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A generalization of a result of Dong and Santos–Sturmfels on the Alexander dual of spheres and balls [☆]

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ABSTRACT

We prove a generalization of a result of Dong and Santos–Sturmfels about the homotopy type of the Alexander dual of balls and spheres. Our results involve *NH*-manifolds, which were recently introduced as the non-pure counterpart of classical polyhedral manifolds. We show that the Alexander dual of an *NH*-ball is contractible and the Alexander dual of an *NH*-sphere is homotopy equivalent to a sphere. We also prove that *NH*-balls and *NH*-spheres arise naturally when considering the double duals of standard balls and spheres.

1. Introduction

Let K be a finite simplicial complex. Fix a ground set of vertices V which contains the set V_K of vertices of K, and let Δ denote the simplex spanned by V. The classical Alexander duality theorem admits a combinatorial formulation in terms of a simplicial

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homotopy representative K^{*_V} of $|\partial\Delta| - |K|$ called the *Alexander dual* of K. In this form the theorem asserts that $H_i(K) \simeq H^{n-i-3}(K^{*_V})$, where n is the cardinality of V and both homology and cohomology groups are reduced (see [16, Theorem 71.1] and [3]). In light of this result, it is natural to ask whether the homotopy type of K can also be deduced from K^{*_V}. Unfortunately in general this is not the case. There are canonical examples of contractible polyhedra and homotopy spheres whose Alexander duals are not respectively contractible or homotopy equivalent to spheres (see [7,17]). Moreover, it can be shown that for any finitely presented group G there is a finite simply connected complex K such that $\pi_1(K^{*_V}) = G$ (see [15]). In 2002 Dong [7] proved that the Alexander dual of a simplicial sphere has again the homotopy type of a sphere, and one year later Santos and Sturmfels [17] showed that the Alexander duals of simplicial balls are contractible spaces.

Theorem 1.1 *(Dong, Santos–Sturmfels). If B is a simplicial ball then B^{*_V} is contractible. If S is a simplicial sphere then S^{*_V} is homotopy equivalent to a sphere.*

This result evidences that a locally well-behaved structure on the complex forces homotopy stableness on its dual and one may ask whether other manifold-like constructions satisfy a similar property. The *NH*-manifolds are natural candidates for this. These complexes were recently introduced in [6] as a generalization of combinatorial manifolds to the non-pure setting (or, more precisely, to the non-necessarily pure setting). The study of *NH*-manifolds was in part motivated by the theory of non-pure shellability due to Björner and Wachs [4]. It was also shown in [6] that they appear when investigating Pachner moves between manifolds: if two polyhedral manifolds (with or without boundary) are PL-homeomorphic then they are related by a finite sequence of factorizations involving *NH*-manifolds.

In this paper we prove a generalization of the results of Dong and Santos–Sturmfels to non-necessarily pure balls and spheres. Dong's approach relies mainly on convexity and Santos–Sturmfels' proof follows from Dong's result on spheres. In contrast to the previous treatments, our results rely on the local nature of the manifolds. We first exhibit a new proof of Dong's and Santos–Sturmfels' results for combinatorial balls and spheres (Theorem 3.9) from which the more general Theorem 1.1 then follows by an argument from Dong's paper (see Remark 3.10 below). Then we use the combinatorial version of Theorem 1.1 and various results on manifolds with few vertices (pure and non-pure) to prove the following generalization to the non-pure setting.

Theorem 1.2. *If B is an NH-ball then B^{*_V} is contractible. If S is an NH-sphere then S^{*_V} is homotopy equivalent to a sphere.*

NH-balls and *NH*-spheres are the non-pure analogues of balls and spheres (see Definition 2.2 below and Fig. 1 for some examples in low dimensions).

The second aim of this article is to use the theory of *NH*-manifolds to analyze the topological and simplicial structures of the Alexander duals of balls and spheres in the following sense. Given a subspace A of the d-sphere S^d, since the complement $B = S^d - A$ is also a subspace of $S^{d'}$ for any $d' \geq d$, it is natural to study the relationship between A and $S^{d'} - B$ (the complement of its complement in a sphere of higher dimension). In the combinatorial setting, this amounts to understanding the *double dual* $L = (K^{*_V})^{*_{V'}}$ where $V_K \subseteq V \subsetneq V'$. We prove that the double duals of balls (resp. spheres) are *NH*-balls (resp. *NH*-spheres).

The rest of the paper is organized as follows. In Section 2 we recall the basic properties of classical combinatorial manifolds and *NH*-manifolds and prove a result on the existence of spines for *NH*-manifolds with boundary. This result is used in the proof of the main theorem but it is also interesting in its own right. In Section 3 we exhibit an alternative and simpler proof of Dong's and Santos–Sturmfels' original results on the Alexander duals of balls and spheres which is based on the local structure of manifolds. In Section 4 we prove the main result (Theorem 1.2) and in Section 5 we show that *NH*-balls and *NH*-spheres appear naturally when considering double duals of classical balls and spheres.

2. Preliminaries

All complexes considered in this paper are finite. Given a set of vertices V, $|V|$ will denote its cardinality. We denote by Δ^d the standard complex consisting of all the faces of a d-simplex; the boundary $\partial\Delta^d$ is the complex of all the proper faces of the simplex. We write $\sigma < \tau$ when σ is a face of τ and $\sigma \prec \tau$ when it is an immediate face (i.e. $\dim(\sigma) = \dim(\tau) - 1$). A simplex is *maximal* in a complex K if it is not a proper face of any other simplex of K. A *ridge* of K is an immediate face of a maximal simplex and two simplices $\sigma, \tau \in K$ are *adjacent* if $\sigma \cap \tau$ is an immediate face of σ or τ. A complex is *pure* if all its maximal simplices have the same dimension. Given a simplex σ, we denote by $\overline{\sigma}$ the simplicial complex spanned by σ (i.e. the simplicial complex containing all its faces). More generally, given a set of vertices V we denote by $\overline{V}$ the simplicial complex consisting of all the subsets of V. The set of vertices of a complex K will be denoted by V_K and Δ_K will stand for $\overline{V}_K$.

As usual $K * L = \{\sigma \cup \tau \,|\, \sigma \in K, \tau \in L\}$ will denote the join of the complexes K and L. As a consequence, if $\emptyset$ is the empty simplex and $\{\emptyset\}$ the complex containing only the empty simplex then $K * \{\emptyset\} = K$ and $K * \emptyset = \emptyset$. For $\sigma \in K$, $lk(\sigma, K) = \{\tau \in K : \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in K\}$ denotes its *link* and $st(\sigma, K) = \overline{\sigma} * lk(\sigma, K)$ its (closed) *star*. The union of two complexes K, L will be denoted by $K + L$. A subcomplex $L \subset K$ is said to be *top generated* if every maximal simplex of L is also maximal in K.

We write $K \simeq_{PL} L$ when K and L are PL-homeomorphic; that is, whenever they have a subdivision in common. We will frequently identify a complex K with its geometric realization and write $K \simeq L$ when K is homotopy equivalent to L. If B is a subcomplex of a simplicial complex K, $K - B$ will denote its complement in K (this corresponds to an open subspace of the geometric realization of K) and we denote by $\overline{K - B}$ its closure (which is a subcomplex of K). Given a vertex v of a simplicial complex K, we denote by $dl_v(K)$ the *deletion* of v which is the subcomplex of K consisting of all the simplices which do not contain the vertex v. Given $t \geq 0$, $\Sigma^t K$ will denote the simplicial t-fold (unreduced) suspension of K. Note that $\Sigma^t K \simeq \partial\Delta^t * K$.

A maximal simplex $\tau \in K$ is *collapsible* in K if there is a ridge $\sigma \prec \tau$ which is not a face of any other simplex of K (i.e. σ is a free face). If τ is collapsible, the operation which transforms K into $K \setminus \{\tau, \sigma\}$ is called an *elementary (simplicial) collapse* and is denoted by $K \searrow^e K \setminus \{\tau, \sigma\}$. It is easy to see that $K \setminus \{\tau, \sigma\} \subset K$ is a strong deformation retract. The inverse operation is called an *elementary (simplicial) expansion*. If there is a sequence $K \searrow^e K_1 \searrow^e \cdots \searrow^e L$ we say that K *collapses* to L (or equivalently, L *expands* to K) and write $K \searrow L$ or $L \nearrow K$. A complex K is *collapsible*, and we write $K \searrow 0$, if it collapses to a single vertex. A complex K is *PL-collapsible* if it has a subdivision which is collapsible.

