means of the characters of a certain finite group. This reduces the problem of computing 2-cocycles of a quandle to an easy manipulation involving cosets in a finite group. Our method is based on a result of Etingof and Graña [9] which relates the 2nd cohomology of a quandle and the first cohomology of an infinite group.

1.3. We now review the basics of our construction. Let X be a finite quandle. Recall that the *enveloping group* of X is the group

$$G_X = \langle x \in X : xy = (x \triangleright y)x \rangle. \tag{1.1}$$

Assume that X is indecomposable and fix $x_0 \in X$. Under the identification $\langle x_0 \rangle \simeq \mathbb{Z}$ we show in Lemma 2.3 that $G_X \simeq N_X \rtimes \mathbb{Z}$, where N_X is the commutator group $[G_X, G_X]$ of G_X. The group G_X acts transitively on X in a natural way, hence so does N_X, see Corollary 2.4. We denote by N_0 the stabilizer of N_X on x_0: this is a finite group *cf.* Lemma 2.1.

Fix an abelian group A and let $M = \mathrm{Fun}(X, A)$ be the right G_X-module of functions $X \to A$, i.e. $(f \cdot x)(y) = f(x \triangleright y)$ for $x, y \in X$ and $f \in M$.

We prove that there is a commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^1(\mathbb{Z}, M^N) \arrow[r, "\sim"] \arrow[d, "\inf"'] & A \arrow[d, "\iota"] & \\
H^2(X, A) \arrow[r, dash, "\sim"] & H^1(G_X, M) \arrow[d, "\mathrm{res}"'] & A \times H^1(N_X, M) \arrow[d, "\pi"] & \\
 & H^1(N_X, M)^{\mathbb{Z}} \arrow[r, equal] \arrow[d] & H^1(N_X, M) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(N_0, A) \\
 & 0 & 0 &
\end{tikzcd}$$

where the isomorphism

$$H^2(X, A) \simeq H^1(G_X, M), \quad q \longmapsto f_q, \tag{1.2}$$

is [9, Corollary 5.4], see also (2.3); inf and res denote the inflation-restriction maps and ι and π denote the canonical inclusion and projection.

See Lemmas 3.1, 3.4 and Proposition 3.7 for a proof of the isomorphisms and the equality in the rows of the diagram. The exactness of the first column is a well-known fact *cf.* Lemma 2.9. We show that it splits in Lemma 2.10.

By diagram chasing, we derive an isomorphism

$$H^2(X, A) \simeq A \times \mathrm{Hom}(N_0, A).$$

From this isomorphism we obtain an explicit description of rack and quandle 2-cocycles with values in any abelian group A, see Theorem 1.1.

We denote by $f \mapsto f_0$ the map $H^1(G_X, M) \to \mathrm{Hom}(N_0, A)$ deduced from the diagram above.

Our first main result reads as follows, see Sect. 3 for a proof.

Theorem 1.1 *Let X be a finite indecomposable quandle, $x_0 \in X$ and A an abelian group with trivial G_X-action. Then*

$$H^2(X, A) \simeq A \times \mathrm{Hom}(N_0, A), \quad q \mapsto (q_{x_0,x_0}, (f_q)_0). \tag{1.3}$$

In particular this shows that the non-constant 2-cocycles on X are controlled by a finite group.

1.4. Our second main result is a precise recipe to reconstruct a cocycle $q \in H^2(X, A)$ from a datum $(a, g) \in A \times \mathrm{Hom}(N_0, A)$. That is, we give a converse to the map in (1.3) to build all explicit 2-cocycles of a given quandle. To do this, we need to introduce some extra notation.

First, we fix a *good coset decomposition*

$$N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0,$$

into N_0-cosets, i.e. the representatives $\sigma_0, \dots, \sigma_k$ are chosen so that:

(1) $\sigma_0 = 1$;
(2) for each $i \in \{0, \dots, k\}$ there is $j \in \{0, \dots, k\}$ such that $x_0 \triangleright \sigma_i = \sigma_j$;
(3) for each $x \in X$ there is $j \in \{0, \dots, k\}$ such that $\sigma_j \triangleright x_0 = x$.

The existence of such a decomposition is given in Proposition 4.1, together with a recursive method for constructing it.

We define $\sigma : N \to \{\sigma_0, \dots, \sigma_k\}$, $\sigma(n) = \sigma_i$ if $n \in \sigma_i N_0$. We set, *cf.* (3.4),

$$c(n) = \sigma(n)^{-1} n \in N_0.$$

Given $y \in X$ and $j \in \{0, \dots, k\}$ such that $\sigma_j \triangleright x_0 = y$ we write

$$\sigma_y := \sigma_j.$$

Our second main result is the following, see sect. 4 for the proof.

Theorem 1.2 *Let X be a finite indecomposable quandle, $x_0 \in X$ and A an abelian group with trivial G_X-action. Let $N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0$ be a good decomposition of N_X into N_0-cosets. For each $a \in A$ and $g \in \mathrm{Hom}(N_0, A)$, the map $q \colon X \times X \to A$ given by*

$$q_{x,y} = a + g(c(x\sigma_y x_0^{-1})) \tag{1.4}$$

is a 2-cocycle of X with values in A.

Combining Theorems 1.1, 1.2 and the isomorphism (1.2), namely

$$q_{x,y} = f_q(x)(y), \quad q \in H^2(X, A),$$

we immediately obtain the following corollary.

Corollary 1.3 *Let X be a finite indecomposable quandle, $x_0 \in X$ and A an abelian group with trivial G_X-action. Let $N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0$ be a good decomposition of N_X into N_0-cosets and let $q \in H^2(X, A)$. Then there exists $a \in A$ and $g \in \mathrm{Hom}(N_0, A)$ such that (1.4) holds for all $x, y \in X$.*

Corollary 1.3 has many applications and can be used for explicit calculations of rack cohomology groups of quandles. In particular, if the commutator subgroup N_X acts regularly on X, then $N_0 = 1$ and hence we obtain the following corollary.

Corollary 1.4 *Let X be a finite indecomposable quandle. If the action of N_X on X is regular, then $H^2(X, \mathbb{C}^\times) \simeq \mathbb{C}^\times$.*

1.5. The paper is organized as follows. Preliminaries on racks and quandles, cohomology theory of groups, and cohomology theories of racks and quandles appear in Sect. 2. Our first main result, Theorem 1.1, is proved in Sect. 3. Theorem 1.2 is proved in Sect. 4. Applications of our theory are given in Sect. 5. These applications include the calculations of the 2nd rack cohomology group of: (a) the quandle associated with the conjugacy class of transpositions, see Theorem 5.5; (b) affine racks of size p and p^2, where p is a prime number, see Propositions 5.8, 5.10, 5.11 and 5.12; and (c) another proof of Eisermann's formula for computing the 2nd quandle homology group of a quandle, see Theorem 5.6.

2 Preliminaries

2.1 Notation

For a set X we denote by $\mathbb{S}_X$ the group of permutations $X \to X$. If X is finite of cardinal $|X| \in \mathbb{N}$, then we identify $\mathbb{S}_{|X|} = \mathbb{S}_X$. For any group G we denote by $[G, G]$ its commutator subgroup and G_{ab} its abelianization, i.e. $G_{\mathrm{ab}} = G/[G, G]$. In addition, $Z(G)$ is the center of G and $G_G(g) = \{h \in G : hg = gh\}$ for $g \in G$. We denote by $\mathrm{Aut}(G)$ the group of automorphisms $G \to G$; if $\gamma \in \mathrm{Aut}(G)$, then $\mathrm{ord}(\gamma)$ is the order of γ.

Let M be an abelian group equipped with a G-action. We denote by $H^n(G, M)$, $n \geq 0$, the nth cohomology group of G with coefficients on M. We denote by $Z^n(G, M)$, resp. $B^n(G, M)$, the groups of cocycles, resp. cobordisms, of G with values on M. We refer the reader to [3] for unexplained notation and terminology.

2.2 Racks

A *rack* is a non-empty set X together with a binary operation $\triangleright : X \times X \to X$ such that the maps $\varphi_x = x \triangleright - : X \to X$, $y \mapsto x \triangleright y$, are bijective for each $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. A *quandle* is a rack that further satisfies $x \triangleright x = x$ for all $x \in X$. A prototypical example of a rack is a group G with $\triangleright$ given by conjugation. A rack is *indecomposable* if the inner group $\mathrm{Inn}(X) = \langle \varphi_x : x \in X \rangle \leq \mathbb{S}_X$ acts transitively on X.

The enveloping group G_X *cf.* (1.1) also acts on X, and this action is readily seen to be transitive when X is indecomposable. The group G_X is infinite. There is a finite analogue of this group, which is constructed as follows: For each x, let $n_x = \operatorname{ord}\varphi_x$. Then the subgroup $Z_X = \langle x^{n_x}, x \in X\rangle \leq G_X$ is normal and the quotient $F_X = G_X/Z_X$ is finite, see [14, §2]. We write $N_X = [G_X, G_X]$ to denote the commutator subgroup of G_X.

Lemma 2.1 [15, Lemma 1.10] *Let X be an indecomposable quandle. Then $N_X \simeq [F_X, F_X]$. In particular, N_X is finite.*

The last claim of Lemma 2.1 also follows from the following result and a theorem of Schur, see for example [23, Theorem 5.32].

