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REGULAR OPTIMAL CONTROL PROBLEMS WITH QUADRATIC FINAL PENALTIES

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1. Introduction

Hamilton's canonical equations (HCEs) have played a central role in Mechanics after (i) their equivalence with the principle of least action, and (ii) the variational calculus leading to the Euler-Lagrange equation, were established and applied (see [1]). Also, since the foundational work of Pontryagin [22], HCEs have been at the core of modern optimal control theory. When the problem concerning an n-dimensional control system and an additive cost objective is regular [19], i.e. when the Hamiltonian $H(t, x, \lambda, u)$ of the problem is smooth enough and can be uniquely optimized with respect to u at a control value $u^0(t, x, \lambda)$ (depending on the remaining variables), then HCEs appear as a set of $2n$ ordinary differential equations whose solutions are optimal state-costate time trajectories.

Concerning the infinite-horizon bilinear-quadratic regulator and change of set-point servo problems, there exists a recent attempt to find the missing initial condition for the costate variable, based on a state-dependent (generalized) algebraic Riccati equation (GARE) with solution $P_\infty(x)$, which allows to integrate the HCEs on-line with the underlying control process [9]. The same approach in a finite time-domain leads to a first-order partial differential equation (PDE) called *'Generalized Differential Riccati Equation'* (GDRE) (see [3], [6], [11]) for a time-state dependent matrix $P(t, x)$, whose solution allows to calculate the missing initial costate $\lambda(0) = 2P(0, x_0)x_0$ and exhibits, for $S = 0$, a limiting behavior (see [19]) similar to that of linear systems with the same cost, i.e.

$$\lim_{T\to\infty} P(0, x) = P_\infty(x) , \tag{1}$$

where T is the duration of each optimization process.

In the general nonlinear finite-horizon optimization set-up, allowing for a free final state, the cost penalty $K(x)$ imposed on the final deviation generates a two-point boundary-value situation. This is often a rather difficult numerical problem to solve. However, in the linear-quadratic regulator (LQR) case there exist well-known methods (see for instance [4], [24]) to transform the boundary-value into a final-value problem, related to the differential Riccati equation (DRE). Motivated

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by the role of Riccati equations, nonlinear situations have been treated for general final penalties by Byrnes [5], who posed a quasilinear first-order vector PDE (also labelled generalized Riccati equation by the author) for the optimal costate "in feedback form", i.e. as a function of the *'event'* (t, x), but with boundary conditions on both t and x. Its usefulness is still under discussion, since a discretization of the state-space is unavoidable for numerical treatment. The same question in the one-dimensional case and for a quadratic $K(x)$ (in this paper it will always be $K(x) = x'Sx$) has been extended to a whole (T, S)-family of problems (see [7], [12]), generating two first-order, quasilinear, uncoupled PDEs with classical initial conditions, where the dependent variables are the missing boundary conditions $x(T)$ and $\lambda(0)$ of the HCEs. This approach has been completely disjoint from Riccati equations, but more in the line of the early invariant-imbedding ideas introduced by Bellman [2], [23]. Analogous ideas were retaken and reformulated for the multidimensional case, in the light of symplectic properties inherent to Hamiltonian dynamics. The resulting matrix and vector PDEs are under review [8] and will be just summarized here, together with still unpublished feedback expressions for the optimal control.

When the H-minimal control u^0 is not explicitly known, then new but similar PDEs appear, involving also the final value $u(T)$ of the optimal control. The discussion of these equations would exceed the scope of this paper (see however [13], [14]).

After the relevant mathematical objects associated with the finite-horizon control problem are presented in Section 2, then the immersion into a family of (T, S)-processes is worked out in Section 3. Afterwards, in Section 4 the main PDEs for the missing boundary conditions are substantiated. A brief discussion on the potentiality for feedback control follows in Section 5, applications are then developed in Section 6, and finally the conclusions and perspectives are summarized.

2. The Hamiltonian formalism

In what follows only initialized, autonomous (for simplicity) control systems of the form

$$\dot{x} = f(x, u), \; x(0) = x_0 \tag{2}$$

will be considered. The state x moves into some region $\mathcal{O}$ of $\mathbb{R}^n$, and the admissible control strategies are the real, piecewise continuous functions of the time-domain $\mathcal{T}$ into some open subset $\mathcal{U}$ of $\mathbb{R}^m$. The right-hand side $f : \mathcal{O} \times \mathcal{U} \to \mathbb{R}^n$ is assumed to be smooth enough as to guarantee existence and uniqueness of solutions to the dynamics' equation (2) in the range of interest. The (finite-horizon) quadratic final penalty optimization context will imply here that a cost functional like

