



Asymptotic behavior for a nonlocal diffusion equation in exterior domains: The critical two-dimensional case [☆]



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ABSTRACT

We study the long time behavior of bounded, integrable solutions to a nonlocal diffusion equation, $\partial_t u = J * u - u$, where J is a smooth, radially symmetric kernel with support $B_d(0) \subset \mathbb{R}^2$. The problem is set in an exterior two-dimensional domain which excludes a hole \mathcal{H} , and with zero Dirichlet data on \mathcal{H} . In the far field scale, $\xi_1 \leq |x|t^{-1/2} \leq \xi_2$ with $\xi_1, \xi_2 > 0$, the scaled function $\log t u(x, t)$ behaves as a multiple of the fundamental solution for the local heat equation with a certain diffusivity determined by J . The proportionality constant, which characterizes the first non-trivial term in the asymptotic behavior of the mass, is given by means of the asymptotic ‘logarithmic momentum’ of the solution, $\lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} u(x, t) \log |x| dx$. This asymptotic quantity can be easily computed in terms of the initial data. In the near field scale, $|x| \leq t^{1/2} h(t)$ with $\lim_{t \rightarrow \infty} h(t) = 0$, the scaled function $t(\log t)^2 u(x, t) / \log |x|$ converges to a multiple of $\phi(x) / \log |x|$, where ϕ is the unique stationary solution of the problem that behaves as $\log |x|$ when $|x| \rightarrow \infty$. The proportionality constant is obtained through a matching procedure with the far field limit. Finally, in the very far field, $|x| \geq t^{1/2} g(t)$ with $g(t) \rightarrow \infty$, the solution is proved to be of order $o((t \log t)^{-1})$.

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1. Introduction

Let $\mathcal{H} \subset \mathbb{R}^N$ be a non-empty bounded open set, which is assumed, without loss of generality, to satisfy

$$B_2(0) \subset \mathcal{H} \subset B_{\mathcal{R}}(0), \quad \mathcal{R} \in (2, \infty). \quad (H_{\mathcal{H}})$$

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We do not assume \mathcal{H} to be connected, so it may represent one or several holes in an otherwise homogeneous medium. Our goal is to describe the long time behavior of solutions to the nonlocal diffusion problem

$$\begin{cases} \partial_t u(x, t) = Lu(x, t) & \text{in } (\mathbb{R}^N \setminus \mathcal{H}) \times \mathbb{R}_+, \\ u(x, t) = 0 & \text{in } \mathcal{H} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

in the critical two-dimensional case $N = 2$, thus completing the study for other dimensions performed by the authors in [7–9]. The nonlocal operator L is defined by $Lg := J * g - g$, where the kernel J is assumed to satisfy

$$J \in C_c^2(\mathbb{R}^N) \quad \text{radially symmetric,} \quad J > 0 \text{ if } |x| < d, \quad J = 0 \text{ if } |x| \geq d, \quad \int_{\mathbb{R}^N} J = 1. \tag{H_J}$$

Diffusion models of this kind have been widely used to model the dispersal of a species taking into account long-range effects [3,5,10], and also to describe phase transitions [1,2,4] and image enhancement [11].

If the initial data u_0 are integrable and identically zero in the hole \mathcal{H} , problem (P) has a unique solution $u \in C([0, \infty); L^1(\mathbb{R}^N))$. This is easily proved using Banach’s fixed point theorem; see [7]. However, in order to prove our asymptotic results for $N = 2$, we will need further hypotheses on u_0 . In the sequel we assume

$$u_0 \geq 0, \quad u_0 = 0 \text{ in } \mathcal{H}, \quad u_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2, \log|x| \, dx), \tag{H_0}$$

plus some extra control of the growth of u_0 at infinity,

$$u_0 \in L^1(\mathbb{R}^2, |x|^2 \, dx). \tag{H_1}$$

An easy modification of the existence proof shows that if (H₀) holds, then

$$u \geq 0, \quad \int_{\mathbb{R}^2} u(x, t) \log|x| \, dx < \infty \quad \text{for every } t > 0, \quad \|u\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)} \leq \|u_0\|_{L^\infty(\mathbb{R}^2)}. \tag{1.1}$$

Moreover, if u_0 satisfies in addition (H₁), then $u(\cdot, t) \in L^1(\mathbb{R}^2, |x|^2 \, dx)$ for every $t > 0$.

COMPARISON WITH THE CASE WITHOUT HOLES. In the absence of holes, $\mathcal{H} = \emptyset$, the mass $M(t) = \int_{\mathbb{R}^N} u(x, t) \, dx$ is conserved. If the initial data are not only integrable, but also bounded, the solution to (P) behaves for large times as the solution, v , to the local heat equation

$$\partial_t v = \mathfrak{q} \Delta v \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \mathfrak{q} := \frac{1}{2N} \int_{\mathbb{R}^N} |z|^2 J(z) \, dz, \tag{1.2}$$

with initial $v(\cdot, 0) = u_0$; see [6,13]. More precisely,

$$\lim_{t \rightarrow \infty} t^{N/2} \max_{x \in \mathbb{R}^N} |u(x, t) - v(x, t)| = 0.$$

Hence, the asymptotic behavior of u can be described in terms of the fundamental (self-similar) solution of (1.2),

$$\Gamma_{\mathfrak{q}}(x, t) = t^{-N/2} U_{\mathfrak{q}} \left(\frac{x}{t^{1/2}} \right), \quad U_{\mathfrak{q}}(y) = (4\pi\mathfrak{q})^{-N/2} e^{-\frac{|y|^2}{4\mathfrak{q}}}. \tag{1.3}$$

Indeed, in self-similar variables we have convergence towards the stationary profile MU_q , where $M = \int_{\mathbb{R}^N} u_0$,

$$\lim_{t \rightarrow \infty} \max_{y \in \mathbb{R}^N} |t^{N/2} u(yt^{1/2}, t) - MU_q(y)| = 0.$$

Thus, there is an asymptotic symmetrization: no matter whether the initial datum is radial or not, the large time behavior of u is given by a radial profile, which, of course, has the same mass as the datum.

The presence of holes introduces a technical difficulty, since Fourier transforms, which were the main tool in [6,13], can no longer be used. But there are differences of a more fundamental nature. On the one hand, mass is not conserved. On the other hand, the presence of the hole breaks (in general) the symmetry of the spatial domain, and an asymptotic symmetrization is no longer possible. It turns out that the asymptotic behavior depends strongly on the spatial dimension. It is already known that there is a big difference between the case of high dimensions, $N \geq 3$, considered in [7], and the one-dimensional case studied in [8,9]. This paper is devoted to the intermediate, critical two-dimensional case. An analogous study for the local heat equation in dimensions $N \geq 2$ was performed in [12].

STATIONARY SOLUTIONS. A main difference between the various dimensions has to do with stationary solutions of the problem, that is, functions ϕ satisfying

$$L\phi = 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{H}, \quad \phi = 0 \quad \text{in } \mathcal{H}. \tag{1.4}$$

When $N \geq 3$, there is a unique solution of this kind approaching the constant 1 at infinity. Such a solution does not exist for low dimensions, $N = 1, 2$. Nevertheless, for $N = 1$ there are stationary solutions that behave linearly at infinity. More precisely, given constants $b_{\pm} \geq 0$, there is a unique solution to (1.4) satisfying

$$(\phi(x) - \max\{b^+x, -b^-x\}) \in L^\infty(\mathbb{R}). \tag{1.5}$$

In the critical two dimensional case there is a stationary solution with a logarithmic behavior at infinity,

$$(\phi(x) - \log|x|) \in L^\infty(\mathbb{R}^2 \setminus \mathcal{H}). \tag{1.6}$$

The construction of such solution, which is unique, is quite involved, and will be the subject of Section 2.

CONSERVATION LAWS AND NON-TRIVIAL ASYMPTOTIC QUANTITIES. Stationary solutions play an important role in our analysis. Indeed, though mass is not conserved, under adequate conditions on the initial data, there is in all dimensions a conservation law of the form

$$\int_{\mathbb{R}^N} u(x, t)\phi(x) dx = \text{constant}$$

for functions ϕ satisfying (1.4) and having for each dimension the right behavior at infinity specified above. In the particular case $N = 2$ we need $u_0 \in L^1(\mathbb{R}^2, \log|x| dx)$.

If the initial data are also bounded, this conservation law yields on the one hand the large time behavior for the mass,

$$M(t) \rightarrow M_\phi^* := \int_{\mathbb{R}^N} u_0 \phi > 0 \quad \text{if } N \geq 3,$$

$$M(t) = \begin{cases} O(t^{-1/2}) & \text{if } N = 1, \\ O((\log t)^{-1}) & \text{if } N = 2, \end{cases}$$

and on the other hand a global size estimate,

$$\|u(\cdot, t)\|_\infty = \begin{cases} O(t^{-N/2}), & N \geq 3, \\ O(t^{-1}) & N = 1, \\ O((t \log t)^{-1}) & N = 2; \end{cases}$$

see Section 3 for the statements concerning the two-dimensional case.

Though in low dimensions there is not a residual asymptotic mass, the conservation laws allow to obtain non-trivial asymptotic quantities, which enter in the characterization of the large time behavior of the solutions. Thus, for $N = 1$ we have a non-trivial limit for the right and left first momenta $M_1^\pm(t) = \int_{\mathbb{R}_\pm} u(x, t)|x| dx$. Indeed,

$$M_1^\pm(t) \rightarrow M_{\phi^\pm}^* := \int_{\mathbb{R}} u_0 \phi^\pm \quad \text{as } t \rightarrow \infty, \tag{1.7}$$

where ϕ^\pm are the solutions to (1.4) satisfying

$$(\phi^\pm(x) - \max\{\pm x, 0\}) \in L^\infty(\mathbb{R}).$$

In the two-dimensional case, the relevant quantity approaching a non-trivial constant is the *logarithmic momentum*, $M_{\log}(t) := \int_{\mathbb{R}^2} u(x, t) \log|x| dx$. Indeed,

$$M_{\log}(t) \rightarrow M_\phi^* := \int_{\mathbb{R}^2} u_0 \phi \quad \text{as } t \rightarrow \infty,$$

where ϕ is the solution to (1.4) satisfying (1.6); see Section 3.