Lemma 2.2 *Let X be a finite indecomposable quandle. Then all conjugacy classes of G_X are finite.*

Proof Since G_X acts transitively on X and the center $Z(G_X)$ is the kernel of this action, it follows that the index $[G_X : Z(G_X)]$ is finite. This implies that all conjugacy classes of G_X are finite as

$$[G_X : C_{G_X}(g)] \leq [G_X : Z(G_X)],$$

where $C_{G_X}(g)$ denotes the centralizer of g in G_X. □

We consider the unique surjective group homomorphism

$$\mathrm{d} : G_X \to \mathbb{Z} \tag{2.1}$$

satisfying $\mathrm{d}(x) = 1$ for all $x \in X$. In particular, this homomorphism shows that G_X is infinite and induces a notion of degree on G_X.

Lemma 2.3 *Let X be an indecomposable finite quandle and $x_0 \in X$. Then the following hold:*

(1) $G_X = \ker \mathrm{d} \rtimes \langle x_0\rangle$.
(2) $\ker \mathrm{d} = N_X$ *if X is indecomposable.*

Proof Since $\ker \mathrm{d}$ is a normal subgroup of G_X, $\ker \mathrm{d}\langle x_0\rangle$ is a subgroup of G_X. It is clear that $\ker \mathrm{d} \cap \langle x_0\rangle = 1$ comparing degrees. Finally $G_X = \ker \mathrm{d}\langle x_0\rangle$ since $x = (xx_0^{-1})x_0 \in \ker \mathrm{d}\langle x_0\rangle$ for all $x \in X$.

It is clear that $N_X \subseteq \ker \mathrm{d}$. Next we prove the equality when X is indecomposable. Let $\ell : G_X \to \mathbb{Z}$ be defined as $\ell(g) = n$, if $g = x_{i_1}^{\epsilon_1} \dots x_{i_n}^{\epsilon_n}$, $\epsilon_i \in \{\pm 1\}$, $i \in \{1, \dots, n\}$, is a a reduced expression of g in terms of the generators of G_X. We show that $\ker \mathrm{d} \subseteq N_X$ by induction on $\ell(g)$, $g \in \ker \mathrm{d}$. If $\ell(g) = 2$, then $g = x_i^{\pm 1} x_j^{\mp 1}$. So we may assume that $g = x_i x_j^{-1}$ (if not, take inverses). Now, as X is indecomposable, there is $h \in G_X$ such that $h \cdot x_j = x_i$. Hence $g = hx_jh^{-1}x_j^{-1} \in N_X$. Now, if $\ell(g) > 2$, then there is a reduced expression of g (or g^{-1}) in which $g = g_1 x_i x_j^{-1} g_2$, $x_i, x_j \in X$ and $g_1, g_2 \in G_X$. Now, on the one hand, $0 = \mathrm{d}(g) = \mathrm{d}(g_1) + \mathrm{d}(g_2)$ and thus $g_1 g_2 \in N_X$ as $\ell(g_1 g_2) < \ell(g)$. On the other, $g = (g_1 x_i x_j^{-1} g_1^{-1})(g_1 g_2)$ and therefore $g \in N_X$. □

Corollary 2.4 *The restriction of the action of G_X on X to N_X is transitive.*

Proof Let $x, y \in X$ and let $g \in G_X$ such that $g \cdot x = y$ and let $\ell = \mathrm{d}(g)$. Then $g' = y^{-\ell} g \in N_X$ by Lemma 2.3 and $g' \cdot x = y$. □

2.3 Rack cohomology

A cohomology theory for racks was introduced in [10] and independently in [12]. A cohomology theory for quandles was developed in [4]. These theories were further developed and generalized for example in [2,18].

We briefly recall these cohomology theories next. Let X be a rack and let M be a right G_X-module. Set $C^n = C^n(X, M) = \mathrm{Fun}(X^n, M)$, $n \geq 0$, the set of functions from X^n to M. Consider the differential $d : C^n \to C^{n+1}$

$$df(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} \Big(f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \\ - f(x_1, \dots, x_{i-1}, x_i \triangleright x_{i+1}, \dots, x_i \triangleright x_{n+1}) \cdot x_i \Big).$$

The *rack cohomology* $H^\bullet(X, M)$ of X with coefficients in M is the cohomology of the complex $(C^\bullet, d)$ [9, Definition 2.3]. The groups of cocycles resp. cobordisms, are denoted by $Z^n(X, M)$, resp. $B^n(G, M)$. When A is an abelian group and no reference to a G_X-action on A is specified, $H^\bullet(X, A)$ stands for the cohomology of X with values in the trivial module $M = A$. If q is a class in $H^2(X, A)$, we set $q_{x,y} := q(x, y)$. Hence $q \in H^2(X, A)$ if and only if

$$q_{x\triangleright y, x\triangleright z} q_{x,z} = q_{x, y\triangleright z} q_{y,z}, \qquad \forall\, x, y, z \in X \tag{2.2}$$

and two classes $q, q' \in H^2(X, A)$ are equivalent if and only if there exists $\gamma : X \to A$ such that $q'_{x,y} = q_{x,y} \gamma(x \triangleright y) \gamma(y)^{-1}$ for all $x, y \in X$.

The *rack homology* $H_\bullet(X, A)$ with values in an abelian group A is defined analogously, by considering the free abelian group $F_n(X)$ on X^n, $n \geq 0$, and setting $C_n(X, A) := F_n(X) \otimes A$. If X is a quandle, then the subgroup $F_n^D(X) \leq F_n(X)$ generated by n-tuples $(x_1, \dots x_n)$ with $x_i = x_{i+1}$ for some i, defines a subcomplex $C_\bullet^D = C_\bullet^D(X, A)$ of $C_\bullet$. The *quandle homology* $H_\bullet^Q(X, A)$ of X is the homology of the quotient complex $C_\bullet^Q = (C_n / C_n^D)_{n \geq 0}$.

In this work we give a description of the group $H^2(X, A)$ of 2-cocycles on X with values in an abelian group A, which allows us to compute cocycles explicitly. We recall next some identifications between the (co)homology theories described above that will be useful for our goal.

Lemma 2.5 [5, Proposition 3.4] $H^2(X, A) \simeq \mathrm{Hom}(H_2(X, \mathbb{Z}), A)$*, via*

$$H^2(X, A) \ni q \mapsto \big([x, y] \mapsto q_{x,y} \big) \in \mathrm{Hom}(H_2(X, \mathbb{Z}), A).$$

The following is a particular case of [19, Theorem 7].

Lemma 2.6 *Assume X is an indecomposable quandle. Then*

$$H_2(X, \mathbb{Z}) \simeq H_2^Q(X, \mathbb{Z}) \times \mathbb{Z}.$$

Explicitly, if $(x, y) \in X^2$, then the isomorphism is induced by the map

$$(x, y) \mapsto \begin{cases} (x, y) \times 0, & \text{if } x \neq y \\ 0 \times 1, & \text{if } x = y. \end{cases}$$

Etingof and Graña found a deep relation between group cohomology and rack cohomology.

Theorem 2.7 [9, Corollary 5.4] *Let X be a finite indecomposable rack and A an abelian group with a trivial G_X-action. Then*

$$H^1(G_X, \mathrm{Fun}(X, A)) \simeq H^2(X, A).$$

This equivalence is given as follows:

(1) If $f \in H^1(G, \mathrm{Fun}(X, A))$ then a 2-cocycle $q^f \in H^2(X, A)$ arises as

$$q^f_{x,y} = f(x)(y), \qquad x, y \in X.$$

(2) Conversely, $q \in H^2(X, A)$ determines $f_q \in H^1(G, \mathrm{Fun}(X, A))$ by extending q recursively via

$$f_q(xy)(z) = q_{x,y\triangleright z} + q_{y,z}, \quad x, y, z \in X. \tag{2.3}$$

Remark 2.8 Let G be a (non-abelian) group and fix $Z^2(X, G) \subset \mathrm{Fun}(X^2, G)$ as the subset of all $q : X^2 \to G$ satisfying (2.2). We say that q is equivalent to q', and we write $q \sim q'$, in $Z^2(X, G)$ if and only if there is $\gamma \in \mathrm{Fun}(X, G)$ such that $q'_{x,y} = \gamma(x \triangleright y)q_{x,y}\gamma(y)^{-1}$. If $H^2(X, G) := Z^2(X, G)/\sim$, then Theorem 2.7 holds, see [9, Remark 5.6].

2.4 Group cohomology

Let G be a group, $N \lhd G$ a normal subgroup and M a right G-module. Recall *cf.* [3, 3.8] that there is a right G/N-action on $H^1(N, M)$, induced by

$$(f \cdot g)(n) = f(gng^{-1}) \cdot g, \quad g \in G, n \in N, f \in H^1(N, M). \tag{2.4}$$

Indeed, let $f \in Z^1(N, M)$. If $g \in N$, then

$$(f \cdot g)(n) = f(gn) - f(g) = f(g) \cdot n + f(n) - f(g)$$

by the cocycle condition. Hence

$$(f \cdot g)(n) - f(n) = f(g) \cdot n - f(g)$$

and thus $f \cdot g = f \in H^1(N, M)$. The *inflation-restriction* sequence is

$$\begin{aligned} 0 \to H^1(G/N, M^N) \xrightarrow{\iota} H^1(G, M) \xrightarrow{r} H^1(N, M)^{G/N} \\ \to H^2(G/N, M^N) \to H^2(G, M) \end{aligned} \tag{2.5}$$

where the *inflation map* $\iota(h)$, $h \in H^1(G/N, M^N)$, is the composition

$$G \twoheadrightarrow G/N \xrightarrow{h} M^N \hookrightarrow M$$

and the *restriction map* $r(g)$, $g \in H^1(G, M)$, is the composition

$$N \hookrightarrow G \xrightarrow{g} M.$$

In the case where $G/N \simeq \mathbb{Z}$ one obtains the following result, see *loc.cit.*

Lemma 2.9 *Assume that $G/N \simeq \mathbb{Z}$. Then*

(1) $H^2(G/N, M^N) = 0$.
(2) $H^1(G/N, M^N) = M^N/\langle m \cdot g - m\rangle$, *(class of)* $f \mapsto$ *(class of)* $f(1)$.