$$\mathcal{J}(T, 0, x_0, u(\cdot)) = \int_0^T L(x(\tau), u(\tau))d\tau + x'(T)Sx(T) \tag{3}$$

has to be minimized on the set of admissible control trajectories, where $\mathcal{T} = [0, T], T < \infty$, L is a nonnegative smooth function called 'the *Lagrangian*' of the problem, and S is a nonnegative-definite symmetric matrix called 'the *final penalty* coefficient'. The '*value function*' $\mathcal{V}$ can always be defined for such a problem, namely

$$\mathcal{V}(t,x) \triangleq \inf_{u(\cdot)} \mathcal{J}(T,t,x,u(\cdot)), t \in [0,T] \tag{4}$$

and, if the problem has a unique solution, then this solution is called 'the *optimal control* strategy' u^*,

$$u^*(\cdot) \triangleq \arg\inf_{u(\cdot)} \mathcal{J}(T,t,x,u(\cdot)) \tag{5}$$

which in turn will generate 'the *optimal state* trajectory'

$$x^*(\cdot) \triangleq \text{solution to (2) with } u(\cdot) = u^*(\cdot) \; . \tag{6}$$

The *Hamiltonian* of such a problem is defined as usual,

$$H(x,\lambda,u) \triangleq L(x,u) + \lambda' f(x,u) \; , \tag{7}$$

where λ is called the 'costate', $\lambda \in \mathbb{R}^n$, (x,λ) ranging in $2n$-dimensional 'phase-space'. Since H is assumed regular, then there exists a unique H-*optimal control*

$$u^0(x,\lambda) \triangleq \arg\min_u H(x,\lambda,u) \; . \tag{8}$$

'Explicitly regular' Hamiltonian means that the function $u^0(x,\lambda)$ is known (not only its existence but also its explicit form) and that it is sufficiently smooth on its variables. The *control Hamiltonian*,

$$\mathcal{H}^0(x,\lambda) \triangleq H(x,\lambda,u^0(x,\lambda)) \; , \tag{9}$$

gives rise to the HCEs (see [22] for general problems; [24], page 406 for the free final state case)

$$\dot{x} = \left(\frac{\partial \mathcal{H}^0}{\partial \lambda}\right)' \triangleq \mathcal{F}(x,\lambda) \; ; \; x(0) = x_0 \; , \tag{10}$$

$$\dot{\lambda} = -\left(\frac{\partial \mathcal{H}^0}{\partial x}\right)' \triangleq -\mathcal{G}(x,\lambda) \; ; \; \lambda(T) = 2Sx(T) \; , \tag{11}$$

that is a $2n$-dimensional ODE for a (Hamiltonian) vector field $\mathcal{X}$,

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \mathcal{F}(x,\lambda) \\ -\mathcal{G}(x,\lambda) \end{pmatrix} \triangleq \mathcal{X}(x,\lambda) \; . \tag{12}$$

Solutions to Eqns. (10, 11) result, under the hypotheses made, the optimal state and costate trajectories (denoted $x^*(t)$ and $\lambda^*(t)$ respectively), which are also related through the value-function by

$$\lambda^*(t) = \left(\frac{\partial \mathcal{V}}{\partial x}(t, x^*(t))\right)' \; . \tag{13}$$

It is useful to remind also that the control Hamiltonian is constant along the optimal trajectories, since

$$\frac{d}{dt}\mathcal{H}^0(x^*(t),\lambda^*(t)) = \left(\frac{\partial \mathcal{H}^0}{\partial x}\right)' \cdot \mathcal{F} + \left(\frac{\partial \mathcal{H}^0}{\partial \lambda}\right)' \cdot [-\mathcal{G}] = 0 \ . \tag{14}$$

3. Imbedding the problem into a (T,S)-family

The following notation for the missing boundary conditions will be used in this Section

$$\begin{aligned} \rho(T,S) &\triangleq x^*(T) & (15)\\ \sigma(T,S) &\triangleq \lambda^*(0) & (16) \end{aligned}$$

(the notation $x^{T,S}(T) = \rho(T,S)$ and $\lambda^{T,S}(0) = \sigma(T,S)$ may eventually emphasize that the quantities refer to a particular (T,S)-problem)

