# STOCHASTIC NONDETERMINISM AND EFFECTIVITY FUNCTIONS 

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#### Abstract

This paper investigates stochastic nondeterminism by relating nondeterministic labelled Markov processes and stochastic effectivity functions to each other; the underlying state spaces are continuous. Both generalizations to labelled Markov transition systems have been proposed recently with differing intentions. It turns out that they display surprising similarities and interesting differences, as we will demonstrate in this paper.


## 1. Introduction

Nondeterministic labelled Markov processes (henceforth abbreviated as NLMPs) propose a model for stochastic non-determinism in which a state is assigned a measurable set of probability distributions as a way of selecting non-deterministically the distribution of the next state in a transition system. As outlined in [4, 5] , this serves as a model for internal nondeterminism; a crucial point here is the question of measurability for the underlying transition law. To give the idea, at the heart of a NLMP over a measurable state space $S$ lies a family of functions $\left(\kappa_{a}\right)_{a \in A}$ indexed by actions from a set $A$, each of which assigns to a state $s$ a set $\kappa_{a}(s)$ of probabilities over $S$, modeling the set of distributions which are possible after action $a$ in state $s$. To obtain meaningful insights into the behavior of such a system, some assumptions on measurability of $s \mapsto \kappa_{a}(s)$ should be imposed. The measurable structure is given by the hit $\sigma$-algebra, a construction very similar to the hyperspace constructions in topology [23, 21]. This construction forms the basis for investigations into system behavior, most notably into different variants of bisimulations. For example, it was shown that the negation-free logic proposed in [26] is a suitable logic for investigating event bisimulation [4, Theorem 4.5]. This demonstrates that NLMPs are an adequate tool for representing and investigating stochastic non-determinism.

In a parallel development, stochastic effectivity functions have been proposed for providing a stochastic interpretation of game logic, continuing the line of research which has been initiated by Parikh [25] and later continued in [27] one one hand, and the stochastic interpretation of propositional dynamic logic, a fragment of game logic, through Kripke models on the other hand [11]. Parikh had observed that neighborhood models are a suitable tool for the variant of modal logics which he had proposed as game logic. A neighborhood model over a set $W$ of worlds is essentially represented by an effectivity function over $W$, i.e., a function $F$ which

[^0]maps $W$ to the set of all upper closed subsets of the power set $\mathcal{P}(W)$, so that for $F(w) \subseteq \mathcal{P}(W)$ has the property that $A \in F(w)$ and $A \subseteq A^{\prime}$ implies $A^{\prime} \in F(w)$ all $w \in W$. Assume that $w \in W$ represents the state of some game. Let $A \in F(w)$ indicate that, when making a move, the player has a strategy to reach a state in $A$. Clearly $F(w)$ is upper closed. Now assume that these games can be combined in various ways, e.g., by composing them sequentially, by choosing one rather than the other, or by iterating them a finite but indefinite number of times. Having only one player is probably not too entertaining, so it is assumed that the game has two players, which are henceforth called Angel and Demon, and which are assumed to take turns. It is assumed that the entire game is determined, indicating that exactly one of both players has a winning strategy [20, Chapter 33]; for symmetry, each game has a dual game which assumes that Angel and Demon change rôles. Determinacy then implies that if Angel does not have a winning strategy for a game, Demon has one for its dual, which means that if Angel does not have a strategy for achieving a state in a set $A$, Demon has a strategy for achieving a state in $W \backslash A$, and vice versa. This implies that we have only to cater for Angel's movements; assume the latter's effectivity function for a game is given by $F$, then $w \mapsto\{W \backslash A \mid A \notin F(w)\}$ will take care of Demon's movements for this game. This scenario will motivate part of our approach, so we will return to it from time to time.
A probabilistic interpretation of game logic assumes that the outcome of a game is modeled in terms of probability distributions over the set $W$ of states or worlds, which means that we assign to each world $w$ an upward closed set $F(w)$ of probability measures on $W$ for modeling a game, and $A \in F(w)$ is intended to represent the assumption that $A$ is a possible set of distributions which Angel can achieve upon playing that game. But since we are in the realm of probabilities, we have to make sure that the probabilities are defined at all, so we have to assume also here the set of worlds $W$ being endowed with a measurable structure, each element of $F(w)$ being a measurable set of probabilities as well. So we set up the scenario of assigning to each world $w$ a set $F(w)$ of measurable sets of probabilities. With this basic assumptions we are able to model some, but not all, constructions in games logic, the important operation of composing games is not among them, however. For modeling this, we want to be able to compose stochastic effectivity functions so that the result can be used for representing sequential compositions; this operation is technically a bit involved, as it requires an additional property on these functions which we call $t$-measurability [12, 13]. It states roughly that we obtain a measurable function when taking quantitative measurements into account; this will then enable us to compose effectivity functions through integration. It turns out that t-measurability can be characterized topologically, that it is a powerful concept for the purposes indicated here, but that it marks on the other hand one of the boundaries between NLMPs and effectivity functions.

Both NLMPs and stochastic effectivity functions are generalizations of Markov kernels, a.k.a. stochastic relations. It is not so straightforward, however, to see a relationship between NLMPs and effectivity functions, and here things start to become interesting. We investigate the interplay between both by looking into conditions under which they generate each other. For example, we devise a mechanism which renders an effectivity function from an NLMP - most of the time, but when it appears to be most obvious, it does not work. We show that in the finitely supported case there are some very close connections between them in Polish spaces; this is established through a selection argument. We review bisimulations for NLMPs and translate these results to effectivity functions. On the other hand, morphisms for effectivity functions give rise to an investigation of morphisms for NLMPs. We show that
porting a concept from one side to the other furthers insight into both ways of modeling stochastic nondeterminism.

Outline. In the next section, we recall some basic material concerning measurable spaces and subprobability measures. Highlighted concepts are the hit $\sigma$-algebra and the space of upper closed sets. In Section 3 we present the two models of stochastic behavior which are studied in this work, namely, nondeterministic kernels and effectivity functions; they have been used to construct nondeterministic Labelled Markov Processes and stochastic game models.

The investigation of bisimulation is taken up in Section 4, and serves as a framework for investigating several issues. In Section 4.2 the relation between non-determinism and effectivity is studied from a measurability point of view, and also the interaction of morphisms and relational bisimulations. A logic for characterizing bisimilarity on finitary effectivity functions is proposed in Section 4.3. Finally, a coalgebraic perspective on bisimulation and behavioral equivalence is developed in Section [5, where the notion of subsystem is developed and plays a central rôle.

Section 6 contains some further directions and offers concluding remarks.

## 2. $\sigma$-Algebras and All That

The reader is briefly reminded of some notions and constructions from measure theory, including the famous $\pi$ - $\lambda$-Theorem and Choquet's representation of integrals as areas; these two tools are used all over the paper. We also introduce invariant sets for an equivalence relation together with the corresponding $\sigma$-algebra.
2.1. Measurability. First we fix some notations. A measurable space $(S, \mathcal{S})$ is a set $S$ with a $\sigma$-algebra $\mathcal{S}$, i.e., $\mathcal{S} \subseteq \mathcal{P}(S)$ is a Boolean algebra which is closed under countable unions. Here $\mathcal{P}(S)$ is the power set of $S$. Given $\mathcal{S}_{0} \subseteq \mathcal{P}(S)$, denote by

$$
\sigma\left(\mathcal{S}_{0}\right):=\bigcap\left\{\mathcal{T} \mid \mathcal{S}_{0} \subseteq \mathcal{T}, \mathcal{T} \text { is a } \sigma \text {-algebra }\right\}
$$

the smallest $\sigma$-algebra containing $\mathcal{S}_{0}$ (the set for which the intersection is constructed is not empty, since it contains $\mathcal{P}(S)$ ). If $S$ is a topological space with topology $\tau$, then the elements of $\sigma(\tau)$ are called the Borel sets of $S$; the $\sigma$-algebra $\sigma(\tau)$ is usually denoted by $\mathcal{B}(S)$. For a measurable space $(S, \mathcal{S})$ and $A \subseteq S$ we define $\mathcal{S} \cap A:=\{B \cap A \mid B \in \mathcal{S}\}$ as the trace of $\mathcal{S}$ on $A$, so that $(A, \mathcal{S} \cap A)$ becomes a measurable space in its own right; note that we do not require $A$ to be a measurable set.
Given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, the product space $(S \times T, \mathcal{S} \otimes \mathcal{T})$ has the Cartesian product $S \times T$ as a carrier set, the product $\sigma$-algebra

$$
\mathcal{S} \otimes \mathcal{T}:=\sigma(\{A \times B \mid A \in \mathcal{S}, B \in \mathcal{T}\})
$$

is the smallest $\sigma$-algebra on $S \times T$ which contains all rectangles $A \times B$ with $A \in \mathcal{S}$ and $B \in \mathcal{T}$. Define for $E \subseteq S \times T$

$$
\begin{array}{rlr}
E^{s}:=\{t \in T \mid\langle s, t\rangle \in E\} & \text { vertical cut, } \\
E_{t}:=\{s \in S \mid\langle s, t\rangle \in E\} & & \text { horizontal cut } 1,
\end{array}
$$

then $E^{s} \in \mathcal{T}$ for all $s \in S$, and $E_{t} \in \mathcal{S}$ for all $t \in T$, provided $E \in \mathcal{S} \otimes \mathcal{T}$. The converse does not hold, that is, a set having all of its cuts measurable is not necessarily measurable 19, Exercise 21.20].
Dually, the coproduct $(S, \mathcal{S}) \oplus(T, \mathcal{T})$ of the measurable spaces has as a carrier set the disjoint union $S \uplus T$ of the carrier sets $S$ and $T$, and as a $\sigma$-algebra

$$
\mathcal{S} \oplus \mathcal{T}:=\{C \subseteq S \uplus T \mid C \cap S \in \mathcal{S} \text { and } C \cap T \in \mathcal{T}\} .
$$

The coproduct of measurable spaces is sometimes also called their sum.
If $(S, \tau)$ and $(T, \theta)$ are topological spaces, then the Borel sets $\mathcal{B}(\tau \times \theta)$ of the product topology may properly contain the product $\mathcal{B}(\tau) \otimes \mathcal{B}(\theta)$. If, however, both spaces are Hausdorff and $\theta$ has a countable basis, then $\mathcal{B}(\tau \times \theta)=\mathcal{B}(\tau) \otimes \mathcal{B}(\theta)$ [3, Lemma 6.4.2]. In particular, the Borel sets of the product of two Polish spaces are generated by Cartesian products of Borel sets from the components (a Polish space is a topological space which has a countable base and for which a complete metric exists). The same applies to analytic spaces (an analytic space is a separable metric space which is the image of a continuous map between Polish spaces). This is so because the topology of these spaces is also countably generated [14, Corollary 2.97]. A standard Borel space is a measurable space the $\sigma$-algebra of which is generated by a Polish topology.

In summary, the observation on products mentioned above suggests that we have to exercise particular care when working with the product of measurable spaces, which carry a topological structure as well.

Given the measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, a map $f: S \rightarrow T$ is said to be $\mathcal{S}$ - $\mathcal{T}$-measurable iff $f^{-1}[D] \in \mathcal{S}$ for all $D \in \mathcal{T}$. Call the measurable map $f:(S, \mathcal{S}) \rightarrow(T, \mathcal{T})$ final iff $\mathcal{T}$ is the largest $\sigma$-algebra $\mathcal{C}$ on $T$ such that $f^{-1}[\mathcal{C}]:=\left\{f^{-1}[C] \mid C \in \mathcal{C}\right\} \subseteq \mathcal{S}$ holds, so that $\mathcal{T}=\left\{B \subseteq T \mid f^{-1}[B] \in \mathcal{S}\right\}$. Hence we may conclude from $f^{-1}[B] \in \mathcal{S}$ that $B \in \mathcal{T}$, if $f$ is onto. An equivalent formulation for finality of $f$ is that a map $g: T \rightarrow U$ is $\mathcal{T}-\mathcal{U}$ measurable if and only if $g \circ f: S \rightarrow U$ is $\mathcal{S}$ - $\mathcal{U}$-measurable, whenever $(U, \mathcal{U})$ is a measurable space. Measurability of real valued maps always refers to the Borel sets on the reals, hence $f: S \rightarrow \mathbb{R}$ is measurable iff $\{s \in S \mid f(s) \bowtie q\} \in \mathcal{S}$ for each rational number $q$, with $\bowtie$ as one of the relations $\leq,<, \geq,>$.
Definition 2.1. Let $\mathcal{A}$ be some family of sets. The hit $\sigma$-algebra $\mathscr{H}(\mathcal{A})$ is the least $\sigma$-algebra on $\mathcal{A}$ containing all sets $\mathrm{H}_{C}:=\{D \in \mathcal{A}: D \cap C \neq \varnothing\}$ with $C \in \mathcal{A}$.

This $\sigma$-algebra will be used when we formulate the measurable structure underlying nondeterministic labelled Markov processes. This is an easy criterion for hit-measurability.
Lemma 2.2. Let $(S, \mathcal{S})$ be a measurable space, $\mathcal{A}$ a $\sigma$-algebra on a set $T$, and a map $\kappa: S \rightarrow$ $\mathcal{A}$. Then $\kappa$ is $\mathcal{S}-\mathscr{H}(\mathcal{A})$ measurable iff for every $C \in \mathcal{A},\{s \in S: \kappa(s) \subseteq C\} \in \mathcal{S}$.