OUTER LIMIT. The non-trivial limit quantities give an indication of the right scalings in order to obtain the asymptotics for the solution far away from the origin. Thus, for $N \geq 3$ the adequate scaling is the one that conserves mass,

$$u^\lambda(x, t) = \lambda^{N/2} u(\lambda^{1/2}x, \lambda t).$$

The scaled solution satisfies

$$\partial_t u^\lambda = L_\lambda u^\lambda \quad \text{for } x \in (\mathbb{R}^N \setminus \mathcal{H}^\lambda), \quad t > 0, \quad \mathcal{H}^\lambda = \{x : \lambda^{1/2}x \in \mathcal{H}\},$$

where L_λ is the operator defined by

$$L_\lambda \varphi(x) = \lambda \int_{\mathbb{R}^N} J_\lambda(x - y)(\varphi(y) - \varphi(x)) dy, \quad J_\lambda(x) = \lambda^{N/2} J(\lambda^{1/2}x).$$

If $\varphi \in C_c^\infty(\mathbb{R}^N)$, an easy computation, which uses the symmetry of the kernel plus Taylor’s expansion, shows that $L_\lambda \varphi$ converges uniformly to $\mathfrak{q} \Delta \varphi$ as $\lambda \rightarrow \infty$, with \mathfrak{q} as in (1.2). Hence, the asymptotic behavior is expected to be given by a multiple of the fundamental solution of the *local* heat equation with diffusivity \mathfrak{q} . Notice that the fundamental solution conserves mass. The proportionality constant should therefore be given by the limit mass, M_ϕ^* . Indeed, we have

$$\lim_{t \rightarrow \infty} t^{N/2} \sup\{|u(x, t) - M_{\phi}^* \Gamma_{\mathfrak{q}}(x, t)| : |x| \geq \delta \sqrt{t}\} = 0 \quad \text{for all } \delta > 0.$$

Notice that the only effect of the holes in this outer limit for large dimensions, $N \geq 3$, is the loss of mass.

For $N = 1$ the right scaling is the one that preserves the first momentum, and the asymptotic behavior is given in terms of the *dipole* solution to the (local) heat equation with diffusivity \mathfrak{q} ,

$$\mathcal{D}_{\mathfrak{q}}(x, t) = \partial_x \Gamma_{\mathfrak{q}}(x, t) = -\frac{x}{2\mathfrak{q}t} \Gamma_{\mathfrak{q}}(x, t).$$

This special solution, which is self-similar, has δ' , the derivative of the Dirac mass, as initial data, and preserves the first momentum. Thus,

$$\lim_{t \rightarrow \infty} t \sup\{|u(x, t) + 2M_{\phi_+}^* \mathcal{D}_{\mathfrak{q}}(x, t)| : x \geq \delta \sqrt{t}\} = 0 \quad \text{for all } \delta > 0,$$

with $M_{\phi_+}^*$ as in (1.7). A similar statement holds in sets of the form $x \leq -\delta t^{1/2}$, substituting $M_{\phi_+}^*$ by $-M_{\phi_-}^*$. Notice that in this one-dimensional case the effect of the hole on the outer limit is more dramatic when compared to the case without holes: it changes both the rate of decay and the limit profile. Moreover, we also lose the symmetry of the asymptotic profile, even when the hole is small, compared to the support of the kernel, and hence does not disconnect the domain.

As for the critical two-dimensional case, the fact that the logarithmic momentum has a nontrivial asymptotic limit does not show directly which is the right scaling. However, the *rescaled mass*, $\log t M(t)$, behaves for large times as twice the logarithmic momentum (it is here that we use the condition (H_1)); see Section 4. Hence, the rescaled mass approaches a non-trivial constant, namely

$$\log t M(t) \rightarrow 2M_{\phi}^*.$$

Therefore, in outer regions the solution is expected to approach a fundamental solution of the local heat equation with variable mass $2M_{\phi}^*/\log t$. As we will prove in Section 4, this is indeed the case.

Theorem 1.1. *Let $N = 2$. Assume that \mathcal{H} and J satisfy, respectively, $(H_{\mathcal{H}})$ and (H_J) . Let u be the solution to (P) with an initial data u_0 satisfying (H_0) – (H_1) . Then,*

$$\lim_{t \rightarrow \infty} t \log t \sup \left\{ \left| u(x, t) - 2M_{\phi}^* \frac{\Gamma_{\mathfrak{q}}(x, t)}{\log t} \right| : |x| \geq \delta \sqrt{t} \right\} = 0 \quad \text{for all } \delta > 0,$$

where $\Gamma_{\mathfrak{q}}$ is given by (1.3), with \mathfrak{q} as in (1.2), $M_{\phi}^* = \int_{\mathbb{R}^2} u_0 \phi$, and ϕ is the unique solution to (1.4) and (1.6).

In this borderline dimension the existence of a hole modifies slightly the rate of decay in the outer limit, but it does not change the rescaled asymptotic profile – except for the proportionality constant –, which is still given by the (radially symmetric) profile of the fundamental solution of the heat equation.

It is worth noticing that the mentioned scaling only gives a non-trivial profile if $0 < \xi_1 \leq |x|t^{-1/2} \leq \xi_2 < \infty$, in the far field scale. In the *very far field*, $|x|t^{-1/2} \rightarrow \infty$, our result says that $t \log t u(x, t) \rightarrow 0$, but not more. Analogous results hold for other dimensions in the very far field.

INNER LIMIT. What happens close to the holes? For large dimensions, $N \geq 3$, solutions still decay as $O(t^{-N/2})$ in the inner region $|x| \leq \delta t^{1/2}$ for some $\delta > 0$. If we scale the solutions accordingly, we get that the new variable $w(x, t) = t^{N/2}u(x, t)$ satisfies

$$\partial_t w = Lw + \frac{Nw}{2t}.$$

Thus, w is expected to converge to a stationary solution, solving (1.4). To determine completely this solution we have to prescribe its behavior at infinity. Since there is an overlapping region between the inner and the outer developments, they can be matched, leading to

$$t^{N/2}(u(x, t) - M_\phi^* \phi(x) \Gamma_q(x, t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ uniformly in } \mathbb{R}^N.$$

In particular,

$$t^{N/2}u(x, t) \rightarrow \frac{M_\phi^* \phi(x)}{(4\pi q)^{N/2}} \quad \text{uniformly in compact subsets of } \mathbb{R}^N.$$

Notice that, as expected, the effect of the hole is more important when we are close to it. In this region, the asymptotic profile does not have radial symmetry, and sees more details of the hole through the function ϕ .

In low dimensions the determination of the inner behavior is by far more involved. The main reason is that solutions do not decay at the same rate everywhere in sets of the form $|x| \leq Dt^{1/2}$. Thus, for $N = 1$ the rates of decay depend on the ratio $t^{3/2}/|x|$. More precisely, in the *near field* scale, $|x| \leq t^{1/2}h(t)$, $\lim_{t \rightarrow \infty} h(t) = 0$, the scaled function $t^{3/2}u(x, t)/|x|$ converges to a multiple of $\phi^*(x)/|x|$, where ϕ^* is a solution to (1.4)–(1.5) for some constants b^\pm . The right choice for the involved constants is, again, obtained through a matching procedure with the outer limit. We obtain

$$\frac{t^{3/2}}{|x|} \left(u(x, t) + 2 \frac{\phi^*(x)}{x} \mathcal{D}_q(x, t) \right) \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ uniformly in } \mathbb{R},$$

where ϕ^* is the solution to (1.4)–(1.5) with $b^\pm = M_{\phi^\pm}^*$. In particular,

$$t^{3/2}u(x, t) \rightarrow \frac{\phi^*(x)}{2q^{3/2}\sqrt{\pi}} \quad \text{uniformly in compact subsets of } \mathbb{R}.$$

Let us remark that in this case the matching is more complicated, since the overlapping region is not so wide as in the case of large dimensions. A main step is obtaining a super-solution giving the right decay rates in the whole inner region and up to the beginning of the far-field scale.

In the two-dimensional case that is the subject of this paper, the rates of decay depend on the ratio $t(\log t)^2 / \log |x|$. Moreover, the scaled function $t(\log t)^2 u(x, t) / \log |x|$ converges in the near field scale, $|x| \leq t^{1/2}h(t)$, $\lim_{t \rightarrow \infty} h(t) = 0$, to a multiple of $\phi(x) / \log |x|$, where ϕ is the unique solution to (1.4) and (1.6). We finally obtain, after matching the inner and the outer developments, our main result, whose proof is completed in Section 5.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the unique solution u to problem (P) satisfies*

$$\frac{t(\log t)^2}{\log |x|} \left(u(x, t) - 4M_\phi^* \frac{\Gamma_q(x, t)\phi(x)}{(\log t)^2} \right) \rightarrow 0 \quad \text{as } t \rightarrow \infty \text{ uniformly in } \mathbb{R}^2. \tag{1.8}$$

As a consequence,

$$\lim_{t \rightarrow \infty} t(\log t)^2 u(x, t) \rightarrow \frac{M_\phi^*}{\pi q} \phi(x) \quad \text{uniformly in compact subsets of } \mathbb{R}^2.$$

Remark. The proof can be easily extended to the case of initial data without sign restrictions. Indeed, if u^\pm are the solutions with initial data $\{u_0\}_\pm$, then, by the linearity of the equation, $u = u^+ - u^-$. Since $M_\phi^* = \int_{\mathbb{R}^2} \{u_0\}_+ \phi - \int_{\mathbb{R}^2} \{u_0\}_- \phi$, the result for general data will follow from the results for u^+ and u^- .

Notice, however, that in the case of initial data with sign changes it may happen that $M_\phi^* = 0$. In this situation our result is not optimal (solutions decay faster), and we should look for a different scaling.

Notice that the decay rate $O((t \log t)^{-1})$ is seen not only in the far field scale, $|x| \approx \xi t^{1/2}$, $\xi \neq 0$, but also in much smaller ‘parabolas’. Indeed, let h be such that $h(t) \rightarrow 0$, $th(t) \rightarrow \infty$, $\frac{\log h(t)}{\log t} \rightarrow -\alpha$, $0 \leq \alpha < 1$, as $t \rightarrow \infty$. Then,

$$\lim_{t \rightarrow \infty} t \log t u(x, t) = (1 - \alpha) \frac{M_\phi^*}{2\pi q} \quad \text{if } |x|^2 = th(t).$$

In particular, this result holds if $h(t) = t^{-\alpha}$ or $h(t) = (\log t)^{-\gamma}$ with $\gamma > 0$ (in which case $\alpha = 0$).

2. The stationary problem

The first aim of this section is to prove the existence of a unique solution ϕ to (1.4) and (1.6). Existence will follow from the fact that there exist ordered sub- and super-solutions with adequate behaviors at infinity. We will also obtain estimates for the first derivatives of ϕ , which will be used later, in Section 5, to determine the inner behavior.

2.1. Sub- and super-solutions

We start by constructing sub- and super-solutions to the stationary problem behaving as $\log|x|$ when $|x| \rightarrow \infty$.