By assuming that the Hamiltonian vector field is at least C^1, then the existence of a flow

$$\phi : \mathbb{R} \times \mathcal{O}_x \times \mathcal{O}_\lambda \to \mathbb{R}^n \times \mathbb{R}^n \ , \tag{17}$$

is guaranteed, where $\mathcal{O}_x, \mathcal{O}_\lambda$ are appropriate regions of $\mathbb{R}^n$. The flow verifies

$$\begin{aligned} D_1\phi(t,x,\lambda) &= \begin{pmatrix} \mathcal{F}(\phi(t,x,\lambda)) \\ -\mathcal{G}(\phi(t,x,\lambda)) \end{pmatrix} = \mathcal{X}(\phi(t,x,\lambda)) \ , & (18)\\ \phi(0,x,\lambda) &= \begin{pmatrix} x \\ \lambda \end{pmatrix} , \forall \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathcal{O}_x \times \mathcal{O}_\lambda \ , & (19) \end{aligned}$$

where D_1 is preferred to the usual $\frac{\partial}{\partial t}$, to avoid confusions.

By calling ϕ^t to the t-advance transformations associated with the flow, the following identities become clear

$$\begin{pmatrix} \rho \\ 2S\rho \end{pmatrix} = \phi^T(x_0,\sigma) = \phi(T,x_0,\sigma) = \begin{pmatrix} \phi_1(T,x_0,\sigma) \\ \phi_2(T,x_0,\sigma) \end{pmatrix} , \tag{20}$$

where ϕ_1, ϕ_2 denote the 'components' of the flow over the state and costate subspaces, respectively. The first component of Eq. (18) reads

$$\begin{gathered} D_1\phi_1(T,x_0,\sigma(T,S)) = \mathcal{F}(\phi(T,x_0,\sigma(T,S))) = \\ = \mathcal{F}(\phi_1(T,x_0,\sigma(T,S)),\phi_2(T,x_0,\sigma(T,S))) = \mathcal{F}(\rho(T,S),2S\rho(T,S)) \triangleq F(\rho,S) \ , \end{gathered} \tag{21}$$

and similarly for the second component, in brief

$$\begin{aligned} G(\rho,S) &\triangleq \mathcal{G}(\rho,2S\rho) \ , & (22)\\ D_1\phi_2 &= -G \ . & (23) \end{aligned}$$

The (phase-space) derivative of the T-advance function will be needed in the sequel, so a special name is given to it and to its partitions

$$V \triangleq D\phi^T(x_0,\sigma) = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} = \begin{pmatrix} \phi_{1_x} & \phi_{1_\lambda} \\ \phi_{2_x} & \phi_{2_\lambda} \end{pmatrix} , \tag{24}$$

where $\phi_{1_x} = \frac{\partial \phi_1}{\partial x}$, and so on. Existence and uniqueness of solutions imply that the inverse of V exists and verifies

$$U \triangleq V^{-1} = D\phi^{-T}(\rho, 2S\rho) \ . \tag{25}$$

4. The main PDEs for missing boundary conditions

Hamiltonian vector fields have flows with the following important properties (see [18], page 378; [20], page 220)

$$\begin{aligned} U_1'U_4 - U_3'U_2 &= I = U_4'U_1 - U_2'U_3 \ , & (26)\\ U_1'U_3 - U_3'U_1 &= 0 = U_2'U_4 - U_4'U_2 \ , & (27) \end{aligned}$$

where the following notation is adopted for submatrices: $U_i' \triangleq (U_i)'$.

Since the same is true for V, its inverse can be calculated in terms of the submatrices V_i, namely

$$V^{-1} = U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \begin{pmatrix} V_4' & -V_2' \\ -V_3' & V_1' \end{pmatrix} \ . \tag{28}$$

Now by deriving the first component of Eq. (20) with respect to T,

$$D_1\rho(T,S) = D_1\phi_1(T, x_0, \sigma(T,S)) + D_3\phi_1(T, x_0, \sigma(T,S))D_1\sigma(T,S)) \ , \tag{29}$$

which will be written (with $\phi_{1_\lambda} \equiv \frac{\partial \phi_1}{\partial \lambda} \equiv D_3\phi_1$) simply as

$$\rho_T = F + \phi_{1_\lambda}\sigma_T \ ; \tag{30}$$

and similarly, for the second component,

$$2S\rho_T = -G + \phi_{2_\lambda}\sigma_T \ . \tag{31}$$

The following operations over Eqns. (30, 31) use the symplectic properties of the vector field

$$V_4'(\rho_T - F) = V_4'V_2\sigma_T = V_2'V_4\sigma_T = V_2'(2S\rho_T + G) \ . \tag{32}$$

By repeating the procedure for the S-derivatives, analogous equations are obtained, namely

$$\begin{aligned} \rho_S &= \phi_{1_\lambda}\sigma_S = V_2\sigma_S \ , & (33)\\ 2(\rho + S\rho_S) &= \phi_{2_\lambda}\sigma_S = V_4\sigma_S \ , & (34) \end{aligned}$$

and from their combination,

$$V_4'\rho_S = V_4'V_2\sigma_S = V_2'V_4\sigma_S = V_2'(2\rho + 2S\rho_S) \ . \tag{35}$$