In particular, the exact sequence (2.5) *reduces to*

$$0 \to H^1(G/N, M^N) \overset{\text{inf}}{\to} H^1(G, M) \overset{\text{res}}{\to} H^1(N, M)^{G/N} \to 0. \tag{2.6}$$

Lemma 2.10 *Assume that N is finite and $G/N \simeq \mathbb{Z}$. Then* (2.6) *splits. A retraction for* $\inf : H^1(G/N, M^N) \to H^1(G, M)$ *is given by*

$$j : H^1(G, M) \to H^1(G/N, M^N), \qquad j(f)(\ell) = \frac{1}{|N|} \sum_{n \in N} \left(f\left(nx_0^\ell\right) - f(n) \right).$$

Proof We need to check that j is well-defined, that is:

(1) If $f \in Z^1(G, M)$, then $j(f)(G/N) \subseteq M^N$.
(2) If $f \in Z^1(G, M)$, then $j(f) \in Z^1(G/N, M^N)$.
(3) If $f \in B^1(G, M)$, then $j(f) \in B^1(G/N, M^N)$.

Let $f \in Z^1(G, M)$ and set $\varphi := j(f)$. For (1), using the cocycle condition,

$$\begin{aligned}
\varphi(\ell) \cdot n &= \frac{1}{|N|} \sum_{m \in N} \left(f\left(mx_0^\ell\right) \cdot n - f(m) \cdot n \right) \\
&= \frac{1}{|N|} \sum_{m \in N} \left(f\left(mx_0^\ell n\right) - f(n) - f(mn) + f(n) \right) \\
&= \frac{1}{|N|} \sum_{m \in N} \left(f\left(mx_0^\ell n x_0^{-\ell} x_0^\ell\right) - f(mn) \right).
\end{aligned}$$

By reordering the sum, $\varphi(\ell) \cdot n = \varphi(\ell)$ for all $n \in N$, $\ell \in \mathbb{Z}$. Hence (1) holds.

In (2), we get

$$\begin{aligned}
\varphi(\ell + r) &= \frac{1}{|N|} \sum_{n \in N} \left(f\left(nx_0^{\ell+r}\right) - f(n) \right) \\
&= \frac{1}{|N|} \sum_{n \in N} \left(f\left(nx_0^\ell\right) \cdot x_0^r + f\left(x_0^r\right) - f(n) \right) \\
&= \frac{1}{|N|} \sum_{n \in N} \left(f\left(nx_0^\ell\right) \cdot x_0^r - f(n) \cdot x_0^r + f(n) \cdot x_0^r + f\left(x_0^r\right) - f(n) \right) \\
&= \varphi(\ell) \cdot r + \frac{1}{|N|} \sum_{n \in N} \left(f\left(nx_0^r\right) - f\left(x_0^r\right) + f\left(x_0^r\right) - f(n) \right) \\
&= \varphi(\ell) \cdot r + \varphi(r).
\end{aligned}$$

Thus (2) holds. If $f \in B^1(G, M)$, then there exists $\psi \in M$ such that $f(g) = \psi \cdot g - \psi$. Hence,

$$\begin{aligned}
j(f)(\ell) &= \frac{1}{|N|} \sum_{n \in N} \left(\psi \cdot nx_0^\ell - \psi - \psi \cdot n + \psi \right) \\
&= \frac{1}{|N|} \sum_{n \in N} \left(\psi \cdot nx_0^\ell - \psi \cdot n \right)
\end{aligned}$$

and thus $j(f)(\ell) = \gamma \cdot \ell - \gamma$ for

$$\gamma = \frac{1}{|N|} \sum_{n \in N} \psi \cdot n \in M^N.$$

This shows (3). Finally we prove that $j \circ \inf = \mathrm{id}$. For this, recall that if $\varphi \in H^1(G/N, M^N)$ and $g \in G$, then $\inf(\varphi)(g) = \varphi(\bar{g})$, where $\bar{g}$ is the class of g in $G/N \simeq \mathbb{Z}$. Then

$$(j \circ \inf)(\varphi)(\ell) = \frac{1}{|N|} \sum_{n \in N} \left(\inf(\varphi) \left(n x_0^{\ell} \right) - \inf(\varphi)(n) \right) = \frac{1}{|N|} \sum_{n \in N} \varphi(\ell) = \varphi(\ell)$$

for all $\ell \in \mathbb{Z}$. This completes the proof. □

3 Proof of Theorem 1.1

Assume that X is a finite indecomposable rack. We write $G = G_X$, $N = [G_X, G_X]$. Let A be an abelian group with trivial G-action and set $M = \mathrm{Fun}(X, A)$. Fix $x_0 \in X$ and $G \simeq N \rtimes \mathbb{Z}$ as in Lemma 2.3. It follows from Lemma 2.10 that

$$0 \to H^1(G/N, M^N) \xrightarrow{\inf} H^1(G, M) \xrightarrow{\mathrm{res}} H^1(N, M)^{G/N} \to 0$$

splits. We first identify the first term of this sequence.

Lemma 3.1 *$H^1(\mathbb{Z}, M^N) \simeq A$, via $f \mapsto f(1)(x_0)$.*

Proof Recall from Lemma 2.9(2) that $H^1(\mathbb{Z}, M^N) \simeq M^N/F$, where F is the submodule generated by $\{\varphi \cdot x_0^p - \varphi : p \in \mathbb{Z}, \varphi \in M^N\}$. Since $X = N \triangleright \{x_0\}$ by Corollary 2.4 and $n \triangleright x_0 = x \in X$ for some $n \in N$,

$$\varphi(x) = \varphi(n \triangleright x_0) = (\varphi \cdot n)(x_0)$$

for all $\varphi \in M$. Hence, if $\varphi \in M^N$, then $\varphi(x) = \varphi(x_0)$, $x \in X$. Consequently, $F = \{0\}$ and $H^1(\mathbb{Z}, M^N) \simeq M^N$. But $M^N \simeq A$ as any $\varphi \in M^N$ is determined by its value $\varphi(x_0) \in A$. Hence the lemma follows. □

As for the third term, we will show in Proposition 3.7 that

$$H^1(N, M)^{\mathbb{Z}} \simeq \mathrm{Hom}(N_0, A). \tag{3.1}$$

To do so, we first need several lemmas.

Lemma 3.2 *The map*

$$Z^1(N, M) \to \mathrm{Hom}(N_0, A), \quad f \mapsto f_0, \tag{3.2}$$

where $f_0(n_0) = f(n_0)(x_0)$ for $n_0 \in N_0$, is well-defined and factors to a map $H^1(N, M) \to$ $\mathrm{Hom}(N_0, A)$.

Proof We first prove that f_0 is indeed a group homomorphism:

$$\begin{aligned} f_0(n_0 n_0') &= f(n_0 n_0')(x_0) = (f(n_0) \cdot n_0')(x_0) + f(n_0')(x_0) \\ &= f(n_0)(n_0' \triangleright x_0) + f(n_0')(x_0) \\ &= f(n_0)(x_0) + f(n_0')(x_0) = f_0(n_0) + f_0(n_0'), \ n_0, n_0' \in N_0. \end{aligned}$$

We now show that the map factors to a map $H^1(N, M) \to \mathrm{Hom}(N_0, A)$. Let $f \in B^1(N, M)$, that is $f(n) = \varphi \cdot n - \varphi$ for some $\varphi \in M$. Then

$$\begin{aligned} f_0(n_0) &= f(n_0)(x_0) = (\varphi \cdot n_0)(x_0) - \varphi(x_0) \\ &= \varphi(n_0 \triangleright x_0) - \varphi(x_0) = \varphi(x_0) - \varphi(x_0) = 0. \end{aligned}$$

This completes the proof. □

Lemma 3.3 *The map* $H^1(N, M) \to \operatorname{Hom}(N_0, A)$*,* $f \mapsto f_0$*, is an injective group homomorphism.*

Proof It is clear that $f \mapsto f_0$ is a group homomorphism.

Let $f \in H^1(N, M)$ be such that $f_0 = 0$. That is, $f(n_0)(x_0) = 0$ for every $n_0 \in N_0$. We claim that there is $\varphi \in M$ such that $f(m) = (\varphi \cdot m) - \varphi$ and thus $f = 0$ in $H^1(N, M)$. Set

$$\varphi(x) := f(n)(x_0) \qquad \text{if } x = n \triangleright x_0.$$

Let us check that this is well-defined: if $x = n \triangleright x_0 = n' \triangleright x_0$, then $n^{-1}n' \in N_0$. Since $f(1) = 0$, one obtains that $f(n^{-1}) = -f(n) \cdot n^{-1}$. Then

$$\begin{aligned} 0 &= f_0(n^{-1}n') = f(n^{-1}n')(x_0) \\ &= -f(n)(n^{-1}n' \triangleright x_0) + f(n')(x_0) = -f(n)(x_0) + f(n')(x_0), \end{aligned}$$

and thus $\varphi(x)$ does not depend on $n \in N$ such that $x = n \triangleright x_0$. Finally for each $m \in N$ and every $x = n \triangleright x_0 \in X$ with $n \in N$,

$$\begin{aligned} (\varphi \cdot m - \varphi)(x) &= \varphi(m \triangleright x) - \varphi(x) = \varphi(m \triangleright n \triangleright x_0) - \varphi(n \triangleright x_0) \\ &= f(mn)(x_0) - f(n)(x_0) = (f(m) \cdot n)(x_0) + f(n)(x_0) - f(n)(x_0) \\ &= f(m)(n \triangleright x_0) = f(m)(x), \end{aligned}$$

and therefore $f = 0$. □

Recall the definition of the $\mathbb{Z}$-action on $H^1(N, M)$ from (2.4).