Lemma 2.1. *Let $N = 2$, and assume that \mathcal{H} and J satisfy, respectively, $(H_{\mathcal{H}})$ and (H_J) . There exist constants $R_0, k_0 > 0$ such that the function*

$$V_-(x) = \log(|x|^2 + k_0) - R_0, \quad x \in \mathbb{R}^2,$$

satisfies

$$-LV_- \leq 0 \quad \text{in } \mathbb{R}^2, \quad V_- \leq 0 \quad \text{in } \mathcal{H}.$$

Proof. Since $V_- \in C^\infty(\mathbb{R}^2)$ and the kernel J is radially symmetric, a simple calculation using Taylor’s expansion shows that, for q as in (1.2),

$$LV_-(x) - q\Delta V_-(x) = \frac{1}{4!} \sum_{|\beta|=4} \int_{\mathbb{R}^2} J(|x-y|) D^\beta V_-(\xi) (x-y)^\beta,$$

for some ξ lying in the segment that joins x and y . We are using the standard multi-index notation. A direct computation yields

$$\Delta V_-(x) = \frac{4k_0}{(|x|^2 + k_0)^2}, \quad |D^\beta V_-(x)| \leq \frac{C}{(|x|^2 + k_0)^2}, \quad |\beta| = 4.$$

Hence, since we only have to take into account the values y such that $|x-y| \leq 1$,

$$|D^\beta V_-(\xi)| \leq \frac{C}{(|x|^2 + k_0)^2}, \quad |\beta| = 4.$$

Therefore, there exists a constant $K > 0$, independent of k_0 and R , such that

$$-LV_-(x) \leq \frac{K - 4k_0}{(|x|^2 + k_0)^2}.$$

The first part of our statement follows taking $k_0 \geq K/4$.

Once k_0 is fixed, since \mathcal{H} is bounded, by choosing R_0 large enough we get $V_- \leq 0$ in \mathcal{H} . \square

The construction of a super-solution, which we do next, is far more involved, and much more complicated than for $N \neq 2$.

Lemma 2.2. *Assume the hypotheses of Lemma 2.1. Given $0 < r_0 < d/4$ such that $B_{2r_0}(0) \subset \mathcal{H}$, there are constants $\kappa > 0$, $\gamma_0 \geq 0$, and $D > 0$ such that for every $\gamma \geq \gamma_0$ there is a locally bounded super-solution V_+ of the stationary problem (1.4) satisfying*

$$\begin{cases} -LV_+(x) \geq \frac{\kappa}{|x|^3}, & x \in \mathbb{R}^2 \setminus \mathcal{H}, \\ V_+(x) \geq \gamma - \gamma_0 \geq 0, & x \in \mathbb{R}^2, \\ V_+(x) = \log(|x| - r_0) + \gamma, & |x| \geq D. \end{cases} \tag{2.1}$$

Proof. We will prove that there exist $\kappa > 0$, $k \geq 2$, and a sequence $0 = a_1 > a_2 > \dots > a_{k+1}$ such that

$$v_+(x) = \begin{cases} \log(|x| - r_0), & |x| \geq D := 2r_0 + k\frac{d}{2}, \\ a_j, & x \in \Gamma_j := \{D - j\frac{d}{2} \leq |x| < D - (j-1)\frac{d}{2}\}, \quad j = 1, \dots, k+1, \\ a_{k+1}, & x \in \Gamma_{k+1} := B_{2r_0}, \end{cases}$$

satisfies

$$-Lv_+(x) \geq \frac{\kappa}{|x|^3} \quad \text{in } \mathbb{R}^2 \setminus B_{2r_0}.$$

Hence, the result will follow just taking $V_+ = v_+ + \gamma$, with $\gamma \geq \gamma_0 := |a_{k+1}|$.

For $|x| \geq 2r_0$, a direct computation yields

$$\Delta(\log(|x| - r_0)) = -\frac{r_0}{|x|(|x| - r_0)^2}, \quad |D^\beta(\log(|x| - r_0))| \leq \frac{C}{(|x| - r_0)^4}, \quad |\beta| = 4.$$

Hence, Taylor’s expansion shows that there exists $k \in \mathbb{N}$, $k \geq 2$, large enough, such that

$$\log(|x| - r_0) - \int_{\mathbb{R}^2} J(x - y) \log(|y| - r_0) dy \geq \frac{r_0|x|}{2(|x| - r_0)^4} \quad \text{if } |x| \geq \underbrace{2r_0 + k\frac{d}{2}}_D - d. \tag{2.2}$$

In particular

$$-Lv_+(x) \geq \frac{r_0}{2|x|^3} \quad \text{if } |x| \geq D + d. \tag{2.3}$$

The integer k may be chosen so that $\log(|x| - r_0) \geq 0$ for $|x| \geq D - d$. Hence, if $0 = a_1 \geq a_2$, for $D \leq |x| \leq D + d$ we have, using (2.2),

$$\begin{aligned}
 -Lv_+(x) &= \log(|x| - r_0) - \int_{\{|x| \geq D\}} J(x - y) \log(|y| - r_0) dy - a_2 \int_{\Gamma_2} J(x - y) dy \\
 &\geq \log(|x| - r_0) - \int_{\mathbb{R}^2} J(x - y) \log(|y| - r_0) dy \geq \frac{r_0|x|}{2(|x| - r_0)^4} \geq \underbrace{\frac{r_0 D}{2(D + d - r_0)^4}}_{c_0}.
 \end{aligned}$$

We will prove that this estimate can be extended to the set $2r_0 \leq |x| \leq D$, so that

$$-Lv_+(x) \geq c_0 \quad \text{if } 2r_0 \leq |x| \leq D + d.$$

From this inequality and (2.3) we conclude (2.1) immediately, just taking $\kappa > 0$ small enough. To prove this claim we start by considering the annulus Γ_1 , and then proceed inductively towards the hole, choosing in each step one of the constants a_j . The *positive* constants

$$\underline{J}^j = \inf \left\{ \int_{\Gamma_{j+1}} J(x - y) dy : x \in \Gamma_j \right\}, \quad j = 2, \dots, k,$$

will play an important role in the process.

Let $x \in \Gamma_1$. We already have $0 = a_1 \geq a_2$, but we are still free to choose a_2 , as long as it is negative. Then, if $a_3 \leq 0$, since J has unit integral,

$$\begin{aligned}
 -Lv_+(x) &= - \int_{\{|x| \geq D\}} J(x - y) \log(|y| - r_0) dy - a_2 \int_{\Gamma_2} J(x - y) dy - a_3 \int_{\Gamma_3} J(x - y) dy \\
 &\geq -\log(D + d - r_0) + |a_2| \underline{J}^2 \geq c_0,
 \end{aligned}$$

if $|a_2|$ is large enough. Notice that a_2 has to be strictly negative.

We have already fixed a_1 and a_2 , and have imposed that $a_3 \leq 0$. If $x \in \Gamma_2$, and $a_4 \leq 0$,

$$\begin{aligned}
 -Lv_+(x) &= a_2 - \int_{\{|x| \geq D\}} J(x - y) \log(|y| - r_0) dy + \sum_{i=2}^4 |a_i| \int_{\Gamma_i} J(x - y) dy \\
 &\geq a_2 - \log\left(D + \frac{d}{2} - r_0\right) + |a_3| \underline{J}^3 \geq c_0,
 \end{aligned}$$

if $|a_3|$ is large enough. Since $\underline{J}^3 < 1$, we have $|a_3| > |a_2|$.

We now use induction. Let $3 \leq j \leq k - 1$. Assume that we have already chosen the constants $0 = a_1 > a_2 > \dots > a_j$, and imposed $a_{j+1} \leq 0$, so that $-Lv_+(x) \geq 0$ for $x \in \cup_{i=1}^{j-1} \Gamma_i$. Let now $x \in \Gamma_j$. If $a_{j+2} \leq 0$,

$$-Lv_+(x) = a_j + \sum_{i=j-2}^{j+2} |a_i| \int_{\Gamma_i} J(x - y) dy \geq a_j + |a_{j+1}| \underline{J}^{j+1} \geq c_0,$$

if $|a_{j+1}|$ is large enough. Since $\underline{J}^{j+1} < 1$, we have $|a_{j+1}| > |a_j|$.

The final step is nearly identical. We have already fixed $\{a_j\}_{j=1}^{k-1}$, and have imposed that $a_{k+1} \leq 0$. Let $x \in \Gamma_k$. Then,

$$-Lv_+(x) = a_k + \sum_{i=k-2}^{k+1} |a_i| \int_{\Gamma_i} J(x - y) dy \geq a_k + |a_{k+1}| \underline{J}^{k+1} \geq c_0,$$

if $|a_{k+1}|$ is large enough. Since $\underline{J}^{k+1} < 1$, we have $|a_{k+1}| > |a_k|$. \square

With the same ideas we can construct another super-solution, that will be of use in Section 5.

Lemma 2.3. *Assume the hypotheses of Lemma 2.1. Let $0 < \nu < 1$. There exist constants $\kappa > 0$, $\gamma_0 \geq 0$, and $D \geq 1$ such that for every $\gamma \geq \gamma_0$ there is a locally bounded super-solution w_ν^+ of the stationary problem (1.4) satisfying*

$$\begin{cases} -Lw_\nu^+(x) \geq \frac{\kappa}{|x|^2(\log|x|)^{2-\nu}}, & x \in \mathbb{R}^2 \setminus \mathcal{H}, \\ w_\nu^+(x) \geq \gamma - \gamma_0 \geq 0, & x \in \mathbb{R}^2, \\ w_\nu^+(x) = (\log|x|)^\nu + \gamma, & |x| \geq D. \end{cases} \tag{2.4}$$

2.2. Existence and uniqueness

The function ϕ solving (1.4) and (1.6) will be obtained as the limit when n tends to infinity of the solutions ϕ_n to

$$L\phi_n = 0 \quad \text{in } B_n(0) \setminus \mathcal{H}, \quad \phi_n = 0 \quad \text{in } \mathcal{H}, \quad \phi_n = \frac{1}{2}V_- \quad \text{in } B_{n+d}(0) \setminus B_n(0),$$

where V_- is the sub-solution constructed in Lemma 2.1. Existence and uniqueness for such problem is a consequence of [7, Lemma 3.1].