2.1. Combinatorial manifolds

We recall some basic definitions and properties of the classical theory of combinatorial manifolds. For a comprehensive exposition of the subject we refer the reader to [9,10,13].

A combinatorial d-ball is a complex which is PL-homeomorphic to Δ^d. A combinatorial d-sphere is a complex PL-homeomorphic to $\partial\Delta^{d+1}$. By convention, $\{\emptyset\} = \partial\Delta^0$ is considered a sphere of dimension -1. A combinatorial d-manifold is a complex M such that $lk(v, M)$ is a combinatorial $(d-1)$-ball or $(d-1)$-sphere for every $v \in V_M$. It is easy to verify that d-manifolds are pure complexes of dimension d. The link of any simplex in a manifold is also a ball or a sphere and the class of combinatorial manifolds is closed under PL-homeomorphisms. In particular, combinatorial balls and spheres are combinatorial manifolds. By a result of J.H.C. Whitehead, combinatorial d-balls are precisely the PL-collapsible combinatorial d-manifolds (see [9, Corollaries III.6 and III.17]). The following classical result will be used frequently in the paper (see [9,10,13]).

Theorem 2.1 *(Newman). If S is a combinatorial d-sphere containing a combinatorial d-ball B then the closure $\overline{S - B}$ is a combinatorial d-ball.*

The boundary ∂M of a combinatorial d-manifold M can be regarded as the set of simplices whose links are combinatorial balls. This coincides with the usual definition of boundary for d-pure complexes as the subcomplex generated by the mod 2 sum of the $(d-1)$-simplices. It is easy to see that ∂M is a $(d-1)$-combinatorial manifold without boundary.

A *weak d-pseudomanifold* without boundary is a d-pure simplicial complex P satisfying that each $(d-1)$-simplex is contained in exactly two d-simplices. It is easy to see that in this case $lk(\sigma, P)$ is a weak $(d - \dim(\sigma) - 1)$-pseudomanifold for every $\sigma \in P$ and that $H_d(P; \mathbb{Z}_2) \neq 0$, since the mod 2 sum of the d-simplices of P is a generating cycle. A *d-pseudomanifold* is a weak d-pseudomanifold with or without boundary (i.e. the $(d-1)$-simplices are contained in *at most* two d-simplices) which is strongly connected; that is, any two d-simplices σ, τ can be connected by a sequence of d-simplices

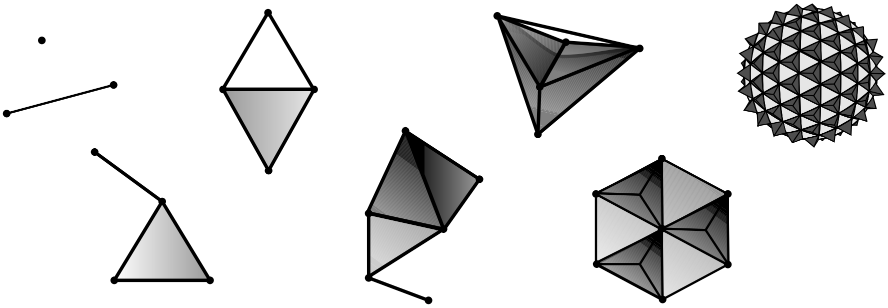

Fig. 1. Examples of *NH*-manifolds. The four figures in the top are *NH*-spheres of dimension 1, 2, 3 and 3 and homotopy dimension 0, 1, 1 and 2 respectively. The bottom figures are *NH*-balls.

$\sigma = \eta_0, \dots, \eta_k = \tau$ such that $\eta_i \cap \eta_{i+1}$ is $(d-1)$-dimensional for each $i = 0, \dots, k-1$ (i.e. η_i and η_{i+1} are adjacent). It is easy to see that a connected combinatorial d-manifold is a d-pseudomanifold.

2.2. Non-pure manifolds

NH-manifolds are the non-pure versions of combinatorial manifolds and play a key role in this work. We give next a brief summary of the subject and refer the reader to [6] for a more detailed exposition.

NH-manifolds have a local structure consisting of Euclidean spaces of varying dimensions. In Fig. 1 we exhibit some examples of *NH*-manifolds.

Definition 2.2. An *NH-manifold* (resp. *NH-ball*, *NH-sphere*) of dimension 0 is a manifold (resp. ball, sphere) of dimension 0. An *NH*-sphere of dimension -1 is, by convention, the complex $\{\emptyset\}$. For $d \geq 1$, we define by induction

- An *NH-manifold* of dimension d is a complex M of dimension d such that $lk(v, M)$ is an *NH*-ball of dimension $0 \leq k \leq d-1$ or an *NH*-sphere of dimension $-1 \leq k \leq d-1$ for all $v \in V_M$.
- An *NH-ball* of dimension d is a PL-collapsible *NH*-manifold of dimension d.
- An *NH-sphere* of dimension d and *homotopy dimension* k is an *NH*-manifold S of dimension d for which there exist a top generated *NH*-ball B of dimension d and a combinatorial k-ball L such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B + L$ is a *decomposition* of S and write $\dim_h(S)$ for the homotopy dimension of S. Note that L is automatically top generated in S.

In [6] it is proved that *NH*-manifolds satisfy many (generalized) results of the classical theory of combinatorial manifolds. Also, by [6, Theorem 3.6], pure *NH*-manifolds are standard combinatorial manifolds. We next summarize the relevant results of this theory that will be used in this article. We need first a definition.

Definition 2.3. An *NH-pseudomanifold* is a simplicial complex with the following two properties.

(1) For each ridge $\sigma \in M$, $lk(\sigma, M)$ is either a point or an *NH*-sphere of homotopy dimension 0.
(2) Given any two maximal simplices $\sigma, \tau \in M$, there exists a sequence $\sigma = \eta_1, \ldots, \eta_s = \tau$ of maximal simplices of M such that η_i and η_{i+1} are adjacent for each $1 \leq i \leq s-1$.

Theorem 2.4. *Let M be an NH-manifold of dimension d, let $\sigma \in M$ and let B_1, B_2 be NH-balls and S_1, S_2 be NH-spheres. Then,*

(1) *$lk(\sigma, M)$ is an NH-ball or an NH-sphere.*
(2) *NH-manifolds, NH-balls and NH-spheres are closed under PL-homeomorphisms.*
(3) *$B_1 * B_2$ and $B_1 * S_2$ are NH-balls. $S_1 * S_2$ is an NH-sphere.*
(4) *If M is connected then it is an NH*-pseudomanifold.

The *pseudoboundary* $\tilde{\partial} M$ of an *NH*-manifold is the set of simplices whose links are *NH*-balls (in general, this is not a simplicial complex). The boundary ∂M is the simplicial complex generated by the simplices in $\tilde{\partial} M$. By [6, Proposition 4.3], $\tilde{\partial} M$ is a complex if and only if M is pure. This implies that boundaryless *NH*-manifolds are classical (combinatorial) manifolds. This shows that, unlike classical manifolds, *NH*-spheres which are non-pure do have boundary. By [6, Lemma 4.8], if $S = B + L$ is a decomposition of an *NH*-sphere then $lk(\sigma, S)$ is an *NH*-sphere with decomposition $lk(\sigma, B) + lk(\sigma, L)$ for every $\sigma \in L$.

Given an *NH*-manifold M, we denote by $\overset{\circ}{M}$ the relative interior of M, which is the set of simplices whose links are *NH*-spheres (of any dimension). The following is a special case of [6, Theorem 6.3], which is a generalization of Alexander's classical theorem on regular expansions (see [13, Theorem 3.9]).