Lemma 3.4 *Assume* X *is a quandle. Then* $H^1(N, M) = H^1(N, M)^{\mathbb{Z}}$.

Proof Let $f \in H^1(N, M)$ and set $g = f - f \cdot x_0$. If $n_0 \in N_0$, then

$$g_0(n_0) = f(n_0)(x_0) - f(x_0 n_0 x_0^{-1})(x_0 \triangleright x_0) = 0.$$

Thus $g_0 = 0$ and hence $f = f \cdot x_0$ for all $f \in H^1(N, M)$ by Lemma 3.3, since the group homomorphism $g \mapsto g_0$ is injective. □

In order to show the surjectivity of the map $f \mapsto f_0$ from Lemma 3.3, we need to fix a decomposition of N into N_0-cosets

$$N = \bigsqcup_{i=0}^{k} \sigma_i N_0,$$

where $\sigma_i \in N$ is a representative, $\sigma_0 N_0 = N_0$. We define

$$\sigma : N \to \{\sigma_0, \dots, \sigma_k\}, \quad \sigma(n) = \sigma_i \quad \text{if } n \in \sigma_i N_0. \tag{3.3}$$

For $n \in N$ we consider $c(n) \in N_0$ defined by

$$n = \sigma(n)c(n). \tag{3.4}$$

Remark 3.5 For all $n \in N$ and $n_0 \in N_0$ it follows that $c(nn_0) = c(n)n_0$. Indeed, $nn_0 = \sigma(n)c(n)n_0 = \sigma(nn_0)c(nn_0)$ and thus the claim holds since each $m \in N$ decomposes uniquely as $m = \sigma(m)c(m)$.

Lemma 3.6 *The map* $H^1(N, M) \to \operatorname{Hom}(N_0, A)$*,* $f \mapsto f_0$*, is surjective.*

Proof Let $g\colon N_0 \to A$ be a group homomorphism; we shall construct an $f \in Z^1(N, M)$ such that $f_0 = g$. We claim that the map $f : N \to M, n \mapsto f(n)$, given by

$$f(n)(x) = g(c(nm)) - g(c(m)) = g(c(nm)c(m)^{-1}), \tag{3.5}$$

where $m \in N$ is such that $x = m \triangleright x_0$, is well-defined. Indeed, if $m' \in N$ also satisfies $x = m' \triangleright x_0$, then $m^{-1}m' \in N_0$ and thus $\sigma(m)^{-1}\sigma(m') \in N_0$. That is $\sigma(m) = \sigma(m')$ and thus $\sigma(nm) = \sigma(nm')$ for every $n \in N$ since $(nm')^{-1}nm \in N_0$. As g is a group homomorphism,

$$g(c(nm)c(m)^{-1}) - g(c(nm')c(m')^{-1}) = g(c(nm)c(m)^{-1}c(m')c(nm')^{-1}).$$

Now, $c(nm)c(m)^{-1}c(m')c(nm')^{-1}$ is, by definition,

$$(\sigma(nm)^{-1}nm)(m^{-1}\sigma(m))(\sigma(m')^{-1}m')(m'^{-1}n^{-1}\sigma(nm')) = 1.$$

Hence $g(c(nm)c(m)^{-1}) - g(c(nm')c(m')^{-1}) = g(1) = 0$ and thus f does not depend on the choice of m.

Now we show that $f \in Z^1(N, M)$. Let $x \in X$, $n, n' \in N$ and $m \in N$ be such that $x = m \triangleright x_0$. On the one hand, we have

$$f(nn')(x) = g(c(nn'm)) - g(c(m)).$$

On the other,

$$\begin{aligned}(f(n) \cdot n')(x) + f(n')(x) &= f(n)(n' \triangleright x) + f(n')(x)\\ &= g(c(nn'm)) - g(c(n'm)) + g(c(n'm)) - g(c(m))\\ &= f(nn')(x).\end{aligned}$$

Finally we see that $g = f_0$, that is $f_0(n) = g(n)$ for $n \in N_0$. Now, if $n \in N_0$, then $c(n) = c(1 \cdot n) = c(1)n$ *cf.* Remark 3.5. Also, as as $x_0 = 1 \triangleright x_0$,

$$\begin{aligned}f_0(n) = f(n)(x_0) &= g(c(n \cdot 1)c(1)^{-1}) = g(c(1 \cdot n)) - g(c(1))\\ &= g(c(1)n) - g(c(1)) = g(c(1)) + g(n) - g(c(1)) = g(n)\end{aligned}$$

and the lemma follows. □

Now we proceed to show (3.1).

Proposition 3.7 *The map $Z^1(N, M) \to \mathrm{Hom}(N_0, A)$ given by $f \mapsto f_0$, where $f_0(n_0) = f(n_0)(x_0)$ for $n_0 \in N_0$, induces a group isomorphism*

$$H^1(N, M)^{\mathbb{Z}} \to \mathrm{Hom}(N_0, A).$$

Proof Lemma 3.4 implies that $H^1(N, M)^{\mathbb{Z}} \simeq H^1(N, M)$ and Lemmas 3.3 and 3.6 yield $H^1(N, M) \simeq \mathrm{Hom}(N_0, A)$, as desired. □

This allows us to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Using the cocycle condition, we get

$$\begin{aligned}j(f)(\ell) &= \frac{1}{|N|}\sum_{n\in N}\left(f(n)\cdot x_0^{\ell} + f\left(x_0^{\ell}\right) - f(n)\right)\\ &= f\left(x_0^{\ell}\right) + \frac{1}{|N|}\sum_{n\in N}\left(f(n)\cdot x_0^{\ell} - f(n)\right).\end{aligned}$$

Hence, as X is a quandle, for each $\ell \in \mathbb{Z}$,

$$j(f)(\ell)(x_0) = f(x_0)(x_0). \tag{3.6}$$

Since $H^1(G/N, M^N) \simeq A$ by Lemma 3.1 and by Proposition 3.7 there exists an isomorphism $\zeta\colon H^1(N, M)^{\mathbb{Z}} \simeq \mathrm{Hom}(N_0, A)$, we write the inflation-restriction sequence (2.6) as

$$0 \to A \xrightarrow{\inf_0} H^1(G, M) \xrightarrow{\mathrm{res}_0} \mathrm{Hom}(N_0, A) \to 0, \tag{3.7}$$

where $\mathrm{res}_0(f) = \mathrm{res}(f)_0$ for all $f \in H^1(G, M)$ and $\inf_0$ is the composition $A \simeq M^N \simeq H^1(G/N, M^N)$. We set $f_0 = \mathrm{res}_0(f)$ by abuse of notation, i.e.

$$f_0(n_0) = f(n_0)(x_0), \quad n_0 \in N_0. \tag{3.8}$$

A retraction for $\inf_0$ is given by the composition

$$j_0\colon H^1(G, M) \xrightarrow{j} H^1(G/N, M^N) \simeq A,$$

using Lemmas 2.10 and 3.1, that is

$$j_0(f) = j(f)(1)(x_0) = f(x_0)(x_0), \tag{3.9}$$

cf. (3.6). Hence $H^1(G, M) \simeq A \times \mathrm{Hom}(N_0, A)$ via

$$f \mapsto (f(x_0)(x_0), f_0). \tag{3.10}$$

This completes the proof. □

4 Proof of Theorem 1.2

In this section we show the Reconstruction Theorem 1.2. We fix $x_0 \in X$ and write $N_0 \leq N_X$ for the stabilizer of x_0 in N_X. By Lemma 2.1, N_0 is a finite group.

The key for the proof of Theorem 1.2 lays in the existence of a particular class of decompositions

$$N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0$$

of N_X into N_0-cosets, which are *good* in our context.

Proposition 4.1 *Let X be a finite indecomposable quandle. Then there exists a decomposition $N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0$ of N_X into N_0-cosets such that the following hold:*

(1) $\sigma_0 = 1$.
(2) For each $i \in \{0, \dots, k\}$ there is $j \in \{0, \dots, k\}$ such that $x_0 \triangleright \sigma_i = \sigma_j$.
(3) For each $x \in X$ there is $j \in \{0, \dots, k\}$ such that $\sigma_j \triangleright x_0 = x$.

Proof Fix a decomposition into cosets $N_X = \bigsqcup_{i=0}^{k} \sigma_i N_0$. Recall from (3.3) and (3.4) the definition of the corresponding assignments

$$\sigma : N \to \{\sigma_0, \dots, \sigma_k\} \quad \text{and} \quad c : N \to N_0.$$

Since $\sigma_0 = 1$, (1) holds. Condition (3) also holds trivially: If $x \in X$, there is $n \in N_X$ is such $n \triangleright x_0 = x$ by Corollary 2.4. Now, there is $j \in \{0, \ldots, k\}$ such that $n \in \sigma_j N_0$, that is $n = \sigma_j n_0$ for some $n_0 \in N_0$. Then $x = n \triangleright x_0 = \sigma_j \triangleright (n_0 \triangleright x_0) = \sigma_j \triangleright x_0$.

