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Some remarks on graded nilpotent Lie algebras and the Toral Rank Conjecture

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If $\mathfrak{n}$ is a $\mathbb{Z}_+^d$-graded nilpotent finite-dimensional Lie algebra over a field of characteristic zero, a well-known result of Deninger and Singhof states that $\dim H^*(\mathfrak{n}) \geq L(p)$ where p is the polynomial associated to the grading and $L(p)$ is the sum of the absolute values of the coefficients of p. From this result they derived the Toral Rank Conjecture (TRC) for 2-step nilpotent Lie algebras. An algebraic version of the TRC states that $\dim H^*(\mathfrak{n}) \geq 2^{\dim(\mathfrak{z})}$ for any finite-dimensional nilpotent Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$. The TRC is more than 25 years old and remains open even for $\mathbb{Z}_+^d$-graded 3-step nilpotent Lie algebras. Investigating to what extent the bound given by Deninger and Singhof could help to prove the TRC in this case, we considered the following two questions regarding a nilpotent Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$: (A) If $\mathfrak{n}$ admits a $\mathbb{Z}_+^d$-grading $\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{Z}_+^d} \mathfrak{n}_\alpha$, such that its associated polynomial p satisfies $L(p) > 2^{\dim \mathfrak{z}}$, does $\mathfrak{n}$ admit a $\mathbb{Z}_+$-grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_k'$ such that its associated polynomial p' satisfies $L(p') > 2^{\dim \mathfrak{z}}$? (B) If $\mathfrak{n}$ is r-step nilpotent admitting a grading $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$ such that its associated polynomial p satisfies $L(p) > 2^{\dim \mathfrak{z}}$, does $\mathfrak{n}$ admit a grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_r'$ such

that its associated polynomial p' satisfies $L(p') > 2^{\dim \mathfrak{z}}$? In this paper we show that the answer to (A) is yes, but the answer to (B) is no.

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1. Introduction

All Lie algebras considered in this paper are finite dimensional over a field of characteristic zero. Let $\mathbb{Z}_+^d = \{(\alpha_1, \ldots, \alpha_d) \neq 0 : \alpha_i \in \mathbb{Z}_{\geq 0}\}$. Given a $\mathbb{Z}_+^d$-graded nilpotent Lie algebra

$$\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{Z}_+^d} \mathfrak{n}_\alpha,$$

(that is $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subset \mathfrak{n}_{\alpha+\beta}$), Deninger and Singhof [1] considered the *polynomial associated* to the grading

$$p(x_1, \ldots, x_d) = \prod_{\alpha \in \mathbb{Z}_+^d} (1 - x_1^{\alpha_1} \cdots x_d^{\alpha_d})^{d_\alpha}, \quad d_\alpha = \dim \mathfrak{n}_\alpha,$$

$$= \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha x_1^{\alpha_1} \cdots x_d^{\alpha_d} \in \mathbb{Z}[x_1, \ldots, x_d];$$

and proved that

$$\dim H^*(\mathfrak{n}) \geq L(p) \tag{1.1}$$

where $L(p) = \sum_{\alpha \in \mathbb{Z}_+^d} |a_\alpha|$ is the *length* of p (see also [3, Theorem 1]).

If $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a 2-step nilpotent Lie algebra, with center $\mathfrak{z}$, then $\mathfrak{n}_1 = \mathfrak{v}$ and $\mathfrak{n}_2 = \mathfrak{z}$ defines a $\mathbb{Z}_+$-grading on $\mathfrak{n}$. Deninger and Singhof considered in [1] this particular instance of their result to obtain the Toral Rank Conjecture (TRC) for 2-step nilpotent Lie algebras.

Recall that the TRC was formulated by Halperin [2] more than 25 years ago and an algebraic version of it is the following.

TRC. *If $\mathfrak{n}$ is a nilpotent Lie algebra with center $\mathfrak{z}$, then*

$$\dim H^*(\mathfrak{n}) \geq 2^{\dim \mathfrak{z}}.$$

This conjecture has a topological origin: the toral rank $r(X)$ of a differentiable manifold X is the dimension of the greatest torus acting freely on X. Originally, the TRC states that the homology of the manifold X has dimension greater than or equal to $2^{r(X)}$. It follows from a theorem of Nomizu [4] that, for compact nilmanifolds, the original TRC would follow from the algebraic version stated above.

This conjecture remains open in general (see [5] or [6] for more information). Although a very particular instance of (1.1) implies the TRC for 2-step nilpotent Lie algebras, the TRC remains open even for $\mathbb{Z}_+^d$-graded 3-step nilpotent Lie algebras

and it is natural to ask to what extent the result of Deninger and Singhof could help to prove the TRC in this case. It might be worth mentioning that sometimes (1.1) is pretty accurate, for instance

$$\dim H^*(\mathfrak{n}) = L(p)$$

for Heisenberg Lie algebras (of arbitrary dimensions) and for every nilpotent Lie algebra of dimension ≤ 6 (when they are given an appropriate grading [3, §§1.4.4 and 1.6.1]).