Theorem 2.5. *Let M be an NH-ball (resp. NH-sphere) and B a combinatorial ball. Suppose $M \cap B \subseteq \partial B$ is an NH-ball or an NH-sphere generated by ridges of M or B and that $(M \cap B)^\circ \subseteq \tilde{\partial} M$. Then*

(1) *$M + B$ is an NH-ball (resp. NH-sphere) if $M \cap B$ is an NH-ball.*
(2) *$M + B$ is an NH-sphere if $M \cap B$ is an NH-sphere.*

It is known that a d-pure complex with $d+2$ vertices is either the boundary of a simplex or the complex obtained by starring a simplex (see [14, Lemma 6]) but for a general d-dimensional complex with $d+2$ vertices not even its homotopy type can be known beforehand. However, when the complex is an *NH*-manifold then it is either contractible or homotopy equivalent to a sphere. Actually, the next stronger result holds.

Proposition 2.6. *If M is an NH-manifold of dimension d and $d+2$ vertices then M is an NH-ball or NH-sphere.*

Proof. By assumption there is a vertex u in M such that $M = \Delta^d + \overline{u} * lk(u, M)$. Since M is an *NH*-manifold, $lk(u, M)$ is an *NH*-ball or an *NH*-sphere. The result follows from Theorem 2.5 applied to the *NH*-ball $\overline{u} * lk(u, M)$ and the combinatorial ball Δ^d. $\square$

The next two results, which are interesting in their own right, will be used in the proof of the main theorem.

Proposition 2.7. *Let M be a connected NH-manifold of dimension d such that $H_d(M; \mathbb{Z}_2) \neq 0$. Then, M is a combinatorial d-manifold without boundary. In particular, if S is an NH-sphere with $\dim_h(S) = \dim(S)$ then S is a combinatorial sphere.*

Proof. By [6, Theorem 3.6] it suffices to prove that M is pure. Let c be a generating d-cycle of $H_d(M; \mathbb{Z}_2)$ and let $K \subset M$ be the subcomplex generated by the d-simplices appearing in c with nonzero coefficients. We shall show that $M = K$. Note that since $K \subset M$ is top generated and M is an *NH*-pseudomanifold (see Theorem 2.4 (4)), K is a weak pseudomanifold without boundary (since c is a cycle). If $M \neq K$, let $\eta \in M - K$ be a maximal simplex adjacent to K and let ρ be the simplex spanned by the vertices in $\overline{\eta} \cap K$. Since by dimensional considerations $\rho \prec \eta$, $lk(\rho, M) = lk(\rho, M - \overline{\eta}) + lk(\rho, \overline{\eta})$ is an *NH*-sphere of homotopy dimension 0. But $lk(\rho, K) \subset lk(\rho, M - \overline{\eta})$ is a weak pseudomanifold without boundary and hence $H_{\dim(lk(\rho,K))}(lk(\rho, K); \mathbb{Z}_2) \neq 0$. This contradicts the fact that $lk(\rho, M - \overline{\eta})$ is an *NH*-ball since a generating cycle in $lk(\rho, K)$ is also generating in $lk(\rho, M - \overline{\eta})$. Note also that $\partial M = \partial K = \emptyset$. $\square$

We say that a subcomplex K of an *NH*-manifold M is a *spine* of M if $\dim(K) < \dim(M)$ and $M \searrow K$. The following is a generalization of a classical result for combinatorial manifolds.

Corollary 2.8 *(Existence of spines for NH-manifolds). Every connected NH-manifold M with non-empty boundary has a spine.*

Proof. Let d be the dimension of M and let Y^d be the d-pure subcomplex of M (i.e. the subcomplex of M generated by the d-simplices). Start collapsing the d-simplices of Y^d and suppose we get stuck before depleting all the d-simplices. Then, there is a boundaryless d-pseudomanifold $L \subset Y^d \subset M$ and hence $0 \neq H_d(L; \mathbb{Z}_2) \subset H_d(M; \mathbb{Z}_2)$, which is a contradiction by Proposition 2.7. $\square$

3. A new proof of Dong's and Santos–Sturmfels' results

In this section we exhibit an alternative and simpler proof of Dong's and Santos–Sturmfels' original results on the Alexander duals of balls and spheres. We study first the relationship between the Alexander dual of a complex relative to its own set of vertices and to a bigger ground set of vertices. The comparison between both duals will be used throughout the paper. From the geometrical point of view, it amounts to analyzing the relationship between the complements of a complex when seen as a subspace of spheres of different dimensions.

Given a simplicial complex K, let $K^* = \{\sigma \subset V_K \,|\, V_K \setminus \sigma \notin K\}$ be the Alexander dual *with respect to the ground set* V_K. More generally, given a vertex set $V \supseteq V_K$, let $K^{*_V} = \{\eta \subset V \,|\, V \setminus \eta \notin K\}$ be the Alexander dual of K *with respect to the ground set* V. Since we need to emphasize the complement of V_K in V, K^{*_V} will be usually denoted by K^τ, where $\tau = V \setminus V_K$. Note that if $\tau = \emptyset$ then $K^\tau = K^*$ is the Alexander dual of K relative to its own set of vertices. Note also that $(\Delta^d)^* = \emptyset$ and $(\partial\Delta^d)^* = \{\emptyset\}$.

Before we compare the Alexander duals of a simplicial complex with respect to different ground sets of vertices, we recall a well-known and standard result which will be used later.

Lemma 3.1. *Let K be a simplicial complex and $A, B \subset K$ subcomplexes such that $K = A + B$.*

(1) *If A and B are contractible then $K \simeq \Sigma(A \cap B)$. If in addition K is acyclic then K is contractible. In particular, acyclic simplicial complexes of dimension d and $d+2$ vertices are contractible.*
(2) *If $A \cap B$ and B are contractible then $K \simeq A$.*

Lemma 3.2. *Let K be a simplicial complex and let τ be a non-empty set disjoint from V_K. Then,*

$$K^\tau = \partial\overline{\tau} * \Delta_K + \overline{\tau} * K^*. \tag{A}$$

Here K^* is considered as a subcomplex of the simplex Δ_K. In particular, we have the following consequences.

(1) *If K is not a simplex or $\dim(\tau) \geq 1$ then $V_{K^\tau} = V_K \cup V_\tau$. If $K = \overline{\eta}$ is a simplex and $\dim(\tau) = 0$ then $\overline{\eta}^\tau = \overline{\eta}$. In any case, $V_K \subseteq V_{K^\tau}$.*
(2) *If K is not a simplex or $\dim(\tau) \geq 1$ then $(K^\tau)^* = K$.*
(3) *If $V_{K^*} \subsetneq V_K$ and $\rho = V_K \setminus V_{K^*}$ then $(K^*)^\rho = K$.*
(4) *If K is not a simplex then $K^\tau \simeq \Sigma^t K^*$ for some $t \geq 0$.*

Proof. To prove (A) set $V = V_K \cup \tau$. Let $\sigma \in K^\tau$ be a maximal simplex, so $V \setminus \sigma \notin K$. If $\tau < \sigma$, say $\sigma = \tau \cup \eta$, then $V \setminus \sigma = V_K \setminus \eta$ and therefore $\sigma = \tau \cup \eta$ is in $\overline{\tau} * K^*$. Any other simplex in K^τ not containing τ lies trivially in $\partial\overline{\tau} * \Delta_K$. For the other inclusion, if $\sigma = \tau \cup \eta$ is maximal and $\eta \in K^*$ then $V \setminus \sigma = V_K \setminus \eta \notin K$, and hence $\sigma \in K^\tau$. If $\sigma \in \partial\overline{\tau} * \Delta_K$ is maximal then in particular $\Delta_K \subset \overline{\sigma}$, and therefore $V \setminus \sigma \subset \tau$. Since no vertex of τ lies in K, $V \setminus \sigma \notin K$ and then $\sigma \in K^\tau$.

Item (1) follows directly from formula (A) and items (2)–(3) from the fact that for a fixed ground set V, $(K^{*_V})^{*_V} = K$. Finally, (4) follows from formula (A) and Lemma 3.1 (1). $\square$

Note that the homotopy equivalence in (4) also holds for $\tau = \emptyset$ taking $t = 0$.

Lemma 3.3. *Let K be a simplicial complex of dimension d that is not a d-simplex. The following statements are equivalent.*

(1) $|V_K| = d + 2$.
(2) $V_{K^*} \neq V_K$.
(3) $K \neq K^{**}$.