For Condition (2), set $S = \{\sigma_1, \ldots, \sigma_k\}$. We define

$$t_j = t_j(S) := \min\{t \geq 1 : x_0^t \triangleright \sigma_j = \sigma_j\}$$

for all $j \in \{1, \ldots, k\}$. Observe that $1 \leq t_j(S) \leq \operatorname{ord} \varphi_{x_0}$, cf. §2.2. For $i \in \{1, \ldots, k\}$ and $t \in \{0, \ldots, t_j(S) - 1\}$ we define $\tau_{j,t} = x_0^t \triangleright \sigma_j$ and let

$$T = \{\tau_{j,t} : 1 \leq j \leq k,\ 1 \leq t < t_j(S)\}.$$

It is clear that $S \subseteq T$, as $\sigma_j = \tau_{j,0}$ by definition, and that if $S = T$, then we are done. Notice that this is not a multi-set: we may have $\tau_{j,t} = \tau_{j',t'}$, for different (j, t), (j', t'). On the other hand, if $t \neq t'$, then $\tau_{j,t} \neq \tau_{j,t'}$ for every j, since $t < t_j(S)$. In other words, there are $r \leq k$, $1 = i_1 < i_2 < \cdots < i_r$ and $s_j \leq t_{i_j}$, $1 \leq j \leq r$ such that

$$T = \{\tau_{i_j,t} : 1 \leq j \leq r,\ 1 \leq t < s_j\}$$

and $\tau_{i_j,t} \neq \tau_{i_{j'},t'}$ if $j \neq j'$ or $t \neq t'$. We reorder the set S so $i_j = j$, $j = 1, \ldots, r$. If $S \neq T$, then we proceed inductively: we order T by:

$$\tau_{i,s} \prec \tau_{j,t} \iff i < j \text{ or } i = j \text{ and } s < t.$$

Let $\tau = \min\{\tau_{j,t} : \tau_{j,t} \notin S\}$ and let ℓ be such that $\sigma(\tau) = \sigma_\ell$, i.e. $\tau_{j,t} = x_0^t \triangleright \sigma_j \in \sigma_\ell N$ and $\tau_{j,t} \neq \sigma_\ell$. Observe that if $\tau = \tau_{j,t}$, then $\ell \neq j$. Set $S_0 = S$ and $T_0 = T$. We make a new choice of representatives replacing the original set S_0 by

$$S_1 = (S_0 \setminus \{\sigma_\ell\}) \cup \{\tau\} = \{\sigma_1, \ldots, \sigma_{\ell-1}, \tau, \sigma_{\ell+1}, \ldots, \sigma_k\}.$$

Define $t_j(S_1)$ and $(T_1, \prec)$ accordingly. We claim that $t_j(S_1) \leq t_j(S_0)$ for all j. Indeed, equality holds if $j \neq \ell$ and it readily follows that

$$t_\ell(S_1) = t_\ell(S_0) - t < t_\ell(S_0).$$

In particular, it follows that $|S| = |S_1| \leq |T_1| < |T_0|$. (This also follows as when constructing T_1 we are removing all the $\tau_{\ell,t}$.) If $T_1 = S_1$, then we are done. Otherwise, we repeat this procedure until we end up with $S_p = T_p$ for some $p > 1$. Then S_p becomes the set of representatives we searched for. □

We say that a decomposition of N_X into N_0-cosets satisfying the conditions in Proposition 4.1 is *good*.

If $N_X = \bigsqcup_{i=0}^k \sigma_i N_0$ is a good decomposition, then for each $y \in X$ we set

$$\sigma_y := \sigma_j. \tag{4.1}$$

for $j \in \{0, \ldots, k\}$ such that $\sigma_j \triangleright x_0 = y$.

Lemma 4.2 *If $N_X = \bigsqcup_{i=0}^k \sigma_i N_0$ is good, then*

$$c(x_0 \triangleright n) = c(n).$$

Proof Indeed, $x_0 \triangleright n = x_0\sigma(n)x_0^{-1}c(n)$, as $c(n) \in N_0$ and $x_0\sigma(n)x_0^{-1} = \sigma_i$, for some $i \in \{0, \ldots, k\}$. □

Recall the definition of the group homomorphism $\mathrm{d} : G_X \to \mathbb{Z}$ from (2.1).

Lemma 4.3 *For each $u \in G_X$ and $y \in X$,*

$$\sigma_{u \triangleright y} = u\sigma_y x_0^{-\mathrm{d}(u)} c\left(u\sigma_y x_0^{-\mathrm{d}(u)}\right)^{-1}.$$

In particular if $n \in N$, then $\sigma_{n \triangleright y} = \sigma\left(n\sigma_y\right)$.

Proof Since

$$\sigma_{u \triangleright y} \triangleright x_0 = u \triangleright y = u \triangleright (\sigma_y \triangleright x_0) = (u\sigma_y) \triangleright x_0 = (u\sigma_y x_0^{-\mathrm{d}(u)}) \triangleright x_0$$

and $u\sigma_y x_0^{-\mathrm{d}(u)} \in N$, it follows that $\sigma_{u \triangleright y} = \sigma(u\sigma_y x_0^{-\mathrm{d}(u)})$. Then

$$u\sigma_y x_0^{-\mathrm{d}(u)} = \sigma_{u \triangleright y} c\left(u\sigma_y x_0^{-\mathrm{d}(u)}\right),$$

and the first claim follows. If $n \in N$, then $\mathrm{d}(n) = 0$ and therefore it follows that $\sigma_{n \triangleright y} = n\sigma_y c\left(n\sigma_y\right)^{-1} = \sigma\left(n\sigma_y\right)$ *cf.* (3.4). □

We can now proceed to prove Theorem 1.2.

Proof of Theorem 1.2 We need to define an inverse to the map (3.10). Fix $a \in A$, $g \in \mathrm{Hom}(N_0, A)$ and set $f : G \to M$ as

$$f(u)(y) := \mathrm{d}(u)\, a + g\left(c\left(u\sigma_y x_0^{-\mathrm{d}(u)}\right)\right),$$

for each $u \in G$. We show that $f \in Z^1(G, M)$ and $f \mapsto (a, g)$ via (3.10).

On the one hand, as $\sigma_{x_0} = \sigma_0 = 1$,

$$f(x_0)(x_0) = a + g\left(c\left(x_0 x_0^{-1}\right)\right) = a.$$

On the other, if $n_0 \in N_0$, then $\mathrm{d}(n_0) = 0$ and thus

$$f_0(n_0) = f(n_0)(x_0) = g(c(n_0)) = g(n_0).$$

Now we check the cocycle condition. First,

$$f(uu')(y) = \mathrm{d}(uu')a + g\left(c\left(uu'\sigma_y x_0^{-\mathrm{d}(uu')}\right)\right).$$

Second,

$$\begin{aligned}
(f(u) \cdot u')(y) + f(u')(y) &= f(u)(u' \triangleright y) + f(u')(y) \\
&= \mathrm{d}(u)a + g\left(c\left(u\sigma_{u' \triangleright y} x_0^{-\mathrm{d}(u)}\right)\right) + \mathrm{d}(u')a + g\left(c\left(u'\sigma_y x_0^{-\mathrm{d}(u')}\right)\right) \\
&= f(uu')(y),
\end{aligned}$$

since A is abelian, d and g are a group homomorphisms and

$$c\left(u\sigma_{u' \triangleright y} x_0^{-\mathrm{d}(u)}\right) = c\left(uu'\sigma_y x_0^{-\mathrm{d}(uu')}\right) c\left(u'\sigma_y x_0^{-\mathrm{d}(u')}\right)^{-1}$$

by Lemma 4.3. Hence $f \in Z^1(G, M)$. □

5 Applications

Our method for computing the 2nd cohomology group of an indecomposable quandle X involves the group N_0, see Sect. 1.3. In several important cases, this group can be obtained applying the following lemma.

Lemma 5.1 *Let X be a finite indecomposable quandle and $x_0 \in X$. Assume that the canonical map $X \to G_X$ is injective. Then*

$$N_0 \simeq [F_X, F_X] \cap C_{F_X}(\psi(x_0)),$$

where $\psi\colon X \to G_X \to F_X$ is the composition of the canonical maps and $C_{F_X}(\psi(x_0))$ is the centralizer of $\psi(x_0)$ in F_X.

Proof Since $X \to G_X$ is injective and X is indecomposable, X can be identified with the conjugacy class of x_0 in G_X. By [15, Lemma 1.8], X can also be identified with the conjugacy class of $\psi(x_0)$ in F_X. From Lemma 2.1 one obtains that $N_X = [G_X, G_X] \simeq [F_X, F_X]$ and thus the claim follows.

Remark 5.2 If X is a conjugation quandle, then the canonical map $X \to G_X$ is injective. Thus Lemma 5.1 gives a nice description of N_0 in the case of finite indecomposable conjugation quandles.

Example 5.3 The claim of Lemma 5.1 does not hold for arbitrary quandles. Let X be the quandle $\{x_1, x_2, x_3, x_4\}$ with the structure given by

$$\varphi_{x_1} = (x_2x_3x_4), \quad \varphi_{x_2} = (x_1x_4x_3), \quad \varphi_{x_3} = (x_1x_2x_4), \quad \varphi_{x_4} = (x_1x_3x_2).$$

This quandle is isomorphic to the conjugacy class of 3-cycles in $\mathbb{A}_4$. Let $f\colon X \times X \to \mathbb{C}^\times$ be the map given by

$$f(x, y) = \begin{cases} 1 & \text{if } x = x_1 \text{ or } y = x_1 \text{ or } x = y, \\ -1 & \text{otherwise.} \end{cases}$$

Then f is a 2-cocycle of X with values in $\{-1, 1\} \simeq \mathbb{Z}_2$, see [2, Example 2.2]. Let $Y = X \times \{-1, 1\}$ be the quandle given by

$$(x, i) \triangleright (y, j) = (x \triangleright y, jf(x, y)), \quad x, y \in X,\ i, j \in \{-1, 1\}.$$

Then the canonical map $Y \to G_Y$ is not injective. Indeed,

$$(x_2, 1)(x_3, -1) = (x_1, 1)(x_2, 1) = (x_3, 1)(x_1, 1) = (x_2, 1)(x_3, 1)$$

implies that $(x_3, -1) = (x_3, 1)$ in G_Y.