Given a $\mathbb{Z}_+^d$-graded nilpotent Lie algebra, it is in general very difficult to consider, in the context of (1.1), all its possible $\mathbb{Z}_+^d$-gradings, even for the 3-step case. Thus we centered our attention to the following two questions regarding a nilpotent Lie algebra $\mathfrak{n}$ with center $\mathfrak{z}$:

(A) If $\mathfrak{n}$ admits a $\mathbb{Z}_+^d$-grading $\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{Z}_+^d} \mathfrak{n}_\alpha$, such that its associated polynomial p satisfies $L(p) > 2^{\dim \mathfrak{z}}$, does $\mathfrak{n}$ admit a $\mathbb{Z}_+$-grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_k'$ such that its associated polynomial p' satisfies $L(p') > 2^{\dim \mathfrak{z}}$?

(B) If $\mathfrak{n}$ is r-step nilpotent and it admits a $\mathbb{Z}_+$-grading $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$ such that its associated polynomial p satisfies $L(p) > 2^{\dim \mathfrak{z}}$, does $\mathfrak{n}$ admit a $\mathbb{Z}_+$-grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_r'$ such that its associated polynomial p' satisfies $L(p') > 2^{\dim \mathfrak{z}}$?

In this short paper we show that the answer to (A) is yes but the answer to (B) is no, giving an example for $r = 3$. More precisely, for question (B), we construct a family $\mathfrak{n} = \mathfrak{n}(n)$, $n \in \mathbb{N}$, of graded 3-step nilpotent Lie algebras with center $\mathfrak{z}$ such that $L(p) < 2^{\dim \mathfrak{z}}$ for all gradings $\mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ of $\mathfrak{n}(n)$ ($n \geq 17$), but admitting a $\mathbb{Z}_+$-grading whose associated polynomial has length greater than $2^{\dim \mathfrak{z}}$.

We do not know an example of a $\mathbb{Z}_+$-graded nilpotent Lie algebra satisfying $L(p) < 2^{\dim \mathfrak{z}}$ for all its possible gradings.

2. The Answer to (A) is Yes

Proposition 2.1. *If $\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{Z}_+^d} \mathfrak{n}_\alpha$ is $\mathbb{Z}_+^d$-graded, with associated polynomial p, then there exists a grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_k'$ such that its associated polynomial p' satisfies $L(p') = L(p)$.*

Proof. If $d = 1$ there is nothing to prove. We now assume that the proposition is proved for $\mathbb{Z}_+^{d-1}$-graded Lie algebras and let $\mathfrak{n} = \bigoplus_{\alpha \in \mathbb{Z}_+^d} \mathfrak{n}_\alpha$ be a $\mathbb{Z}_+^d$-graded nilpotent Lie algebra with associated polynomial p. Let us fix $m \in \mathbb{N}$ and consider the $\mathbb{Z}_+^{d-1}$-grading of $\mathfrak{n}$ defined by

$$\mathfrak{n} = \bigoplus_{\beta \in \mathbb{Z}_+^{d-1}} \mathfrak{n}_\beta,$$

where $\mathfrak{n}_\beta = \bigoplus_{t=0}^{\lfloor \frac{\beta_1}{m} \rfloor} \mathfrak{n}_{(\beta_1 - mt, \beta_2, \dots, \beta_{d-1}, t)}$. It is straightforward to see that, if p_m is the polynomial associated to this $\mathbb{Z}_+^{d-1}$-grading, then

$$p_m(x_1, \dots, x_{d-1}) = p(x_1, \dots, x_{d-1}, x_1^m).$$

It is clear that

$$L(p_m) = \sum_{\beta \in \mathbb{Z}_+^{d-1}} \left| \sum_{t=0}^{\lfloor \frac{\beta_1}{m} \rfloor} a_{(\beta_1 - mt, \beta_2, \dots, \beta_{d-1}, t)} \right|$$

and, if $m > \max\{\alpha_1 : a_{(\alpha_1, \dots, \alpha_d)} \neq 0\}$, then $L(p_m) = L(p)$, since $a_{(\beta_1 - mt, \dots, \beta_{d-1}, t)} \neq 0$ only if $t = \lfloor \frac{\beta_1}{m} \rfloor$.

By the induction hypothesis there exists a grading $\mathfrak{n} = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_k'$ such that its associated polynomial p' satisfies $L(p') = L(p_m) = L(p)$. This completes the induction step. $\qquad\square$

3. The Answer to (B) is No

In this section we construct a family $\mathfrak{n} = \mathfrak{n}(n)$, $n \in \mathbb{N}$, of graded 3-step nilpotent Lie algebras with center $\mathfrak{z}$ such that $L(p) < 2^{\dim \mathfrak{z}}$ for all gradings $\mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ of $\mathfrak{n}(n)$, for all $n \geq 17$, but admitting a $\mathbb{Z}_+$-grading whose associated polynomial has length greater than $2^{\dim \mathfrak{z}}$. In fact, for all $n \in \mathbb{N}$, the Lie algebra $\mathfrak{n}(n)$ admits a $\mathbb{Z}_+^{n+1}$-grading such that its associated polynomial p satisfies $L(p) \geq 2^{\dim \mathfrak{z}}$ (see Remark 3.3). It follows from Proposition 2.1 that $\mathfrak{n}(n)$ admits a grading $\mathfrak{n}(n) = \mathfrak{n}_1' \oplus \mathfrak{n}_2' \oplus \cdots \oplus \mathfrak{n}_k'$ such that its associated polynomial p' satisfies $L(p') \geq 2^{\dim \mathfrak{z}}$.