Proposition 2.1. *Let $N = 2$, and assume that \mathcal{H} and J satisfy, respectively, $(H_{\mathcal{H}})$ and (H_J) . There exists a solution to (1.4) and (1.6).*

Proof. Let V_- and V_+ be the sub- and super-solutions constructed in Lemmas 2.1 and 2.2. Taking $\gamma \geq \gamma_0$ large enough, we have $\frac{1}{2}V_- \leq V_+$ in \mathbb{R}^2 . Thus, the comparison principle gives

$$\frac{1}{2}V_- \leq \phi_n \leq V_+ \quad \text{in } B_n.$$

Moreover, by applying the comparison principle once again, we find that $\phi_n \leq \phi_{n+1}$ in B_n . In particular, there exists $\phi = \lim_{n \rightarrow \infty} \phi_n$. It is easily seen that this monotone limit solves (1.4) and satisfies

$$\frac{1}{2} \log(|x|^2 + k_0) - \frac{1}{2}R_0 = \frac{1}{2}V_-(x) \leq \phi(x) \leq V_+(x) = \log(|x| - r_0) + C, \quad x \in \mathbb{R}^2,$$

for some constant $C > 0$, which implies (1.6). \square

Uniqueness follows from the fact that the unique bounded solution of (1.4) is $\phi \equiv 0$.

Proposition 2.2. *Let $N = 2$, and assume that \mathcal{H} and J satisfy, respectively, $(H_{\mathcal{H}})$ and (H_J) . The only bounded solution to (1.4) is $\phi = 0$.*

Proof. The function $\phi_\varepsilon = \phi - \varepsilon V_+$ satisfies $L\phi_\varepsilon \geq 0$ in $\mathbb{R}^2 \setminus \mathcal{H}$, and reaches its maximum at some finite point \bar{x} , since by construction $\phi_\varepsilon(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. A standard (for nonlocal operators) argument shows that if $\phi_\varepsilon(\bar{x}) > 0$ we reach a contradiction. Indeed, if this happens, $\bar{x} \in \mathbb{R}^2 \setminus \mathcal{H}$ and ϕ_ε is constant in $B_d(\bar{x})$. We can thus propagate the maximum to the whole connected component of $\mathbb{R} \setminus \mathcal{H}$ where \bar{x} lies, which leads to a contradiction for points near the boundary of this component, at a distance of it less than d . Then, passing to the limit as $\varepsilon \rightarrow 0$, we obtain $\phi \leq 0$. The same argument applied to $-\phi$ leads to $\phi \geq 0$. \square

Remark. This result is also true for $N = 1$; see [9]. However, it fails for $N \geq 3$. Indeed, in large dimensions there are stationary solutions that take a constant value, different from zero, at infinity; see [7].

2.3. *Estimates for the derivatives*

To prove them, we use that ϕ solves a problem of the form

$$\partial_t u - Lu = f \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^2. \tag{2.5}$$

By the variations of constants formula, solutions to (2.5) can be written in terms of the fundamental solution $F = F(x, t)$ of the operator $\partial_t - L$ in the whole space, which can be decomposed as

$$F(x, t) = e^{-t}\delta(x) + W(x, t),$$

where δ is the Dirac mass at the origin and W is a nonnegative smooth function defined via its Fourier transform,

$$\widehat{W}(\xi, t) = e^{-t} \left(e^{J(\xi)t} - 1 \right);$$

see [6]. Thus,

$$\begin{aligned} u(x, t) &= e^{-t}u_0(x) + \int_{\mathbb{R}^2} W(x - y, t)u_0(y) dy \\ &\quad + \int_0^t e^{-(t-s)} f(x, s) ds + \int_0^t \int_{\mathbb{R}^2} W(x - y, t - s)f(y, s) dy ds. \end{aligned}$$

Therefore, estimates for solutions to (2.5), and in particular for ϕ , will follow if we have good estimates for the right hand side of the equation, f , and for the regular part, W , of the fundamental solution.

The asymptotic convergence of W to the fundamental solution of the local heat equation with diffusivity q yields a first class of estimates. Indeed, for every multi-index $\beta \in \mathbb{N}^2$ there is a constant C such that

$$|D_x^\beta W(x, t) - D_x^\beta \Gamma_q(x, t)| \leq Ct^{-\frac{|\beta|+3}{2}}, \quad x \in \mathbb{R}^2, \quad t > 0; \tag{2.6}$$

see [13]. Hence, in particular,

$$|D_x^\beta W(x, t)| \leq Ct^{-\frac{|\beta|+2}{2}}, \quad x \in \mathbb{R}^2, \quad t > 0. \tag{2.7}$$

These estimates give the right order of time decay, and will prove to be useful later, in Sections 4 and 5. However, they do not take into account the spatial structure of W , and are quite poor for $t \approx 0$; hence they are not enough for our present goal. Instead, we will use the pointwise estimate

$$|D_x^\beta W(x, t)| \leq \frac{Ct}{1 + |x|^{4+|\beta|}}, \quad x \in \mathbb{R}^2, \quad t > 0, \tag{2.8}$$

and the integral estimate

$$\int_{\mathbb{R}^2} |D_x^\beta W(x, t)| dx \leq Ct^{-|\beta|/2}, \quad x \in \mathbb{R}^2, \quad t > 0, \tag{2.9}$$

both of them valid for all $\beta \in \mathbb{N}^2$. These estimates were proved in [14] through a comparison argument, using that W is a solution to

$$\begin{cases} \partial_t W(x, t) - LW(x, t) = e^{-t} J(x) & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ W(x, 0) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.10}$$

Lemma 2.4. *Assume the hypotheses in Proposition 2.1. Let ϕ be the solution to (1.4) and (1.6). There exists a constant $C > 0$ such that*

$$|\nabla\phi(x)| \leq \frac{C}{|x|} \quad \text{in } \mathbb{R}^2 \setminus \mathcal{H}.$$

Proof. The result will follow from an analogous estimate for $\psi(x) = \phi(x) - g(x)$, where $g(x) = \frac{1}{2} \log(1 + |x|^2)$. The function ψ , which is bounded, is a solution to (2.5) with a right hand side

$$f(x) = - \underbrace{\mathcal{X}_{\mathcal{H}}(x)(J * \phi)(x)}_{f_1(x)} - \underbrace{Lg(x)}_{f_2(x)},$$

and initial data $u_0(x) = \psi(x)$.

Let $\psi^\lambda(x) = \psi(\lambda x)$. By differentiating the equation for ψ given by the variations of constants formula, we get for $|x| \geq 1/2$ and λ large enough so that $\lambda x \notin \mathcal{H}$,

$$\begin{aligned} \nabla\psi^\lambda(x) &= \underbrace{\frac{\lambda}{1 - e^{-t}} \int_{\mathbb{R}^2} \nabla W(\lambda x - y, t) \psi(y) dy}_{\mathcal{A}(x,t)} \\ &\quad - \underbrace{\frac{\lambda}{1 - e^{-t}} \int_0^t \int_{\mathcal{H}} \nabla W(\lambda x - y, t - s) (J * \phi)(y) dy ds}_{\mathcal{B}(x,t)} \\ &\quad - \underbrace{\frac{\lambda}{1 - e^{-t}} \int_0^t \int_{\mathbb{R}^2} \nabla W(\lambda x - y, t - s) f_2(y) dy ds}_{\mathcal{C}(x,t)} - \underbrace{\lambda \nabla f_2(\lambda x)}_{\mathcal{D}(x,t)}. \end{aligned}$$

The first term is easily estimated using (2.9) with $|\beta| = 1$, since ψ is bounded,

$$|\mathcal{A}(x, t)| \leq \frac{C\lambda t^{-1/2}}{1 - e^{-t}} \leq C \quad \text{if we take } t = \lambda^2, \lambda \geq 1.$$

As for the second term, the key is that, since \mathcal{H} is bounded, there exists λ_0 such that for $y \in \mathcal{H}$ and $\lambda \geq \lambda_0$ there holds that $|\lambda x - y| \geq \frac{1}{2}\lambda|x|$. Hence, using that $J * \phi$ is bounded in \mathcal{H} , and estimate (2.8) with $|\beta| = 1$, we have

$$|\mathcal{B}(x, t)| \leq C|\mathcal{H}| \frac{\lambda}{1 - e^{-t}} \frac{2^5}{\lambda^5|x|^5} \int_0^t (t - s) ds = \frac{C\lambda^{-4}t^2}{(1 - e^{-t})|x|^5} \leq C \quad \text{if } |x| \geq 1/2, \lambda \geq \lambda_0, t = \lambda^2.$$

In order to control the third term we decompose the integral in two parts and use the bound $|f_2(x)| \leq \frac{C}{(1+|x|^2)^2}$, which was obtained in the course of the proof of Lemma 2.1. We have,

$$\begin{aligned}
|\mathcal{C}(x, t)| &\leq \underbrace{\frac{\lambda}{1-e^{-t}} \int_0^t \int_{\{|y| \leq \frac{\lambda}{2}|x|\}} |\nabla W(\lambda x - y, t-s)| \frac{dy}{(1+|y|^2)^2} ds}_{C_1(x,t)} \\
&\quad + \underbrace{\frac{\lambda}{1-e^{-t}} \int_0^t \int_{\{|y| \geq \frac{\lambda}{2}|x|\}} |\nabla W(\lambda x - y, t-s)| \frac{dy}{(1+|y|^2)^2} ds}_{C_2(x,t)}.
\end{aligned}$$

The first integral is easily controlled thanks to (2.8) with $|\beta| = 1$,

$$C_1(x, t) \leq \frac{C\lambda}{1-e^{-t}} \frac{t^2}{\lambda^5|x|^5} \int_{\{|y| \leq \frac{\lambda}{2}|x|\}} \frac{dy}{(1+|y|^2)^2} \leq C \quad \text{if } |x| \geq \frac{1}{2}, \lambda \geq 1, \text{ and } t = \lambda^2.$$

On the other hand, using the estimates (2.7) and (2.8) with $|\beta| = 1$, we get

$$\begin{aligned}
C_2(x, t) &\leq C \frac{\lambda}{1-e^{-t}} \left(\int_0^{t-1} (t-s)^{-3/2} ds + \int_{t-1}^t (t-s) ds \right) \int_{\{|y| \geq \frac{\lambda}{2}|x|\}} \frac{dy}{(1+|y|^2)^2} \\
&\leq C \frac{\lambda^{-1}}{(1-e^{-t})|x|^2} \leq C \quad \text{if } |x| \geq \frac{1}{2}, \lambda \geq 1, t = \lambda^2.
\end{aligned}$$

We finally estimate \mathcal{D} . To this aim we notice that $\partial_{x_i} f_2(x) = L\left(\frac{x_i}{1+|x|^2}\right)$. Hence, since

$$\left| \Delta \left(\frac{x_i}{1+|x|^2} \right) \right| \leq \frac{C}{(1+|x|^2)^2}, \quad \left| D^\beta \left(\frac{x_i}{1+|x|^2} \right) \right| \leq \frac{C}{(1+|x|^2)^2}, \quad |\beta| = 4,$$

using Taylor's expansion we get $|\partial_{x_i} f_2(x)| \leq \frac{C}{(1+|x|^2)^2}$, which implies

$$|\mathcal{D}(x, t)| \leq \frac{C\lambda}{(1+\lambda^2|x|^2)^2} \leq C \quad \text{if } |x| \geq 1/2.$$

In conclusion, $|\nabla \psi^\lambda(x)| \leq C$ if $|x| \geq \frac{1}{2}$ and $\lambda \geq \lambda_0$. This yields, by taking $|y| \geq \lambda_0$, $x = y/|y|$ and $\lambda = |y|$,

$$|y| |\nabla \psi(y)| = |\nabla \psi^\lambda(x)| \leq C.$$

Recalling the definition of ψ , we see that this implies that

$$|\nabla \phi(y)| \leq \frac{C}{|y|} \quad \text{if } |y| \geq \lambda_0.$$

The lemma is proved, since $\phi \in C^1(\mathbb{R}^2 \setminus \mathcal{H})$. \square

Remark. As a corollary of Lemma 2.4 we have that

$$|\phi(y) - \phi(x)| \leq \frac{C}{|x|} \quad \text{if } x \in \mathbb{R}^2 \setminus \mathcal{H}, |x-y| < d. \quad (2.11)$$

Indeed, applying the Mean Value Theorem,

$$|\phi(y) - \phi(x)| \leq \frac{Cd}{|\xi|} \leq \frac{Cd}{|x| - d} \leq \frac{2Cd}{|x|} \quad \text{if } x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x| \geq 2d, \quad |x - y| < d.$$

The estimate for $|x| \leq 2d$ is immediate, since ϕ is locally bounded.