Proof. Suppose that $|V_K| = d + 2$ and let $\sigma \in K$ be a d-simplex. Then the only vertex $v \in V_K \setminus \sigma$ is not in V_{K^*}. Conversely, if $w \in V_K \setminus V_{K^*}$ then $V_K \setminus \{w\} \in K$. Since K is not a d-simplex, $|V_K| \geq d + 2$. Since $V_K \setminus \{w\} \in K$ and $\dim(K) = d$, $|V_K| \leq d + 2$. This proves that (1) and (2) are equivalent.

(2) implies (3) since $V_{K^{**}} \subseteq V_{K^*} \subseteq V_K$. Also, (3) implies (2) since if $V_{K^*} = V_K$, $K^{**} = K$. $\square$

Corollary 3.4. *Let K be a simplicial complex and let τ be a non-empty set disjoint from V_K. Then,*

(1) *If K is not a simplex or* $\dim(\tau) \geq 1$ *then* $|V_{K^\tau}| = \dim(K^\tau) + 2$.
(2) *The subcomplexes $\partial\overline{\tau} * \Delta_K, \overline{\tau} * K^* \subset K^\tau$ in formula (A) of Lemma 3.2 are top generated.*

Proof. If K is not a simplex, item (1) follows directly from Lemmas 3.2 and 3.3. If $K = \overline{\eta}$ is a simplex and $\dim(\tau) \geq 1$ then $K^\tau = \partial\overline{\tau} * \overline{\eta}$ which has dimension $\dim(\tau) + \dim(\eta)$ and $\dim(\tau) + 1 + |\eta| = \dim(\tau) + \dim(\eta) + 2$ vertices.

For (2), simply notice that $(\partial\overline{\tau} * \Delta_K) \cap (\overline{\tau} * K^*) = \partial\overline{\tau} * K^*$ and that K^* is always properly contained in Δ_K. $\square$

Remark 3.5. Lemma 3.2 and Corollary 3.4 state that every complex is the Alexander dual of a complex of dimension d and $d + 2$ vertices for some $d \geq 0$. Note that $d = \dim(K^\tau) = \dim(\overline{\tau}) + |V_K| - 1$, so d can be very big with respect to $\dim(K)$.

Lemma 3.6. *Let K be a complex of dimension d and $d+2$ vertices. Then, for every vertex $u \in V_K \setminus V_{K^*}$ we have that $K^* = (lk(u, K))^\tau$ where $\tau = V_K \setminus V_{st(u,K)}$.*

Proof. By the assumption we can write $K = \Delta^d + \overline{u} * lk(u, K)$. Let τ be as in the claim. Then,

$$\begin{aligned}\sigma \in (lk(u,K))^\tau &\Leftrightarrow V_{lk(u,K)} \cup \tau \setminus \sigma \notin lk(u,K) \\ &\Leftrightarrow V_{lk(u,K)} \cup (V_K \setminus V_{st(u,K)}) \setminus \sigma \notin lk(u,K) \\ &\Leftrightarrow V_K \setminus \{u\} \setminus \sigma \notin lk(u,K) \\ &\Leftrightarrow V_K \setminus \sigma \notin K \\ &\Leftrightarrow \sigma \in K^*. \qquad \square\end{aligned}$$

Lemma 3.7. *Let $K \neq \Delta^d$ be a simplicial complex of dimension d and let $v \in V_K$. Then,*

(1) $lk(v, K^*) = (dl_v(K))^*$.
(2) $lk(v, K) = (dl_v(K^*))^\tau$ *where* $\tau = V_{dl_v(K)} \setminus V_{dl_v(K^*)}$.
(3) *If v is not isolated and $lk(v, K)$ is not a simplex then $dl_v(K^*) \simeq \Sigma^t lk(v, K)^*$ for some $t \geq 0$.*
(4) *If $lk(v, K)$ is a simplex then $dl_v(K^*)$ is also a simplex.*

Proof. For (1),

$$\sigma \in lk(v, K^*) \Leftrightarrow \{v\} \cup \sigma \in K^* \Leftrightarrow V_K \setminus \sigma \notin dl_v(K) \Leftrightarrow \sigma \in (dl_v(K))^*.$$

To prove (2), let $\eta = \{x\}$ for some $x \notin V_K$. Since $K \neq \Delta^d$, $(K^\eta)^* = K$ and by (1),

$$lk(v, K) = lk(v, (K^\eta)^*) = (dl_v(K^\eta))^*.$$

Note that $K^\eta = \Delta_K + \overline{\eta} * K^*$, and then

$$dl_v(K^\eta) = dl_v(\Delta_K) + dl_v(\overline{\eta} * K^*) = \overline{V_K \setminus \{v\}} + \overline{\eta} * (dl_v(K^*)).$$

Now Lemma 3.6 implies that

$$(dl_v(K^\eta))^* = lk(\eta, dl_v(K^\eta))^\tau = (dl_v(K^*))^\tau$$

where $\tau = V_{dl_v(K^\eta)} \setminus V_{st(\eta, dl_v(K^\eta))} = V_{dl_v(K)} \setminus V_{dl_v(K^*)}$. This proves (2).

To prove (3), consider the Alexander duals of the complexes in the equality (2) to yield

$$lk(v, K)^* = ((dl_v(K^*))^\tau)^*.$$

When $\tau \neq \emptyset$, this equals $dl_v(K^*)$ by Lemma 3.2 (2), which settles the result with $t = 0$. Note that, by hypothesis, $dl_v(K^*) = \Delta^r$ and $\dim(\tau) = 0$ cannot simultaneously hold.

Suppose now that $\tau = \emptyset$. Denote $T = dl_v(K^*)$. If $\dim(T) \neq |V_T| - 2$ then $lk(v, K)^* = T^{**} = T$ by Lemma 3.3 and the result holds with $t = 0$. If $\dim(T) = |V_T| - 2$ then $\rho = V_T \setminus V_{T^*} \neq \emptyset$ and

$$T = (T^*)^\rho = \partial\overline{\rho} * \Delta_{T^*} + \overline{\rho} * T^{**} = \partial\overline{\rho} * \Delta_{T^*} + \overline{\rho} * lk(v, K)^*.$$

Since by hypothesis $\Delta_{T^*} = \Delta_{lk(v,K)} \neq \emptyset$ and $T^{**} = lk(v, K)^* \neq \emptyset$,

$$dl_v(K^*) = T \simeq \Sigma(\partial\overline{\rho} * lk(v, K)^*) \simeq \Sigma^t lk(v, K)^*.$$

To prove (4) note that, by (2), $(dl_v(K^*))^\tau = lk(v, K)$. Since it is a simplex, $((dl_v(K^*))^\tau)^* = \emptyset$. By Lemma 3.2 (2) $dl_v(K^*)$ is a simplex. □

It is straightforward that σ is a free face of τ in K if and only if $V \setminus \tau$ is a free face of $V \setminus \sigma$ in K^*. In particular, we have the following known result (see also [11]).

Lemma 3.8. *Let L be a subcomplex of K. Then $K \searrow L$ if and only if $K^* \nearrow L^\tau$ where $\tau = V_K \setminus V_L$. In particular, if L^* is contractible or homotopy equivalent to a sphere then so is K^*.*

We are now able to give an alternative proof of Dong's and Santos–Sturmfels' original results.

Theorem 3.9 *(Dong, Santos–Sturmfels). If $B \neq \Delta^d$ is a combinatorial d-ball then B^τ is contractible. If S is a combinatorial d-sphere then S^τ is homotopy equivalent to a sphere.*

Proof. By Lemma 3.2 (4) it suffices to prove the result for $\tau = \emptyset$. We first prove it for a combinatorial ball B by induction on $d \geq 1$. If $d = 1$ then B collapses to a 1-ball with two edges (whose Alexander dual is a vertex) and the result follows from Lemma 3.8. Now, let $d \geq 2$. If $|V_B| = d+2$, take $u \in V_B \setminus V_{B^*}$. If $lk(u, B)$ is not a simplex, Lemma 3.6 and Lemma 3.2 (4) imply $B^* \simeq \Sigma^t lk(u, B)^*$, which is contractible by induction since $lk(u, B)$ is a ball. If $lk(u, B)$ is a simplex, the result follows immediately.