3.1. *Definition of the family $\mathfrak{n}(n)$*

In what follows, if A is a set, $\langle A \rangle$ will denote the free vector space with A as a basis. For each positive integer n, let

$$E_n = \langle \{e_i : i = 1, \dots, n\} \rangle, \quad U_n = \langle \{u_i : i = 1, \dots, n\} \rangle,$$
$$X_n = \langle \{x_i : i = 1, \dots, n\} \rangle, \quad Y_n = \langle \{y_i : i = 1, \dots, n\} \rangle.$$

Since n will be fixed most of the time, we will use E, U, X and Y to denote the spaces E_n, U_n, X_n and Y_n. We will define on the vector space

$$\mathfrak{n} = \mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3,$$

where

$$\mathfrak{n}_1 = E \oplus \langle \{a, b, x\} \rangle,$$
$$\mathfrak{n}_2 = \langle \{u, y\} \rangle \oplus \Lambda^2 E \oplus \langle \{c\} \rangle \oplus X,$$
$$\mathfrak{n}_3 = U \oplus Y \oplus \langle \{f, h\} \rangle,$$

and let

$$\mathfrak{B}_1 = \{e_1, e_2, \ldots, e_n, a, b, x\},$$
$$\mathfrak{B}_2 = \{u, y\} \cup \{e_i \wedge e_j : 1 \leq i < j \leq n\} \cup \{c\} \cup \{x_i : 1 \leq i \leq n\},$$
$$\mathfrak{B}_3 = \{u_1, \ldots, u_n, y_1, \ldots, y_n, f, h\}$$

be ordered basis of $\mathfrak{n}_1$, $\mathfrak{n}_2$ and $\mathfrak{n}_3$, respectively (we choose the lexicographic order for $e_i \wedge e_j$). Now

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3$$

is an ordered basis of $\mathfrak{n}$. It is clear that

$$d_1 = \dim(\mathfrak{n}_1) = n + 3,$$
$$d_2 = \dim(\mathfrak{n}_2) = \frac{n(n+1)}{2} + 3,$$
$$d_3 = \dim(\mathfrak{n}_3) = 2(n+1).$$

We now define the Lie bracket of $\mathfrak{n}$ in terms of this basis as shown in the following table:

$[\mathfrak{n},\mathfrak{n}]$	e_1	$\cdots$	e_n	a	b	x	u	y	$\Lambda^2 E$	c	X	U	Y	f	h
e_1	0	$\cdots$	$e_1 \wedge e_n$	0	0	x_1	u_1	y_1	0	0	0	0	0	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
e_n	$e_n \wedge e_1$	$\cdots$	0	0	0	x_n	u_n	y_n	0	0	0	0	0	0	0
a	0	$\cdots$	0	0	c	0	0	f	0	h	0	0	0	0	0
b	0	$\cdots$	0	$-c$	0	0	h	h	0	0	0	0	0	0	0
x	$-x_1$	$\cdots$	$-x_n$	0	0	0	f	h	0	0	0	0	0	0	0
u	$-u_1$	$\cdots$	$-u_n$	0	$-h$	$-f$	0	0	0	0	0	0	0	0	0
y	$-y_1$	$\cdots$	$-y_n$	$-f$	$-h$	$-h$	0	0	0	0	0	0	0	0	0
$\Lambda^2 E$	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0
c	0	$\cdots$	0	$-h$	0	0	0	0	0	0	0	0	0	0	0
X	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0
U	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0
Y	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0
f	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0
h	0	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0

Remark 3.1. Suppose that we change the basis $B = \{e_i\}$ of E by $B' = \{e_i'\}$, and we define accordingly $x_i' = [e_i', x]$, $y_i' = [e_i', y]$, $u_i' = [e_i', u]$. If we consider the following new ordered basis of $\mathfrak{n}_1$, $\mathfrak{n}_2$ and $\mathfrak{n}_3$ respectively:

$$\mathfrak{B}_1' = \{e_1', e_2', \ldots, e_n', a, b, x\},$$

$$\mathfrak{B}'_2 = \{u, y\} \cup \{e'_i \wedge e'_j : 1 \leq i < j \leq n\} \cup \{c\} \cup \{x'_i : 1 \leq i \leq n\},$$

$$\mathfrak{B}'_3 = \{u'_1, \ldots, u'_n, y'_1, \ldots, y'_n, f, h\},$$

then the above bracket-table looks the same for the new basis $\mathfrak{B}' = \mathfrak{B}'_1 \cup \mathfrak{B}'_2 \cup \mathfrak{B}'_3$ of $\mathfrak{n}$. From now on, we will think of $\mathfrak{B}$ as a map that assigns a basis B' of E to $\mathfrak{B}(B') = \mathfrak{B}'$.