Fix $y_0 \in Y$. A straighforward calculation shows that $F_Y \simeq \mathbf{SL}(2, 3)$ and $[F_Y, F_Y] \cap C_{F_Y}(\psi(y_0)) \simeq \mathbb{Z}_2$. However, since $[F_Y, F_Y]$ and Y both have eight elements, N_0 is the trivial group.

5.1 Transpositions in $\mathbb{S}_n$

Let $X = (12)^{\mathbb{S}_n}$ be the quandle of transpositions in the symmetric group $\mathbb{S}_n$. For $n \geq 4$ a non-constant 2-cocycle $\chi \in H^2(X, \mathbb{C}^\times)$ was constructed in [21]. This cocycle is given by

$$\chi(\sigma, \tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\tau = (ij)$, $1 \leq i < j \leq n$.

Lemma 5.4 *Let $X = (12)^{\mathbb{S}_n}$, $n \geq 4$, and fix $x_0 = (12) \in X$.*

(1) $F_X \simeq \mathbb{S}_n$. Hence $N_X \simeq \mathbb{A}_n$.
(2) $N_0 \simeq \mathbb{Z}_2 \ltimes \mathbb{A}_{n-2}$. In particular, $N_0/[N_0, N_0] \simeq \mathbb{Z}_2$.

Proof Recall that $\mathbb{S}_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i < n-1,$$
$$\sigma_k\sigma_j = \sigma_j\sigma_k, \quad 1 \leq j, k < n, \ |j-k| > 1,$$
$$\sigma_i^2 = 1, \quad 1 \leq i < n.$$

Set $\iota : X \hookrightarrow \mathbb{S}_n$ the canonical inclusion let $\varphi : \langle X \rangle \to \mathbb{S}_n$ the unique group homomorphism with $\varphi_{|X} = \iota$. This is in fact an epimorphism. Observe that $\varphi(x^2) = \iota(x)^2 = 1$ and

$$\varphi(xy) = \iota(x)\iota(y) = \iota(x)\iota(y)\iota(x)^{-1}\iota(x) = \iota(x \triangleright y)\iota(x) = \varphi((x \triangleright y)x).$$

Thus, φ factors through $\phi : F_X \twoheadrightarrow \mathbb{S}_n$. Now, set S be the free group on $s_1, \dots, s_{n-1}$ and let $\psi' : S \to F_X$ be the group epihomomorphism given by $s_i \mapsto (i\; i+1)$. Now ψ' factors through $\psi : \mathbb{S}_n \twoheadrightarrow F_X$ and it is clear that ϕ and ψ are inverses to each other.

Let us prove the second claim. By the first part, we identify N with $\mathbb{A}_n$. Consider $\mathbb{A}_{n-2} \leq \mathbb{A}_n$ as those permutations fixing 1 and 2 and set $t = (12)(34)$. Then $t\sigma t^{-1} \in \mathbb{A}_{n-2}$ for all $\sigma \in \mathbb{A}_{n-2}$. Clearly $\langle t \rangle \ltimes \mathbb{A}_{n-2} \leq N_0$.

Since $\mathbb{A}_n$ is generated by $\{(34\ell) \mid 1 \leq \ell \leq n, \ \ell \neq 3, 4\}$, the group N is generated by the subgroups $\mathbb{A}_{n-2}$ and $\mathbb{A}_4 \simeq \langle (134), (234) \rangle$. Notice that $\langle (134), (234) \rangle \cap N_0 \simeq \langle t \rangle$. We have $|\langle t \rangle \ltimes \mathbb{A}_{n-2}| = (n-2)!$ and

$$\{\sigma(12)\sigma^{-1} : \sigma \in \mathbb{A}_n\} = (12)^{\mathbb{S}_n}.$$

Thus $|N_0| = |N|/|(12)^{\mathbb{S}_n}| = (n-2)!$ and hence $N_0 = \langle t \rangle \ltimes \mathbb{A}_{n-2}$.

Finally, since the commutator subgroup of some group $A \ltimes B$ is the group generated by $[A, A] \cup [A, B] \cup [B, B]$ and $N_0 = \langle t \rangle \ltimes \mathbb{A}_{n-2}$, it follows that $[N_0, N_0] \simeq \mathbb{A}_{n-2}$ and hence $N_0/[N_0, N_0] \simeq \mathbb{Z}_2$. □

Theorem 5.5 *Let $n \geq 4$ and $X = (12)^{\mathbb{S}_n}$ be the conjugacy class of transpositions. Then $H^2(X, \mathbb{C}^\times) \simeq \mathbb{C}^\times \times \langle \chi \rangle$.*

Proof Set $x_0 = (12) \in \mathbb{S}_n$. Since $N_0 \simeq \mathbb{Z}_2 \ltimes \mathbb{A}_{n-2}$ and $N_0/[N_0, N_0] \simeq \mathbb{Z}_2$ by Lemma 5.4, it follows that $\mathrm{Hom}(N_0, \mathbb{C}^\times) \simeq \mathbb{Z}_2$. Applying the isomorphism (1.3) of Theorem 1.1 to the 2-cocycle χ given in (5.1),

$$\chi \mapsto (-1, (f_\chi)_0),$$

where $(f_\chi)_0 \colon N_0 \to \mathbb{C}^\times$, $n_0 \mapsto f_\chi(n_0)(x_0)$, $n_0 \in N_0$. Now the claim follows since $(f_\chi)_0$ generates $\mathrm{Hom}(N_0, \mathbb{C}^\times)$. Indeed, $f_\chi \neq 1$ since

$$\begin{aligned}(f_\chi)_0((12)(34)) &= f_\chi((12)(34))(12) \overset{(2.3)}{=} \chi((12), (34) \triangleright (12))\chi((34), (12)) \\ &= \chi((12), (12))\chi((34), (12)) = -1.\end{aligned}$$

This completes the proof. □

5.2 Eisermann formula

We give a new proof of a formula of Eisermann as a consequence of our results.

Theorem 5.6 [8, Theorem 1.12] *Let X be a finite indecomposable quandle and $x_0 \in X$. Then*

$$H_2^Q(X,\mathbb{Z}) \simeq \big([G_X, G_X] \cap C_{G_X}(x_0)\big)_{\mathrm{ab}} \simeq (N_0)_{\mathrm{ab}},$$

where N_0 is the stabilizer of a given $x_0 \in X$ of the action of $[G_X, G_X]$ on X.

Proof The claim follows by "chasing" the chain of equivalences

$$A \times \mathrm{Hom}(N_0, A) \simeq H^2(X, A) \simeq \mathrm{Hom}(H_2(X,\mathbb{Z}), A)$$

given by the application of Theorem 1.2 and Lemma 2.5. More explicitly, if $(a, g) \in A \times \mathrm{Hom}(N_0, A)$, then it defines $q \in H^2(X, A)$ via (1.4), which in turn defines a morphism $H_2(X,\mathbb{Z}) \to A$ by Lemma 2.5:

$$[x, y] \mapsto q_{x,y} = a + g\left(c\left(x\sigma_y x_0^{-1}\right)\right) \in A,$$

cf. Theorem 1.2. Now, $H_2(X,\mathbb{Z}) \simeq H_2^Q(X,\mathbb{Z}) \times \mathbb{Z}$ by Lemma 2.6 and so this assignment becomes a map in $\mathrm{Hom}(H_2^Q(X,\mathbb{Z}) \times \mathbb{Z}, A)$:

$$([x, y], \ell) \longmapsto \ell a + g\left(c\left(x\sigma_y x_0^{-1}\right)\right).$$

Thus we see that the restriction of this map to $H_2^Q(X,\mathbb{Z}) \times \{0\}$ gives an equivalence $\mathrm{Hom}(H_2^Q(X,\mathbb{Z}), A) \simeq \mathrm{Hom}(N_0, A) \simeq (N_0)_{\mathrm{ab}}$ for any abelian group A. Hence we derive Eisermann's formula $H_2^Q(X,\mathbb{Z}) \simeq (N_0)_{\mathrm{ab}}$. □

If we combine this fact with Lemma 2.6, we obtain the following.

Corollary 5.7 *Let X be a finite indecomposable quandle, $x_0 \in X$. Then $H_2(X,\mathbb{Z}) \simeq (N_0)_{\mathrm{ab}} \times \mathbb{Z}$.*

5.3 Affine quandles

Let L be an abelian group and $\gamma \in \mathrm{Aut}(L)$. The *affine* (or *Alexander*) quandle $\mathrm{Aff}(L, \gamma)$ is the set L together with the action

$$x \triangleright y = \gamma(y) + x - \gamma(x), \quad x, y \in L.$$

In [7] Clauwens described the enveloping group of an affine quandle; we review his construction next. Set

$$\begin{aligned} \tau_\gamma &: L \otimes_{\mathbb{Z}} L \to L \otimes_{\mathbb{Z}} L, \quad (x, y) \mapsto (x, y) - (y, \gamma(x)), \\ S(L, \gamma) &:= \operatorname{coker} \tau_\gamma = L \otimes_{\mathbb{Z}} L / \langle (x, y) - (y, \gamma(x)) \rangle. \end{aligned} \tag{5.2}$$

We write $[x, y] \in S(L, \gamma)$ for the class of an element $x \otimes y \in L \otimes_{\mathbb{Z}} L$. Set $X = \mathrm{Aff}(L, \gamma)$; then G_X is the set $L \rtimes \mathbb{Z} \times S(L, \gamma)$ with multiplication

$$(x, m, [p, q])\,(y, n, [r, s]) = \left(x + \gamma^m(y), m + n, [p + r + x, q + s + \gamma^m(y)]\right),$$

for $m, n \in \mathbb{Z}$, $x, y \in L$, $[p, q], [r, s] \in S(L, \gamma)$.