Proof. Just note that $\{s \in S: \kappa(s) \subseteq C\}=S \backslash\{s \in S: \kappa(s) \cap T \backslash C \neq \varnothing\}$.
Some notation concerning binary relations will be needed. Let $R$ a binary relation over $S$. A set $Q$ is $R$-closed if $x \in Q$ and $x R y$ imply $y \in Q$; this is the appropriate generalization

[^1]of invariance for equivalence relations. $\Sigma_{R}(\mathcal{S}), \Sigma_{R}(S)$ or $\Sigma_{R}$ will denote the $\sigma$-algebra of $R$-closed sets in $\mathcal{S}$. If $\mu, \mu^{\prime}$ are measures defined on $\mathcal{S}$, we write $\mu \bar{R} \mu^{\prime}$ if they coincide over $\Sigma_{R}(S)$. Lastly, let $\mathcal{D}$ be a subset of $\mathcal{P}(S)$, the powerset of $S$. The relation $\mathcal{R}(\mathcal{D})$ is given by:
\[

$$
\begin{equation*}
\langle s, t\rangle \in \mathcal{R}(\mathcal{D}) \quad \Longleftrightarrow \quad \forall Q \in \mathcal{D}: s \in Q \Leftrightarrow t \in Q \tag{1}
\end{equation*}
$$

\]

Then the following observation, which is proved in [30, Lemma 3.1.6], is sometimes helpful

Lemma 2.3. $\mathcal{R}(\mathcal{D})=\mathcal{R}(\sigma(\mathcal{D}))$.
If $R$ is an equivalence relation, then an $R$-closed set is the union of equivalence classes; we call an $R$-closed set in this case $R$-invariant. As usual,

$$
\operatorname{ker}(f):=\left\{\left\langle s, s^{\prime}\right\rangle \mid f(s)=f\left(s^{\prime}\right)\right\}
$$

is the kernel of $f$.
We know that in general the image of a Borel set is not Borel; for surjective maps and invariant Borel sets, however, we can establish measurability using Lusin's classic Separation Theorem.

Lemma 2.4. Let $X, Y$ be Polish, $f: X \rightarrow Y$ Borel measurable and onto, and assume $A \in \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))$. Then $f[A]$ is a Borel set in $Y$.

Proof. See 13, Corollary 2.6].

These are two easy consequences.
Corollary 2.5. Let $X, Y$ be Polish, $f: X \rightarrow Y$ measurable and onto, then $\Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))=$ $\left\{f^{-1}[B] \mid B \in \mathcal{B}(Y)\right\}$.

Proof. Since $f$ is Borel measurable, and since $f^{-1}[B]$ is an ker $(f)$-invariant Borel set of $X$, we obtain $\Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X)) \supseteq\left\{f^{-1}[B] \mid B \in \mathcal{B}(Y)\right\}$. On the other hand, if $A \subseteq X$ is $f$-invariant, we have $f^{-1}[f[A]]=A$. Since $f[A]$ is a Borel set by Lemma [2.4, the other inclusion follows.

Given a map $f: X \rightarrow Y$, we will sometimes need to extend this to a map $f \times i d_{[0,1]}$, which sends $\langle x, q\rangle$ to $\langle f(x), q\rangle$, and we will have to know something about the kernel of this map, for which obviously

$$
\langle x, q\rangle \operatorname{ker}\left(f \times i d_{[0,1]}\right)\left\langle x^{\prime}, q^{\prime}\right\rangle \text { iff } f(x)=f\left(x^{\prime}\right) \text { and } q=q^{\prime}
$$

holds. The $f \times i d_{[0,1]}$-invariant sets are described here.
Lemma 2.6. Let $X$ and $Y$ be Polish and $f: X \rightarrow Y$ be measurable and onto. Then

$$
\Sigma_{\operatorname{ker}\left(f \times i d_{[0,1]}\right)}(\mathcal{B}(X \otimes[0,1]))=\Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X)) \otimes \mathcal{B}([0,1]) .
$$

Proof. See [13, Lemma 2.10].

We will now explore some structural properties which are induced by surjective and measurable maps. It is somewhat surprising that this induces an isomorphism on the set of all subprobabilities. Looking at the proof, it is even more surprising that this is a consequence of Lusin's Theorem. The next lemma is a step towards the isomorphism.
Lemma 2.7. Let $X$ and $Y$ be Polish, $f: X \rightarrow Y$ onto and measurable. Then the $\sigma$-algebras $\Sigma_{\mathrm{ker}(f)}(\mathcal{B}(X))$ and $\mathcal{B}(Y)$ are isomorphic as Boolean $\sigma$-algebras.

Proof. By Lemma [2.4, $f[D] \in \mathcal{B}(Y)$, whenever $D \in \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))$. Define

$$
\psi: \begin{cases}\mathcal{B}(Y) & \rightarrow \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X)) \\ D & \mapsto f^{-1}[D]\end{cases}
$$

Then $\psi(D) \in \Sigma_{\mathrm{ker}(f)}(\mathcal{B}(X))$ on account of $f$ being measurable, and $\psi$ is injective because $f$ is onto. Now $C=f^{-1}[f[C]]$ for $C \in \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))$, and $f[C] \in \mathcal{B}(Y)$, hence $\psi$ is onto as well. We know that $f\left[f^{-1}[D]\right]=D$, since $f$ is onto, hence $f: \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X)) \rightarrow \mathcal{B}(Y)$ and $\psi: \mathcal{B}(Y) \rightarrow \Sigma_{\text {ker }(f)}(\mathcal{B}(X))$ are inverse to each other.

This will have some interesting consequences for the measurable structure of the set of all subprobability measures. They are introduced next.

We write $\mathbb{S}(S, \mathcal{S})$ for the set of all subprobability measures on the measurable space $(S, \mathcal{S})$. This space is made a measurable space upon taking as a $\sigma$-algebra

$$
\begin{equation*}
w(\mathcal{S}):=\sigma\left(\left\{\boldsymbol{\beta}_{(S, \mathcal{S})}(A, \bowtie q) \mid A \in \mathcal{S}, q \in[0,1]\right\}\right) \tag{2}
\end{equation*}
$$

Here

$$
\boldsymbol{\beta}_{(S, \mathcal{S})}(A, \bowtie q):=\boldsymbol{\beta}_{\mathcal{S}}(A, \bowtie q):=\{\mu \in \mathbb{S}(S, \mathcal{S}) \mid \mu(A) \bowtie q\}
$$

is the set of all subprobabilities on $(S, \mathcal{S})$ which evaluate on the measurable set $A$ as $\bowtie q$, where $\bowtie$ is one of the relations $\leq,<, \geq,>$. This $\sigma$-algebra is sometimes called the weak-* ${ }^{*} \sigma$ algebra.

A morphism $f:(S, \mathcal{S}) \rightarrow(T, \mathcal{T})$ in the category of measurable spaces induces a map $\mathbb{S} f$ : $\mathbb{S}(S, \mathcal{S}) \rightarrow \mathbb{S}(T, \mathcal{T})$ upon setting

$$
(\mathbb{S} f)(\nu)(B):=\nu\left(f^{-1}[B]\right)
$$

for $B \in \mathcal{B}(T, \mathcal{T})$; as usual, $\mathbb{S} f$ is sometimes written as $\mathbb{S}(f)$. Because

$$
(\mathbb{S} f)^{-1}\left[\boldsymbol{\beta}_{\mathcal{T}}(B, \bowtie q)\right]=\boldsymbol{\beta}_{\mathcal{S}}\left(f^{-1}[B], \bowtie q\right)
$$

this map is $w(\mathcal{S})-w(\mathcal{T})$-measurable as well. Thus $\mathbb{S}$ is an endofunctor on the category of measurable spaces with measurable maps as morphisms; in fact, it is the functorial part of a monad which is sometimes called the Giry monad [16].

A $\mathcal{S}$-w( $\mathcal{T})$-measurable map $K: S \rightarrow \mathbb{S}(T, \mathcal{T})$ is called a sub Markov kernel, or sometimes a stochastic relation and denoted by $K:(S, \mathcal{S}) \rightsquigarrow(T, \mathcal{T})$. From the definition it is apparent that a map $K: S \rightarrow \mathbb{S}(T, \mathcal{T})$ is a stochastic relation iff these conditions are satisfied:
(1) the map $s \mapsto K(s)(D)$ is measurable for each $D \in \mathcal{T}$,
(2) the map $D \mapsto K(s)(D)$ is a subprobability measure on $\mathcal{T}$ for each $s \in S$.

Returning to invariant sets, we note

Lemma 2.8. Let $X$ and $Y$ be Polish, $f: X \rightarrow Y$ onto and measurable. Then there exists for $\nu \in \mathbb{S}(Y)$ a measure $\mu \in \mathbb{S}\left(X, \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))\right)$ such that $\nu=\mathbb{S}(f)(\mu)$.

Proof. Put $\mu(A):=\nu(f[A])$ for $A \in \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))$, then $\mu \in \mathbb{S}\left(S, \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))\right)$ by Lemma 2.7 , and plainly $\mathbb{S}(f)(\mu)=\nu$.

It follows from Corollary 2.5 that a surjective map $f: X \rightarrow Y$ between Polish spaces is final, and we obtain from [13, Corollary 2.7] that $\mathbb{S}(f): \mathbb{S}\left(X, \Sigma_{\operatorname{ker}(f)}(\mathcal{B}(X))\right) \rightarrow \mathbb{S}(Y)$ is a Borel isomorphism, when both spaces carry the weak $\sigma$-algebra.

It may be interesting to compare the proof just given with the one provided for the same fact in [10, Proposition 1.101]. That proof is based on an observation on the universally measurable right inverse of a measurable map [1, Theorem 3.4.3], which in turn is based on a selection argument. The proof presented here makes heavy use of the finality of surjective measurable maps which is based essentially on Lusin's Separation Theorem.

Upper Closed Sets. Call a subset $V$ of the powerset of some set upper closed iff $A \in V$ and $A \subseteq B$ implies $B \in V$. Denote by

$$
\mathbb{V}(S):=\{V \in w(\mathcal{S}) \mid V \text { is upper closed }\}
$$

all upper closed subsets of the weakly measurable sets of $\mathbb{S}(S, \mathcal{S})$. If $f: S \rightarrow T$ is $\mathcal{S}$ - $\mathcal{T}$ measurable, define

$$
\mathbb{V}(f)(V):=\left\{W \in w(\mathcal{T}) \mid(\mathbb{S} f)^{-1}[W] \in V\right\}
$$

Then $\mathbb{V}(f): \mathbb{V}(S) \rightarrow \mathbb{V}(T)$.

Some Conventions. From now on, we will not write down explicitly the $\sigma$-algebra $\mathcal{S}$ of a measurable space $(S, \mathcal{S})$, unless there is good reason to do so; if we need the $\sigma$-algebra of the measurable space $S$, we refer to it as $\mathscr{S}(S)$. Furthermore the space $\mathbb{S}(S)$ of all subprobabilities will be understood to carry the weak-*- $\sigma$-algebra $w(\mathscr{S}(S))$ always. When we consider $w(S)$ as a measurable space, its $\sigma$-algebra will always be the hit $\sigma$-algebra. And whenever we consider a measurable map $f: S \rightarrow T$ where $T$ has a hit $\sigma$-algebra, we will say that $f$ is hit-measurable. In an unambiguous context, e.g., whenever $S$ is a topological space with $\mathscr{S}(S)$ the Borel sets $\mathcal{B}(S)$ of $S$, we will write $w(S)$ rather than $w(\mathscr{S}(S))$. We will sometimes write $\Sigma_{\rho}$ for $\Sigma_{\rho}(\mathcal{S})$, and $\Sigma_{\mathrm{ker}(f)}$ will be abbreviated as $\Sigma_{f}$.
2.2. Some Indispensable Tools. We post here for the reader's convenience some measure theoretic tools which will be used all over; the reader may wish to consult [14] for more information and a tutorial on measures. Fix a set $S$.

Dynkin's $\pi$ - $\lambda$-Theorem. This technical tool is most useful when it comes to determine the $\sigma$-algebra generated by a family of sets [21, Theorem 10.1].

Theorem 2.9. Let $\mathcal{A}$ be a family of subsets of $S$ that is closed under finite intersections. Then $\sigma(\mathcal{A})$ is the smallest family of subsets containing $\mathcal{A}$ which is closed under complementation and countable disjoint unions.

Choquet's Representation. The following condition on product measurability and an associated integral representation attributed to Choquet is used [3, Corollary 3.4.3]. Assume that $(S, \mathcal{S})$ is a measurable space.
Theorem 2.10. Let $f: S \rightarrow \mathbb{R}_{+}$be measurable and bounded, then

$$
C_{\bowtie}(f):=\left\{\langle s, r\rangle \in S \times \mathbb{R}_{+} \mid f(s) \bowtie r\right\} \in \mathcal{S} \otimes \mathcal{B}\left(\mathbb{R}_{+}\right) .
$$

If $\mu$ is a $\sigma$-finite measure on $\mathcal{S}$, then

$$
\begin{equation*}
\int_{S} f(s) \mu(d x)=\int_{0}^{\infty} \mu(\{s \in S \mid f(x)>t\}) d t=(\mu \otimes \lambda)\left(C_{>}(f)\right) . \tag{3}
\end{equation*}
$$

with $\mu \otimes \lambda$ as the product of $\mu$ with Lebesgue measure $\lambda$.
For $S$ an interval in $\mathbb{R}$, the set $C_{>}(f)=\left\{\langle s, t\rangle \in S \times \mathbb{R}_{+} \mid 0 \leq t<f(s)\right\}$ may be visualized as the area between the $x$-axis and the graph of $f$. Hence formula (3) specializes to the Riemann integral, if $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is Riemann integrable, and $\mu$ is also Lebesgue measure.

Measurable Selections. Given a measurable space $S$ and a Polish space $X$, let $F(s) \subseteq X$ be a non-empty closed subset of $X$ for all $s \in S$. Call a sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ of measurable maps $f_{n}: S \rightarrow X$ a Castaing representation for $F$ iff the set $\left\{f_{n}(s) \mid n \in \mathbb{N}\right\}$ is dense in $F(s)$ for each $s \in S$. The Kuratowski and Ryll-Nardzewski Selection Theorem [3, Corollary 6.9.4] states a condition on the existence of a Castaing representation for $F$.
Theorem 2.11. Given a measurable space $S$ and a Polish space $X$, let $F(s) \subseteq X$ be a nonempty closed subset of $X$ for all $s \in S$ such that the set $\{s \in S \mid F(s) \cap G \neq \varnothing\} \in \mathscr{S}(S)$ for all $G \subseteq X$ open. Then $F$ has a Castaing representation.