Proposition 3.2. *For every $n \in \mathbb{N}$, $\mathfrak{n}(n)$ is a graded 3-step nilpotent Lie algebra with*

$$d_1 = n + 3, \quad d_2 = \frac{n(n+1)}{2} + 3, \quad d_3 = 2n + 2, \quad \dim \mathfrak{z} = \frac{(n+4)(n+1)}{2}.$$

Proof. The only basis elements t, v, w such that $[[t, v], w] \neq 0$ are $t = b$ and $v = w = a$. Therefore, the Jacobi's identity is trivially satisfied in $\mathfrak{n}$. $\qquad\square$

Remark 3.3. Let $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_n\}$ be the canonical basis of $\mathbb{Z}_+^{n+1}$. The Lie algebra $\mathfrak{n}(n)$ admits the following $\mathbb{Z}_+^{n+1}$-grading:

$$\deg(e_i) = \epsilon_i, \quad \deg(x_i) = \epsilon_0 + \epsilon_i, \quad \deg(u_i) = 2\epsilon_0 + \epsilon_i, \quad \deg(y_i) = 2\epsilon_0 + \epsilon_i,$$

for $i = 1, \ldots, n$;

$$\deg(a) = \deg(b) = \deg(x)$$

$$= \frac{\deg(c)}{2} = \frac{\deg(u)}{2} = \frac{\deg(y)}{2} = \frac{\deg(h)}{3} = \frac{\deg(f)}{3} = \epsilon_0,$$

and finally $\deg(e_i \wedge e_j) = \epsilon_i + \epsilon_j$, for $1 \leq i < j \leq n$. Then the polynomial associated to this grading is

$$p(x_0, x_1, \ldots, x_n) = (1 - x_0)^3 (1 - x_0^2)^3 (1 - x_0^3)^2$$

$$\times \prod_{i=1}^{n} (1 - x_i)(1 - x_0 x_i)(1 - x_0^2 x_i)^2 \prod_{1 \leq i < j \leq n} (1 - x_i x_j).$$

If $x_j = \sqrt{-1}$ for $j = 1, \ldots, n$ and $x_0 = e^{\sqrt{-1}\frac{\pi}{4}}$ then

$$L(p) \geq |p(x_0, x_1, \ldots, x_n)| > 2^{\frac{(n+4)(n+1)}{2}} = 2^{\dim(\mathfrak{z})}.$$

3.2. *Derivations of $\mathfrak{n}(n)$*

Let $\mathrm{Der}(\mathfrak{n})$ be the Lie algebra of derivations of $\mathfrak{n}$. In this subsection we will describe some properties of the matrices corresponding to elements in $\mathrm{Der}(\mathfrak{n})$ associated to a basis $\mathfrak{B}(B)$, where B is a basis of E.

Definition 3.4. We denote by $\mathrm{Der}(\mathfrak{n})_0$ the subalgebra of derivations D such that

- $D(E) \subset E$ and
- $D(a) = D(b) = D(x) = D(u) = D(y) = 0$.

It is clear that there is Lie algebra isomorphism

$$\mathfrak{gl}(E) \to \mathrm{Der}(\mathfrak{n})_0$$

$$A \mapsto D_A.$$

The matrix $[D_A]_{\mathfrak{B}(B)}$ of D_A in the basis $\mathfrak{B}(B)$ is block-diagonal, where the blocks corresponding to each subspace are described by the following table:

E	a	b	x	u	y	$\Lambda^2 E$	c	X	U	Y	f	h	
$[A]_B$	0	0	0	0	0	$[\Lambda^2 A]_{\Lambda^2 B}$	0	$[A]_B$	$[A]_B$	$[A]_B$	0	0	(3.1)

Here, $\Lambda^2 B = \{e_i \wedge e_j : 1 \le i < j \le n\}$ and $\Lambda^2 A$ is the linear map on $\Lambda^2 E$ defined by $\Lambda^2 A(e_i \wedge e_j) = A(e_i) \wedge e_j + e_i \wedge A(e_j)$.

Definition 3.5. We denote by $\mathrm{Der}(\mathfrak{n})_1$ the set of derivations D such that $D(E) \subset W$, where $W = \langle \{a, b, x, u, y\} \rangle \oplus \Lambda^2 E \oplus \langle \{c\} \rangle \oplus X \oplus U \oplus Y \oplus \langle \{f, h\} \rangle$.

Proposition 3.6. *If B is a basis of E and $D \in \mathrm{Der}(\mathfrak{n})_1$, then the matrix of D in the basis $\mathfrak{B}(B)$ is lower triangular.*

Proof. We need to check that for every element $w \in \mathfrak{B}(B)$, the coordinates of Dw are zero on the basis vectors located left to w, according to the order in $\mathfrak{B}(B)$. We will use the following notation: if $v \in \mathfrak{n}$ and $w \in \mathfrak{B}(B)$, $\lambda_w(v)$ will be the w coordinate of v.