3. A conservation law and some by-products

Solutions to (P) satisfy a conservation law that will allow us to prove that the mass has a logarithmic decay rate. This is later used to obtain the decay rate of solutions, a super-solution that gives the right rates of decay in inner regions, the asymptotic limit of the ‘logarithmic’-momentum, and bounds for the first and second momenta when (H_1) holds.

Proposition 3.1. *Let $N = 2$. Assume that \mathcal{H} , J and u_0 satisfy $(H_{\mathcal{H}})$, (H_J) and (H_0) respectively. Let ϕ be the solution to (1.4) and (1.6), and u the solution to problem (P). Then,*

$$M_\phi(t) := \int_{\mathbb{R}^2} u(x, t)\phi(x) \, dx = \underbrace{\int_{\mathbb{R}^2} u_0\phi}_{M_\phi^*} \quad \text{for every } t \geq 0. \tag{3.1}$$

The proof, based on Fubini’s Theorem, is entirely similar to the one for dimensions $N=1$ or $N \geq 3$; see [7,9]. Hence, we omit it.

A first consequence of this conservation law is a bound for the mass at time t which in particular shows that the mass decays to 0 as $t \rightarrow \infty$.

Proposition 3.2. *If in addition to the assumptions of Proposition 3.1 we have also $u_0 \in L^\infty(\mathbb{R}^2)$, then there exist $t_0 > 0, C_0 > 0$ such that*

$$M(t) := \int_{\mathbb{R}^2} u(x, t) \, dx \leq \frac{C_0}{\log t} \quad \text{if } t \geq t_0. \tag{3.2}$$

Proof. The solution, u_C , of the Cauchy problem with initial condition u_0 is a super-solution of the Cauchy–Dirichlet problem (P), since $u_C(\cdot, t) > 0$ for $t > 0$. Therefore, the asymptotic behavior of u_C , proved in [13], provides an estimate of the time decay rate of u : $u(x, t) \leq u_C(x, t) \leq Ct^{-1}$. Though this rate is not optimal, as will turn out later, it is enough to get the decay rate of the mass. Indeed, let $\sigma > 0$ such that $\log|x| \leq \sigma\phi(x)$ for $x \notin \mathcal{H}$ (recall that $\phi > 0$ on $\partial\mathcal{H}$). Then,

$$M(t) \leq \int_{|x|^2 \leq \delta(t)t} u(x, t) \, dx + \sigma \int_{|x|^2 \geq \delta(t)t} \frac{u(x, t)\phi(x)}{\log|x|} \, dx \leq Ct^{-1}\delta t + \frac{\sigma}{\frac{1}{2}\log(\delta t)} M_\phi^*$$

if $\delta(t)t \geq \mathcal{R}^2$, with \mathcal{R} as in $(H_{\mathcal{H}})$. Taking $\delta(t) = 1/\log t$ we get that both terms on the right hand side of this estimate almost balance, and (3.2) follows. \square

Remark. We will see later that $\log t M(t)$ has a nontrivial limit as $t \rightarrow \infty$. Thus, the decay rate in Proposition 3.2 is optimal.

With this estimate for the decay rate of the mass we can improve the decay rate of $u(\cdot, t)$.

Proposition 3.3. *Under the assumptions of Proposition 3.2, there exists a constant $C_1 > 0$ such that*

$$u(x, t) \leq \frac{C_1}{t \log t} \quad \text{if } t \geq t_0. \tag{3.3}$$

Proof. We estimate u at time t by the solution of the Cauchy problem with initial data $u(x, t/2)$. Thus, using the bound (2.7) with $|\beta| = 0$ for W , and the bound for the mass (3.2), we get

$$\begin{aligned} u(x, t) &\leq u_C(x, t) = e^{-t/2}u(x, t/2) + \int_{\mathbb{R}^2} W(x - y, t/2)u(y, t/2) dy \\ &\leq e^{-t/2}\|u_0\|_{L^\infty(\mathbb{R}^2)} + Ct^{-1}M(t/2) \leq \frac{C}{t \log t} \quad \text{if } t \geq t_0. \quad \square \end{aligned}$$

As we will see in Section 4, the bound provided by (3.3) gives the optimal global decay rate for the solution. However, u decays faster than $O((t \log t)^{-1})$ in inner regions. Once we have at hand the global decay rate, this last assertion is proved by means of comparison with a suitable super-solution that we construct next. A similar super-solution will be used later in the study of the inner limit. With that application in mind, we keep a parameter $\gamma \geq 2$ that will be set equal to 2 in the estimate of u .

Lemma 3.1. *Let $0 < a < \mathfrak{q} = \frac{1}{4} \int_{\mathbb{R}^2} J(z)|z|^2 dz$, and let V_+ be the super-solution (depending on the radius $r_0 > 0$) to the stationary problem constructed in Lemma 2.2. There exist $b, T > 0$ such that the function*

$$V(x, t) = \frac{\Gamma_a(x, t)(V_+(x) + b)}{\log^\gamma t}, \quad \Gamma_a(x, t) = \frac{e^{-\frac{|x|^2}{4at}}}{4\pi at},$$

satisfies

$$(\partial_t V - LV)(x, t) \geq \frac{1}{10} \left(\frac{\mathfrak{q}}{a} - 1\right) \frac{\Gamma_a(x, t)(V_+(x) + b)}{t \log^\gamma t} \quad \text{in } 4r_0^2 \leq |x|^2 \leq 2at, \quad t \geq T. \tag{3.4}$$

Proof. We have

$$\begin{aligned} (\partial_t V - LV)(x, t) &= \frac{V_+(x) + b}{\log^\gamma t} (\partial_t \Gamma_a - L\Gamma_a)(x, t) \\ &\quad - \gamma \frac{\Gamma_a(x, t)(V_+(x) + b)}{t \log^{\gamma+1} t} - \frac{\Gamma_a(x, t)LV_+(x)}{\log^\gamma t} \\ &\quad - \underbrace{\frac{1}{\log^\gamma t} \int_{\mathbb{R}^2} J(x - y)(\Gamma_a(y, t) - \Gamma_a(x, t))(V_+(y) - V_+(x)) dy}_{\mathcal{A}(x, t)}. \end{aligned} \tag{3.5}$$

In order to estimate $\partial_t \Gamma_a - L\Gamma_a$, we observe that Taylor’s expansion yields

$$L\Gamma_a(x, t) = \mathfrak{q}\Delta\Gamma_a(x, t) + \frac{1}{4!} \sum_{|\beta|=4} \int_{\mathbb{R}^2} J(|x - y|)D_x^\beta \Gamma_a(\xi, t)(x - y)^\beta dy$$

for some ξ lying in the segment that joins x and y . Hence, since $\partial_t \Gamma_a = a\Delta\Gamma_a$,

$$(\partial_t \Gamma_a - L\Gamma_a)(x, t) = \left(1 - \frac{\mathfrak{q}}{a}\right)\partial_t \Gamma_a(x, t) - \frac{1}{4!} \sum_{|\beta|=4} \int_{\mathbb{R}^2} J(|x - y|)D_x^\beta \Gamma_a(\xi, t)(x - y)^\beta dy.$$

To deal with the main term, we compute

$$\partial_t \Gamma_a(x, t) = \left(\frac{|x|^2}{4at} - 1 \right) \frac{\Gamma_a(x, t)}{t} \leq -\frac{\Gamma_a(x, t)}{2t} < 0 \quad \text{if } |x|^2 \leq 2at.$$

An estimate for the error term follows from

$$|D_x^\beta \Gamma_a|(\xi, t) \leq \frac{C\Gamma_a(\xi, t)}{t^2} \leq \frac{C\Gamma_a(x, t)}{t^2} \quad \text{if } |\beta| = 4, |x|^2 \leq 2at.$$

Thus, since $0 < a < \mathfrak{q}$, for some t_0 large enough we have the lower bound

$$(\partial_t \Gamma_a - L\Gamma_a)(x, t) \geq \frac{1}{2} \left(\frac{\mathfrak{q}}{a} - 1 \right) \frac{\Gamma_a(x, t)}{t} - \frac{C\Gamma_a(x, t)}{t^2} \geq \frac{1}{4} \left(\frac{\mathfrak{q}}{a} - 1 \right) \frac{\Gamma_a(x, t)}{t}, \quad t \geq t_0. \tag{3.6}$$

Let us now bound the integral term $\mathcal{A}(x, t)$. On the one hand,

$$|V_+(y) - V_+(x)| \leq \log(D + d - r_0) - a_{k+1} \quad \text{if } x, y \in B_{D+d}(0).$$

On the other hand, by the Mean Value Theorem,

$$|V_+(y) - V_+(x)| = |\nabla V_+(\xi) \cdot (y - x)| \leq \frac{d}{|\xi| - r_0} \leq \frac{d}{|x| - d - r_0} \leq \frac{C}{|x|} \quad \text{if } x, y \in \mathbb{R}^2 \setminus B_D(0).$$

The difference between the values of Γ_a at x and y is also estimated thanks to the Mean Value Theorem, using that

$$|\nabla \Gamma_a(\xi, t)| = \frac{|\xi| \Gamma_a(\xi, t)}{2at} \leq \frac{C|x| \Gamma_a(x, t)}{t}.$$

Gathering all this information, we obtain

$$|\mathcal{A}(x, t)| \leq C_{d,a} \frac{\Gamma_a(x, t)}{t \log^\gamma t} \quad \text{for } 4r_0^2 \leq |x|^2 \leq 2at. \tag{3.7}$$

If we use estimates (3.6) and (3.7), together with (2.1), in (3.5), we get

$$\begin{aligned} (\partial_t V - LV)(x, t) &\geq \frac{\Gamma_a(x, t)}{t \log^\gamma t} \left(\frac{1}{4} \left(\frac{\mathfrak{q}}{a} - 1 \right) (V_+(x) + b) - \gamma \frac{V_+(x) + b}{\log t} - C_{d,a} \right) \\ &\geq \frac{1}{10} \left(\frac{\mathfrak{q}}{a} - 1 \right) \frac{\Gamma_a(x, t) (V_+(x) + b)}{t \log^\gamma t} \end{aligned}$$

if b is large enough and $t \geq T$, with T large. \square

Now we can obtain the time decay rates in all near field scales.