Now, by using $V' = \begin{pmatrix} V_1' & V_3' \\ V_2' & V_4' \end{pmatrix}$ and properties in Eqns. (26, 27), together with results in Eqns. (32, 35), then a condensed form for previous identities is obtained

$$V' \begin{pmatrix} 2S\rho_T + G & 2\rho + 2S\rho_S \\ F - \rho_T & -\rho_S \end{pmatrix} = \begin{pmatrix} \sigma_T & \sigma_S \\ 0 & 0 \end{pmatrix} , \tag{36}$$

which means that there really are only two independent first-order vector PDEs for ρ, σ, namely

$$\begin{aligned} V_1'(2S\rho_T + G) + V_3'(F - \rho_T) &= \sigma_T \ , & (37) \\ V_1'(2\rho + 2S\rho_S) - V_3'\rho_S &= \sigma_S \ . & (38) \end{aligned}$$

Notice that only V_1 and V_3 are involved, although a pair of equivalent PDEs can be obtained involving V_2 and V_4. In the one-dimensional case these equations can be uncoupled, obtaining (see [7])

$$\begin{aligned} \rho\rho_T - (SF + \frac{G}{2})\rho_S &= \rho F \ , & (39) \\ \rho\sigma_T - (SF + \frac{G}{2})\sigma_S &= 0 \ , & (40) \end{aligned}$$

but for $n > 1$ a more involved treatment is needed, as will be shown below.

For an n-dimensional state space it will be convenient to assign a name to the combined variable $v \triangleq \begin{pmatrix} x \\ \lambda \end{pmatrix}$. Eq. (18) can be written

$$D_1\phi(t, v) = \mathcal{X}(\phi(t, v)) \ , \tag{41}$$

and by taking derivatives on both members with respect to v, i.e. $D_2 = \frac{\partial}{\partial v}$, and then interchanging the order of derivation, the '*variational equation*' (see for instance [16], page 299) is obtained, namely

$$D_1[D_2\phi(t, v)] = D\mathcal{X}(\phi(t, v)) \cdot D_2\phi(t, v) \ , \tag{42}$$

or, by abusing notation ($V(t) \triangleq D\phi^t$, the symbol V is reserved for $V(T)$; $\mathcal{A}(t) \triangleq D\mathcal{X} \circ \phi^t$)

$$\dot{V}(t) = \mathcal{A}(t)V(t) \ , \tag{43}$$

with the initial condition

$$V(0) = I \ . \tag{44}$$

Actually, this means that $V = \Phi(T, 0)$, the fundamental solution of (43), which verifies (see [24], page 488)

$$\frac{\partial \Phi(T, 0)}{\partial T} = \mathcal{A}(T)\Phi(T, 0) \ , \tag{45}$$

i.e. the following identity is established

$$V_T(x_0, \sigma) = D\mathcal{X}(\rho, 2S\rho) \cdot V(x_0, \sigma) \ , \tag{46}$$

or, in short, reserving the symbol $\mathcal{A}$ for $\mathcal{A}(T)$,

$$V_T = \mathcal{A}V \ . \tag{47}$$

Now, inspired in the treatment of the LQR in Hamiltonian form (see [4], [24]), two auxiliary matrices will be defined

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \triangleq U \begin{pmatrix} I \\ 2S \end{pmatrix} = \begin{pmatrix} V_4' - 2V_2'S \\ -V_3' + 2V_1'S \end{pmatrix} , \tag{48}$$

and deriving them with respect to T, S and using Eqns. (28, 47, 48),

$$\begin{pmatrix} \alpha_T \\ \beta_T \end{pmatrix} = \begin{pmatrix} V_2'\mathcal{A}_3' + V_4'\mathcal{A}_4' - 2\left(V_2'\mathcal{A}_1' + V_4'\mathcal{A}_2'\right)S \\ -\left(V_1'\mathcal{A}_3' + V_3'\mathcal{A}_4'\right) + 2\left(V_1'\mathcal{A}_1' + V_3'\mathcal{A}_2'\right)S \end{pmatrix} , \tag{49}$$

$$\begin{pmatrix} \alpha_S \\ \beta_S \end{pmatrix} = \begin{pmatrix} -2V_2' \\ 2V_1' \end{pmatrix} , \tag{50}$$