Since D is a derivation, we know that

$$D(\mathfrak{n}') \subset \mathfrak{n}', \quad D([\mathfrak{n}, \mathfrak{n}']) \subset [\mathfrak{n}, \mathfrak{n}'], \quad D(\mathfrak{z}(\mathfrak{n})) \subset \mathfrak{z}(\mathfrak{n}).$$

In what follows, we will omit the parenthesis and write Dv for $D(v)$.

(1) From the definition of $\mathfrak{n}$, it is clear that $[\mathfrak{n}, \mathfrak{n}'] = \langle h \rangle$, so $Dh = \lambda_h(Dh)h$.
(2) Since $c = [a, b]$ we have

$$Dc = [Da, b] + [a, Db]$$

$$= \lambda_a(Da)c - (\lambda_u(Da) + \lambda_y(Da))h + \lambda_b(Db)c + \lambda_y(Db)f + \lambda_c(Db)h$$

$$= (\lambda_a(Da) + \lambda_b(Db))c + \lambda_y(Db)f + (-\lambda_u(Da) - \lambda_y(Da) + \lambda_c(Db))h.$$

This is what we need for Dc.
(3) Since $f = [x, u] = [a, y]$, then $[Dx, u] + [x, Du] = [Da, y] + [a, Dy]$. In addition,

$$[Dx, u] = \sum \lambda_{e_i}(Dx)u_i + \lambda_b(Dx)h + \lambda_x(Dx)f,$$

$$[x, Du] = -\sum \lambda_{e_i}(Du)x_i + \lambda_y(Du)h + \lambda_u(Du)f,$$

$$[Da, y] = \sum \lambda_{e_i}(Da)y_i + (\lambda_b(Da) + \lambda_x(Da))h + \lambda_a(Da)f,$$
$$[a, Dy] = \lambda_b(Dy)c + \lambda_c(Dy)h + \lambda_y(Dy)f,$$

and therefore

$$\lambda_b(Dy) = 0 \quad \text{and} \quad \lambda_{e_i}(Dx) = \lambda_{e_i}(Du) = \lambda_{e_i}(Da) = 0$$
$$\text{for all } 1 \le i \le n. \tag{3.2}$$

We also conclude that

$$Df = (\lambda_b(Dx) + \lambda_y(Du))h + (\lambda_x(Dx) + \lambda_u(Du))f$$
$$= (\lambda_b(Da) + \lambda_x(Da) + \lambda_c(Dy))h + (\lambda_a(Da) + \lambda_y(Dy))f.$$

This is what we need for Df.

(4) Since $D \in \mathrm{Der}(\mathfrak{n})_1$, we know that $D(e_i) \in W$ and hence, for all $1 \le i < j \le n$,

$$D(e_i \wedge e_j) = [D(e_i), e_j] + [e_i, D(e_j)]$$
$$= -\lambda_x(De_i)x_j - \lambda_u(De_i)u_j - \lambda_y(De_i)y_j$$
$$+ \lambda_x(De_j)x_i + \lambda_u(De_j)u_i + \lambda_y(De_j)y_i,$$

and this is what we need to prove for $D(e_i \wedge e_j)$.

(5) Let $1 \le i \le n$. Since $[a, e_i] = 0$, then $[Da, e_i] + [a, D(e_i)] = 0$ and

$$0 = \sum_j \lambda_{e_j}(Da)e_j \wedge e_i - \lambda_x(Da)x_i - \lambda_y(Da)y_i - \lambda_u(Da)u_i$$
$$+ \lambda_b(De_i)c + \lambda_y(De_i)f + \lambda_c(De_i)h,$$

therefore

$$0 = \lambda_{e_j}(Da) \quad \text{for all } 1 \le i \le n,$$
$$0 = \lambda_x(Da) = \lambda_y(Da) = \lambda_u(Da),$$
$$0 = \lambda_b(De_i) = \lambda_y(De_i) = \lambda_c(De_i) \quad \text{for all } 1 \le i \le n.$$

Hence we proved, among other things, what is needed for Da.

(6) Since $[b, e_i] = 0$, then $[Db, e_i] + [b, D(e_i)] = 0$ and taking into account that $\lambda_y(De_i) = 0$ (see (5)) we obtain

$$0 = \sum_j \lambda_{e_j}(Db)e_j \wedge e_i - \lambda_x(Db)x_i - \lambda_u(Db)u_i$$
$$- \lambda_y(Db)y_i - \lambda_a(De_i)c + \lambda_u(De_i)h$$

and hence

$$0 = \lambda_{e_j}(Db) \quad \text{for all } 1 \le j \le n,$$
$$0 = \lambda_x(Db) = \lambda_y(Db) = \lambda_u(Db),$$
$$0 = \lambda_a(De_i) = \lambda_u(De_i) \quad \text{for all } 1 \le i \le n.$$

This is almost what we needed for Db. We now combine this and results from (2) and (5) to obtain

$$Dc = [Da, b] + [a, Db] = (\lambda_a(Da) + \lambda_b(Db))c + \lambda_c(Db)h,$$

and hence $[Dc, b] = 0$. Since $[b, c] = 0$, it follows that $[Db, c] = 0$ and thus $\lambda_a(Db) = 0$. This completes what we need for Db.