## 3. The Main Characters: Nondeterministic Kernels And Effectivity Functions

In this section we introduce the notion of stochastic nondeterminism we are interested in, suitably defined so that measurable concerns arising in a continuous setting are taken care of. A model supporting this kind of behavior is that of nondeterministic labelled Markov process. Afterwards, we move a step forward in complexity and consider stochastic effectivity functions, which provide the building blocks for game models as developed in [12].
3.1. Nondeterministic Labelled Markov Processes. Nondeterministic labelled Markov processes (NLMP) were developed in [5, 33] as a nondeterministic variant of the labelled Markov processes studied by various authors [7, 6, 8], a development which generalized the Markov decision processes investigated by Larsen and Skou [22] for discrete state spaces to general measurable spaces.

Labelled Markov processes (LMP) are exactly the stochastic Kripke frames: that is, a state space $S$ with a labelled family of stochastic relations or Markov kernels $K_{a}: S \rightsquigarrow S$ working as probabilistic transitions for each action $a$. NLMPs arise naturally, e.g., by abstraction or underspecification of LMP. The nondeterminism makes its appearance in internal form, in such a way that there is an additional branching apart from that given by labels or actions. More precisely, NLMPs allow, for each state $s$ and each action $a$, a possibly infinite set $\kappa_{a}(s)$ of
probabilistic behaviors. Models combining probabilistic behavior with internal nondeterminism were studied previously in the literature; probabilistic automata (as proposed by Segala [28, 29, 31]) provide an example, although these specific models have only countable, hence discrete, state spaces.

As in the case of LMP, the main ingredient of the definition of their nondeterministic counterpart is the corresponding notion of nondeterministic kernels $\kappa_{a}$; this requires a suitable notion of measurability as well. The one we choose, hit-measurability, specializes to the usual notion of measurability for the deterministic case. A model for this approach comes from classical topology, where the transition from a topological space to its hyper space of closed subsets is supported by the Vietoris topology and its variants [21].

In this paper, the focus will be on nondeterministic kernels, which are defined as follows.
Definition 3.1. Let $S$ be a measurable space. A nondeterministic kernel on $S$ is a hitmeasurable map $\kappa: S \rightarrow w(S)$. We call $\kappa$ image-finite if the sets $\kappa(s)$ are finite for all $s \in S$, image-countable maps are defined analogously.

Consequently, $\kappa(s)$ is for each $s \in S$ a measurable subset of $\mathbb{S}(S)$ such that $\{s \in S \mid \kappa(s) \cap G \neq$ $\varnothing\} \in \mathscr{S}(S)$ for every measurable subset $G$ of $\mathbb{S}(S)$.

Having defined our kernels, we fix a set $L$ of labels and write down a formal definition of NLMP:

Definition 3.2. A nondeterministic labelled Markov process (NLMP) is a tuple $\mathbf{S}=\left(S,\left\{\kappa_{a}\right.\right.$ : $a \in L\})$ where the state space $S$ is a measurable space, and for each label $a \in L, \kappa_{a}$ is a nondeterministic kernel on $S$. We call $\mathbf{S}$ image-finite (image-countable) if all of its kernels $\kappa_{a}$ are.

In [5, 4] the problem of defining appropriate notions of bisimulation for NLMP was addressed. State bisimulation for NLMP (called traditional bisimulation in [4]) is a generalization of both the standard notion of bisimulation for nondeterministic processes, e.g. Kripke models, and of probabilistic bisimulation of Larsen and Skou [22], later studied in a more general measure theoretic context by Panangaden et al, see [24]. An alternative notion, event bisimulation puts its emphasis on families of measurable subsets of the state space that are "respected" by the kernels; in the deterministic case, event bisimulations are included in the investigation of the lifting of countably generated equivalence relations to the space of probabilities [9].
Event bisimulations for a nondeterministic kernel are defined as sub- $\sigma$-algebras on its state space. They yield a relation on the state space by the construction 1 , and, by extension, a relation on its subprobabilities.

Definition 3.3. Let $S$ be a measurable space.
(1) An event bisimulation on the nondeterministic kernel $\kappa: S \rightarrow w(S)$ is a sub- $\sigma$-algebra $\mathcal{A}$ of $\mathscr{S}(S)$ such that $\kappa:(S, \mathcal{A}) \rightarrow(w(S), \mathscr{H}(w(\mathcal{A})))$ is measurable. We also say that a relation $R$ is an event bisimulation if there is an event bisimulation $\mathcal{A}$ such that $R=\mathcal{R}(\mathcal{A})$.
(2) A relation $R$ is a state bisimulation on $\kappa: S \rightarrow w(S)$ if is symmetric and for all $s, t \in S$, s $R$ t implies that for all $\mu \in \kappa(s)$ there exists $\mu^{\prime} \in \kappa(t)$ such that $\mu \bar{R} \mu^{\prime}$.

We say that $s, t \in S$ are state (resp., event-) bisimilar, denoted by $s \sim_{\mathrm{s}} t\left(s \sim_{\mathrm{e}} t\right)$, if there is a state (event) bisimulation $R$ such that $s R t$.

State bisimulations for NLMP are exactly the relations on the state-space that satisfy the above definition simultaneously for each kernel $\kappa_{a}$ of the NLMP. This holds analogously for event bisimulations. Since $\mathcal{R}(\mathcal{A})$ is an equivalence, every event bisimulation yields an equivalence relation. This is not true for state bisimulations; in case $R$ is a state bisimulation that is actually an equivalence relation, we will say that $R$ is a state bisimulation equivalence.

While state bisimulations will have a straightforward generalization for effectivity functions, the rôle of event-based bisimulations will become apparent only when subsystems are treated in Section 5 .
3.1.1. The Category of Nondeterministic Kernels. In this section we will give a categorical presentation of nondeterministic kernels. It is then immediate how to extend this to obtain a category of NLMPs. Recall that the map $S \mapsto \mathscr{S}(S)$ can be regarded as a subfunctor of the contravariant power set functor, acting on arrows by taking inverse images. Formally, given a measurable $f: S \rightarrow T$, we define $\mathscr{S} f: \mathscr{S}(T) \rightarrow \mathscr{S}(S)$ as

$$
\mathscr{S} f(Q):=f^{-1}[Q] .
$$

Then the map $S \mapsto w(S)$ is the composition $\mathscr{S} \circ \mathbb{S}$ of the Giry functor $\mathbb{S}$ and $\mathscr{S}$. On arrows it behaves like this:

$$
w(f)(H)=(\mathbb{S} f)^{-1}[H]
$$

for $H$ a measurable set of measures.
Definition 3.4. The category $\mathscr{N} \mathscr{K}$ of nondeterministic kernels has as objects all hit-measurable maps $\kappa: S \rightarrow w(S)$ for some measurable space $S$. Given such $\kappa$ and $\kappa^{\prime}: T \rightarrow w(T)$, a $\mathscr{N} \mathscr{K}$ morphism is a measurable map $f: S \rightarrow T$ such that the following diagram commutes:

that is, for all $s \in S$,

$$
\begin{equation*}
\kappa(s)=(\mathbb{S} f)^{-1}\left[\kappa^{\prime}(f(s))\right] . \tag{4}
\end{equation*}
$$

It is easy to see that this defines a category.
Then the category of NLMPs (for a fixed set of labels $L$ ) has as objects all NLMP, and given $\mathbf{S}=\left(S,\left\{\kappa_{a}: a \in L\right\}\right)$ and $\mathbf{S}^{\prime}=\left(S^{\prime},\left\{\kappa_{a}^{\prime}: a \in L\right\}\right)$, a morphism $f: \mathbf{S} \rightarrow \mathbf{S}^{\prime}$ is a measurable map $f: S \rightarrow S^{\prime}$ such that $f: \kappa_{a} \rightarrow \kappa_{a}^{\prime}$ is a $\mathscr{N} \mathscr{K}$-morphism for all $a$.

It should be noted that although a nondeterministic kernel on $S$ is a map from $S$ to $w(S)$, we can't say that it is a coalgebra: In spite of $w(S)$ being a measurable space (with the hit $\sigma$-algebra), $w$ is not an endofunctor on the category of measurable spaces.
The first satisfactory test for this notion of morphism is that so related nondeterministic kernels are state bisimilar (cf. zig-zag morphisms for LMP).

Definition 3.5. Let $\kappa: S \rightarrow w(S)$ and $\kappa^{\prime}: T \rightarrow w(T)$ be nondeterministic kernels with $S$ and $T$ disjoint. The direct sum $\kappa \oplus \kappa^{\prime}$ is the nondeterministic kernel defined on the direct sum space $S \oplus T$ by the following stipulation

$$
\left(\kappa \oplus \kappa^{\prime}\right)(x):= \begin{cases}\kappa(x) & x \in S \\ \kappa^{\prime}(x) & x \in T\end{cases}
$$

The proof of the following proposition is included in the more general Theorem 4.5 and thus we omit it; its statement formalizes the relation between morphisms and bisimulations.
Proposition 3.6. (Notation as in Definition 3.4.) Let $G:=\{\langle s, f(s)\rangle \mid s \in S\}$ be the graph of $f$ with converse $G^{\smile}$ Then $R:=G \cup G \smile$ is a state bisimulation on $S \oplus T$.

Proof. It's easy to see that $R$-closed subsets $A$ of the disjoint union $S \oplus T$ are characterized by

$$
\begin{equation*}
A=f^{-1}[A] \uplus f[A] . \tag{5}
\end{equation*}
$$

To check that $R$ is a state bisimulation, we should see that for any $s R t$, and for every $\mu \in \kappa(s)$ there exists a $\nu \in \kappa^{\prime}(t)$ such that $\mu R \nu$.
Assume $s \in S, s R t$ and $\mu \in \kappa(s)$. Then $t=f(s) \in T$ and we must find $\nu$ as above. We have a perfect candidate for that. By Equation (4), $\mathbb{S} f(\mu) \in \kappa^{\prime}(f(s))=\kappa^{\prime}(t)$. Let's check $\mu \bar{R} \mathbb{S} f(\mu)$. Take a $R$-closed set $A$. We should have

$$
\mu(A)=\mathbb{S} f(\mu)(A)=\mu\left(f^{-1}[A]\right)
$$

the last equality by definition of $\mathbb{S} f$. Now $\mu(A)=\mu\left(f^{-1}[A]\right)+\mu(f[A])$ by Equation (5). But since $\mu$ is supported on $S$, we have $\mu(f[A])=0$. Hence $\mu(A)=\mu\left(f^{-1}[A]\right)$ and we have our result. The other way is symmetric.
3.2. Stochastic Effectivity Functions. While nondeterministic kernels assign to each state $s$ in a measurable state space $S$ a measurable set $\kappa(s)$ of subprobabilities in a measurable way, effectivity functions use a different approach. In order to explain it, we assume that we see player Angel in state $s$, and we assume that Angel has a strategy to achieve a set $A$ of probabilities over $S$ when playing game $\gamma$. We deal with sets of probabilities as the possible distribution of the new state after $\gamma$ because we want to model the game semantics probabilistically. Hence the new distribution may be an element of $A$. But if $B$ is a set of probabilities which contains $A$, Angel has a strategy to achieve $B$ as well, so the portfolio, which we define as the family of all sets for which Angel has a strategy in state $s$ playing $\gamma$, is upward closed. We want the sets of measures to be measurable themselves, hence the portfolio is an upper closed subset of $w(\mathscr{S}(S))$, so that we obtain a map $S \rightarrow \mathbb{V}(S)$. But this is not yet sufficient, a further requirement has to be added. Before stating it, we mention that we will have good reasons to separate the domain from the space underlying the range for an effectivity function. Thus we assume that there is a second measurable space $T$ for which the effectivity function determines possible sets of distributions as quantitative assessments. It helps to visualize $S$ as the space of inputs, and $T$ as the space of outputs, respectively.
Let $H \in w(\mathscr{S}(T)) \otimes \mathcal{B}([0,1])$ be a measurable subset of $\mathbb{S}(T) \times[0,1]$ indicating such a quantitative assessment of subprobabilities. A typical example could be

$$
\left\{\langle\mu, q\rangle \mid \mu \in \boldsymbol{\beta}_{T}(A, \geq q), 0 \leq q \leq 1\right\}=\{\langle\mu, q\rangle \mid \mu(A) \geq q, 0 \leq q \leq 1\}
$$

for some $A \in \mathscr{S}(T)$, combining all subprobabilities and reals such that the probability for the given set $A$ of states do not lie below this value. Consider a map $P: S \rightarrow \mathbb{V}(T)$, fix some real $q$ and consider the horizontal section $H_{q}=\{\mu \mid\langle\mu, q\rangle \in H\}$ of $H$ at $q$, viz., the set of all measures evaluated through $q$. We ask for all states $s$ such that $H_{q}$ is effective for $s$, i.e., $\left\{s \in S \mid H_{q} \in P(s)\right\}$ to be a measurable subset of $S$. It turns out, however, that this is not yet enough, we also require the real components being captured through a measurable set as well - after all, the real component will be used to be averaged over later on, so it should behave decently. This idea is captured in the following definition.