The rack X identifies with the subset $L \rtimes \{1\} \times 0$ with the rack action given by conjugation:

$$\begin{aligned}(x, 1, 0)(y, 1, 0) &= (x + \gamma(y), 2, [x, \gamma(y)]) = (x + \gamma(y), 2, [x \triangleright y, \gamma(x)])\\ &= (x + \gamma(y) - \gamma(x) + \gamma(x), 2, [x \triangleright y, \gamma(x)])\\ &= (x \triangleright y, 1, 0)(x, 1, 0)\end{aligned}$$

since $[x \triangleright y, \gamma(x)] = [\gamma(y), \gamma(x)] + [x, \gamma(x)] - [\gamma(x), \gamma(x)] = [x, \gamma(y)]$, as $[x, \gamma(x)] = [\gamma(x), \gamma(x)]$. We fix $x_0 = (0, 1, 0)$; then

$$N_X = L \times \{0\} \times S(L, \gamma), \quad N_0 = \{0\} \times \{0\} \times S(L, \gamma). \tag{5.3}$$

Let $\{x_0, x_1, \dots, x_n\}$ be an enumeration of the elements of L. In particular,

$$N_X = \bigsqcup_{i\in\{0,\dots,n\}} \sigma_i N_0 \simeq L \times \operatorname{coker}\ \tau_\gamma, \quad \sigma_i = (x_i, 0, 0), \tag{5.4}$$

is a good decomposition of N into N_0-cosets, *cf.* Proposition 4.1. Indeed,

(1) $\sigma_0 = (0, 0, 0)$ coincides with the unit element in G_X;
(2) fix $j \in \{0, \dots, n\}$ and let $k \in \{0, \dots, n\}$ be such that $x_k = \gamma(x_j)$. Then $x_0 \triangleright \sigma_j = (0, 1, 0)(x_j, 0, 0)(0, -1, 0) = (\gamma(x_j), 0, 0) = \sigma_k$; and
(3) if $i \in \{0, \dots, n\}$ and $x_j = (1 - \gamma)^{-1}(x_i)$, then $\sigma_j \triangleright x_0 = x_i$.

Recall from (4.1) the definition of the elements σ_y, $y \in X$, and from (3.4) the map $c\colon N_X \to N_0$. We see from Item (3) above that in this case

$$\sigma_y = \left((1 - \gamma)^{-1}(y), 0, 0\right), \quad y \in X.$$

As a direct consequence of Theorem 1.2, we obtain the following.

Proposition 5.8 *Let L be an abelian group, $\gamma \in \operatorname{Aut}(L)$ and $X = \operatorname{Aff}(L, \gamma)$ be the corresponding affine quandle and set $\Gamma = S(L, \gamma)$ as in (5.2). Fix $x_0 = 0 \in X$ and let A be an abelian group with trivial G_X-action. Consider a decomposition of N_X into N_0-cosets as in (5.4). For each $a \in A$ and $g \in \operatorname{Hom}(\Gamma, A)$, the map $q\colon X \times X \to A$ given by*

$$q_{x,y} = a + \sum_{0<j<\operatorname{ord}(\gamma)} g\left([x, \gamma^j(y)]\right) \tag{5.5}$$

is a 2-cocycle of X and any $q \in H^2(X, A)$ arises in this way.

Proof By Theorem 1.2 and Corollary 1.3, any 2-cocycle is of the form

$$q_{x,y} = a + g(c(x\sigma_y x_0^{-1})).$$

for some $a \in A$ and $g \in \operatorname{Hom}(\Gamma, A)$. Using the identifications above, we have

$$\begin{aligned}x\sigma_y x_0^{-1} &= (x, 1, 0)((1 - \gamma)^{-1}(y), 0, 0)(0, -1, 0)\\ &= (x + \gamma(1 - \gamma)^{-1}(y), 0, [x, \gamma(1 - \gamma)^{-1}(y)])\\ &= \sigma_k(0, 0, [x, \gamma(1 - \gamma)^{-1}(y)]) \in \sigma_k N_0\end{aligned}$$

for $k \in \{0, \dots, n\}$ such that $x + \gamma(1 - \gamma)^{-1}(y) = x_k$. Hence

$$\gamma(1 - \gamma)^{-1}(y) = (1 - \gamma)^{-1}(y) - y = \sum_{0<j<\operatorname{ord}(\gamma)} \gamma^j(y)$$

and the result follows. □

If $L = \mathbb{F}_q$ is the finite field of q elements and γ is the multiplication by some $1 \neq \omega \in \mathbb{F}_q^{\times}$, we write $\mathrm{Aff}(q, \omega) = \mathrm{Aff}(L, \gamma)$.

Lemma 5.9 *Let p be a prime number and $1 \neq \omega \in \mathbb{F}_p^{\times}$, set $X = \mathrm{Aff}(p, \omega)$. Then $G_X \simeq L \rtimes \mathbb{Z}$, $N_X \simeq L$ and N_0 is trivial.*

Proof Indeed, $S(L, \gamma)$ is a quotient of $\mathbb{Z}_p \simeq \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and we have that $0 \neq (1 - \omega) \otimes 1 \in \mathrm{Im}(\tau_\gamma)$, hence $S(L, \gamma) = 0$ and the lemma follows. □

We recover the following result from [13, Lemma 5.1].

Proposition 5.10 $H^2(\mathrm{Aff}(p, \omega), \mathbb{C}^{\times}) \simeq \mathbb{C}^{\times}$.

Proof It follows from Theorem 1.1, using Lemma 5.9. □

5.4 Indecomposable quandles of size p^2

Let p be a prime number and let X be an indecomposable quandle of size p^2. By [13], X is one of the following affine quandles (L, γ) in the following list:

$$L = \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad \gamma_{\alpha,\beta}(x, y) = (\alpha\, x, \beta\, y), \quad \alpha, \beta \in \mathbb{Z}_p^{*}\backslash\{1\}; \tag{5.6}$$

$$L = \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad \gamma_{\alpha}(x, y) = (\alpha\, x, \alpha\, y + x), \quad \alpha \in \mathbb{Z}_p^{*}\backslash\{1\}; \tag{5.7}$$

$$L = \mathbb{F}_{p^2}, \quad \gamma_\alpha(x) = \alpha\, x, \quad \alpha \in \mathbb{F}_{p^2}\backslash\mathbb{F}_p; \tag{5.8}$$

$$L = \mathbb{Z}_{p^2}, \quad \gamma_\alpha(x) = \alpha\, x, \quad \alpha \not\equiv 0, 1\,(p). \tag{5.9}$$

We identify $\mathbb{F}_{p^2} \simeq \mathbb{F}_p \oplus \mathbb{F}_p$ as abelian groups for notational reasons. For $\alpha = (\alpha_0, \alpha_1) \in \mathbb{F}_p^2$ we set

$$d_\alpha := (1 - \alpha_0 + \alpha_1)(1 - \alpha_0 - \alpha_1)\left(1 - \alpha_0^2 + \alpha_1^2\right). \tag{5.10}$$

Assume $\alpha \in \mathbb{F}_{p^2}\backslash\mathbb{F}_p$, so $\alpha_1 \neq 0$. If $d_\alpha = 0$, then $\alpha_0 \neq 1$ and we set

$$t_\alpha := \left(\alpha_0 - \alpha_0^2 + \alpha_1^2\right)(1 - \alpha_0)^{-1}, \quad s_\alpha := (1 - \alpha_0)\alpha_1^{-1}. \tag{5.11}$$

Proposition 5.11 *The 2nd homology groups of the indecomposable quandles of order p^2 are as follows:*

$$H_2\left((\mathbb{Z}_p \oplus \mathbb{Z}_p, \gamma_{\alpha,\beta}), \mathbb{Z}\right) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Z}_p, & \text{if } \alpha\beta = 1, \\ \mathbb{Z}, & \text{if } \alpha\beta \neq 1. \end{cases}$$

$$H_2\left((\mathbb{Z}_p \oplus \mathbb{Z}_p, \gamma_{\alpha}), \mathbb{Z}\right) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Z}_p, & \text{if } \alpha^2 = 1, \\ \mathbb{Z}, & \text{if } \alpha^2 \neq 1. \end{cases}$$

$$H_2\left((\mathbb{F}_{p^2}, \gamma_{\alpha}), \mathbb{Z}\right) \simeq \begin{cases} \mathbb{Z} \times \mathbb{Z}_p, & \text{if } d_\alpha = 0, \\ \mathbb{Z}, & \text{if } d_\alpha \neq 0. \end{cases}$$

$$H_2\left((\mathbb{Z}_{p^2}, \gamma_{\alpha}), \mathbb{Z}\right) \simeq \mathbb{Z}.$$

Proof By Corollary (5.7), if $X = (L, \gamma)$ and $\tau_\gamma : L \otimes_{\mathbb{Z}} L \to L \otimes_{\mathbb{Z}} L$ as in (5.2), then

$$H_2(X, \mathbb{Z}) = (N_0)_{\mathrm{ab}} \times \mathbb{Z} = \mathrm{coker}\ \tau_\gamma \times \mathbb{Z}$$