(7) We now consider the cases of x, u, y.

(i) We first check that the a and b coordinates of Dx, Du, Dy are zero.
We start with Dx: since $[b, x] = 0$, we have $[Dx, b] + [x, Db] = 0$, and since $\lambda_c([x, Db]) = 0$, we obtain $\lambda_c([Dx, b]) = 0$. This implies that $\lambda_a(Dx) = 0$. Repeating this argument and observing that $[a, x] = 0$, we will get that Dx has no b coordinate.
If now we do it considering $[a, u] = 0$, we will get that Du has no b coordinate.
We notice that the same argument, always analyzing the c coordinate, can be repeated using $[u, b] = h$ and $[b, y] = h$ respectively to conclude that Du and Dy do not have a coordinates.
We have already seen in (3.2) that Dy does not have a b coordinate.

(ii) Let us consider now the x and u coordinates of Dy: $[u, y] = 0$ implies $[Du, y] + [u, Dy] = 0$ and, since we know that the a coordinate of Du is 0, the f coordinate of $[Du, y]$ is 0, and then $[u, Dy]$ has no f coordinate, which implies that the x coordinate of Dy is 0.
Again, the same argument considering $[x, y] = h$ leads us to conclude that the u coordinate of Dy is 0.

(iii) We consider now the x coordinate of Du. Being $[u, y] = 0$, $[Du, y] + [u, Dy] = 0$. Looking at the h coordinate of this sum, we get

$$0 = \lambda_b(Du) + \lambda_x(Du) - \lambda_b(Dy).$$

We have just seen that $\lambda_b(Dy) = 0 = \lambda_b(Du)$, then $\lambda_x(Du) = 0$, as we need.

(iv) Finally, we just need to prove that the u and e_i coordinates of Dy are zero. Since $[x, y] = h$, and recalling that $\lambda_u(Dx) = 0$ and $\lambda_{e_i}(Dx) = 0$, we have

$$\lambda_h(Dh)h = D(h) = [Dx, y] + [x, Dy]$$

$$= \lambda_x(Dx)h - \sum \lambda_{e_i}(Dy)x_i + \lambda_u(Dy)f + \lambda_y(Dy)h,$$

so $\lambda_u(Dy) = 0$ and $\lambda_{e_i}(Dy) = 0$ for all $1 \le i \le n$.

(8) For any $1 \le i \le n$, we have that

$$D(x_i) = D([x, e_i]) = [Dx, e_i] + [x, D(e_i)]$$

$$= -\lambda_x(Dx)x_i - \lambda_u(Dx)u_i - \lambda_y(Dy)y_i + \lambda_u(Dx)f + \lambda_y(Dx)h.$$

On the other hand, since we know from (7.iii) that $\lambda_x(Du) = 0$,

$$D(u_i) = D([u, e_i]) = [Du, e_i] + [u, D(e_i)]$$
$$= -\lambda_u(Du)u_i - \lambda_y(Du)y_i - \lambda_x(De_i)f - \lambda_b(De_i)h.$$

Finally, having in mind that the x and u coordinates of Dy are 0 (see (7.ii)),

$$D(y_i) = D([y, e_i]) = [Dy, e_i] + [y, D(e_i)]$$
$$= -\lambda_y(Dy)y_i - \lambda_a(De_i)f - (\lambda_b(De_i) + \lambda_x(De_i))h.$$

With this we conclude the cases of x_i, u_i and y_i and the proof is complete. □

Proposition 3.7. *Let $D \in \mathrm{Der}(\mathfrak{n})_1$. For each element $v \in \mathfrak{B}(B)$, we denote by λ_v the diagonal coefficient of the matrix of D corresponding to the vector v. Then:*

(a) $\lambda_{e_i} = 0, 1 \le i \le n$;
(b) $\lambda_a = \lambda_b = \lambda_x$;
(c) $\lambda_h = \lambda_f = 3\lambda_a$;
(d) $\lambda_y = \lambda_u = \lambda_c = 2\lambda_a$.

Proof. (a) This is obvious from the definition. We will prove next (b), (c) and (d). From the previous proposition we know that the matrix of D is lower triangular. Hence:

(1) $[x, u] = f$ implies $[D(x), u] + [x, D(u)] = D(f)$, and thus $\lambda_x + \lambda_u = \lambda_f$.
(2) $[a, y] = f$ implies $[D(a), y] + [a, D(y)] = D(f)$, and thus $\lambda_a + \lambda_y = \lambda_f$.
(3) $[a, c] = h$ implies $[D(a), c] + [a, D(c)] = D(h)$, and thus $\lambda_a + \lambda_c = \lambda_h$.
(4) $[b, y] = h$ implies $[D(b), y] + [b, D(y)] = D(h)$. We know, from (6) in the proof of the previous proposition, that the u coordinate of $D(b)$ is 0, and thus $\lambda_b + \lambda_y = \lambda_h$.
(5) $[b, u] = h$ implies $[D(b), u] + [b, D(u)] = D(h)$. Also from (6) in the proof of the previous proposition, we know that the y coordinate of $D(b)$ is 0, and thus $\lambda_b + \lambda_u = \lambda_h$.
(6) $[x, y] = h$ implies $[D(x), y] + [x, D(y)] = D(h)$, and thus $\lambda_x + \lambda_y = \lambda_h$.
(7) $[a, b] = c$ implies $[D(a), b] + [a, D(b)] = D(c)$, and thus $\lambda_a + \lambda_b = \lambda_c$.