Definition 3.7. Call a map $P: S \rightarrow \mathbb{V}(T)$ t-measurable iff

$$
\left.\left\{\langle s, q\rangle \mid H_{q} \in P(s)\right\} \in \mathscr{S}(S) \otimes \mathcal{B}([0,1])\right\}
$$

whenever $H \in w(\mathscr{S}(T)) \otimes \mathcal{B}([0,1])$.
Thus if $P$ is t-measurable then we know in particular that all pairs of states and numerical values indicating the effectivity of the evaluation of a measurable set $A \in \mathscr{S}(T)$, i.e., the set $\left\{\langle s, q\rangle \mid \boldsymbol{\beta}_{\mathscr{L}(T)}(A, \bowtie q) \in P(s)\right\}$ is always a measurable subset of $S \otimes[0,1]$. This is so because we know that

$$
\left\{\langle\mu, q\rangle \mid \mu \in \boldsymbol{\beta}_{\mathscr{S}(T)}(A, \bowtie q), 0 \leq q \leq 1\right\}=\{\langle\mu, q\rangle \in \mathbb{S}(\mathscr{S}(T)) \times[0,1] \mid \mu(A) \bowtie q\},
$$

and the latter set is a member of $w(\mathscr{S}(T)) \otimes \mathcal{B}([0,1])$ by Theorem 2.10.
For illustration, a description of t-measurability as Borel measurability in terms of a compact Hausdorff topology on the Borel sets of $\mathbb{S}(T)$ is given. We assume for this the target space $T$ to be countably generated. Let $X$ be a measurable space with a countably generated $\sigma$-algebra, say, $\mathscr{S} X=\sigma(\{G \mid G \in \mathcal{G}\})$ with $\mathcal{G}$ countable. We define on $\mathcal{P}(w(X))$ the Priestley topology $\tau$ which has as a sub basis

$$
\begin{aligned}
\mathfrak{B}:= & \left\{\left\|\boldsymbol{\beta}_{X}(A, \leq q)\right\| \mid A \in \mathcal{G}, q \in[0,1] \text { rational }\right\} \cup \\
& \left\{-\left\|\boldsymbol{\beta}_{X}(A, \leq q)\right\| \mid A \in \mathcal{G}, q \in[0,1] \text { rational }\right\}
\end{aligned}
$$

with

$$
\begin{aligned}
\|W\| & :=\{\mathcal{T} \subseteq \mathcal{P}(w(\mathbb{S}(X))) \mid W \in \mathcal{T}\} \\
-\|W\| & :=\{\mathcal{T} \subseteq \mathcal{P}(w(\mathbb{S}(X))) \mid W \notin \mathcal{T}\} .
\end{aligned}
$$

We obtain from Goldblatt's Theorem 17]
Theorem 3.8. $\mathcal{P}(w(\mathscr{S}(X)))$ is a compact Hausdorff space with the Priestley topology $\tau$ generated by $\mathfrak{B} . \mathbb{V}(X)$ is compact, hence Borel measurable.

Now let $P: S \rightarrow \mathbb{V}(T)$ be a map such that the $\sigma$-algebra on $T$ is countably generated, then we have

Lemma 3.9. If $P$ is $t$-measurable, then $P$ is $\mathscr{S}(S)-\mathcal{B}(\mathbb{V}(T))$-measurable.
Proof. 1. Define $\mathfrak{B}$ as a sub-basis for the Priestly topology on $\mathcal{P}(w(\mathbb{S}(T)))$ with countable generator $\mathcal{G}$ for $\mathscr{S}(T)$, as above. We have to show that $P^{-1}[H] \in \mathscr{S}(S)$, provided $H \in \mathcal{B}(\mathbb{V}(T))$. Because $\mathbb{V}(T)$ is compact, hence closed, and because the Borel sets of $\tau$ are countably generated (since the topology is), the Borel sets of $\mathbb{V}(T)$ are countably generated with $\mathfrak{B} \cap \mathbb{V}(S)$ as its generator. Thus it is sufficient to show that $P^{-1}[H] \in \mathscr{S}(S)$, provided $H \in \mathfrak{B}$. This
is enough since the set $\left\{H \in \mathcal{B}(\mathbb{V}(T)) \mid P^{-1}[H] \in \mathscr{S}(S)\right\}$ is a $\sigma$-algebra; if it contains $\mathfrak{B}$, it contains all its countable unions, so in particular all open sets.
2. Now take $A \in \mathcal{G}$ and $q \in[0,1] \cap \mathbb{Q}$, then

$$
\begin{aligned}
P^{-1}\left[\left\|\boldsymbol{\beta}_{T}(A, \geq q)\right\|\right] & =\left\{s \in S \mid P(s) \in\left\|\boldsymbol{\beta}_{T}(A, \geq q)\right\|\right\} \\
& =\left\{s \in S \mid \boldsymbol{\beta}_{T}(A, \geq q) \in P(s)\right\} \\
& =\left\{\langle s, r\rangle \mid \boldsymbol{\beta}_{T}(A, \geq r) \in P(s)\right\}_{q} \in \mathscr{S}(S) .
\end{aligned}
$$

By the same argument is can be shown that

$$
P^{-1}\left[-\left\|\boldsymbol{\beta}_{T}(A, \geq q)\right\|\right]=\left\{\langle s, r\rangle \mid \boldsymbol{\beta}_{T}(A, \geq r) \notin P(s)\right\}_{q} \in \mathscr{S}(S) .
$$

Returning to the general discussion, we introduce stochastic effectivity functions now.
Definition 3.10. Given measurable spaces $S$ and $T, a$ stochastic effectivity function $P$ : $S \rightarrow T$ from $S$ to $T$ is a $t$-measurable map $P: S \rightarrow \mathbb{V}(T)$.

These are easy examples for stochastic effectivity functions, see [13, Section 3].
Example 3.11. Let $K_{n}: S \rightsquigarrow T$ be a stochastic relation for each $n \in \mathbb{N}$. Then

$$
\begin{aligned}
& s \mapsto\left\{A \in w(\mathscr{S} T) \mid K_{n}(s) \in A \text { for all } n \in \mathbb{N}\right\} \\
& s \mapsto\left\{A \in w(\mathscr{S} T) \mid K_{n}(s) \in A \text { for some } n \in \mathbb{N}\right\}
\end{aligned}
$$

define stochastic effectivity functions $S \rightarrow T$. In particular,

$$
s \mapsto\{A \in w(\mathscr{S} T) \mid K(s) \in A\}
$$

defines a stochastic effectivity function for a stochastic relation $K: S \rightsquigarrow S$. Conversely, if $P(s)$ is always a principal ultrafilter in $w(\mathscr{S} T)$ for each $s \in S$ and effectivity function $P: S \rightarrow T$, then $P$ is derived from a stochastic relation. The interplay between stochastic relations and effectivity functions generated by them is investigated in greater detail in [12]
Let $S$ be finite, say, $S=\{1, \ldots, n\}$, and assume we have a transition system $\rightarrow_{S}$ on $S$, hence a relation $\rightarrow_{S} \subseteq S \times S$. Put $\operatorname{succ}(s):=\left\{s^{\prime} \in S \mid s \rightarrow_{S} s^{\prime}\right\}$ as the set of a successor states for state $s$. Define for $s \in S$ the set of weighted successors

$$
\ell(s):=\left\{\sum_{s^{\prime} \in \operatorname{succ}(s)} \alpha_{s^{\prime}} \cdot e_{s^{\prime}} \mid \mathbb{Q} \ni \alpha_{s^{\prime}} \geq 0 \text { for } s^{\prime} \in \operatorname{succ}(s), \sum_{s^{\prime} \in \operatorname{succ}(s)} \alpha_{s^{\prime}} \leq 1\right\} .
$$

The upward closed set

$$
P(s):=\{A \in \mathcal{B}(\mathbb{S}(\{1, \ldots, n\})) \mid \ell(s) \subseteq A\}
$$

yields a stochastic effectivity function $S \rightarrow S$. -
The last example will be scrutinized below, the first ones indicate that there is some easy relationship between countable sets of effectivity functions. This is supported by the following observation, the proof of which is straightforward.

Lemma 3.12. Let $S$ and $T$ be measurable spaces, and $\mathfrak{P}=\left\{P_{\gamma} \mid \gamma \in \Gamma\right\}$ is a countable family of t-measurable maps from $S$ to $\mathbb{V}(T)$. Then the maps $\bigcup \mathfrak{P}$ and $\bigcap \mathfrak{P}$ defined as

$$
\begin{aligned}
& (\bigcup \mathfrak{P})(s):=\bigcup\left\{P_{\gamma}(s) \mid \gamma \in \Gamma\right\}, \\
& (\bigcap \mathfrak{P})(s):=\bigcap\left\{P_{\gamma}(s) \mid \gamma \in \Gamma\right\}
\end{aligned}
$$

for each $s \in S$, are $t$-measurable.
Morphisms for effectivity functions are defined as pairs of measurable maps which preserve the structure. To be specific

Definition 3.13. Let $P: S \rightarrow T$ and $Q: M \rightarrow N$ be stochastic effectivity functions, then $a$ pair $(f, g)$ of measurable maps $f: S \rightarrow M$ and $g: T \rightarrow N$ is called a morphism $(f, g): P \rightarrow Q$ iff this diagram commutes


By expanding, we obtain

$$
G \in Q(f(s)) \Longleftrightarrow(\mathbb{S} g)^{-1}[G] \in P(s)
$$

for all $s \in S$ and all $G \in w(\mathscr{S}(N))$. A pair $(\alpha, \beta)$ of equivalence relations $\alpha$ on $S$ and $\beta$ on $T$ is called a congruence for $P: S \rightarrow T$ iff we can find a stochastic effectivity function $P_{\alpha, \beta}: S / \alpha \rightarrow T / \beta$ such that the pair $\left(\eta_{\alpha}, \eta_{\beta}\right)$ of factor maps is a morphism $P \rightarrow P_{\alpha, \beta}$. Thus we obtain

$$
G \in P_{\alpha, \beta}\left([s]_{\alpha}\right) \Longleftrightarrow\left(\mathbb{S} \eta_{\beta}\right)^{-1}[G] \in P(s),
$$

whenever $G \in w(\mathscr{S}(T / \beta))$. Thus if $\alpha$ cannot distinguish the elements $s, s^{\prime} \in S$, $\beta$ cannot distinguish the elements in $P(s)$ from those in $P\left(s^{\prime}\right)$. It will be necessary to distinguish the domain from the space underlying the range for an effectivity function, as we will see when discussing subsystems in Section 5 .

If $S=T$ and $M=N$, however, we talk about a morphism $f: P \rightarrow Q$ resp. about a congruence $\alpha$ on $P$. Then we have, e.g., for the congruence $\alpha$ the equivalence

$$
G \in P_{\alpha}\left([s]_{\alpha}\right) \Longleftrightarrow\left(\mathbb{S} \eta_{\alpha}\right)^{-1}[G] \in P(s) .
$$

Of course, one could still work in this situation with a morphism $(f, g): P \rightarrow Q$. But this would not make too much sense, since one wants to study in this case the effect the map $f: S \rightarrow M$ has on the effectivity function, so one wants to know how $P$ operates on $\mathbb{V}(f)$; a similar argument holds for congruences. The situation changes, however, when we have different measurable structures on the same carrier set.
Stochastic effectivity functions with their morphisms constitute a category $\mathscr{E} \mathscr{F}$. Morphisms and congruences for stochastic effectivity functions are studied in depth in [13].
Just to provide an illustration, we establish
Proposition 3.14. Let $X$ and $M$ be standard Borel spaces with stochastic effectivity functions $P$ on $X$ and $Q$ on $M$, and assume that $f: P \rightarrow Q$ is a surjective morphism. Then $\operatorname{ker}(f)$ is a congruence for $f$.

Proof. We show that $f \times i d_{[0,1]}: X_{f} \times[0,1] \rightarrow M \otimes[0,1]$ is final, then we can apply 13, Proposition 3.14]. But finality of this compound map follows from finality of $f$ together with the observation that $M \otimes[0,1]$ is standard Borel:

$$
\begin{aligned}
\left(f \times i d_{[0,1]}\right)^{-1}[\mathcal{B}(M \otimes[0,1])] & =\left(f \times i d_{[0,1]}\right)^{-1}[\mathcal{B}(M) \otimes \mathcal{B}([0,1])], & & \text { since } M \text { is standard Borel } \\
& =f^{-1}[\mathcal{B}(M)] \otimes \mathcal{B}([0,1]) & & \\
& =\Sigma_{f} \otimes \mathcal{B}([0,1]), & & \text { since } f \text { is final. }
\end{aligned}
$$

This implies the assertion.
This indicates that there is a close interplay between congruences and surjective morphisms, which is actually what one would expect when studying both.

Duality. As an illustration for the manipulation of effectivity functions, we introduce dual effectivity functions. When interpreting game logics through effectivity functions, one assumes that the game is determined, which means that either Angel or Demon has a winning strategy. This assumption entails that if Angel has no strategy for achieving a set $A$, Demon will have a strategy for achieving the complement $A^{c}$ of $A$, so Angel is not effective for $A$ iff Demon is effective for $A^{c}$. This can be modeled through the dual $\partial P$ of the effectivity function $P$, so that $\partial P$ will encode the sets for which Demon is effective. Because it does not cost much more, we define the dual for effectivity functions $S \rightarrow T$ rather than for $S \rightarrow S$.
Definition 3.15. Let $P: S \rightarrow T$ be given. The dual $\partial P$ of $P$ is defined by

$$
\partial P(s):=\left\{D \in w(T) \mid D^{c} \notin P(s)\right\} .
$$

It is straightforward to see that $\partial$ is an involutive automorphism of the category of $\mathscr{E} \mathscr{F}$, when defined as the identity on $\mathscr{E} \mathscr{F}$-morphisms. Just for the record:
Proposition 3.16. $\partial$ is an endofunctor of the category $\mathscr{E} \mathscr{F}$ such that $\partial^{2}=\mathrm{Id}_{\mathscr{E} F}$.
The dual of $P$ will be helpful when interpreting the box operator in the two level logic discussed in Section 4.3,

## 4. Bisimulations

State bisimulations on effectivity functions are defined as binary relations on the base space. We follow the lines of the notion of bisimulation for games as spelled in [27], but extending it to continuous spaces. For motivation, we assume first that we are in a discrete setting and consider an effectivity function $Q$ from $S$ to the set $\{V \subseteq \mathcal{P}(S) \mid V$ is upper closed $\}$. This gives a scenario very similar to the one considered in [27]. We call a symmetric relation $R \subseteq S \times S$ a state bisimulation for $Q$ iff we have for each $\langle s, t\rangle \in R$ the following: Given a set $A \in Q(s)$, we can find a set $B \in Q(t)$ such that for each element of $B$ there exists a related element of $A$, so that we can find for each $t^{\prime} \in B$ an element $s^{\prime} \in A$ with $\left\langle s^{\prime}, t^{\prime}\right\rangle \in R$. This is the generalization of Milner's definition to upward closed sets. We are working with states on the one hand, and with distributions over states on the other hand, so we have to adapt the definition to our scenario. The straightforward way out is to define the relation for distributions, giving us the relation $\bar{R}$ on $\mathbb{S}(S)$. This observation suggests the following definition of state bisimulation.