Suppose $|V_B| \geq d+3$ and let $v \in \partial B$. Now, $dl_v(B^*)$ is contractible by Lemma 3.7 (4) or Lemma 3.7 (3) and induction. Since $B^* = (dl_v(B^*)) + st(v, B^*)$ is acyclic by Alexander duality, B^* is contractible by Lemma 3.1 (1).

Now let S be a combinatorial sphere. We may assume that $|V_S| \geq d+3$. We proceed again by induction on d. Let $d \geq 1$ and $v \in V_S$. By Lemma 3.7 (1), $lk(v, S^*) = (dl_v(S))^*$ which is contractible by Theorem 2.1 and the previous case. Since $S^* = (dl_v(S^*)) + st(v, S^*)$ where $(dl_v(S^*)) \cap st(v, S^*) = lk(v, S^*)$ is contractible, $S \simeq dl_v(S^*) \simeq \Sigma^t lk(v, S)^*$ by Lemma 3.1 (2) and Lemma 3.7 (3). The result now follows by the inductive hypothesis on the $(d-1)$-sphere $lk(v, S)$. □

Remark 3.10. Recall that for each dimension $d \geq 5$ there are simplicial d-spheres which are not combinatorial spheres. In particular, no subdivision of them is combinatorially equivalent to the boundary of a $(d+1)$-polytope. However, the more general formulation of Theorem 3.9 for simplicial balls and spheres stated in Theorem 1.1 follows from our combinatorial proof, using an argument of Dong [7] (see also [17]). Concretely, given a simplicial sphere S, the cases $|V_S| - \dim(S) \geq 5$ are a consequence of Alexander duality: S^* is simply connected since it contains the complete 2-skeleton of $\overline{V_{S^*}}$ and a simply connected space with the homology of a sphere is homotopy equivalent to one. On the other hand, the cases $2 \leq |V_S| - \dim(S) \leq 4$ are polytopal which, in turn, are combinatorial (see [8,14]). A similar argument can be made for simplicial balls.

4. Proof of the main result

In this section we generalize Dong's result on the Alexander dual of simplicial spheres [7] and Santos–Sturmfels' result on simplicial balls [17] to the more general setting of *NH*-spheres and *NH*-balls. First we need some results on manifolds with few vertices.

Theorem 4.1. *(See [5, Theorem A].) Let M be a boundaryless combinatorial d-manifold with n vertices. If*

$$n < 3\left\lceil \frac{d}{2} \right\rceil + 3$$

then M is a combinatorial d-sphere. Also, if $d = 2$ and $n = 6$ then M is either PL-homeomorphic to the 2-sphere or to the projective plane $\mathbb{R}P^2$.

The following is an immediate consequence of this result.

Corollary 4.2. *Let M be a combinatorial d-manifold with boundary with n vertices. If*

$$n < \min\left\{3\left\lceil \frac{d-1}{2} \right\rceil + 3, 3\left\lceil \frac{d}{2} \right\rceil + 2\right\}$$

then M is a combinatorial d-ball. The result is also valid if $d = 3$ and $n = 6$.

Proof. By Theorem 4.1 ∂M is a combinatorial $(d-1)$-sphere. This includes the case $d = 3$ and $n = 6$ since $\mathbb{R}P^2$ cannot be the boundary of a compact manifold. Take $u \notin V_M$ and build $N = M + \overline{u} * \partial M$ where $M \cap \overline{u} * \partial M = \partial M$. It is easy to see that N is a boundaryless combinatorial d-manifold. Now, since $|V_N| < 3\lceil \frac{d}{2} \rceil + 3$, N is a combinatorial d-sphere by Theorem 4.1 and $M = \overline{N - \overline{u} * \partial M}$ is a combinatorial d-ball by Theorem 2.1. □

Proposition 4.3. *Let B be an NH-ball of dimension d and $n \leq d + 3$ vertices. Then, the d-pure subcomplex $Y^d \subset B$ is a combinatorial d-ball.*

Proof. Since B is acyclic, by Theorem 2.4 (4) Y^d is a weak d-pseudomanifold with boundary. We may assume $d \geq 2$ and $|V_{Y^d}| = d + 3$ since the cases $d = 0, 1$ and $|V_{Y^d}| = d + 1$ are trivial and, if $|V_{Y^d}| = d + 2$, Y^d is an elementary starring of a simplex by [14, Lemma 6]. Note that Y^d is necessarily connected. We first prove that Y^d is a combinatorial manifold. Let $v \in V_{Y^d}$. By the same reasoning as above we may assume $|V_{lk(v,B)}| = d+2$. If $lk(v, B)$ is an *NH*-ball then $lk(v, Y^d)$ is a combinatorial $(d-1)$-ball by inductive hypothesis since $lk(v, Y^d)$ is the $(d-1)$-pure part of $lk(v, B)$. Suppose $lk(v, B)$ is an *NH*-sphere. If $\dim_h(lk(v, B)) = d - 1$ then $lk(v, B) = lk(v, Y^d)$ is a combinatorial $(d - 1)$-sphere by Proposition 2.7. Otherwise, $lk(v, Y^d)$ is the $(d - 1)$-pure part of the *NH*-ball in any decomposition of $lk(v, B)$ and the result follows again by induction. This shows that Y^d is a combinatorial d-manifold.

Suppose $d = 2$. Note that Y^d is $\mathbb{Z}_2$-acyclic since it is connected, it has non-empty boundary and it is contained in the acyclic complex B. On the other hand, any $\mathbb{Z}_2$-acyclic complex with 5 vertices is collapsible (see for example [1, Theorem 1]).

For $d \geq 3$, Y^d is a combinatorial d-ball by Corollary 4.2. □

Note that the bound $n \leq d + 3$ in Proposition 4.3 is sharp since the *NH*-ball with vertex set $\{a, b, c, d, e, f, g\}$ and maximal simplices $\{a, b, c, d\}$, $\{d, e, f, g\}$, $\{a, d, e\}$ and $\{b, d, g\}$ is a counterexample to the case $n = d + 4$.

Proposition 4.4. *Any NH-ball B of dimension $d \geq 2$ and $d + 3$ vertices collapses to a complex of dimension $d - 2$.*

Proof. We first show that all the maximal $(d - 1)$-simplices in B can be collapsed. Let Y^{d-1} be the subcomplex of B generated by the maximal $(d-1)$-simplices and let Y^d be the d-pure part of B. By the previous proposition, Y^d is a combinatorial ball. Suppose that, for a given collapsing order, not all the $(d-1)$-simplices in Y^{d-1} can be collapsed. Let K be the subcomplex of Y^{d-1} generated by the $(d - 1)$-simplices that remain after the collapsing. By assumption, $K \neq \emptyset$. Note that K is a weak $(d - 1)$-pseudomanifold with boundary by Theorem 2.4 (4) but it has no free $(d-2)$-faces in B. Then $\partial K \subset Y^d$. Therefore, if c denotes the formal sum of the $(d-1)$-simplices of K then $c \in H_{d-1}(B, Y^d)$. Since B and Y^d are contractible, $H_{d-1}(B, Y^d) = 0$. This implies that c is not a generating cycle, which is a contradiction since the $(d - 1)$-simplices of c are maximal. This shows that we can collapse all the maximal $(d-1)$-simplices in B. On the other hand, since Y^d is a combinatorial d-ball with $d+3$ vertices or less, it is vertex decomposable by [12, 5.7]. In particular Y^d is collapsible. Then we can make the collapses in order of decreasing dimension and collapse the d-simplices and the $(d - 1)$-simplices of Y^d afterwards to obtain a $(d - 2)$-dimensional complex. □

Corollary 4.5. *Any NH-ball B of dimension $d \geq 3$ and $d + 2$ vertices collapses to a complex of dimension $d - 3$.*

Proof. We proceed by induction on d. If $d = 3$ then B is collapsible since it is acyclic and has few vertices (see [1, Theorem 1]). Let $d \geq 4$ and write $B = \Delta^d + st(u, B)$ where $u \notin \Delta^d$. Now, $\Delta^d \cap st(u, B) = lk(u, B) \subset \Delta^d$ is an *NH*-ball since B is one. Also, $\dim(lk(u, B)) \leq d - 1$ and $|V_{lk(u,B)}| \leq d + 1$. Let $m = |V_{lk(u,B)}| - \dim(lk(u, B))$. If