We compute coker τ_γ case by case. We will use the identifications

$$\begin{aligned}\left(\mathbb{Z}_p \oplus \mathbb{Z}_p\right) \otimes_{\mathbb{Z}} \left(\mathbb{Z}_p \oplus \mathbb{Z}_p\right) \simeq \mathbb{Z}_p^4, &\quad (a,b)\otimes(c,d) \mapsto (ac, ad, bc, bd)\\ \mathbb{F}_{p^2} \otimes_{\mathbb{Z}} \mathbb{F}_{p^2} \simeq \mathbb{F}_p^2 \otimes_{\mathbb{F}_p} \mathbb{F}_p^2 \simeq \mathbb{F}_p^4, &\quad (a,b)\otimes(c,d) \mapsto (ac, ad, bc, bd)\\ \mathbb{Z}_{p^2} \otimes_{\mathbb{Z}} \mathbb{Z}_{p^2} \simeq \mathbb{Z}_{p^2}, &\quad a\otimes b \mapsto ab.\end{aligned} \tag{5.12}$$

Case (5.6): We have that $\tau_{\alpha,\beta} := \tau_{\gamma_{\alpha,\beta}}$ is

$$\tau_{\alpha,\beta}((a,b)\otimes(c,d)) = (a,b)\otimes(c,d) - (c,d)\otimes(\alpha\, a, \beta\, b).$$

With the identifications above this yields

$$\tau_{\alpha,\beta} : \mathbb{Z}_p^4 \to \mathbb{Z}_p^4, \quad (x,y,z,w) \mapsto ((1-\alpha)x, y - \beta\, z, z - \alpha\, y, (1-\beta)\, w).$$

Next, we compute the image $I_{\alpha,\beta}$ of this map: For $(a,b,c,d) \in \mathbb{Z}_p^4$ to be in this subgroup, we need $x = a(1-\alpha)^{-1}$, $w = d(1-\beta)^{-1}$ (recall $\alpha, \beta \neq 1$) and y, z to be a solution of $y - \beta\, z = b$, $-\alpha\, y + z = c$. This system has always a solution if $\alpha\beta \neq 1$. If $\alpha\beta = 1$, then

$$I_{\alpha,\beta} = \{(a, b, -\alpha\, b, d) | a, b, d \in \mathbb{Z}_p\} \simeq \mathbb{Z}_p^3, \text{ hence}$$

$$\operatorname{coker}\ \tau_\alpha = \begin{cases} 0, & \text{if } \alpha\beta \neq 1,\\ \mathbb{Z}_p, & \text{if } \alpha\beta = 1.\end{cases}$$

In case (5.7), we have $\tau_\alpha := \tau_{\gamma_\alpha} : \mathbb{Z}_p^4 \to \mathbb{Z}_p^4$ is given by

$$(x,y,z,w) \mapsto \big((1-\alpha)x, y - \alpha\, z + x, z - \alpha\, y, (1-\alpha)w - y\big).$$

For (a,b,c,d) to be in the image I_α of τ_α, we need $x = a(1-\alpha)^{-1}$ (recall $\alpha \neq 1$) and (y,z,w) to be a solution of

$$y - \alpha\, z = b - a(1-\alpha)^{-1}, \quad -\alpha\, y + z = c, \quad -y + (1-\alpha)w = d.$$

This system has always a solution if $\alpha^2 \neq 1$. If $\alpha^2 = 1$, then

$$I_\alpha = \{(a, b, \alpha\, b - \alpha(1-\alpha)a, d) | a, b, d \in \mathbb{Z}_p\} \simeq \mathbb{Z}_p^3, \text{ hence}$$

$$\operatorname{coker}\ \tau_\alpha = \begin{cases} 0, & \text{if } \alpha^2 \neq 1,\\ \mathbb{Z}_p, & \text{if } \alpha^2 = 1.\end{cases}$$

In case (5.8), if $\alpha = (\alpha_0, \alpha_1) \in \mathbb{F}_p^2 \backslash \mathbb{F}_p$ (hence $\alpha_1 \neq 0$), then the map $\tau_\alpha \in \operatorname{End}(\mathbb{F}_p^4)$ is represented by the matrix

$$[\tau_\alpha] = \begin{pmatrix} 1-\alpha_0 & 0 & -\alpha_1 & 0\\ -\alpha_1 & 1 & -\alpha_0 & 0\\ 0 & -\alpha_0 & 1 & -\alpha_1\\ 0 & -\alpha_1 & 0 & 1-\alpha_0 \end{pmatrix},$$

with $\det[\tau_\alpha] = d_\alpha$, see (5.10). Let I_α denote the image of this map. Now, the rank of this matrix is ≥ 3, as $\det \begin{pmatrix} 0 & -\alpha_1 & 0\\ 1 & -\alpha_0 & 0\\ -\alpha_0 & 1 & -\alpha_1 \end{pmatrix} = -\alpha_1^2 \neq 0$. Hence,

$$\operatorname{coker}\ \tau_\alpha = \begin{cases} 0, & \text{if } \det[\tau_\alpha] \neq 0,\\ \mathbb{Z}_p, & \text{if } \det[\tau_\alpha] = 0.\end{cases}$$

If $\det[\tau_\alpha] = 0$, i.e. $d_\alpha = 0$, then we set $t_\alpha, s_\alpha \in \mathbb{F}_p$ as in (5.11) and thus

$$I_\alpha = \{(a, b, c, -t_\alpha(a+b) - s_\alpha c) | a, b, c \in \mathbb{Z}_p\} \simeq \mathbb{Z}_p^3.$$

In case (5.9), $\tau_\alpha : \mathbb{Z}_{p^2} \to \mathbb{Z}_{p^2}$ is $x \mapsto (1-\alpha)x$; hence coker $\tau_\alpha = 0$. □

5.5 Explicit cocycles

Next we apply Proposition 5.8 to compute all non-constant 2-cocycles for the affine quandles X described in (5.6)–(5.9). More precisely, we focus on those affine quandles in that list admitting a non-constant 2-cocyle, as stated in Proposition 5.11:

$$L = \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad \gamma_\alpha(x, y) = (\alpha\, x, \alpha^{-1}\, y), \quad \alpha \in \mathbb{Z}_p^* \backslash \{1\}; \tag{5.13}$$

$$L = \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad \gamma(x, y) = (-x, x - y); \tag{5.14}$$

$$L = \mathbb{F}_{p^2}, \quad \gamma_\alpha(x) = \alpha\, x, \quad \alpha \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p, d_\alpha = 0. \tag{5.15}$$

Recall our identification $\mathbb{F}_{p^2} \simeq \mathbb{F}_p^2$, $x \mapsto (x_0, x_1)$, and $t_\alpha, s_\alpha \in \mathbb{Z}_p$ from (5.11). For $x, y \in L$ and $j \in \mathbb{N}$ we set, for X is as in (5.13),

$$\zeta_j(x, y) = \alpha^j x_2 y_1 + \alpha^{1-j} x_1 y_2;$$

for X is as in (5.14),

$$\zeta_j(x, y) = (j + 2(-1)^j) x_1 y_1 + (-1)^j (x_1 y_2 - x_2 y_1);$$

and, for X is as in (5.15),

$$\zeta_j(x, y) = x_1 (\alpha^j y)_1 + t_\alpha \left(x_0 (\alpha^j y)_0 + x_0 (\alpha^j y)_1 \right) + s_\alpha\, x_1 (\alpha^j y)_0.$$

Next, we define the map $\langle\, ,\, \rangle : L \times L \to \mathbb{Z}$ as

$$\langle x, y \rangle = \sum_{0 < j < \mathrm{ord}(\gamma)} \zeta_j(x, y), \quad x, y \in L.$$

Notice that $\mathrm{ord}(\gamma) = p - 1$, $2p$ (or 2 if $p = 2$) or $p^2 - 1$ according to whether X is as in (5.13), (5.14) or (5.15), respectively.

Proposition 5.12 *Let $X = (L, \gamma)$ be an indecomposable affine rack of order p^2. If $q \in H^2(X, \Bbbk^*)$ is non-constant, then X belongs to the list (5.13)–(5.15) and there are $0 < \ell < p$ and $\lambda \in \Bbbk^*$ such that*

$$q_{x,y} = \lambda \exp\left(\frac{2\pi i \ell}{p} \langle x, y \rangle \right), \quad x, y \in X. \tag{5.16}$$

Proof Fix $x_0 = 0 \in L$ and a good decomposition $N \simeq L \times$ coker τ_γ of N_X into N_0-cosets, see (5.4). In this case, $N_0 = x_0 \times$ coker $\tau_\gamma \simeq \mathbb{Z}_p$, by Proposition 5.11. More precisely, if we denote by $\varphi : N_0 \to \mathbb{Z}_p$ this isomorphism, then it follows from the proof of Proposition 5.11 that, for t_α, s_α as in (5.11):

$$\varphi\left([(a, b), (c, d)]\right) = \begin{cases} bc + \alpha ad \in \mathbb{Z}_p, & X \text{ as } (5.13); \\ bc + ad - 2ac \in \mathbb{Z}_p, & X \text{ as } (5.14); \\ bd + t_\alpha(ac + ad) + s_\alpha\, bc \in \mathbb{Z}_p, & X \text{ as } (5.15). \end{cases} \tag{5.17}$$

On the other hand, if $g \in \mathrm{Hom}(N_0, \Bbbk^*)$, then there is $0 \le \ell < p$ such that g is the morphism g_ℓ given by $1 \mapsto \exp\left(\frac{2\pi \mathrm{i}\ell}{p}\right)$. By Proposition 5.8, any $q \in H^2(X, \Bbbk^*)$ is thus of the form

$$q_{x,y} = \lambda \prod_{0<j<\mathrm{ord}(\gamma)} \exp\left(\frac{2\pi \mathrm{i}\ell}{p}\varphi([x, \gamma^j(y)])\right), \quad x, y \in X,$$

for some $\lambda \in \Bbbk^*$, $\ell \in \mathbb{Z}$. Hence the result follows as $\zeta_j(x, y) \in \mathbb{Z}$ is a representative of $\varphi([x, \gamma^j(y)])$, for each $x, y \in X$, via (5.17). $\square$

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