Definition 4.1. Let $P: S \rightarrow S$ be an effectivity function. A relation $R \subseteq S \times S$ is a state bisimulation on $P$ if it is symmetric and for all $s, t \in S$ such that $s R t$, we can find for each $A \in P(s)$ a set $B \in P(t)$ with this property: for each $\nu \in B$ there exists $\mu \in A$ such that $\mu \bar{R} \nu$.

In the following subsections we will investigate how this new notion interacts with morphisms and provide a logical characterization for effectivity functions satisfying certain finiteness assumptions.
4.1. Effectivity Function Morphisms Induce State Bisimulations. In this section, we introduce strong morphisms and show that the graph of a strong morphism induces a bisimulation on the direct sum of two effectivity functions. This result is analogous to Proposition 3.6, and the argument is very similar, so that it can be considered as a generalization. The generalization will be helpful in Section 4.2, where the relation between effectivity functions and nondeterministic kernels will be studied in detail.

Definition 4.2. Let $P: S \rightarrow S$ and $Q: T \rightarrow T$ be effectivity functions with disjoint state spaces $S$ and $T$. The direct sum $P \oplus Q$ is the effectivity function defined on the direct sum space $S \oplus T$ by the following stipulation

$$
(P \oplus Q)(x):= \begin{cases}P(x) & x \in S \\ Q(x) & x \in T\end{cases}
$$

In the case of effectivity functions, a strict notion of morphism seems to be necessary to obtain the results pertaining bisimulations. This is inspired by the transition from homomorphisms to strong homomorphisms for Kripke models in modal logics by saying "that relational links are preserved from the source model to the target and back again" [2, p. 58]. We do not work with relational links but rather with upward closed subsets, so this strengthening happens like this:

Definition 4.3. Let $P: S \rightarrow S$ and $Q: T \rightarrow T$ be effectivity functions. A measurable map $f: S \rightarrow T$ is a strong morphism from $P$ to $Q$ if it is surjective and the following holds for all $s \in S$ and all $A \in w(S):$

$$
\begin{equation*}
A \in P(s) \Longleftrightarrow(\mathbb{S} f)^{-1}[B] \subseteq A \text { for some } B \in Q(f(s)) \tag{6}
\end{equation*}
$$

Next we will prove two features of strong morphisms in the setting of standard Borel spaces. The first one shows that strong morphisms are indeed effectivity function morphisms, while the second expresses a form of "backwards surjectivity" of strong morphisms.

Lemma 4.4. Let $P: S \rightarrow S$ and $Q: T \rightarrow T$ be effectivity functions on the standard Borel spaces $S$ and $T$, and assume that $f: S \rightarrow T$ is a strong morphism from $P$ to $Q$. Then
(1) $f$ is an effectivity function morphism from $P$ to $Q$;
(2) The following holds for all $s \in S$ and $A \in w(S)$ :

$$
\begin{equation*}
A \in P(s) \Longrightarrow B \subseteq(\mathbb{S} f)[A] \text { for some } B \in Q(f(s)) \tag{7}
\end{equation*}
$$

Proof. For the first item, we have to check that

$$
W \in Q(f(x)) \Longleftrightarrow(\mathbb{S} f)^{-1}[W] \in P(x)
$$

for all $s \in S$ and all $W \in w(\mathscr{S}(T))$. Assume first that $W \in Q(f(s))$. In the definition of strong morphism, take $A:=(\mathbb{S} f)^{-1}[W]$. We have, trivially, $(\mathbb{S} f)^{-1}[W] \subseteq A$; hence by (6) we obtain $A \in P(s)$, that is $(\mathbb{S} f)^{-1}[W] \in P(s)$. For the other direction, suppose $(\mathbb{S} f)^{-1}[W] \in P(s)$. By (6) we know there exists some $B \in Q(f(s))$ such that $(\mathbb{S} f)^{-1}[B] \subseteq(\mathbb{S} f)^{-1}[W]$. By the proof of Lemma 2.7, we know that this is equivalent to $B \subseteq W$ since $f$ is surjective. But then $W \in Q(f(s))$ by upper-closedness.
For the second item, assume $A \in P(s)$. Since $f$ is strong, there must exist some $B \in Q(f(s))$ such that $(\mathbb{S} f)^{-1}[B] \subseteq A$. But then we might apply $\mathbb{S} f$ on both sides of this inclusion and obtain

$$
\begin{equation*}
B \subseteq \mathbb{S} f\left[(\mathbb{S} f)^{-1}[B]\right] \subseteq \mathbb{S} f[A], \tag{8}
\end{equation*}
$$

where the first inclusion holds since $\mathbb{S} f$ is onto by Lemma [2.8.
We require in both parts of the proof that we work in standard Borel spaces, because we want to have that the image of a surjective Borel map between standard Borel spaces under the subprobability functor is onto again.
Theorem 4.5. Let $S$ and $T$ be standard Borel spaces, let $P: S \rightarrow \mathbb{V}(S)$ and $Q: T \rightarrow \mathbb{V}(T)$ be effectivity functions, and $f: P \rightarrow Q$ a strong morphism. The relation $R:=G \cup G \mathcal{}$, where $G=\{\langle s, f(s)\rangle: s \in S\}$ is the graph of $f$, is a state bisimulation on $P \oplus Q$.

Proof. It's easy to see that $R$-closed subsets $A$ of the disjoint union $S \oplus T$ are characterized by

$$
\begin{equation*}
A=f^{-1}[A] \uplus f[A] . \tag{9}
\end{equation*}
$$

Note that pairs in $R$ have exactly one component in $S$ and the other in $T$. To check that $R$ is a state bisimulation, we first consider the case $s R t$ with $s \in S$ and $t \in T$. We have to show

$$
\forall X \in(P \oplus Q)(s) \exists Y \in(P \oplus Q)(t): \forall \nu \in Y \exists \mu \in X .(\mu \bar{R} \nu) .
$$

In the case considered we have $t=f(s)$ and we might simply write

$$
\forall X \in P(s) \exists Y \in Q(f(s)): \forall \nu \in Y \exists \mu \in X .(\mu \bar{R} \nu) .
$$

Let $X \in P(s)$. Since $f$ is strong, there must exist some $Y \in Q(f(s))$ such that $Y \subseteq(\mathbb{S} f)[X]$ by (77). Now take any $\nu \in Y$. By the previous inclusion, there exists some $\mu \in X$ such that $\mathbb{S} f(\mu)=\nu$. Let's check $\mu \bar{R} \nu$. Take a $R$-closed set $A$. We should have

$$
\mu(A)=\nu(A)=\mathbb{S} f(\mu)(A)=\mu\left(f^{-1}[A]\right),
$$

where the last equality holds by definition of $\mathbb{S} f$. Now $\mu(A)=\mu\left(f^{-1}[A]\right)+\mu(f[A])$ by Equation (99). But since $\mu$ is supported on $S$, we have $\mu(f[A])=0$. Hence $\mu(A)=\mu\left(f^{-1}[A]\right)$ and we have proved this case.
Now assume $t R s$. As before, $t=f(s)$ and we have to prove

$$
\forall Y \in Q(f(s)) \exists X \in P(s): \forall \mu \in X \exists \nu \in Y .(\mu \bar{R} \nu)
$$

Let $Y \in Q(f(s))$. Since $f$ is a morphism, $(\mathbb{S} f)^{-1}[Y] \in P(s)$ and we choose this preimage as our $X$. Now take $\mu \in X$. By definition of $X, \nu:=\mathbb{S} f(\mu) \in Y$, and we can perform the same calculation that solved the other case.

It should be emphasized that in this proof only the consequences of the strength of $f$ appearing in Lemma 4.4 are used. Nevertheless, the full notion will play a rôle while representing nondeterministic kernels as effectivity functions in the next section.

It is helpful at this point of the development to compare morphisms with strong morphisms. Let $f: P \rightarrow Q$ be a morphism, and $F$ be a strong morphism between $P$ and $Q$. The morphism $f$ satisfies the condition

$$
G \in Q(f(s)) \text { iff } \mathbb{S}(f)^{-1}[G] \in P(s),
$$

or, equivalently,

$$
Q(f(s))=\left\{G \mid \mathbb{S}(f)^{-1}[G] \in P(s)\right\}
$$

so that $Q(f(s))$ is completely determined by $P(s)$ through $f$. The strong morphism $F$, however, has to satisfy the condition

$$
A \in P(s) \text { iff } \exists B \in Q(F(s)): \mathbb{S}(F)^{-1}[B] \subseteq A,
$$

which means that $P(s)$ is the smallest upper closed subset of $\mathscr{S}(w(S))$ which contains the set $\left\{\mathbb{S}(F)^{-1}[B] \mid B \in Q(F(s))\right\}$ of inverse images of elements in $Q(F(s))$ under $\mathbb{S}(F)$. This is the manner in which the back condition for Kripke models carries over.
4.2. Generation From Kernels. In this section we will study the problem of generating an effectivity function from a nondeterministic kernel. Essentially we ask for conditions under which a hit-measurable map gives rise to a t-measurable map.

Given a measurable set of measures on $S$, one can build a canonical upper-closed set in $\mathbb{V}(S)$ : the principal filter generated by it in the ordered set $(w(S), \subseteq)$.

Definition 4.6. Let $S$ be a measurable space. Given $W \in w(S)$, the filter generated by $W$ is the set

$$
\lfloor W\rfloor:=\{U \in w(S) \mid W \subseteq U\} .
$$

If $P: S \rightarrow \mathbb{V}(S)$ and there exists $\kappa: S \rightarrow w(S)$ such that $P(s)=\lfloor\kappa(s)\rfloor$ for all $s \in S$, we say that $P$ is filter-generated by $\kappa$. We will also say that $P$ is based on $\kappa$.

Finally, call a measurable map $\kappa: S \rightarrow w(S)$ a generating kernel if $\lfloor\kappa\rfloor$ is $t$-measurable.
We can state more formally the issues we are concerned with in this section:
(1) Showing that $\kappa$ is a nondeterministic kernel if an effectivity function $P$ is filtergenerated by $\kappa$.
(2) Investigating whether or not $\lfloor\kappa\rfloor$ is an effectivity function whenever $\kappa$ is a nondeterministic kernel.

We'll see that although we can easily answer the first question in the affirmative, the second one is a little more tricky to deal with.

First, we'll check that filter-generated effectivity functions are always based on a nondeterministic kernel.

Proposition 4.7. Let $\kappa: S \rightarrow w(S)$. If $\lfloor\kappa\rfloor$ is $t$-measurable, then $\kappa$ is hit-measurable.

Proof. Since the complements of hit-sets generate $\mathscr{H}(S)$, it suffices to show that the set $\kappa^{-1}\left[w(S) \backslash \mathrm{H}_{A}\right]$ is measurable for arbitrary $A \in w(S)$.

$$
\begin{aligned}
\kappa^{-1}\left[w(S) \backslash \mathrm{H}_{A}\right] & =\{s \mid \kappa(s) \cap A=\varnothing\} \\
& =\{s \mid \kappa(s) \subseteq(\mathbb{S}(S) \backslash A)\} \\
& =\{s \mid \mathbb{S}(S) \backslash A \in\lfloor\kappa(s)\rfloor\} \\
& =\left\{\langle s, q\rangle \mid((\mathbb{S}(S) \backslash A) \times[0,1])_{q} \in\lfloor\kappa(s)\rfloor\right\}_{0} .
\end{aligned}
$$

But this last set is measurable, because it is the cut of a measurable set (the latter, by $t$-measurability of $\lfloor\kappa\rfloor$ ).

We now state a criterion for a mapping $\kappa: S \rightarrow w(S)$ to generate an effectivity function. It actually provides a representation of t-measurability as a parametrized form of hit-measurability, relating these forms of stochastic nondeterminism. We will see in Proposition 4.13, however, that t-measurability is much stricter a condition than being generated by a nondeterministic kernel in this way.

Proposition 4.8. Let $\kappa: S \rightarrow w(S)$. $\lfloor\kappa\rfloor$ is $t$-measurable if and only if the map $\bar{\kappa}(s, q):=$ $\kappa(s) \times\{q\}\left(\right.$ i.e., $\left.\kappa \times \operatorname{id}_{[0,1]}\right)$ is $\mathscr{S}(S)-\mathscr{H}(w(S) \otimes \mathcal{B}([0,1]))$ measurable.

Proof. Suppose $H \in w(S) \otimes \mathcal{B}([0,1])$. Then

$$
\bar{\kappa}(s, q) \subseteq H \Longleftrightarrow \kappa(s) \times\{q\} \subseteq H \Longleftrightarrow \kappa(s) \subseteq H_{q} \Longleftrightarrow H_{q} \in\lfloor\kappa(s)\rfloor,
$$

hence $\{\langle s, q\rangle: \bar{\kappa}(s, q) \subseteq H\}=\left\{\langle s, q\rangle: H_{q} \in\lfloor\kappa(s)\rfloor\right\}$, and the equivalence follows from Lemma 2.2 .