$m = 1$ then $lk(u, B)$ is a simplex and $B \searrow \Delta^d \searrow 0$. For $m = 2, 3, 4$ we use the inductive hypothesis, Proposition 4.4 or Corollary 2.8 respectively to show that $lk(u, B)$ collapses to a complex of dimension $\dim(lk(u, B)) - (5-m) = |V_{lk(u,B)}| - m - (5-m) = |V_{lk(u,B)}| - 5 \leq d+1-5 = d-4$. Therefore, $\overline{u} * lk(u, B) = st(u, B)$ collapses to a complex of dimension $d-3$. Finally, if $m \geq 5$ then $\dim(lk(u, B)) \leq |V_{lk(u,B)}| - 5 \leq d - 4$ and $\dim(st(u, B)) \leq d - 3$. In any case we can collapse afterwards the i-simplices of Δ^d ($i = d, d-1, d-2$) in order of decreasing dimension to obtain a $(d-3)$-dimensional complex. $\square$

It is not completely clear how sharp the bounds in Proposition 4.4 and Corollary 4.5 are. For example, from [1, Theorem 1] one can see that the conditions on the number of vertices can be relaxed for low dimensional cases. On the other hand, in [2] there is an example of a non-collapsible 3-ball with $3 + 12$ vertices. Note that such a ball cannot collapse to a 1-complex.

We are now ready to prove Theorem 1.2. The proof is split into two theorems.

Theorem 4.6. *Let B be an NH-ball and let τ be a (possibly empty) set. Then, B^τ is contractible.*

Proof. By Lemma 3.2 (4) we only need to prove the case $\tau = \emptyset$. It suffices to prove that B^* is simply connected. Let $d = \dim(B)$ and $n = |V_B|$. We can assume that $d \geq 2$ since in lower dimensions all *NH*-balls are combinatorial. We can also assume that $2 \leq n - d \leq 4$ by the argument of Dong (see Remark 3.10).

If $n \leq 7$, B^* is collapsible since it is acyclic and it has few vertices [1, Theorem 1]. For $n \geq 8$, by Proposition 4.4 and Corollaries 2.8 and 4.5 there exists a subcomplex $K \subset B$ such that $B \searrow K$ with $V_K = V_B$ and $|V_K| - \dim(K) = 5$. Therefore $B^* \nearrow K^*$, and since $|V_K| - \dim(K) = 5$, K^* is simply connected. $\square$

Theorem 4.7. *Let S be an NH-sphere and let τ be a (possibly empty) set. Then, S^τ is homotopy equivalent to a sphere.*

Similarly as in the proof for *NH*-balls, we only need to prove the case $\tau = \emptyset$ and $2 \leq |V_S| - \dim(S) \leq 4$ (see Remark 3.10). We can suppose also that $\dim_h(S) < \dim(S)$ by Proposition 2.7 and Theorem 3.9. The 1-dimensional case is easy to verify.

The proof of Theorem 4.7 will be divided in the following four cases. Let $d = \dim(S) \geq 2$, $n = |V_S|$ and $k = \dim_h(S)$. We handle each case separately.

(A) $n = d + 2$ and $k = d - 1$.
(B) $n = d + 2$ and $k = d - 2$.
(C) $n = d + 3$ and $k = d - 1$.
(D) Remaining cases.

Proof of Case (D). We will show that $S \searrow K$ with $|V_K| - \dim(K) = 5$. The result will follow immediately from Lemma 3.8 and the fact that K^* is simply connected. The case $n = d + 4$ follows directly from Corollary 2.8 by collapsing (only) the d-simplices of S.

Suppose now that $n = d+2$ or $d+3$ and let $S = B + L$ be a decomposition. We first analyze the case $n = 5$. In this situation, $|V_L| = 1$. If $d = 2$ then B is acyclic with four vertices and if $d = 3$ then $B = \Delta^3$. Similarly as in the 1-dimensional case, $S \searrow S^0$ and the result follows from Lemma 3.8.

Suppose $n = d+3$ with $n \geq 6$. The complex B in the decomposition of S is an *NH*-ball of dimension d and $|V_B| \in \{d+1, d+2, d+3\}$. The case $|V_B| = d+1$ corresponds to $B = \Delta^d$ and in the other cases B is in the conditions of Corollary 4.5 and Proposition 4.4 respectively. In any case, B collapses to a $(d-2)$-dimensional complex T. Moreover, since $d \geq 3$, we can arrange the collapses in order of decreasing dimension to get $V_T = V_B$ by collapsing only the d and $(d-1)$-dimensional simplices. Since $\dim(L) \leq d-2$ and it is top generated, the collapses in $B \searrow T$ can be carried out in S and therefore $S \searrow K = T + L$, which is a complex with the desired properties.

The case $n = d+2$ with $n \geq 6$ follows similarly as in the previous case by showing that S collapses to a $(d-3)$-dimensional complex with the same vertices. □

Proof of Case (A). We proceed by induction. Write $S = \Delta^d + \overline{u} * lk(u, S)$ with $u \notin V_{\Delta^d}$. Note that $lk(u, S)$ is an *NH*-sphere of homotopy dimension $d-2$ and dimension $d-2$ or $d-1$. By Lemma 3.6 and Lemma 3.2 (4) it suffices to show that $lk(u, S)^*$ is homotopy equivalent to a sphere. If $\dim(lk(u, S)) = d-2$ then $lk(u, S)$ is pure by Proposition 2.7 and the result follows from Theorem 3.9. If $\dim(lk(u, S)) = d-1$ then $|V_{lk(u,S)}| = d+1$ and the result follows by the inductive hypothesis. □

In order to prove the Cases (B) and (C) we need some additional results.

Lemma 4.8. *Let $S = B + L$ be a decomposition of an NH-sphere. If $v \in V_L$ then $dl_v(S)$ is contractible.*

Proof. If v is in the interior of L then $dl_v(L)$ deformation retracts to $\partial L \subset B$ and, hence, $dl_v(S) \simeq B$. Otherwise, v is in $\partial L \cap \tilde{\partial} B$ and $dl_v(S) = dl_v(B) + dl_v(L)$ with $dl_v(B) \cap dl_v(L) = \partial(dl_v(L))$. Since $v \in \partial L \cap \tilde{\partial} B$, it follows that $dl_v(B)$ and $dl_v(L)$ are contractible. On the other hand, $\partial(dl_v(L))$ is contractible by Theorem 2.1. Hence, $dl_v(S)$ is contractible. □

Lemma 4.9. *Let $S = B + L$ be a decomposition of an NH-sphere of dimension $d \geq 1$ satisfying the hypotheses of Case* (C)*. If $lk(v, S)$ is a combinatorial $(d-2)$-sphere then $dl_v(S)$ is an NH-ball.*

Proof. We proceed by induction in d. The case $d = 1$ is straightforward. Let $d \geq 2$. We prove first that $dl_v(S)$ is an *NH*-manifold.

Let w be a vertex of $dl_v(S)$. We have to show that its link is an *NH*-sphere or an *NH*-ball. If w is not in $st(v, S)$ then $lk(w, dl_v(S)) = lk(w, S)$ which is an *NH*-ball or *NH*-sphere. Suppose w is in $st(v, S)$. We will show first that $lk(w, S)$ is an *NH*-sphere of homotopy dimension $d-2$. We prove this in various steps. Note that this is clear if $w \in V_L$, so we may suppose $w \notin V_L$.

Step 1. We first prove that if $v \notin V_L$ then there is a d-simplex in $st(w,S)$ which is adjacent to a $(d-1)$-simplex of L. Let $A = \overline{V_S \setminus \{v,w\}}$. Since $v, w \notin V_L$, $L \subset A$ and therefore it is not a simplex of S because L is top generated in S. Since $\dim(S) = d$ and $st(v,S)$ is $(d-1)$-pure, w is a face of a d-simplex ρ not containing v. Since any two $(d-1)$-faces of A are adjacent, ρ is adjacent to some $(d-1)$-simplex of L.