From (3) and (7), we have $2\lambda_a + \lambda_b = \lambda_h$ and combining with (5), we obtain

$$2\lambda_a = \lambda_u. \tag{3.3}$$

Substituting in (1), $\lambda_x + 2\lambda_a = \lambda_f$.
From (4) and (5), we have

$$\lambda_y = \lambda_u, \tag{3.4}$$

from (1) and (6), we have

$$\lambda_h = \lambda_f, \tag{3.5}$$

from (2), (4) and (6), $\lambda_a = \lambda_b = \lambda_x$, and this proves (b). From this and (3.5) we obtain (c), that is, $\lambda_h = \lambda_f = 3\lambda_a$.

From (3) and (4) it follows $\lambda_y = \lambda_c$ and from this and (3.4) and (3.3), we obtain (d). This ends the proof of the proposition. $\square$

As a consequence, we obtain the following proposition that describes the Levi decomposition of $\mathrm{Der}(\mathfrak{n})$.

Proposition 3.8. *Let* $\mathrm{Der}(\mathfrak{n})$ *be the Lie algebra of derivations of* $\mathfrak{n}$, *and let* $\mathrm{Der}(\mathfrak{n})_0$ *and* $\mathrm{Der}(\mathfrak{n})_1$ *be the Lie subalgebras of* $\mathrm{Der}(\mathfrak{n})$ *defined previously. Then*:

(a) $\mathrm{Der}(\mathfrak{n})_0$ *is a Lie subalgebra of* $\mathrm{Der}(\mathfrak{n})$ *isomorphic to* $\mathfrak{gl}(E)$;
(b) $\mathrm{Der}(\mathfrak{n})_1$ *is a solvable ideal of* $\mathrm{Der}(\mathfrak{n})$;
(c) $\mathrm{Der}(\mathfrak{n}) = \mathrm{Der}(\mathfrak{n})_0 \oplus \mathrm{Der}(\mathfrak{n})_1$.

Proof. (a) This has been already discussed when we defined $\mathrm{Der}(\mathfrak{n})_0$, and (b) is a consequence of the fact that the matrix of any $D \in \mathrm{Der}(\mathfrak{n})_1$ is lower triangular in any basis $\mathfrak{B}(B)$ of $\mathfrak{n}$.

To prove (c), let us see first that the sum is direct. If $D \in \mathrm{Der}(\mathfrak{n})_0 \cap \mathrm{Der}(\mathfrak{n})_1$, then $D(E) \subset E$, $D(E) \subset W$ and hence $D(E) = 0$. Since E and a, b, x, u, y generate $\mathfrak{n}$ as a Lie algebra, it follows that $D = 0$.

Now, we will see that $\mathrm{Der}(\mathfrak{n})_0 + \mathrm{Der}(\mathfrak{n})_1 = \mathrm{Der}(\mathfrak{n})$. Given $D \in \mathrm{Der}(\mathfrak{n})$, let $A = p_E \circ D|_E \in \mathfrak{gl}(E)$ where p_E the projection over E with respect to the decomposition $\mathfrak{n} = E \oplus W$. Let $D_0 \in \mathrm{Der}(\mathfrak{n})_0$ be the derivation associated to A. Since the matrix of D_0 in a basis $\mathfrak{B}$ is of the form (3.1), it follows that $D_1 = D - D_0 \in \mathrm{Der}(\mathfrak{n})_1$ This proves (c). $\square$

Proposition 3.9. *Let* $D \in \mathrm{Der}(\mathfrak{n})$ *be a diagonalizable derivation with eigenvalues* 1, 2 *and* 3; *then the dimension of the eigenspaces are* d_1, d_2 *and* d_3 *respectively* (*see Proposition* 3.2). *In particular, if* $\tilde{\mathfrak{n}}_1 \oplus \tilde{\mathfrak{n}}_2 \oplus \tilde{\mathfrak{n}}_3$ *is any grading of* $\mathfrak{n}(n)$, *then* $\dim \tilde{\mathfrak{n}}_i = d_i$, $i = 1, 2, 3$.

Proof. Suppose $D = D_A + D_1$ where $D_A \in \mathrm{Der}(\mathfrak{n})_0$ and $D_1 \in \mathrm{Der}(\mathfrak{n})_1$. Since D is diagonalizable, then A is diagonalizable as well. Then we can choose a basis B of E such that the matrix of D_A in the basis $\mathfrak{B} = \mathfrak{B}(B)$ is diagonal (see (3.1)). Since D_1 has a lower triangular matrix in the basis $\mathfrak{B}$, the matrix of D in this basis is lower triangular.