When one restricts the state spaces to be standard Borel, there is a tight connection between the category of generating kernels and the effectivity functions generated as filters by them. The definition of $\lfloor\cdot\rfloor$ can be extended to encompass arrows by stipulating $\lfloor f\rfloor:=f$ for every $\mathscr{N} \mathscr{K}$-morphism, rendering it a functor with some special properties. We will state this more formally in the next theorem.

For its proof we need a lemma as a preparation. Here we go:
Lemma 4.9. Let $S$ and $T$ be standard Borel spaces, and $\kappa: S \rightarrow w(S)$ and $\kappa^{\prime}: T \rightarrow w(T)$ generating kernels. A surjective measurable map $f: S \rightarrow T$ is a strong morphism between $\lfloor\kappa\rfloor$ and $\left\lfloor\kappa^{\prime}\right\rfloor$ if and only if $f$ is a $\mathscr{N} \mathscr{K}$-morphism between $\kappa$ and $\kappa^{\prime}$.

Proof. We will show that the two notions involved are the same by successively considering equivalent formulations. We start by writing down the definition of strong morphism between $\lfloor\kappa\rfloor$ and $\left\lfloor\kappa^{\prime}\right\rfloor$ : for all $A$ and $s$,

$$
A \in\lfloor\kappa(s)\rfloor \text { iff } \exists B \in\left\lfloor\kappa^{\prime}(f(s))\right\rfloor:(\mathbb{S} f)^{-1}[B] \subseteq A
$$

By applying the definition of $\lfloor\cdot\rfloor$, this is equivalent to

$$
\kappa(s) \subseteq A \text { iff } \exists B: \underline{\kappa^{\prime}(f(s)) \subseteq B} \&(\mathbb{S} f)^{-1}[B] \subseteq A
$$

Since $\mathbb{S} f$ is onto, we know by the proof of Lemma 2.8 that the underlined condition is equivalent to $(\mathbb{S} f)^{-1}\left[\kappa^{\prime}(f(s))\right] \subseteq(\mathbb{S} f)^{-1}[B]$. Hence the last displayed formula is equivalent to the next one.

$$
\kappa(s) \subseteq A \text { iff } \exists B:(\mathbb{S} f)^{-1}\left[\kappa^{\prime}(f(s))\right] \subseteq(\mathbb{S} f)^{-1}[B] \&(\mathbb{S} f)^{-1}[B] \subseteq A
$$

This in turn is equivalent to

$$
\kappa(s) \subseteq A \text { iff }(\mathbb{S} f)^{-1}\left[\kappa^{\prime}(f(s))\right] \subseteq A,
$$

because of transitivity of the subset relation (for $\Rightarrow$ ) and by taking $B:=\kappa^{\prime}(f(s)$ ) (for $\Leftarrow$ ). Finally, this is the same as

$$
\kappa(s)=(\mathbb{S} f)^{-1}\left[\kappa^{\prime}(f(s))\right]
$$

since $A$ is universally quantified. This last equation is the definition of $\mathscr{N} \mathscr{K}$-morphism.
We are ready now to enter into a discussion of the functorial properties of $\lfloor\cdot\rfloor$. By Lemma 4.9, we conclude at once that $L \cdot\rfloor$ is well defined on arrows and is full, because there is a one-one correspondence between $\mathscr{N} \mathscr{K}$-arrows and strong morphisms. Finally, $\lfloor\cdot\rfloor$ trivially preserves composition of morphisms since it is the identity on arrows. This establishes

Theorem 4.10. $\lfloor\cdot\rfloor$ is a full functor between the category of generating kernels over standard Borel spaces and the one of filter-generated effectivity functions over standard Borel spaces with strong morphisms.

It can actually be proved that $\lfloor\cdot\rfloor$ is an isomorphism between the categories above.
In the next proposition we state that $\lfloor\cdot\rfloor$ preserves and reflects state bisimulations. In order to make a distinction, we will use the shorter name $\mathscr{N} \mathscr{K}$-bisimulations for the state bisimulations on nondeterministic kernels. The proof is fairly straightforward and requires only the definition of a bisimulation.

Proposition 4.11. Let $\kappa: S \rightarrow w(S)$ be a generating kernel and $R$ a binary relation on $S$. $R$ is a $\mathscr{N} \mathscr{K}$-bisimulation on $\kappa$ if and only if it is a bisimulation on $\lfloor\kappa\rfloor$.

A very simple observation in the game theoretic context is that when generating effectivity function $\lfloor\kappa\rfloor$ from $\kappa$, we end up with an effectivity function where there are essentially no choices for Angel: we may say that $\lfloor\cdot\rfloor$ demonizes the set of measures $\kappa(s)$ for each $s$, since Demon is effective for each and every element of $\kappa(s)$. An alternative representation would be given by ascribing each measure as effective for Angel; that is, each singleton $\{\mu\}$ with $\mu \in \kappa(s)$ should be effective for Angel. Then this alternative representation would be given by the angelization functor

$$
(\mathfrak{A} \kappa)(s):=\bigcup\{\lfloor\{\mu\}\rfloor \mid \mu \in \kappa(s)\} .
$$

This functor is defined as the identity on arrows. One may ask whether a nondeterministic kernel $\kappa$ is generating by $\lfloor\cdot\rfloor$ if and only if $\mathfrak{A} \kappa$ is t-measurable. The answer follows easily by using duals. We omit the proof of the following straightforward proposition.

Proposition 4.12. The functors $\lfloor\cdot\rfloor$ and $\mathfrak{A}$ satisfy $\partial \mathfrak{A}=\lfloor\cdot\rfloor$ and $\partial\lfloor\cdot\rfloor=\mathfrak{A}$.
A property expressed in terms of $\mathfrak{A}$ may be rephrased as one about $\lfloor\cdot\rfloor$, and vice versa. This also holds for state bisimulations, though since it is not yet apparent how to express them in terms of arrows, a proof is needed.

The following fairly surprising observation indicates that t-measurability is much stricter that hit-measurability: only image-countable kernels can be represented as effectivity functions.

Proposition 4.13. Let $S$ be a Polish space, $\kappa: S \rightarrow w(S)$, and assume there exists $s_{0} \in S$ such that $\kappa\left(s_{0}\right)$ is uncountable. Then $\lfloor\kappa\rfloor$ is not $t$-measurable.

Proof. Let $A:=\kappa\left(s_{0}\right)$. Since $S$ is Polish, $\mathbb{S}(S)$ is standard Borel. $A \subseteq \mathbb{S}(S)$ is an uncountable Borel set, and hence the trace $(A, w(S) \cap A)$ is an uncountable standard Borel space. Then we know there exists a measurable $H \subseteq A \times[0,1]$ such that $\pi_{2}[H]$ is an analytic non-Borel subset of $[0,1]$ (i.e., $\pi_{2}[H] \notin \mathcal{B}([0,1])$ ), see [30, Theorem 4.1.5].

If $\lfloor\kappa\rfloor$ were t-measurable, then $\left\{\langle s, q\rangle:(S \times[0,1] \backslash H)_{q} \in\lfloor\kappa(s)\rfloor\right\}$ should belong to $\mathcal{B}(S) \otimes$ $\mathcal{B}([0,1])$, since $H^{c}=S \times[0,1] \backslash H \in w(S) \otimes \mathcal{B}([0,1])$. Therefore, the set appearing in the following calculation should be measurable:

$$
\begin{aligned}
\left(\left\{s_{0}\right\} \times[0,1]\right) \backslash\left\{\langle s, q\rangle \mid\left(H^{\mathrm{c}}\right)_{q} \in\lfloor\kappa(s)\rfloor\right\} & =\left\{\left\langle s_{0}, q\right\rangle \mid\left(H^{\mathrm{c}}\right)_{q} \notin\left\lfloor\kappa\left(s_{0}\right)\right\rfloor\right\} \\
& =\left\{\left\langle s_{0}, q\right\rangle \mid \kappa\left(s_{0}\right) \nsubseteq\left(H^{\mathrm{c}}\right)_{q}\right\} \\
& =\left\{\left\langle s_{0}, q\right\rangle \mid A \nsubseteq\left(H_{q}\right)^{\mathrm{c}}\right\} \\
& =\left\{\left\langle s_{0}, q\right\rangle \mid A \cap H_{q} \neq \varnothing\right\} \\
& =\left\{\left\langle s_{0}, q\right\rangle \mid \exists \mu \in A:\langle\mu, q\rangle \in H\right\} \\
& \stackrel{(*)}{=}\left\{s_{0}\right\} \times \pi_{2}[H] .
\end{aligned}
$$

where the equality $\left({ }^{*}\right)$ holds since $\pi_{1}[H] \subseteq A$. Since the last set has a non-Borel cut, it does not belong to $\mathcal{B}(S) \otimes \mathcal{B}([0,1])=\mathcal{B}(S \times[0,1])$, so $\lfloor\kappa\rfloor$ is not t-measurable.

We will show now that every image-finite kernel is generating. For this, will need the following lemma, in which we check that every image-finite kernel is given by stochastic relations. This is actually an easy application of measurable selections.

Lemma 4.14. Let $S$ be a Polish space and $\kappa: S \rightarrow w(S)$ be a nondeterministic kernel such that for all $s \in S, \kappa(s)$ is finite. Then there exists a sequence $\left(K_{n}\right)_{n \in \mathbb{N}}$ of stochastic relations $K_{n}: S \rightsquigarrow S$ such that $\kappa(s)=\left\{K_{n}(s) \mid n \in \mathbb{N}\right\}$ for all $s$.

Proof. The sets $\kappa(s)$ are closed for all $s \in S$. Since $\kappa$ is hit-measurable, we infer that for each open subset $G$ of $\mathbb{S}(S)$ the set $\{s \in S \mid \kappa(s) \cap G \neq \varnothing\}$ is measurable, thus there exists a Castaing representation $\left(K_{n}\right)_{n \in \mathbb{N}}$ for $\kappa$ by the Selection Theorem 2.11. Clearly, each $K_{n}$ is a stochastic relation $S \rightsquigarrow S$.

Theorem 4.15. Let $S$ be a Polish. Every image-finite nondeterministic kernel $\kappa: S \rightarrow w(S)$ is generating.

Proof. By Lemma 4.14, there are stochastic relations $K_{n}: S \rightsquigarrow S(n \in \mathbb{N})$ such that

$$
\lfloor\kappa(s)\rfloor=\bigcap\left\{\left\lfloor\left\{K_{n}(s)\right\}\right\rfloor \mid n \in \mathbb{N}\right\}
$$

for all $s \in S$. Since each $\left\lfloor\left\{K_{n}\right\}\right\rfloor$ is t-measurable, by Lemma $3.12\lfloor\kappa\rfloor$ is t-measurable.
4.3. Two-Level Logic. We now introduce a modal logic, $2 \mathcal{L}$, that characterizes state bisimilarity for finitary effectivity functions. As the logic for NLMP appearing in [4], it is divided in two levels: one devoted to states and another one to measures. Two operators serve as a bridge between the levels.

In terms of a game theoretic scenario, we want to express in this way that in a particular state, Angel has a strategy to attain a set of measures. This is done by using a modality $\diamond$. On the other hand, we have quantitative operators of the form $[\varphi \bowtie q$ ], that allow us to single out measures, which assign a value $\bowtie q$ to the set of states defined by the state formula $\varphi$.

The $\diamond$-modality is patterned after a necessity operator in traditional modal logics (compare 18,32$]$ ), but instead of asserting that Angel can ensure a set of states, it does so with respect a set of measures. There is an intimate relation between $2 \mathcal{L}$ and the stochastic version of game logic developed in [12], but we won't focus on that matter in the present paper.

In the following, $\varphi$ will range over state formulas and $\psi$ will range over measure formulas. The logic is given by the following grammar,

$$
\begin{align*}
& \varphi::==\top\left|\varphi_{1} \wedge \varphi_{2}\right| \diamond \psi \mid \square \psi  \tag{10}\\
& \psi::==\psi_{1} \wedge \psi_{2}\left|\psi_{1} \vee \psi_{2}\right|[\varphi \bowtie q],
\end{align*}
$$

where $\bowtie$ is $<$ or $>$ and $0 \leq q<1$ is rational.
Fix a standard Borel space $S$ and let $P: S \rightarrow S$ be an effectivity function. The interpretation of the propositional operators over $S$ is straightforward, so we focus on the other connectives. We define inductively

$$
\begin{aligned}
& \llbracket \varphi \rrbracket:=\{s \in S \mid s \models \varphi\}, \\
& \llbracket \psi \rrbracket:=\{\mu \in \mathbb{S}(S) \mid \mu \models \psi\}
\end{aligned}
$$

for state formulas $\varphi$ and measure formulas $\psi$, then

$$
\begin{aligned}
s \models \diamond \psi & \Longleftrightarrow \llbracket \psi \rrbracket \in P(s) \\
s \models \square \psi & \Longleftrightarrow \llbracket \psi \rrbracket \in \partial P(s) \\
\mu \models[\varphi \bowtie q] & \Longleftrightarrow \mu(\llbracket \varphi \rrbracket) \bowtie q .
\end{aligned}
$$

As usual, $\square$ is defined dually from $\diamond$, using duality on $P$ here. We will call $2 \mathcal{L}$-formulas the ones obtained from the first line of productions (10), i.e., the state-formulas. It is plain that there are only countably many $2 \mathcal{L}$-formulas.
Our next step will be to prove that for every effectivity function $P$, the relation of logical equivalence is smooth, in the sense made precise by the following

Definition 4.16. An equivalence relation $\rho$ on the standard Borel space $S$ is called smooth (or countably generated) iff there exists a countable set $\left(C_{n}\right)_{n \in \mathbb{N}}$ of Borel sets such that

$$
s \rho t \quad \text { if and only if } \quad \forall n \in \mathbb{N}: s \in C_{n} \Longleftrightarrow t \in C_{n} \text {. }
$$

These relations have some desirable properties, among others it is known that the factor space becomes an analytic space, and, more important for us, that

$$
\begin{equation*}
\Sigma_{\rho}(\mathcal{B}(S))=\sigma\left(\left\{C_{n} \mid n \in \mathbb{N}\right\}\right) . \tag{11}
\end{equation*}
$$

This is established through the Blackwell-Mackey Theorem, see [30, Theorem 4.5.7], and [14, Proposition 2.108] for a discussion; note again that this relies on $S$ being a standard Borel space, because the Blackwell-Mackey Theorem makes use of Suslin's Theorem, which is not available in general measurable spaces. The characterization of invariant sets shows that a smooth relation permits a concise description of its invariant Borel sets.