Corollary 3.1. *Let $0 < a < \mathfrak{q}$. Under the assumptions of Proposition 3.2, there exist constants $C, b, T > 0$ such that*

$$u(x, t) \leq C \frac{e^{-\frac{|x|^2}{4at}} (V_+(x) + b)}{t(\log t)^2} \quad \text{if } |x|^2 \leq 2at, \quad t \geq T.$$

Proof. Once we have established the global decay rate (3.3), the result follows by comparison with a large enough multiple of the super-solution V constructed in Lemma 3.1 with $\gamma = 2$, taking $T \geq t_0$, since

$$V(x, t) \geq \frac{\kappa}{t \log t} \quad \text{if } 2at \leq |x|^2 \leq 2at + d$$

for a certain constant $\kappa > 0$, and

$$V(x, T) \geq c > 0 \quad \text{if } |x|^2 \leq 2aT. \quad \square$$

A fourth by-product of the conservation law is the large time behavior of the “logarithmic momentum”.

Corollary 3.2. *Under the assumptions of Proposition 3.2,*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} u(x, t) \log |x| \, dx = M_\phi^*.$$

Proof. Using the conservation law (3.1), together with the mass decay estimate (3.2) and the estimate (1.6) for the stationary solution, we immediately obtain

$$\left| \int_{\mathbb{R}^2} u(x, t) \log |x| \, dx - M_\phi^* \right| \leq \int_{\mathbb{R}^2} u(x, t) |\log |x| - \phi(x)| \leq \frac{C}{\log t} \quad \text{if } t \geq t_0,$$

from where the result follows. \square

As mentioned in the Introduction, if u_0 satisfies (H_1) , the second momentum of $u(\cdot, t)$ is finite for all times. We now obtain an estimate for its growth, which follows from the mass estimate (3.2). This estimate for the second momentum will be used later to study the limit of the rescaled mass.

Lemma 3.2. *If, in addition to the assumptions of Proposition 3.2, u_0 satisfies (H_1) , then there are constants $t_1 > 0$ and $C_2 > 0$ such that*

$$M_2(t) := \int_{\mathbb{R}^2} u(x, t) |x|^2 \, dx \leq C_2 \frac{t}{\log t} \quad \text{for } t \geq t_1. \tag{3.8}$$

Proof. There holds,

$$\begin{aligned} M_2'(t) &= \int_{\mathbb{R}^2} \partial_t u(x, t) |x|^2 \, dx = \int_{\mathbb{R}^2 \setminus \mathcal{H}} Lu(x, t) |x|^2 \, dx \\ &= \int_{\mathbb{R}^2} Lu(x, t) |x|^2 \, dx - \int_{\mathcal{H}} \int_{\mathbb{R}^2} J(x - y) u(y, t) |x|^2 \, dy \, dx \\ &\leq \int_{\mathbb{R}^2} Lu(x, t) |x|^2 \, dx = \int_{\mathbb{R}^2} u(x, t) L|x|^2 \, dx \\ &= c \int_{\mathbb{R}^2} u(x, t) \leq \frac{cC_0}{\log t} \quad \text{for } t \geq t_0. \end{aligned}$$

Integrating this inequality, we obtain (3.8), for some $t_1 \geq t_0$. \square

The control of the growth of the second momentum yields in turn a control on the growth of the first momentum.

Lemma 3.3. *Under the assumptions of Lemma 3.2,*

$$M_1(t) := \int_{\mathbb{R}^2} u(x, t)|x| dx \leq C_3 \frac{t^{1/2}}{\log t} \quad \text{for } t \geq t_1. \tag{3.9}$$

Proof. Using (3.2) and (3.8), we get

$$M_1(t) \leq t^{1/2} \int_{|x| \leq t^{1/2}} u(x, t) dx + \frac{1}{t^{1/2}} \int_{|x| \geq t^{1/2}} |x|^2 u(x, t) dx \leq \frac{C_0 t^{1/2}}{\log t} + \frac{C_2 t^{1/2}}{\log t}. \quad \square$$

4. Outer limit

We have now all the necessary tools to obtain the outer limit, Theorem 1.1. The first step is to determine the asymptotic value of the rescaled mass in terms of the initial condition.

Proposition 4.1. *Under the assumptions of Theorem 1.2,*

$$\log t \int_{\mathbb{R}^2} u(x, t) dx \rightarrow 2M_\phi^* \quad \text{as } t \rightarrow \infty.$$

The result follows immediately from Corollary 3.2, combined with the following lemma, that states that the scaled mass behaves for large times as twice the logarithmic momentum.

Proposition 4.2. *Under the assumptions of Theorem 1.2,*

$$\lim_{t \rightarrow \infty} \log t \int_{\mathbb{R}^2} u(x, t) dx = \lim_{t \rightarrow \infty} 2 \int_{\mathbb{R}^2} u(x, t) \log |x| dx.$$

To prove this proposition we consider separately the regions $\{|x|^2 \leq t \log t\}$ and $\{|x|^2 \geq t \log t\}$.

Lemma 4.1. *Under the assumptions of Proposition 3.2,*

$$\underbrace{\int_{\{|x|^2 \leq t \log t\}} u(x, t) (\log t - \log |x|^2) dx}_{\mathcal{A}(t)} = O\left(\frac{\log \log t}{\log t}\right) \quad \text{as } t \rightarrow \infty.$$

Proof. We have

$$|\mathcal{A}(t)| \leq \underbrace{\int_{\{|x|^2 \leq \frac{t}{\log t}\}} u(x, t) \left| \log \frac{|x|^2}{t} \right| dx}_{\mathcal{A}_1(t)} + \underbrace{\int_{\{\frac{t}{\log t} \leq |x|^2 \leq t \log t\}} u(x, t) \left| \log \frac{|x|^2}{t} \right| dx}_{\mathcal{A}_2(t)}.$$

Notice that $\left| \log \frac{|x|^2}{t} \right| \leq C \log t$ in $\{4 \leq |x|^2 \leq t/\log t, t \geq e\}$. Therefore, since $u(x, t) = 0$ for $|x| \leq 2$, using the size estimate (3.3) we get

$$|\mathcal{A}_1(t)| \leq \frac{C \log t}{t \log t} \int_{\{|x|^2 \leq \frac{t}{\log t}\}} dx = \frac{C}{\log t}, \quad t \geq \max\{t_0, e\}.$$

Using now that $\left| \log \frac{|x|^2}{t} \right| \leq C \log \log t$ in $\{t/\log t \leq |x|^2 \leq t \log t, t \geq e\}$, together with the decay of the mass (3.2), we obtain

$$|\mathcal{A}_2(t)| \leq C \log \log t \int_{\mathbb{R}^2} u(x, t) dx \leq \frac{C \log \log t}{\log t}, \quad t \geq \max\{t_0, e\}. \quad \square$$

In the case of the second region, both the integral corresponding to the mass and the one corresponding to the logarithmic momentum converge separately to 0.

Lemma 4.2. *Under the assumptions of Lemma 3.2,*

$$\begin{aligned} \log t \int_{\{|x|^2 \geq t \log t\}} u(x, t) dx &= O\left(\frac{1}{\log t}\right), \\ \int_{\{|x|^2 \geq t \log t\}} u(x, t) \log |x|^2 dx &= O\left(\frac{1}{\log t}\right), \end{aligned} \quad \text{as } t \rightarrow \infty.$$

Proof. First, using Lemma 3.2,

$$0 \leq \log t \int_{\{|x|^2 \geq t \log t\}} u(x, t) dx \leq \frac{\log t}{t \log t} \int_{\mathbb{R}^2} u(x, t) |x|^2 dx \leq \frac{C_2}{\log t} \quad t \geq \max\{t_1, 1\}.$$

As for the second integral, we use that the function $r \rightarrow \frac{r}{\log r}$ is nondecreasing for $r \geq e$. Hence, taking t_2 such that $t_2 \log t_2 = e^2$, and using again Lemma 3.2, we have

$$\begin{aligned} \int_{\{|x|^2 \geq t \log t\}} u(x, t) \log |x|^2 dx &= \int_{\{|x|^2 \geq t \log t\}} u(x, t) \frac{|x|^2}{\log |x|^2} dx \leq \frac{\log(t \log t)}{t \log t} \int_{\mathbb{R}^2} u(x, t) |x|^2 dx \\ &\leq C_2 \frac{\log(t \log t)}{t \log t} \frac{t}{\log t} \leq \frac{C}{\log t}, \quad t \geq \max\{t_1, t_2\}. \quad \square \end{aligned}$$

Once we have identified the limit of the rescaled mass, we can proceed to obtain the outer limit. Let us remark that, although we are dealing with the outer region, the proof requires the refined bound in the inner region provided by Corollary 3.1.

The idea is that $\log t u(\cdot, t) \approx v(\cdot, t)$ for large times in the outer region, where $v = v(x, s)$ is the solution to

$$\partial_s v - \mathfrak{q} \Delta v = 0, \quad x \in \mathbb{R}^2, \quad s > \tau(t) := t/(\log t)^{1/2}, \quad v(x, \tau(t)) = \log \tau(t) u(x, \tau(t)), \quad x \in \mathbb{R}^2. \quad (4.1)$$

Lemma 4.3. *Under the assumptions of Theorem 1.1, the solution u to Problem P satisfies*

$$\lim_{t \rightarrow \infty} t \sup \left\{ \left| \log t u(x, t) - v(x, t) \right| : |x| \geq \delta \sqrt{t} \right\} = 0 \quad \text{for all } \delta > 0,$$

where v is the solution to (4.1).