We will now make use of Proposition 3.7, we notice that this proposition is true for any basis of type $\mathfrak{B}(B)$ (see also Remark 3.1).

As in Proposition 3.7, for each $v \in \mathfrak{B}$, we denote by λ_v the diagonal coefficient of the matrix of D corresponding to the vector v. It is clear that $\{\lambda_v : v \in \mathfrak{B}\}$ are

the eigenvalues of D counted with multiplicity. Now, λ_v is either equal to 1, 2 or 3 for all $v \in \mathfrak{B}$. This, together with Proposition 3.7 and the shape of the matrix of D_A, implies that

$$\lambda_a = \lambda_b = \lambda_x = 1, \quad \lambda_y = \lambda_u = \lambda_c = 2, \quad \lambda_h = \lambda_f = 3.$$

Finally, since D is a Lie algebra homomorphism, we obtain that

$$\lambda_{u_i} = \lambda_{e_i} + \lambda_u, \quad \lambda_{y_i} = \lambda_{e_i} + \lambda_y, \quad \lambda_{x_i} = \lambda_{e_i} + \lambda_x, \quad \lambda_{e_i \wedge e_j} = \lambda_{e_i} + \lambda_{e_j}$$

and hence $\lambda_{e_i} = 1$, $\lambda_{u_i} = 3$, $\lambda_{y_i} = 3$, $\lambda_{y_i} = 2$ and $\lambda_{e_i \wedge e_j} = 2$. Counting the number of eigenvalues, we obtain that the multiplicities of 1, 2 are 3 are respectively d_1, d_2 and d_3. $\qquad\square$

Proposition 3.10. *If $n \geq 17$, then $L(p) < 2^{\dim \mathfrak{z}}$ for all gradings $\mathfrak{n}(n) = \mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3$ of $\mathfrak{n}(n)$.*

Proof. As a consequence of Proposition 3.9, the numbers d_1, d_2 and d_3 are independent of the grading of $\mathfrak{n}(n)$. Since

$$d_1 = n + 3, \quad d_2 = 3 + \frac{n(n+1)}{2}, \quad d_3 = 2(n+1), \quad \dim \mathfrak{z} = \frac{(n+4)(n+1)}{2},$$

we need to prove that $L\left(\frac{(1-x)^{n+3}(1-x^2)^{\frac{n(n+1)}{2}+3}(1-x^3)^{2n+2}}{2^{\frac{(n+4)(n+1)}{2}}}\right) < 1$ for $n \geq 17$.

We have checked this for $n = 17, \ldots, 200$ computationally, thus we must prove it for $n > 200$. We start by rearranging the factors of the polynomial in the following way:

$$\frac{(1-x)^n(1-x^2)^{2n}(1-x^3)^{2n}}{2^{4n}} \cdot \frac{(1-x^2)^{\frac{n(n-3)}{2}+3}}{2^{\frac{n(n-3)}{2}+3}} \cdot 2(1-x)^3(1-x^3)^2.$$

Since $L\left(\frac{(1-x^2)^{\frac{n(n-3)}{2}+3}}{2^{\frac{n(n-3)}{2}+3}}\right) \leq 1$, $L(2(1-x)^3(1-x^3)^2) \leq 64$, and $L(pq) \leq L(p)L(q)$, we only need to show that

$$L(p_n) < \frac{1}{64}, \quad \text{where } p_n(x) = \frac{(1-x)^n(1-x^2)^{2n}(1-x^3)^{2n}}{2^{4n}},$$

for $n \geq 200$. We checked computationally that $L(p_n) < \frac{1}{2}$ for $n = 30, \ldots, 180$. Now, if $n > 180$, then $n - 150 > 30$, and arguing by induction, we obtain

$$L(p_n) \leq L(p_{30})^5 L(p_{n-150}) \leq \left(\frac{1}{2}\right)^6 = \frac{1}{64}. \qquad\square$$

References

[1] Ch. Deninger and W. Singhof, On the cohomology of nilpotent Lie algebras, *Bul. Soc. Math. France* **116** (1988) 3–14.

[2] S. Halperin, Rational homotopy and torus actions, in *Aspects of Topology*, London Mathematical Society Lecture Note Series, Vol. 93 (Cambridge Unviresity Press, 1985), pp. 293–305.

[3] L. Magnin, *Adjoint and Trivial Cohomology Tables for Indecomposable Nilpotent Lie Algebras of Dimension* ≤ 7 *over* $\mathbb{C}$, 2nd Corrected Edition 2007, Online e-book (2008), http://magnin.perso.math.cnrs.fr/.

[4] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* (2) **59** (1954) 531–538.

[5] H. Pouseele and P. Tirao, Constructing Lie algebra homology classes, *J. Algebra* **292**(2) (2005) 585–591.

[6] L. Yu, Small covers and Halperin–Carlsson conjecture, *Pacific J. Math.* **256**(2) (2012) 489–507.