This is a necessary step towards the proof of completeness for bisimilarity.
Lemma 4.17. For every $2 \mathcal{L}$-formula $\varphi, \llbracket \varphi \rrbracket$ is measurable.

Proof. We will see this by structural induction, proving that formulas of both productions are measurable in the respective spaces ( $S$ and $\mathbb{S}(S)$ ).
Again, the only nontrivial cases are the ones with new operators. Assume $\llbracket \psi \rrbracket$ is measurable, i.e., $\llbracket \psi \rrbracket \in w(S)$.

$$
\llbracket \Delta \psi \rrbracket=\{s \mid s \models \diamond \psi\}=\{s \mid \llbracket \psi \rrbracket \in P(s)\}=\left\{\langle s, q\rangle \mid(\llbracket \psi \rrbracket \times[0,1])_{q} \in P(s)\right\}_{0},
$$

but this last set is measurable since $P$ is t-measurable.
Now assume inductively that $\llbracket \varphi \rrbracket$ is measurable. We have

$$
\llbracket[\varphi \bowtie q] \rrbracket=\{\mu \mid \mu(\llbracket \varphi \rrbracket) \bowtie q\}=\boldsymbol{\beta}_{\mathscr{S} S}(\varphi, \bowtie q)
$$

again a measurable set (now in $\mathbb{S}(S)$ ).
Corollary 4.18. For every effectivity function $P: S \rightarrow S$, the relation of $2 \mathcal{L}$-equivalence,

$$
s \approx t \Longleftrightarrow \forall \varphi \in 2 \mathcal{L} .(s \models \varphi \Leftrightarrow t \models \varphi),
$$

is smooth.
Proof. Immediate by the previous lemma and the observation that there are only countably many $2 \mathcal{L}$-formulas.

In the next lemma we show that state bisimulations preserve $2 \mathcal{L}$-formulas.
Lemma 4.19. Let $P: S \rightarrow S$ be an effectivity function, and let $R$ be a state bisimulation for $P$. For every $2 \mathcal{L}$-formula $\varphi, \llbracket \varphi \rrbracket$ is $R$-closed.

Proof. 0. The proof is done by showing inductively that
(1) $\llbracket \varphi \rrbracket$ is $R$-closed,
(2) $\llbracket \psi \rrbracket$ is $\bar{R}$-closed.

Since the family of $R$-closed sets are preserved by arbitrary Boolean operations, we only need to consider the modality $\diamond$ and the $[\varphi \bowtie q$ ] construction.

1. Assume inductively that $\llbracket \psi \rrbracket$ is $\bar{R}$-closed. Let's show $\llbracket \diamond \psi \rrbracket$ is $R$-closed. Suppose $s \in \llbracket \Delta \psi \rrbracket$, i.e., $\llbracket \psi \rrbracket \in P(s)$, and $s R t$. Since $R$ is a bisimulation, there exists $Y \in P(t)$ such that for any given $\nu \in Y$ there exists $\mu \in \llbracket \psi \rrbracket$ with $\mu \bar{R} \nu$. Since $\llbracket \psi \rrbracket$ is $\bar{R}$-closed, we have $\nu \in \llbracket \psi \rrbracket$ for all $\nu \in Y$ and then $\llbracket \psi \rrbracket \in P(t)$. But this means $t \in \llbracket \diamond \psi \rrbracket$, and we have this case.
2. Now assume inductively that $\llbracket \varphi \rrbracket$ is $R$-closed; we'll see that $\llbracket[\varphi \bowtie q] \rrbracket$ is $\bar{R}$-closed. For this, assume $\mu \models[\varphi \bowtie q]$ and $\mu \bar{R} \nu$. But then $\mu(\llbracket \varphi \rrbracket)=\nu(\llbracket \varphi \rrbracket)$ and hence,

$$
\mu(\llbracket \varphi \rrbracket) \bowtie q \Longleftrightarrow \nu(\llbracket \varphi \rrbracket) \bowtie q .
$$

This implies $\nu \models[\varphi \bowtie q]$.
We obtain immediately through Theorem 4.5 and Lemma 4.19,
Corollary 4.20. Strong morphisms preserve $2 \mathcal{L}$-formulas.
We are now in a position to establish a partial converse of the previous lemma. $2 \mathcal{L}$ is actually complete for state bisimilarity on effectivity functions, which are finitary in the sense made precise below. This concept generalizes the notion of uniformly finitary effectivity functions, already used in the non-probabilistic context 27].

Definition 4.21. An effectivity function $P: S \rightarrow S$ is
(1) finitary if for each $s \in S$ there exists a finite family $E(s):=\left\{\kappa_{i}(s): i=1, \ldots, n_{s}\right\}$ with each $\kappa_{i}(s) \subseteq w(S)$ finite such that

$$
P(s)=\bigcup\left\{\left\lfloor\kappa_{i}(s)\right\rfloor: i=1, \ldots, n_{s}\right\} .
$$

(2) finitely supported if it is finitary with $n_{s}=1$ for all $s \in S$.

It is immediate from the definition that finitely supported effectivity functions are exactly the ones which are filter-generated by an image-finite kernel. Also, it may be interesting to note that both finitary and filter-generated are related to the notion of core-completeness appearing in [18].
The next theorem generalizes one of the main results of [5, 4], the logical characterization of bisimilarity for image-finite nondeterministic kernels, since every such kernel encodes an effectivity function by Theorem 4.15. In the following, we use $\llbracket 2 \mathcal{L} \rrbracket$ to denote the set of all extensions of $2 \mathcal{L}$-formulas, that means,

$$
\llbracket 2 \mathcal{L} \rrbracket=\{\llbracket \varphi \rrbracket \mid \varphi \in 2 \mathcal{L}\} .
$$

Theorem 4.22. Let $P: S \rightarrow S$ be a finitary effectivity function. Two states that satisfy exactly the same $2 \mathcal{L}$-formulas are bisimilar.

Proof. Since by Corollary 4.18 the equivalence relation $\approx$ is smooth, we obtain $\Sigma_{\approx}(\mathscr{S}(S))=$ $\sigma(\llbracket 2 \mathcal{L} \rrbracket)$ from (11). The result will follow if we show that $\approx$ is a state bisimulation.

We proceed by way of contradiction. That is, assume $s \approx t$ and there exists $A \in P(s)$ so that for any given $B \in P(t)$ there exists $\nu \in B$ such that $\mu \not \approx \nu$ holds for all $\mu \in A$. Since $P$ is finitary, there must exist a finite $A_{0}=\left\{\mu_{i}: i \in I\right\} \in P(s)$ such that $A_{0} \subseteq A$. Also, there are finitely many (finite) $B_{j}$ such that $P(t)=\bigcup_{j}\left\lfloor B_{j}\right\rfloor$. We enumerate the $\nu$ 's accordingly. Hence we can find $A_{0} \in P(s)$ such that for all $B_{j} \in P(t)$ the following holds

$$
\begin{equation*}
\exists \nu_{j} \in B_{j} \forall \mu_{i} \in A_{0} \cdot\left(\mu_{i} \not \approx \nu_{j}\right) . \tag{12}
\end{equation*}
$$

Now, $\mu_{i} \not \approx \nu_{j}$ if and only if there exists some $Q \in \Sigma_{\approx}(\mathscr{S}(S))$ such that $\mu_{i}(Q) \neq \nu_{j}(Q)$. By (11), we may choose $Q \in \sigma(\llbracket 2 \mathcal{L} \rrbracket)$ and hence $\mu_{i}$ and $\nu_{j}$ differ on $\sigma(\llbracket 2 \mathcal{L} \rrbracket)$. But since $\llbracket 2 \mathcal{L} \rrbracket$ is a
generator of $\sigma(\llbracket 2 \mathcal{L} \rrbracket)$ which is closed under finite intersections, the $\pi-\lambda$ Theorem 2.9 ensures that there are $2 \mathcal{L}$-formulas $\varphi_{i j}$ witnessing the fact that $\mu_{i}$ and $\nu_{j}$ are different:

$$
\nu_{j}\left(\llbracket \varphi_{i j} \rrbracket\right) \neq \mu_{i}\left(\llbracket \varphi_{i j} \rrbracket\right) .
$$

Without loss of generality we might state that we can find $A_{0} \in P(s)$ such that for all $B_{j} \in P(t)$ this is true:

$$
\exists \nu_{j} \in B_{j} \forall \mu_{i} \in A_{0} \cdot \nu_{j}\left(\llbracket \varphi_{i j} \rrbracket\right) \bowtie_{i j} q_{i j} \bowtie_{i j} \mu_{i}\left(\llbracket \varphi_{i j} \rrbracket\right)
$$

for some $q_{i, j}$. This can be expressed by a $2 \mathcal{L}$-formula, as follows:

$$
t \models \square \bigvee_{j} \bigwedge_{i}\left[\varphi_{i j} \bowtie_{i j} q_{i j}\right]
$$

But then $s$ does not satisfy this formula, and we reach a contradiction since we assumed $s \approx t$.

The proof shows a strong resemblance to the familiar proof of the Hennessy-Milner Theorem, see, e.g. [2, p. 69], by exploiting the assumption of being finitely supported, identifying for each of these finite cases a finite number of culprits and then, using the logic's finitary constructors, constructing one violating formula.

Since $2 \mathcal{L}$-equivalence is smooth, we obtain the following
Corollary 4.23. For every finitary effectivity function $P$, state bisimilarity on $P$ is smooth.
In the next section, we will make a first approach into the study of the notions of bisimilarity and behavioral equivalence from a coalgebraic perspective. The main tool would be that of subsystems, a refined version of event bisimulations suited for effectivity functions. The link to the material in the present section is given by the last Corollary, since smooth bisimulations are proved to induce subsystems.

## 5. Subsystems: A Coalgebraic Approach

Given a measurable space $X$, assume that $\mathcal{C}$ is a sub- $\sigma$-algebra of $\mathscr{S}(X)$; denote the measurable space $(X, \mathcal{C})$ by $X_{\mathcal{C}}$, so that $\mathscr{S}\left(X_{\mathcal{C}}\right)=\mathcal{C}$. Since $\mathscr{S}\left(X_{\mathcal{C}}\right) \subseteq \mathscr{S}(X)$, we see that the identity $i_{\mathcal{C}}: X \rightarrow X_{\mathcal{C}}$ is measurable. Now assume that $P: X \rightarrow X$ is an effectivity function. The restriction of $P$ to $\mathcal{C}$ focuses $P$ on the events described in $\mathcal{C}$, provided this is algebraically possible. This leads to the notion of a subsystem, formally:

Definition 5.1. Given a stochastic effectivity function $P$ on the measurable space $X$, a sub-$\sigma$-algebra $\mathcal{C} \subseteq \mathscr{S}(X)$ defines a subsystem of $P$ iff we can find a stochastic effectivity function $P_{\mathcal{C}}$ on $X_{\mathcal{C}}$ such that $i_{\mathcal{C}}$ defines a morphism $P \rightarrow P_{\mathcal{C}}$.

Assume that $\mathcal{C}$ defines a subsystem of $P$, then this diagram commutes


This means that we have

$$
P_{\mathcal{C}}(x)=\left\{D \in w(\mathscr{S} X) \mid\left(\mathbb{S} i_{\mathcal{C}}\right)^{-1}[D] \in P(x)\right\}
$$

for each $x \in X$. Hence this can be used as a definition for $P_{\mathcal{C}}$. Note that $\left(\mathbb{S i}_{\mathcal{C}}\right)(\mu)$ is the restriction of subprobability $\mu \in \mathbb{S}(X)$ to the $\sigma$-algebra $\mathcal{C}$. The true catch is of course that $P_{\mathcal{C}}$ has to be t-measurable.

So one might ask whether there exist subsystems at all. Before answering this, we shall make a brief excursion into congruences for stochastic relations. Let $X$ and $Y$ be standard Borel spaces and $K: X \rightsquigarrow Y$ be a stochastic relation. A congruence $(\alpha, \beta)$ for $K$ is a pair of smooth equivalence relations $\alpha$ on $X$ and $\beta$ on $Y$ such that there exists a relation $K_{\alpha, \beta}: X / \alpha \rightsquigarrow Y / \beta$ such that this diagram commutes:


It is not difficult to see that this condition is equivalent to saying that $K^{\prime}:\left(X, \Sigma_{\alpha}(X)\right) \rightsquigarrow$ $\left(Y, \Sigma_{\beta}(Y)\right)$ is a stochastic relation, where $K^{\prime}(x)$ is the restriction of $K(x)$ to the $\sigma$-algebra $\Sigma_{\beta}(Y)$ of $\beta$-invariant sets 14, Exercise 21]. Stating this formally, $K^{\prime}(x)=\left(\mathbb{S}\left(i_{\beta}\right) \circ K\right)(x)$, where $i_{\beta}: y \mapsto y$ is the injection, which yields a measurable map $(Y, \mathcal{B}(Y)) \rightarrow\left(Y, \Sigma_{\beta}(Y)\right)$. Putting $X_{\alpha}:=\left(X, \Sigma_{\alpha}(X)\right)$, similarly, for $Y_{\beta}$, this is equivalent to saying that this diagram

commutes. This is the diagram for stochastic relations which corresponds to diagram (13). Thus the subsystems for stochastic relations are exactly the congruences (and hence, for relations of the form $K: S \rightsquigarrow S$, the event bisimulations).

Returning to the main stream of our discussion, we observe that surjective morphisms provide a rich source of subsystems for the case of standard Borel spaces. Before we can state and prove this, we need this auxiliary technical statement.