then the following identities are obtained

$$V_1' = \frac{1}{2}\beta_S \ , \ V_2' = -\frac{1}{2}\alpha_S \ , \ V_3' = \beta_S S - \beta \ , \ V_4' = \alpha - \alpha_S S \ , \tag{51}$$

$$\boxed{\begin{pmatrix} \alpha_T \\ \beta_T \end{pmatrix} = \begin{pmatrix} \alpha_S \mathcal{M} - \alpha\mathcal{N} \\ \beta_S \mathcal{M} - \beta\mathcal{N} \end{pmatrix}} , \tag{52}$$

where the new matrices $\mathcal{M}, \mathcal{N}$ take the form

$$\mathcal{M} \triangleq \frac{1}{2}(2\mathcal{A}_1'S - \mathcal{A}_3') + S(2\mathcal{A}_2'S - \mathcal{A}_4') \ , \tag{53}$$

$$\mathcal{N} \triangleq 2\mathcal{A}_2'S - \mathcal{A}_4' \Rightarrow \mathcal{M} = \frac{1}{2}(2\mathcal{A}_1'S - \mathcal{A}_3') + S\mathcal{N} \ . \tag{54}$$

Since for a process of zero duration, $\phi^T = \phi^0 = id$, then in such case $V = D\phi^T = I = U$, and therefore the main matrix PDEs in Eq. (52) are subject to the initial conditions

$$\alpha(0, S) = I \ , \ \beta(0, S) = 2S \ . \tag{55}$$

Now, Eqns. (53, 54) still include the unknown final state ρ inside the $\mathcal{A}_i'$s so the (matrix) PDEs in Eq. (52) can not be solved alone. But, having found expressions for the partitions of V in terms of the auxiliary matrices α, β and their derivatives, (vector) Eqns. (37, 38) turn into

$$\begin{pmatrix} \sigma_T \\ \sigma_S \end{pmatrix} = \begin{pmatrix} \beta_S(SF + \frac{G}{2}) - \beta(F - \rho_T) \\ \beta_S\rho + \beta\rho_S \end{pmatrix} , \tag{56}$$

which become solvable, at least in principle, when coupled to the matrix PDEs for α, β, and subject to initial conditions

$$\rho(0, S) = x_0 \ , \tag{57}$$

$$\sigma(0, S) = 2Sx_0 \ . \tag{58}$$

In short, the problem requires to solve in parallel two matrix first-order PDEs for (α, β) , and another two vector first-order PDEs for (ρ, σ), all meeting appropriate initial conditions. If instead of V_1, V_3 the remaining submatrices V_2, V_4 were chosen, then Eqns. (32, 35) take also a condensed form, namely

$$\left(\begin{array}{c} 0 \\ 0 \end{array}\right)=\left(\begin{array}{c} \alpha_S(SF+\frac{G}{2})-\alpha(F-\rho_T) \\ \alpha_S\rho+\alpha\rho_S \end{array}\right) , \tag{59}$$

but σ can not be directly recuperated from them.

Concerning the existence and uniqueness of solutions to the coupled system of Eqns. (52, 56), there exist only local results (see [15], page 51), although the field of vector and matrix PDEs integration is in active development (see for instance [25]).

5. Feedback

Let us denote as $\sigma(T,S,x_0)$ the optimal initial costate corresponding to a (T,S)-problem with initial state x_0. Smooth dependence on initial conditions for ODEs [16], extensive to first-order quasilinear PDEs [15], guarantees smooth dependence of σ on x_0. Neither the matrix nor the vector PDEs developed in the previous section depend explicitly on x_0. The initial state x_0 only affects solutions through the conditions (57, 58). As a consequence, numerical software can sometimes handle this dependence "analytically", i.e. considering x_0 as a dummy variable when solving for ρ,σ. So it will be assumed that $\sigma(T,S,x_0)$ is available for some appropriate open set $\mathcal{O}(\tilde{x}_0)\subset\mathbb{R}^n$ containing the "expected perturbations" from the optimal state trajectory $\{x^*(t),t\in[0,T]\}$ starting at the original fixed initial condition $\tilde{x}_0=x^*(0)$. Under these assumptions it is clear that the optimal costate trajectory must also verify (analogously to the Dynamic Programming Principle for the value function)

$$\lambda^*(t)=\sigma(T-t,S,x^*(t))\ \forall t\in[0,T] . \tag{60}$$

Therefore, if at some intermediate time t the measured (or observed) state is $x(t)$, possibly different from the expected but still inside $\mathcal{O}(\tilde{x}_0)$, a new optimal control problem starting at $x(t)$ as initial condition (and duration $T-t$) may be considered to cope with state perturbations, and it follows that the new optimal control can be expressed in feedback form as