Proof. Since u is a solution to the Cauchy problem (2.5) with right hand side $f = -\chi_{\mathcal{H}}(J * u)$, and initial data $u(\cdot, \tau(t))$ at time $s = \tau(t)$, then

$$\begin{aligned}
 t(\log t u(x, t) - v(x, t)) &= \underbrace{t(\log t - \log \tau(t))u(x, t)}_{\mathcal{A}(x,t)} + \underbrace{t \log \tau(t) e^{-(t-\tau(t))}u(x, \tau(t))}_{\mathcal{B}(x,t)} \\
 &\quad + \underbrace{t \log \tau(t) \int_{\mathbb{R}^2} (W(x - y, t - \tau(t)) - \Gamma_{\mathfrak{q}}(x - y, t - \tau(t)))u(y, \tau(t)) dy}_{\mathcal{C}(x,t)} \\
 &\quad - \underbrace{t \log \tau(t) \int_{\tau(t)}^t e^{-(t-s)}\chi_{\mathcal{H}}(x)(J * u(\cdot, s))(x) ds}_{\mathcal{D}(x,t)} \\
 &\quad - \underbrace{t \log \tau(t) \int_{\tau(t)}^t \int_{\mathcal{H}} W(x - y, t - s)(J * u(\cdot, s))(y) dy ds}_{\mathcal{E}(x,t)}.
 \end{aligned}$$

We have, using (3.3), the L^∞ bound in (1.1), estimate (2.6) with $|\beta| = 0$, the bound for the mass (3.2), and hypotheses ($H_{\mathcal{H}}$),

$$\begin{aligned}
 |\mathcal{A}(x, t)| &= \frac{1}{2} t \log \log t u(x, t) \leq C_1 \log \log t / (2 \log t), \\
 |\mathcal{B}(x, t)| &\leq t \log \tau(t) e^{-(t-\tau(t))} \|u_0\|_{L^\infty(\mathbb{R}^2)}, \\
 |\mathcal{C}(x, t)| &\leq Ct \log(\tau(t))(t - \tau(t))^{-3/2} M(\tau(t)) \leq Ct(t - \tau(t))^{-3/2}, \\
 \mathcal{D}(x, t) &= 0 \quad \text{if } |x|^2 \geq \delta^2 t, \quad t \geq \mathcal{R}^2 / \delta^2,
 \end{aligned}$$

which implies the uniform convergence to 0 of all these terms as $t \rightarrow \infty$. In order to check that also \mathcal{E} converges to 0, we observe that $u(z, s) \leq \frac{C}{s(\log s)^2}$ for $|z| \leq \mathcal{R} + d$; see ($H_{\mathcal{H}}$) and Corollary 3.1. Therefore, $(J * u(\cdot, s)) \leq \frac{C}{s(\log s)^2}$ in \mathcal{H} . Now, by using that $W(x, t) \leq Ct/|x|^4$, see (2.8), we get, for $|x|^2 \geq \delta^2 t$ with t large enough so that $|x - y| \geq \frac{1}{2}|x|$ for every $y \in \mathcal{H}$,

$$|\mathcal{E}(x, t)| \leq t \log \tau(t) \frac{C}{|x|^4} \int_{\tau(t)}^t \frac{t - s}{s \log^2 s} ds \leq \frac{Ct \log \tau(t)(t - \tau(t))^2}{\tau(t)(\log \tau(t))^2 |x|^4} \leq \frac{C_\delta t}{\tau(t) \log \tau(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad \square$$

In view of Lemma 4.3, Theorem 1.1 will follow from the large time behavior for $v(\cdot, t)$ given next. Notice that the result is not immediate, since the initial data for v moves with t .

Lemma 4.4. *Under the assumptions of Theorem 1.1, the solution v to (4.1) satisfies*

$$\lim_{t \rightarrow \infty} t \sup\{|v(x, t) - 2M_\phi^* \Gamma_{\mathfrak{q}}(x, t)| : x \in \mathbb{R}^2\} = 0.$$

Proof. We have

$$t |v(x, t) - 2M_\phi^* \Gamma_{\mathfrak{q}}(x, t)| = t \underbrace{|v(x, t) - \log \tau(t) M(\tau(t)) \Gamma_{\mathfrak{q}}(x, t - \tau(t))|}_{\mathcal{A}(x,t)}$$

$$\begin{aligned}
 &+ \underbrace{t \log \tau(t) M(\tau(t)) |\Gamma_q(x, t - \tau(t)) - \Gamma_q(x, t)|}_{\mathcal{B}(x,t)} \\
 &+ \underbrace{t |\Gamma_q(x, t)| |\log \tau(t) M(\tau(t)) - 2M_\phi^*|}_{\mathcal{C}(x,t)}.
 \end{aligned}$$

Notice that $t\Gamma_q(x, t) \in L^\infty(\mathbb{R}^2)$, and

$$\lim_{t \rightarrow \infty} t \sup\{|\Gamma_q(x, t) - \Gamma_q(x, t - \tau(t))| : x \in \mathbb{R}^2\} = 0.$$

Hence, [Propositions 3.2 and 4.1](#) imply, respectively, that \mathcal{B} and \mathcal{C} converge, uniformly in x , to 0 as $t \rightarrow \infty$. Therefore, since $t - \tau(t) \geq t/2$ if t is large enough, the result will follow if we prove that

$$\lim_{t \rightarrow \infty} (t - \tau(t)) \sup\{|v(x, t) - \log \tau(t) M(\tau(t)) \Gamma_q(x, t - \tau(t))| : x \in \mathbb{R}^2\} = 0.$$

This is what we do next.

Since v is a solution to [\(4.1\)](#), its Fourier transform satisfies

$$\hat{v}(\xi, s) = \hat{v}(\xi, \tau(t)) e^{-4\pi^2 q(s - \tau(t))|\xi|^2}.$$

Moreover, $\hat{v}(0, \tau(t)) = \log \tau(t) M(\tau(t))$. Therefore, performing the change of variables $\eta = (s - \tau(t))^{1/2} \xi$, we have, for some function $R(t)$ that will be conveniently chosen later,

$$\begin{aligned}
 &(s - \tau(t)) \left| v(x, s) - \log \tau(t) M(\tau(t)) \Gamma_q(x, s - \tau(t)) \right| \\
 &\leq (s - \tau(t)) \int_{\mathbb{R}^2} \left| (\hat{v}(\xi, \tau(t)) - \hat{v}(0, \tau(t))) e^{-4\pi^2 q(s - \tau(t))|\xi|^2} \right| d\xi \\
 &\leq \sup_{|\xi| \leq R(t)} |\hat{v}(\xi, \tau(t)) - \hat{v}(0, \tau(t))| \int_{\mathbb{R}^N} e^{-4\pi^2 q|\eta|^2} d\eta \\
 &\quad + 2 \log \tau(t) M(\tau(t)) \int_{|\eta| \geq (s - \tau(t))^{1/2} R(t)} e^{-4\pi^2 q|\eta|^2} d\eta.
 \end{aligned}$$

On the other hand, using the Mean Value Theorem and [\(3.9\)](#), we obtain that

$$\sup_{|\xi| \leq R(t)} |\hat{v}(\xi, \tau(t)) - \hat{v}(0, \tau(t))| \leq 2R(t) \log \tau(t) \int_{\mathbb{R}^2} |x| |u(x, \tau(t))| dx \leq CR(t)(\tau(t))^{1/2}.$$

Thus, taking $s = t$, and considering [Proposition 3.2](#), we arrive at

$$\begin{aligned}
 &(t - \tau(t)) |v(x, t) - \log \tau(t) M(\tau(t)) \Gamma_q(x, t - \tau(t))| \\
 &\leq CR(t)(\tau(t))^{1/2} + C \int_{|\eta| \geq (t - \tau(t))^{1/2} R(t)} e^{-4\pi^2 q|\eta|^2} d\eta \rightarrow 0 \quad \text{as } t \rightarrow \infty,
 \end{aligned}$$

if $R(t)(\tau(t))^{1/2} \rightarrow 0$ and $(t - \tau(t))^{1/2} R(t) \rightarrow \infty$ as $t \rightarrow \infty$. These two conditions are fulfilled if, for example,

$$R(t) = t^{-1/2}(\log t)^\alpha, \quad 0 < \alpha < 1/4. \quad \square$$

5. Inner limit

The goal of this section is to complete the proof of [Theorem 1.2](#). In view of [Theorem 1.1](#), what is left is to show that the limit [\(1.8\)](#) is valid uniformly in an inner set of the form $\{|x|^2 < \delta t\}$ for some $\delta > 0$. Since $|\phi(x)/\log|x|| \leq C$ in $\mathbb{R}^2 \setminus \mathcal{H}$, this will follow from the next result, if we take into account [\(2.6\)](#) with $|\beta| = 0$.

Theorem 5.1. *Under the assumptions of [Theorem 1.2](#), for every $0 < a < \mathfrak{q}$ we have*

$$\lim_{t \rightarrow \infty} t(\log t)^2 \sup \left\{ \frac{1}{\log|x|} \left| u(x, t) - 4M_\phi^* \frac{\phi(x)W(x, t)}{(\log t)^2} \right| : x \in \mathbb{R}^2 \setminus \mathcal{H}, |x|^2 \leq 2at \right\} = 0. \tag{5.1}$$

The advantage of this formulation in terms of W is that it is more straightforward to apply the nonlocal operator L to W than to $\Gamma_{\mathfrak{q}}$.

In order to prove [\(5.1\)](#) we perform a comparison argument in $\{|x|^2 \leq 2at\}$ with suitable barriers ω_ε^\pm which are fast enough ε -close to the asymptotic limit as $t \rightarrow \infty$,

$$\omega_\varepsilon^\pm(x, t) = \underbrace{4M_\phi^* \frac{\phi(x)W(x, t)}{(\log t)^2}}_{v(x, t)} \pm R_\varepsilon(x, t), \quad R_\varepsilon \geq 0, \quad \lim_{t \rightarrow \infty} t(\log t)^2 \sup_{x \in \mathbb{R}^2} \left| \frac{R_\varepsilon(x, t)}{\log|x|} \right| \leq \varepsilon. \tag{5.2}$$

We start by estimating how far is v from being a solution, since this will be needed to prove that ω_ε^+ and ω_ε^- are respectively a super- and a sub-solution.