Step 2. We now prove that the inclusion induces an isomorphism $H_{d-1}(dl_w(S)) \simeq H_{d-1}(dl_w(B))$. On one hand, the induced homomorphism $H_{d-1}(dl_w(B)) \to H_{d-1}(dl_w(S))$ is injective since $(dl_w(S)) - (dl_w(B)) = dl_w(L)$ is $(d-1)$-dimensional. In order to prove that it is also surjective we show that any $(d-1)$-cycle in $dl_w(S)$ cannot contain a $(d-1)$-simplex of L. Suppose $\sigma \in L$ is a non-trivial factor in a $(d-1)$-cycle c of $dl_w(S)$. Then every $(d-1)$-simplex in L appears in c since c is a cycle and L is a top generated combinatorial $(d-1)$-ball. If $v \in V_L$ then every $(d-1)$-simplex of $st(v,S)$ appears in c since $st(v,S)$ is also a top generated $(d-1)$-ball. In this case, at least one $(d-1)$-simplex of $st(v,S)$ belongs to $st(w,S)$, contradicting the fact that c is a cycle in $dl_w(S)$. On the other hand, if $v \notin V_L$ then by Step 1 there exists a maximal $(d-1)$-simplex $\tau \in L$ with a boundary $(d-2)$-face $\eta < \rho \in st(w,S)$ with $\dim(\rho) = d$. Let $\{z\} = lk(\eta, \overline{\tau})$. Note that there are no d-simplices outside $st(w,S)$ containing the $(d-2)$-simplex η since $|V_S| = d+3$ and, by hypothesis and construction, neither v, w nor z may belong to one such a d-simplex. Since S is an NH-manifold, τ is the only maximal $(d-1)$-simplex containing η, and then $\partial c \neq 0$ in $dl_w(S)$, which is a contradiction.

Step 3. We prove that $lk(w,S)$ is an NH-sphere of homotopy dimension $d-2$. We claim first that $H_{d-1}(dl_w(S)) = 0$. By Step 2 it suffices to show that $H_{d-1}(dl_w(B)) = 0$. From the Mayer–Vietoris sequence applied to $B = dl_w(B) + st(w,S)$ and the fact that $lk(w,B) = lk(w,S)$ (here we use that $w \notin V_L$) it follows that $H_{d-1}(dl_w(B)) \simeq H_{d-1}(lk(w,S))$. If $H_{d-1}(lk(w,S)) \neq 0$ then $lk(w,S)$ is $(d-1)$-pure by Proposition 2.7, which is a contradiction since $st(w,S)$ contains at least a $(d-1)$-simplex. Thus, the claim is proved.

If we now consider the Mayer–Vietoris sequence for $S = dl_w(S) + st(w,S)$ in degree $d-1$ one has that $\mathbb{Z} \simeq H_{d-1}(S) \to H_{d-2}(lk(w,S))$ is injective, so $H_{d-2}(lk(w,S)) \neq 0$ and therefore $lk(w,S)$ is an NH-sphere of homotopy dimension $d-2$.

Finally if $\dim(lk(w,S)) = d-2$ then $lk(w,S)$ is a combinatorial $(d-2)$-sphere by Proposition 2.7 and therefore $lk(w, dl_v(S)) = dl_v(lk(w,S))$ is a combinatorial $(d-2)$-ball by Theorem 2.1. Suppose that $\dim(lk(w,S)) = d-1$. If $|V_{lk(w,S)}| = d+1$ then we may write $lk(w,S) = \Delta^{d-1} + st(v, lk(w,S))$ since v is not a vertex of a d-simplex in S. In this case $dl_v(lk(w,S)) = \Delta^{d-1}$. If $|V_{lk(w,S)}| = d+2$ we may apply the inductive hypothesis since $lk(v, lk(w,S)) = lk(w, lk(v,S))$ is a combinatorial $(d-3)$-sphere, and conclude that $dl_v(lk(w,S))$ is an NH-ball. This proves that $dl_v(S)$ is an NH-manifold.

We prove now that $dl_v(S)$ is an NH-ball. Note that $\dim(dl_v(S)) = d$ and $|V_{dl_v(S)}| = d+2$, so by Proposition 2.6 we only need to prove that it is acyclic, and this follows immediately from the Mayer–Vietoris sequence applied to $S = dl_v(S) + st(v,S)$. $\square$

Lemma 4.10. *Let $S = B+L$ be a decomposition of an NH-sphere satisfying the hypotheses of Case* (C). *If there is a vertex v in V_L such that $\dim(dl_v(S)) = d$ and there is a non-edge $\{u, w\}$ of S with $u, w \neq v$ then $(dl_v(S))^*$ is contractible.*

Proof. By Lemma 3.3, $|V_{S^*}| = d + 3$. By hypothesis $\sigma = V_S \setminus \{u, w\}$ is a d-simplex of S^* and, since $v \neq u, w$, v is a vertex of σ. Therefore, $\dim(lk(v, S^*)) = d - 1$. On the other hand, there exists a d-simplex $\eta \in S$ with $v \notin \eta$. Hence $\{v, a\} = V_S \setminus \eta$ is not in S^*. Therefore, $|V_{lk(v,S^*)}| \leq d + 1$. If $|V_{lk(v,S^*)}| = d$ then $(dl_v(S))^* = lk(v, S^*)$ is a $(d-1)$-simplex. If $|V_{lk(v,S^*)}| = d+1$ then $lk(v, S^*) = (dl_v(S))^*$ is acyclic by Lemma 4.8 and Alexander duality, and therefore contractible by Lemma 3.1 (1). □

Lemma 4.11. *Let S be an NH-sphere satisfying the hypotheses of Case* (C). *Then, for any decomposition $S = B + L$ there exists $z \in V_L$ such that $(dl_z(S))^*$ is contractible.*

Proof. We proceed by induction in d. The 1-dimensional case is straightforward. Let $d \geq 2$ and let $u \in V_L$. If $\dim(lk(u, S)) = d - 2$ then $lk(u, S)$ is a combinatorial $(d-2)$-sphere and the result follows from Lemma 4.9 and Theorem 4.6. Suppose $\dim(lk(u, S)) = d - 1$. We analyze the two possible cases $|V_{lk(u,S)}| = d + 1$ or $|V_{lk(u,S)}| = d + 2$.

If $|V_{lk(u,S)}| = d+1$, let $w \in V_S$ such that $\{u, w\} \notin S$. Let σ be a d-simplex containing u and let $\{v\} = V_S \setminus (\sigma \cup \{w\})$. Since L is top generated, either $v \in V_L$ or $w \in V_L$. If $v \in V_L$ then Lemma 4.10 implies that $(dl_v(S))^*$ is contractible. Assume then that $v \notin V_L$ (and hence $w \in V_L$). We may assume $\dim(lk(w, S)) = d - 1$ since otherwise w is the desired vertex by Lemma 4.9 and Theorem 4.6 again. Let μ be a d-simplex containing w. Since L is top generated, $w \in V_L$ and $v \notin V_L$, $\mu = \{w, v\} \cup \eta$ with $\eta \prec \sigma \setminus \{u\}$. Let $z \neq u$ be the only vertex in η not in μ. Then, $z \in V_L$ and it fulfills the hypotheses of Lemma 4.10. Therefore $(dl_z(S))^*$ is contractible.