$$\boxed{u^*(t)=u^0(x(t),\sigma(T-t,S,x(t)))}. \tag{61}$$

6. Applications

6.1. **The (constant coefficient) LQR problem revisited.** The Hamiltonian form of the LQR problem (with linear dynamics $f=Ax+Bu$ and quadratic Lagrangian $L=x'Qx+u'Ru$) reads

$$\dot{v}=\left(\begin{array}{c} \dot{x} \\ \dot{\lambda} \end{array}\right)=\left(\begin{array}{cc} A & -\frac{1}{2}W \\ -2Q & -A' \end{array}\right)\left(\begin{array}{c} x \\ \lambda \end{array}\right)=\mathbf{H}v , \tag{62}$$

where $W \triangleq BR^{-1}B'$. Therefore in this case the HCEs become a linear, time-constant dynamical system or vector field $\mathcal{X}(v) = \mathbf{H}v$, whose flow verifies

$$\phi^T(v) = e^{\mathbf{H}T}v \; , \tag{63}$$

and consequently

$$\begin{aligned} V &= D\phi^T = \phi^T = e^{\mathbf{H}T} \; , & (64)\\ U &= V^{-1} = e^{-\mathbf{H}T} \; . & (65)\end{aligned}$$

The following identities are easily obtained

$$\begin{aligned} V_T &= \mathbf{H}e^{\mathbf{H}T}, \; D\mathcal{X} = \mathbf{H} = \mathcal{A} \text{ (time-constant)} & (66)\\ \mathcal{A}' &= \begin{pmatrix} A' & -2Q \\ -\frac{1}{2}W & -A \end{pmatrix} = \begin{pmatrix} \mathcal{A}_1' & \mathcal{A}_3' \\ \mathcal{A}_2' & \mathcal{A}_4' \end{pmatrix} & (67)\\ \mathcal{N} &= A - WS, \; \mathcal{M} = A'S + SA + Q - SWS \; . & (68)\end{aligned}$$

Therefore, Eqns. (52) for α, β can be integrated alone, since they do not depend on ρ, σ. Actually, from

$$\begin{pmatrix} x_0 \\ \sigma \end{pmatrix} = e^{-\mathbf{H}T}\begin{pmatrix} \rho \\ 2S\rho \end{pmatrix} = U\begin{pmatrix} I \\ 2S \end{pmatrix}\rho = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\rho \; , \tag{69}$$

it follows that no further equations are needed for ρ, σ. Since α is always invertible (see [24], p.371), then the missing boundary conditions result

$$\begin{aligned} \rho &= \alpha^{-1}x_0, & (70)\\ \sigma &= \beta\rho \; . & (71)\end{aligned}$$

Illustrations can be found in [10]. From Eq. (13) for the LQR case, the initial costate has here the form

$$\sigma = \lambda^*(0) = \left(\frac{\partial \mathcal{V}}{\partial x}(0, x^*(0))\right)' = 2P(0)x_0 \; ; \tag{72}$$

where P is in turn the numerical solution of the DRE, i.e. the final-value matrix ODE

$$\dot{\pi} = \pi W \pi - \pi A - A'\pi - Q \; ; \; \pi(T) = S \; . \tag{73}$$

Therefore, from Eq. (71), for each (T, S)-problem the Riccati matrix $P(t)$ should also verify

$$P(0) = \frac{1}{2}\beta(T, S)\left[\alpha(T, S)\right]^{-1} \; . \tag{74}$$

The method based on PDEs for missing boundary conditions avoid solving DRE for each particular (T, S)-problem, and storing, necessarily as an approximation, the Riccati matrix $P(t)$ for the values of $t \in [0, T]$ chosen by the numerical integrator, possibly different from the time instants for which the control $u(t)$ is constructed. Instead, the HCEs (62) can be integrated with initial conditions

$$x(0) = x_0, \; \lambda(0) = \sigma(T, S) \tag{75}$$

and the optimal trajectories $x^*(t), \lambda^*(t)$ obtained for $0 \leq t \leq T$, which allows to generate the optimal control at each time

$$u^*(t) = u^0\left(x^*(t), \lambda^*(t)\right) = -\frac{1}{2}R^{-1}B'\lambda^*(t) , \tag{76}$$

or, in this case, the feedback form, which becomes directly available due to the linear dependence of Eqns. (70, 71) on initial conditions,

$$u^*(t) = -\frac{1}{2}R^{-1}B'\beta(T-t,S)\left[\alpha(T-t,S)\right]^{-1} x . \tag{77}$$