Lemma 5.1. *There exists a constant $C > 0$ such that*

$$|\partial_t v - Lv|(x, t) \leq \frac{C}{t^2(\log t)^2} \quad \text{if } x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq 2at, \quad t > 1.$$

Proof. A straightforward computation shows that, for $x \in \mathbb{R}^2 \setminus \mathcal{H}$,

$$\begin{aligned} \frac{1}{4M_\phi^*}(\partial_t v - Lv)(x, t) &= \frac{(\partial_t W - LW)(x, t)\phi(x)}{(\log t)^2} - \frac{2\phi(x)W(x, t)}{t(\log t)^3} \\ &\quad + \underbrace{\frac{1}{(\log t)^2} \int_{\mathbb{R}^2} J(x-y)(W(y, t) - W(x, t))(\phi(y) - \phi(x)) dy}_{\mathcal{A}(x, t)}. \end{aligned}$$

In order to estimate the integral term, we write it as

$$\begin{aligned} \mathcal{A}(x, t) &= \nabla \Gamma_{\mathfrak{q}}(x, t) \cdot \int_{\mathbb{R}^2} J(x-y)(y-x)(\phi(y) - \phi(x)) dy \\ &\quad + (\nabla W(x, t) - \nabla \Gamma_{\mathfrak{q}}(x, t)) \cdot \int_{\mathbb{R}^2} J(x-y)(y-x)(\phi(y) - \phi(x)) dy \\ &\quad + \frac{1}{2} \sum_{|\beta|=2} \int_{\mathbb{R}^2} J(x-y) D_x^\beta W(\xi, t) (y-x)^\beta (\phi(y) - \phi(x)) dy. \end{aligned}$$

Estimating the factors involving derivatives of W by [\(2.6\)](#) with $|\beta| = 1$ and [\(2.7\)](#) with $|\beta| = 2$, the gradient of $\Gamma_{\mathfrak{q}}$ by $|\nabla \Gamma_{\mathfrak{q}}(x, t)| \leq C|x|/t^2$, and the difference of the values of ϕ at y and x by [\(2.11\)](#), we get that $|\mathcal{A}(x, t)| \leq C/t^2$ if $x \in \mathbb{R}^2 \setminus \mathcal{H}$. Therefore, since W is a solution to problem [\(2.10\)](#), we conclude that, for $x \in \mathbb{R}^2 \setminus \mathcal{H}, |x|^2 \leq 2at$,

$$\begin{aligned}
 |\partial_t v - Lv|(x, t) &\leq 4M_\phi^* \left(\frac{e^{-t} J(x)\phi(x)}{(\log t)^2} + 2\frac{\phi(x)W(x, t)}{t(\log t)^3} + \frac{C}{t^2(\log t)^2} \right) \\
 &\leq C \left(\frac{e^{-t}}{(\log t)^2} + \frac{|\log |x||}{t^2(\log t)^3} + \frac{1}{t^2(\log t)^2} \right) \leq \frac{C}{t^2(\log t)^2}. \quad \square
 \end{aligned}$$

We now turn our attention to the correction term R_ε . Notice that, in addition to having the required decay property specified in (5.2), R_ε should be such that ω_ε^+ and ω_ε^- are respectively a super- and a sub-solution. To this aim, we need $\partial_t R_\varepsilon - LR_\varepsilon$ to be strictly positive, and with a slower decay than $|\partial_t v - Lv|$.

Guided by our experience with the non-critical cases $N \neq 2$, our first attempt is to take a function in separated variables. A good try is to pick $\nu \in (0, 1)$, and take

$$R_\varepsilon = \frac{\varepsilon}{C_\nu} \underbrace{\frac{w_\nu^+(x)}{t(\log t)^2}}_{w_\nu(x, t)}, \quad C_\nu = \sup_{x \in \mathbb{R}^2 \setminus \mathcal{H}} \left| \frac{w_\nu^+(x)}{\log |x|} \right|,$$

where w_ν^+ is the super-solution to the stationary problem (1.4) given by Lemma 2.3 for this value of ν . Notice that this choice leads to the right decay for R_ε . But we still have to check whether ω_ε^+ is a super-solution. Hence, we have to compute $\partial_t w_\nu - Lw_\nu$.

Using (2.4), we get, for a certain constant κ ,

$$(\partial_t w_\nu - Lw_\nu)(x, t) = -\frac{w_\nu^+(x)}{t^2(\log t)^2} (1 + 2(\log t)^{-1}) + \frac{\kappa}{|x|^2(\log |x|)^{2-\nu} t (\log t)^2}. \tag{5.3}$$

Therefore, using again (2.4), we obtain that there is a constant $\sigma > 0$ such that

$$(\partial_t w_\nu - Lw_\nu)(x, t) \geq \frac{(\kappa t - \sigma|x|^2(\log |x|)^2)}{|x|^2(\log |x|)^{2-\nu} t^2 (\log t)^2}, \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad t \text{ large enough.}$$

Since $|x| \log |x| \leq (\delta t)^{1/2}$ if $|x| \leq \frac{(\delta t)^{1/2}}{\log t}$ and $t \geq \max\{\delta, e\}$, taking $\delta = \kappa/(2\sigma)$ we finally obtain, for some large enough T ,

$$(\partial_t w_\nu - Lw_\nu)(x, t) \geq \frac{\kappa}{2\delta t^2(\log t)^{2-\nu}}, \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq \frac{\delta t}{(\log t)^2}, \quad t \geq T,$$

and we are done in this region. Unfortunately, we can not even guarantee that w_ν is as super-solution in the set $\left\{ \frac{\delta t}{(\log t)^2} \leq |x|^2 \leq 2at \right\}$, hence the need to look for an alternative.

A second possibility is to take $R_\varepsilon = KV$, where V is the super-solution that was constructed, for a fixed $\gamma > 2$, in Lemma 3.1, and K is a positive constant to be fixed. This choice also has the right decay specified in (5.2). Moreover, we see from (3.4), that

$$\partial_t V - LV \geq \frac{C}{t^2(\log t)^{\gamma-1}}, \quad |x|^2 \geq \frac{\delta t}{(\log t)^2}, \quad t \geq T.$$

Hence, if $\gamma \in (0, 3)$, this alternative works in this region. However, our estimate from below for $\partial_t V - LV$ decays too fast in the set $\left\{ x \in \mathbb{R}^2 \setminus \mathcal{H} : |x|^2 \leq \frac{\delta t}{(\log t)^2} \right\}$, and is not able to control the contribution of v in order to make ω_ε^+ a super-solution there.

What will work is a combination of the two possibilities that we have considered,

$$R_\varepsilon = R_{\varepsilon, K}(x, t) = \varepsilon w_\nu(x, t) + KV(x, t), \quad K \geq 1, \quad \varepsilon > 0.$$

Indeed, since V is a super-solution, by the very same computation performed above we have

$$\partial_t \omega_\varepsilon^+ - L\omega_\varepsilon^+ \geq \frac{\kappa}{2\delta t^2(\log t)^{2-\nu}}, \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq \frac{\delta t}{(\log t)^2}, \quad t \geq T.$$

On the other hand, we see from (5.3) and (2.4) that, for some large enough T ,

$$|\partial_t w_\nu - Lw_\nu|(x, t) \leq \frac{C}{t^2(\log t)^{2-\nu}}, \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad \frac{\delta^2 t}{(\log t)^2} \leq |x|^2 \leq 2at, \quad t \geq T.$$

Thus, using (3.4), we see that there is a large enough time $T > 0$ such that, for every $K \geq 1$,

$$\partial_t \omega_\varepsilon^+ - L\omega_\varepsilon^+ \geq \left(\frac{C \log |x|}{t^2(\log t)^\gamma} - \frac{C}{t^2(\log t)^2} - \frac{\varepsilon C}{t^2(\log t)^{2-\nu}} \right), \quad x \in \mathbb{R}^2 \setminus \mathcal{H} : |x|^2 \leq 2at, \quad t \geq T.$$

Therefore, if we first choose $2 < \gamma < 3$, and then $0 < \nu < 3 - \gamma$, we finally obtain that ω_ε^+ is a super-solution in $\{x \in \mathbb{R}^2 \setminus \mathcal{H} : |x|^2 \leq 2at, t \geq T_\varepsilon\}$ for some $T_\varepsilon > 0$ large enough. Analogously, for every $\varepsilon > 0$ and $K \geq 1$, ω_ε^- is a sub-solution in the same set.

Proof of Theorem 5.1. Let $\varepsilon > 0$. The outer behavior, Theorem 1.1, together with (2.6) with $|\beta| = 0$, implies that there exists a time t_ε such that

$$t(\log t) \left| u(x, t) - 2M_\phi^* \frac{W(x, t)}{\log t} \right| < 2\varepsilon, \quad 2at \leq |x|^2 \leq 4at, \quad t \geq t_\varepsilon.$$

Since $\frac{\log |x|}{\log t} \rightarrow \frac{1}{2}$ and $\frac{\phi(x)}{\log |x|} \rightarrow 1$ uniformly in $\{2at \leq |x|^2 \leq 4at\}$ as $t \rightarrow \infty$,

$$\left| u(x, t) - 4M_\phi^* \frac{\phi(x)W(x, t)}{(\log t)^2} \right| \leq 4\varepsilon \frac{\log |x|}{t(\log t)^2} \leq C\varepsilon 4M_\phi^* \frac{\phi(x)W(x, t)}{(\log t)^2} \quad \text{if } 2at \leq |x|^2 \leq 4at, \quad t > \bar{t}_\varepsilon.$$

Thus, if ε is small,

$$(1 - C\varepsilon)(v(x, t) - R_{\varepsilon, K}(x, t)) \leq u(x, t) \leq (1 + C\varepsilon)(v(x, t) + R_{\varepsilon, K}(x, t)), \quad 2at \leq |x|^2 \leq 4at, \quad t > \bar{t}_\varepsilon,$$

since $R_\varepsilon = R_{\varepsilon, K} \geq 0$. On the other hand, if we take $K \geq 1$ large enough,

$$(1 - C\varepsilon)(v - R_{\varepsilon, K})(x, \bar{t}_\varepsilon) \leq u(x, \bar{t}_\varepsilon) \leq (1 + C\varepsilon)(v + R_{\varepsilon, K})(x, \bar{t}_\varepsilon), \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq 4a\bar{t}_\varepsilon.$$

Hence, by the comparison principle we get,

$$(1 - C\varepsilon)(v - R_{\varepsilon, K}) \leq u \leq (1 + C\varepsilon)(v + R_{\varepsilon, K}) \quad x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq 2at, \quad t \geq \bar{t}_\varepsilon.$$

Therefore, using the decay properties of $R_{\varepsilon, K}$, we conclude that

$$\limsup_{t \rightarrow \infty} t(\log t)^2 \sup \left\{ \frac{1}{\log |x|} |u(x, t) - v(x, t)| : x \in \mathbb{R}^2 \setminus \mathcal{H}, \quad |x|^2 \leq 2at \right\} \leq \tilde{C}\varepsilon.$$

Since ε is arbitrary, we get the desired result. \square

Conflict of interest

The authors declare that they have no conflict of interest.

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