Suppose finally that $|V_{lk(u,S)}| = d + 2$. From the decomposition $lk(u, S) = lk(u, B) + lk(u, L)$ there exists $y \in V_{lk(u,L)}$ such that $(dl_y(lk(u, S)))^*$ is contractible by the inductive hypothesis. If u is not a vertex of $(dl_y(S))^*$ then $dl_u(dl_y(S)) = \Delta^d$. In this case, we can write $dl_y(S) = \Delta^d + \overline{u} * lk(u, dl_y(S))$ and we have $(dl_y(S))^* = (dl_y(lk(u, S)))^\tau$ by Lemma 3.6. If $dl_y(lk(u, S))$ is not a simplex then $(dl_y(lk(u, S)))^\tau \simeq \Sigma^t(dl_y(lk(u, S)))^*$, which is contractible by Lemma 3.2 (4), and if $dl_y(lk(u, S)) = \Delta^r$ then $\tau \neq \emptyset$ and $(dl_y(lk(u, S)))^\tau = \partial\overline{\tau} * \Delta^r$. In either case, y is the desired vertex. Assume $u \in V_{(dl_y(S))^*}$. Then we have a non-trivial decomposition

$$(dl_y(S))^* = dl_u((dl_y(S))^*) + st(u, (dl_y(S))^*),$$

where $dl_u((dl_y(S))^*) \cap st(u, (dl_y(S))^*) = lk(u, (dl_y(S))^*)$. Since neither $dl_y(S)$ nor $dl_y(lk(u, S))$ are simplices and u is not isolated in $dl_y(S)$, $dl_u((dl_y(S))^*) \simeq \Sigma^t lk(u, dl_y(S))^*$ by Lemma 3.7 (3). The result then follows from Lemma 3.1 (1) and Lemma 4.8. □

Proof of Cases (B) and (C). We prove (B) and (C) together by induction in d. Let $S = B + L$ be a decomposition.

If $d = 2$, B is collapsible since it is acyclic and has few vertices. Then $S \searrow S^0$ for (B) and $S \searrow S^1$ for (C). The results then follow in both cases from Lemma 3.8.

Let $d \geq 3$. Suppose first that S satisfies the hypotheses of (B). Write $S = \Delta^d + \overline{v} * lk(v, S)$. Then $S^* = lk(v, S)^\tau$ for $\tau = V_S \setminus V_{st(v,S)}$ by Lemma 3.6. Since $lk(v, S)$ is an NH-sphere of dimension $\leq d-1$, the result follows from Theorem 3.9, Cases (A) and (D) or the inductive hypothesis on (B) and (C).

Finally suppose S satisfies the hypotheses of (C). By Lemma 4.11 there exists $v \in V_L$ such that $(dl_v(S))^*$ is contractible. Write $S^* = dl_v(S^*) + st(v, S^*)$ where $dl_v(S^*) \cap st(v, S^*) = lk(v, S^*) = (dl_v(S))^*$. By Lemma 3.1 (2) and Lemma 3.7 (3), $S^* \simeq dl_v(S^*) \simeq \Sigma^t lk(v, S)^*$. Note that v is not isolated and $lk(v, S)$ is not a simplex because $v \in V_L$. Since $lk(v, S)$ is an NH-sphere of dimension $\leq d-1$, $lk(v, S)^*$ is homotopy equivalent to a sphere by Theorem 3.9, Cases (A) and (D) or inductive hypothesis on (B) and (C). $\square$

5. Alexander double duals of balls and spheres

Suppose A is a subspace of the d-sphere S^d. The complement $B = S^d - A$ is also a subspace of $S^{d'}$ for any $d' \geq d$ and taking into account that $S^d - B = A$ it is natural to ask what kind of relationship exists between A and $S^{d'} - B$. In the simplicial setting this amounts to understanding the similarities between a complex K and $(K^\tau)^\sigma$ for $\tau \cap V_K = \emptyset$ and $\sigma \cap V_{K^\tau} = \emptyset$. We call the complex $(K^\tau)^\sigma$ a *double dual* of K. When $\tau = \sigma = \emptyset$ we call $(K^*)^* = K^{**}$ the *standard double dual* of K.

Double duals share many of the properties of the original complexes. For example, from Lemma 3.2 and Lemma 3.3 it is easy to see that $(K^\tau)^\sigma \simeq \Sigma^t K$ for some $t \geq 0$ if $|V_K| \geq d + 3$. Also, it can be shown that a complex K is shellable if and only if $(K^\tau)^\sigma$ is shellable. Indeed, by Lemma 3.2 (2) the only non-trivial case is $\sigma \neq \emptyset$. Lemma 3.2 implies that $(K^\tau)^\sigma = \partial\overline{\sigma} * \Delta_{K^\tau} + \overline{\sigma} * K$. If $F_1, \ldots, F_t$ is a shelling order for K then a shelling order for $\partial\overline{\sigma} * \Delta_{K^\tau}$ followed by $\sigma \cup F_1, \ldots, \sigma \cup F_t$ is a shelling for $(K^\tau)^\sigma$. On the other hand, if $(K^\tau)^\sigma$ is shellable then $K = lk(\sigma, (K^\tau)^\sigma)$ is shellable.

We are mainly interested in double duals of combinatorial balls and spheres. We will show that they are NH-balls and NH-spheres.

Lemma 5.1. *Let K be a simplicial complex. If $V_K \subseteq V$ and $\eta \neq \emptyset$ is a simplex, then*

$$L = \partial\overline{\eta} * \overline{V} + \overline{\eta} * K$$

is an NH-ball (resp. NH-sphere) if and only if K is an NH-ball (resp. NH-sphere). Here K is viewed as a subcomplex of the simplex $\overline{V}$.

Proof. Put $\Delta = \overline{V}$. If L is an NH-ball or NH-sphere then $K = lk(\eta, L)$ is either an NH-ball or NH-sphere by Theorem 2.4 (1). Since $\partial\overline{\eta} * \Delta$ and $\overline{\eta} * K$ are collapsible and $(\partial\overline{\eta} * \Delta) \cap (\overline{\eta} * K) = \partial\overline{\eta} * K$, K is an NH-ball if L is one and an NH-sphere if L is one.

Suppose K is an NH-ball or NH-sphere. By Theorem 2.4 (3), $\partial\overline{\eta} * \Delta$ is a combinatorial ball, $\overline{\eta} * K$ is an NH-ball and $(\partial\overline{\eta} * \Delta) \cap (\overline{\eta} * K) = \partial\overline{\eta} * K$ is an NH-ball or NH-sphere according to K. We use Theorem 2.5 to prove that L is an NH-ball or NH-sphere. Note that $\partial\overline{\eta} * K$ is trivially contained in $\partial(\partial\overline{\eta} * \Delta)$ and it is generated by ridges of $\overline{\eta} * K$. Also, if $\rho \in (\partial\overline{\eta} * K)^{\circ}$ and $\hat{\eta}$ denotes the barycenter of η then

$$lk(\rho, \overline{\eta} * K) \simeq_{PL} lk(\rho, \hat{\eta} * \partial\overline{\eta} * K) = \hat{\eta} * lk(\rho, \partial\overline{\eta} * K)$$

which is an NH-ball by Theorem 2.4 (2). This implies that $\partial\overline{\eta} * K \subset \tilde{\partial}(\overline{\eta} * K)$. By Theorem 2.5, L is an NH-ball or NH-sphere. $\square$

Theorem 5.2. *Let K be a simplicial complex and let τ be a (possibly empty) set disjoint from V_K and σ a (possibly empty) set disjoint from $V_{K^{\tau}}$. Then K is an NH-ball (resp. NH-sphere) if and only if $(K^{\tau})^{\sigma}$ is an NH-ball (resp. NH-sphere).*

Proof. We first prove the case $\tau = \sigma = \emptyset$. By Lemma 3.3 we may assume $|V_K| = \dim(K) + 2$. Let $\rho = V_K \setminus V_{K^*}$. Since $\rho \neq \emptyset$, $K = (K^*)^{\rho} = \partial\overline{\rho} * \Delta_{K^*} + \overline{\rho} * K^{**}$ by Lemma 3.2 (3). The result now follows from the previous lemma.

If K is a simplex and $\dim(\tau) = 0$ the result is trivial. For the remaining cases we have

$$(K^{\tau})^{\sigma} = \begin{cases} \partial\overline{\sigma} * \Delta_{K^*} + \overline{\sigma} * K^{**} & \tau = \emptyset\,,\ \sigma \neq \emptyset \\ K & \tau \neq \emptyset\,,\ \sigma = \emptyset \\ \partial\overline{\sigma} * \Delta_{K^{\tau}} + \overline{\sigma} * K & \tau \neq \emptyset\,,\ \sigma \neq \emptyset \end{cases}$$

and the result follows from the previous lemma and the case $\tau, \sigma = \emptyset$. $\square$

Corollary 5.3. *The double duals of combinatorial balls are NH-balls. The double duals of combinatorial spheres are NH-spheres.*

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