As a side-product, an alternative formula for the Riccati matrix results:

$$P(t) = \frac{1}{2}\beta(T-t,S)\left[\alpha(T-t,S)\right]^{-1} \ \forall t \in [0,T]. \tag{78}$$

6.2. **Bilinear systems and quadratic costs.** The bilinear-quadratic case (with $x \in \mathbb{R}^n, u \in \mathbb{R}$) will be used to illustrate the application of previous results to nonlinear systems. The dynamics and trajectory cost will be, respectively,

$$f(x,u) = Ax + (b+Nx)u \ , \ L(x,u) = x'Qx + ru^2 \ . \tag{79}$$

The H-optimal control is readily obtained (see [24])

$$u^0(x,\lambda) = -\frac{1}{2r}\lambda'(b+Nx) \ , \tag{80}$$

and then the control Hamiltonian takes the form

$$\mathcal{H}^0(x,\lambda) = x'Qx + \lambda'Ax - \frac{1}{4r}[\lambda'(b+Nx)]^2 \ . \tag{81}$$

The HCEs are therefore

$$\dot{x} = Ax - \frac{1}{2r}[\lambda'(b+Nx)](b+Nx) = Ax - \frac{1}{2}\bar{W}(x)\lambda \ , \tag{82}$$

$$\dot{\lambda} = -2Qx - A'\lambda + \frac{\lambda'(b+Nx)}{2r}N'\lambda = -2Qx - \left[\bar{A}(x,\lambda)\right]'\lambda \ , \tag{83}$$

where the x-dependent matrix $\bar{W}(x)$ is clearly a generalization of the $W = BR^{-1}B'$ defined for linear systems, and analogously for $\bar{A}(x,\lambda)$,

$$\bar{W}(x) \triangleq \frac{1}{r}(b+Nx)(b+Nx)' \text{ (symmetric)}, \tag{84}$$

$$\bar{A}(x,\lambda) \triangleq A - \frac{\lambda'(b+Nx)}{2r}N = A + u^0 N \ , \tag{85}$$

allowing to write the vector field $\mathcal{X}$ and its derivative $D\mathcal{X}$ in concise expressions, namely

$$\mathcal{X}(x,\lambda) = \begin{pmatrix} A & -\frac{1}{2}\bar{W}(x) \\ -2Q & -\left[\bar{A}(x,\lambda)\right]' \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix} , \tag{86}$$

$$D\mathcal{X}(x,\lambda) = \begin{pmatrix} \hat{A}(x,\lambda) & -\frac{1}{2}\bar{W}(x) \\ -2\tilde{Q}(\lambda) & -\left[\hat{A}((x,\lambda)\right]' \end{pmatrix} , \tag{87}$$

where new generalizations of LQR matrices appear,

$$\hat{A}(x,\lambda) \triangleq \bar{A}(x,\lambda) - \frac{1}{2r}(b+Nx)\lambda' N \,, \tag{88}$$

$$\tilde{Q}(\lambda) \triangleq Q + \frac{1}{4r}N'\lambda\lambda' N \quad \text{(symmetric)} \,. \tag{89}$$

The matrix $\mathcal{A}$ can be evaluated by looking to the final conditions, i.e.

$$\mathcal{A} = \mathcal{A}(T) = D\mathcal{X}(\rho, 2S\rho) = \begin{pmatrix} \hat{A} & -\frac{1}{2}\bar{W} \\ 2\tilde{Q} & -\hat{A}' \end{pmatrix} \,, \text{ where} \tag{90}$$

$$\bar{W} = \frac{1}{r}(b+N\rho)(b+N\rho)' \,, \tag{91}$$

$$\hat{A} = A - \frac{1}{r}[\rho' S(b+N\rho)N + (b+N\rho)\rho' SN] \,, \tag{92}$$

$$\tilde{Q} = Q + \frac{1}{r}N' S\rho\rho' SN \,. \tag{93}$$

In conclusion, the relevant objects read in this case

$$\mathcal{N} = 2\mathcal{A}_2' S - \mathcal{A}_4' = \hat{A} - \bar{W}S \,, \tag{94}$$

$$\mathcal{M} = \mathcal{A}_1' S - \frac{1}{2}\mathcal{A}_3' + S\mathcal{N} = \hat{A}' S + S\hat{A} + \tilde{Q} - S\bar{W}S \,, \tag{95}$$

$$F = \left[A - \frac{1}{r}(b+N\rho)(b+N\rho)' S\right]\rho \,, \tag{96}$$

$$G = 2\left[Q + A' S - \frac{\rho' S(b+N\rho)}{r}N' S\right]\rho \,. \tag{97}$$

The following checking procedure can be performed over numerical solutions. It is known (see [6]) that the value function verifies, for the finite-horizon bilinear-quadratic problem,

$$\frac{\partial \mathcal{V}_{T,S}}{\partial x}(t,x) = 2\left[P_{T,S}(t,x)\right]x \,, \tag{98}$$

for some matrix $P_{T,S}(t,x)$ solution of a generalized Riccati differential equation (GDRE), actually a first-order PDE in the variables (t,x) that in the one-dimensional case takes the form

$$[p_t + F(x,p)\cdot p_x]x + \left[pF(x,p) + \frac{1}{2}G(x,p)\right] = 0 \,, \tag{99}$$

with F, G as defined in Eqns. (21, 22), respectively.

It is also known (see [9], and [19] for the linear analogue) that for $S=0$ the solutions to GDRE are compatible with solutions $P_\infty(x)$ to the generalized algebraic Riccati equation (GARE) arising in the infinite-horizon case, namely

$$A'P + PA + Q - PW(x)P = 0 \,, \tag{100}$$

via the limiting behavior

$$\lim_{T\to\infty} P_{T,0}(0,x) = 2P_\infty(x) \; . \tag{101}$$

Numerical calculations performed for increasing time-spans (approximately) confirm the asymptotic result

$$\begin{aligned} \sigma_\infty &= \lim_{T\to\infty} \sigma(T,0) = \lim_{T\to\infty} \frac{\partial \mathcal{V}_{T,0}}{\partial x}(0,x_0) = \\ &= \lim_{T\to\infty} 2\left[P_{T,0}(0,x_0)\right] x_0 = 2\left[P_\infty(x_0)\right] x_0 \; . \end{aligned} \tag{102}$$

7. Conclusions and perspectives

The solutions to the PDEs established in the previous Sections allow to transform the classical boundary-value problem posed for Hamilton equations in $2n$ dimensions, into an initial-value situation when the Hamiltonian is regular. This allows in turn to numerically integrate the original HCEs on-line with the control process, and to continuously construct the manipulated variable $u^*(t) = u^0(x^*(t), \lambda^*(t))$ from the state and costate values provided by this integration, since the H-optimal control function $u^0(\cdot,\cdot)$ is known. The on-line accessibility to an accurate value for the optimal state is most valuable in practical situations, since physical states of nonlinear control systems are hardly available at all desired times. Sometimes even a feedback control form can be constructed from the solutions to the quasilinear PDEs.

The PDEs' method solves a whole family of (T,S) problems, avoiding additional off-line calculations and burdensome storing of information for each particular situation, as in methods of the DRE or GDRE type. The numerical integration of the new PDEs is relatively simple when only scalar values for S are admitted, which is enough in many practical situations. Also, solutions providing the missing boundary conditions can be checked in several ways, and eventually iterated until convergence before using them to start controlling in real time.

Having the values of ρ, σ for a wide range of T, S parameter values may be helpful at the design stage. From one side, the values of T, S can be reconsidered by the designer when acknowledging the final values of the state $\rho(T,S)$ that will be obtained under present conditions. And if a change in the parameter values is decided, then it will not be necessary to perform additional calculations to manage the new situation. Besides, the value of $\sigma(T,S)$ is an accurate measure of the 'marginal cost' of the process, i.e. it measures how much the optimal cost would change under perturbations, which can also influence the decision on adopting the final T, S values.

Other than the possibility of integrating HCEs on-line with the real plant (and constructing the optimal control for the model in real time), or even the possibility of generating a feedback law as we shall see, the approach presented in this paper may also be useful when studying input-output $\mathcal{L}$-stability of control systems, since the trajectory cost

$$\int_0^T \left[\|y(t)\|^2 - \gamma^2 \|u(t)\|^2 \right] dt \tag{103}$$

may be analyzed in this set-up for variable gain γ, even for nonlinear observation functions

$$y(t) = h(x(t), u(t)) \tag{104}$$

and nonlinear dynamics (see [17], [21]), provided the resulting Hamiltonian is regular. Therefore, these PDEs seem to provide a novel environment where to explore the balance ‘performance *versus* stability’.

Other aspects of this approach deserve research. For instance, the curves $\sigma(., S)$ are potentially a safeguard against Hamiltonian systems’ instabilities (their linearizations have eigenvalues with positive real parts, because those associated with λ are symmetric to those corresponding to x). Therefore, it will probably add to robustness to construct the control by imposing a bound on costates $\lambda(.)$, for instance impeding the costates to trespass the reverse $\sigma(., S)$ curve starting from $\sigma(\bar{T}, S)$ when a finite horizon of duration $\bar{T}$ is being optimized.

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