Lemma 5.2. Let $X$ and $Y$ be standard Borel and $f: X \rightarrow Y$ be measurable and surjective. Then $\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}[D]$ is $\mathbb{S} f$-invariant for each measurable subset $D$ of $\mathbb{S}\left(X_{\Sigma_{f}}\right)$.

Proof. We look at all sets which satisfy the assertion and show that these sets comprise all measurable subsets $D$ of $\mathbb{S}\left(X_{\Sigma_{f}}\right)$.
In fact, let

$$
\mathcal{H}:=\left\{D \subseteq \mathbb{S}(X f) \mid\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}[D] \text { is } \mathbb{S} f \text {-invariant }\right\}
$$

then this is a $\sigma$-algebra (see the proof of [30, Lemma 3.1.6]), hence it is enough to show that $\boldsymbol{\beta}_{\Sigma_{f}}(A, q) \in \mathcal{H}$ whenever $A \in \Sigma_{f}$. Then it will follow that the $\sigma$-algebra generated by these sets is contained in $\mathcal{H}$, and these are all measurable subsets of $\mathbb{S}\left(X_{\Sigma_{f}}\right)$.

Now assume that $\left(\mathbb{S i}_{\Sigma_{f}}\right)(\mu) \in \boldsymbol{\beta}_{\Sigma_{f}}(A, q)$, and assume that $\left\langle\mu, \mu^{\prime}\right\rangle \in \operatorname{ker}(\mathbb{S} f)$, hence that $(\mathbb{S} f)(\mu)=(\mathbb{S} f)\left(\mu^{\prime}\right)$. Because $\left(\mathbb{S i}_{\Sigma_{f}}\right)^{-1}\left[\boldsymbol{\beta}_{\Sigma_{f}}(A, q)\right]$ equals $\boldsymbol{\beta}_{X}(A, q)$, and because $A=f^{-1}[G]$ for some Borel set $G \subseteq Y$ by Corollary [2.5, we infer that

$$
\mu^{\prime}(A)=\mu^{\prime}\left(f^{-1}[G]\right)=(\mathbb{S} f)\left(\mu^{\prime}\right)(G)=(\mathbb{S} f)(\mu)(G)=\mu(A) \geq q
$$

hence $\left(\mathbb{S} i_{\Sigma_{f}}\right)\left(\mu^{\prime}\right) \in \boldsymbol{\beta}_{\Sigma_{f}}(A, q)$. Thus $\boldsymbol{\beta}_{\Sigma_{f}}(A, q) \in \mathcal{H}$, and we are done.

This permit us to show that surjective morphisms define in fact subsystems on their domains, i.e., on the effectivity functions which serve as their source.

Proposition 5.3. Let $X$ and $Y$ be standard Borel spaces and $P: X \rightarrow X$ resp. $Q: Y \rightarrow Y$ be stochastic effectivity functions. A surjective morphism $f: P \rightarrow Q$ defines a subsystem $\Sigma_{f}$ of $P$. In particular, defining

$$
P_{f}(x):=\left\{A \in w\left(\Sigma_{f}\right) \mid \mathbb{S}\left(i_{\Sigma_{f}}\right)^{-1}[A] \in P(x)\right\} .
$$

yields an effectivity function $P_{f}: X \rightarrow\left(X, \Sigma_{f}\right)$, and $\left(i d_{X}, i d_{\Sigma_{f}}\right): P \rightarrow P_{f}$ is a morphism.

Proof. 1. Given $H \in w\left(\Sigma_{f}\right) \otimes[0,1]$, we have to show that the set

$$
T_{H}:=\left\{\langle x, t\rangle \mid\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right] \in P(x)\right\}
$$

is a member of $\Sigma_{f} \otimes \mathcal{B}([0,1])$. Since $P$ is an effectivity function, we know that $T_{H}$ is a Borel set in $X \times[0,1]$, so by Lemma 2.6 it is enough to show that $T_{H}$ is $\left(f \times i d_{[0,1]}\right)$-invariant.
2. Given $t \in[0,1]$, we know that $\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right] \in w(\mathcal{B}(X))$, by construction, and this set is $\mathbb{S} f$ invariant by Lemma 5.2. Now for showing the invariance property of $T_{H}$, we take $\langle x, t\rangle \in T_{H}$ with $f(x)=f\left(x^{\prime}\right)$. Since $\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right] \in P(x)$, and since this set is $\mathbb{S}(f)$-invariant, we know that we can write $\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right]=(\mathbb{S} f)^{-1}[G]$ for some $G \in \mathcal{B}(Y)$. Hence

$$
\begin{aligned}
\left(\mathbb{S} i_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right] \in P(x) & \Leftrightarrow(\mathbb{S} f)^{-1}[G] \in P(x) & & \\
& \Leftrightarrow G \in Q(f(x)) & & \text { since } f: P \rightarrow Q \text { is a morphism } \\
& \Leftrightarrow G \in Q\left(f\left(x^{\prime}\right)\right) & & \text { since } f(x)=f\left(x^{\prime}\right) \\
& \Leftrightarrow\left(\mathbb{S i}_{\Sigma_{f}}\right)^{-1}\left[H_{t}\right] \in P\left(x^{\prime}\right) & &
\end{aligned}
$$

Thus $\left\langle x^{\prime}, t\right\rangle \in T_{H}$.

Again, the situation is compared to stochastic relations and, by implication, to their event bisimulations. Congruences and kernels of surjective morphisms are closely related in this case, and the discussion above has shown that congruences are nothing but subsystems in disguise (or vice versa). This means that surjective morphisms are in any case a resource from which we may harvest subsystems. Of course, in the context of NLMPs and effectivity functions, this demands further investigations into the way subsystems integrate into this coalgebraic context.
5.1. Cospans For Finitely Generated Effectivity Functions. We will define behavioral equivalence and coalgebraic bisimulations now and investigate their relationship for finitely generated effectivity functions.

Definition 5.4. Let $P: S \rightarrow T$ and $Q: X \rightarrow Y$ be stochastic effectivity functions.

- Call $P$ and $Q$ behaviorally equivalent iff there exists a mediating effectivity function $M$ and surjective morphisms $P \xrightarrow{(f, g)} M \stackrel{(j, \ell)}{\stackrel{( }{l}} Q$.
- Call $P$ and $Q$ bisimilar iff there exists a mediating effectivity function $M$ and morphisms $P \stackrel{(f, g)}{\stackrel{(j, \ell)}{\longrightarrow}} Q$.

Hence behavioral equivalence is given through a co-span of morphisms, bisimilarity by a span, as tradition demands. We will focus now on behavioral equivalent effectivity functions $P$ and $Q$ for which the respective domains and ranges are identical. It will be shown that we can find for finitely supported effectivity functions bisimilar, albeit closely related functions, which are defined on a subsystem. Finitely supported functions live on a finite support set, which we show to be given by a countable set of stochastic relations; this can be considered to be a version of the notion of compactly generated for effectivity functions.

We assume in this subsection that all spaces are standard Borel.

Preliminary Considerations. Let $f: S \rightarrow U$ and $g: T \rightarrow U$ be both measurable and surjective maps, and consider

$$
\begin{equation*}
W:=\{\langle s, t\rangle \mid f(s)=g(t)\}=(f \times g)^{-1}\left[\Delta_{U}\right] \tag{14}
\end{equation*}
$$

with $\Delta_{U}:=\{\langle u, u\rangle \mid u \in U\}$ as the diagonal of $U$; intuitively, $\Delta_{U}$ is a true copy of $U$, slightly tilted. Because $\Delta_{U}$ is closed in the underlying topological space, $W$ is measurable. Under the projections $\pi_{1}, \pi_{2}:\langle u, u\rangle \mapsto u$, with inverse $d, \Delta_{U}$ is homeomorphic to $U$, so that $\mathcal{B}\left(\Delta_{U}\right)=d^{-1}[\mathcal{B}(U)]$, consequently, $w\left(\Delta_{U}\right)=(\mathbb{S} d)^{-1}[\mathcal{B}(\mathbb{S}(U))]$. We have also

$$
\begin{aligned}
\left(f \circ \pi_{1, S}\right)(s, t) & =\left(\pi_{1, U} \circ f \times g\right)(s, t), \\
\left(g \circ \pi_{2, T}\right)(s, t) & =\left(\pi_{2, U} \circ f \times g\right)(s, t)
\end{aligned}
$$

for $\langle s, t\rangle \in S \times T$.
Endow $W$ with the trace of the $\sigma$-algebra $\Sigma_{f} \otimes \Sigma_{g}$, hence

$$
\begin{aligned}
\mathscr{S}(W) & =(f \times g)^{-1}[\mathcal{B}(S \times S)] \cap(f \times g)^{-1}\left[\Delta_{U}\right] \\
& =(f \times g)^{-1}\left[\mathscr{S}\left(\Delta_{U}\right)\right]
\end{aligned}
$$

Finitely Supported Functions. A finitely supported effectivity function on $S$ can be represented through stochastic relations, which are obtained as measurable selections.

Proposition 5.5. Let $P: S \rightarrow S$ be finitely supported by $E$, then there exists a countable set $\mathcal{K}$ of stochastic relations $S \rightsquigarrow S$ such that $E(s)=\mathcal{K}(s):=\{K(s) \mid K \in \mathcal{K}\}$.

Proof. Since $S$ is standard Borel, we find a Polish topology which generates the $\sigma$-algebra; assume that $S$ is endowed with this topology, then $\mathbb{S}(S)$ is a Polish space as well under the weak topology. We take this topology. Let $G \subseteq \mathbb{S}(S)$ be open, then

$$
\{s \in S \mid E(s) \cap G \neq \varnothing\}=\bigcup_{e \in E}\{s \in S \mid e(s) \in G\} \in \mathcal{B}(S)
$$

Thus there exists a Castaing representation $\left(K_{n}\right)_{n \in \mathbb{N}}$ for $E$ by the Selection Theorem 2.11. Because $E$ is finite, we have $E(s)=\left\{K_{n}(s) \mid n \in \mathbb{N}\right\}$.

Note that although $\left(K_{n}\right)_{n \in \mathbb{N}}$ is a countable sequence of stochastic relations, the representation $E(s)=\left\{K_{n}(s) \mid n \in \mathbb{N}\right\}$ together with the finiteness of $E(s)$ implies that the set $\left\{K_{n}(s) \mid\right.$ $n \in \mathbb{N}\}$ is finite for each state $s$, but that we cannot conclude that $\left(K_{n}\right)_{n \in \mathbb{N}}$ is a finite sequence.

Now assume that the finitely supported effectivity functions $P$ and $Q$ are behaviorally equivalent, so that we have a cospan $P \stackrel{f}{\rightarrow} M \stackrel{g}{\leftarrow} Q$ and $f$ and $g$ surjective morphisms. The co-span expands to


Let $\mathcal{K}$ and $\mathcal{L}$ be the countable sets of stochastic relations associated with $P$ resp. $Q$ according to Proposition 5.5. We will show now that $M$ is finitely supported, and that the pointwise images of $\mathcal{K}$ and $\mathcal{L}$ under $\mathbb{S} f$ resp. $\mathbb{S} g$ coincide whenever the image of $f$ and $g$ are the same.

Proposition 5.6. In the notation above, $M$ is finitely supported, and $(\mathbb{S} f)[\mathcal{K}(s)]=(\mathbb{S} g)[\mathcal{L}(t)]$, whenever $f(s)=g(t)$.

Proof. Because $f: P \rightarrow M$ is a morphism, we have $G \in M(f(s))$ iff $(\mathbb{S} f)^{-1}[G] \in P(s)$, which in turn is equivalent to $\mathcal{K}(s) \subseteq(\mathbb{S} f)^{-1}[G]$. Hence $(\mathbb{S} f)[\mathcal{K}(s)] \in M(f(s))$. Similarly, we conclude $(\mathbb{S} g)[\mathcal{L}(t)] \in M(g(t))$. Because $f$ and $g$ both are onto, we find for each $u \in U$ some $s \in S$ and $t \in T$ with $f(s)=u=g(t)$. From this we conclude that $M(u)$ is supported both by $(\mathbb{S} f)[\mathcal{K}(s)]$ and by $(\mathbb{S} g)[\mathcal{L}(t)]$.

Before defining an effectivity function, we briefly investigate the $\mathbb{S}$-image of the projections $\pi_{S}$ and $\pi_{T}$. We show that they induce surjective maps, so that we can be sure that each element of the corresponding range occurs as an image. This is of technical use when investigating the system dynamics for the mediator.
Lemma 5.7. $\mathbb{S}\left(\pi_{S}\right): \mathbb{S}\left(W, \Sigma_{f \times g} \cap W\right) \rightarrow \mathbb{S}\left(S, \Sigma_{f}\right)$ and $\mathbb{S}\left(\pi_{T}\right): \mathbb{S}\left(W, \Sigma_{f \times g} \cap W\right) \rightarrow \mathbb{S}\left(T, \Sigma_{f}\right)$ are both onto.

Proof. We have a look at $\pi_{S}$ only, the argumentation is symmetric for its step twin $\pi_{T}$. Define $\psi: \mathscr{S}(Y) \rightarrow \Sigma_{f}$ through $D \mapsto f^{-1}[D]$, then $\psi$ induces a bijection $\Psi: \mathbb{S}(Y) \rightarrow \mathbb{S}\left(S, \Sigma_{f}\right)$ by Lemma 2.7. Given $\mu \in \mathbb{S}\left(S, \Sigma_{f}\right)$, define

$$
\nu:=(\mathbb{S} f)(\mu)
$$

hence $\nu \in \mathbb{S}(U)$. Because $e_{1} \circ f \times g: W \rightarrow U$ is onto, we find $\nu^{\prime} \in \mathbb{S}\left(W, \Sigma_{f \times g} \cap W\right)$ with

$$
\nu=\mathbb{S}\left(e_{1} \circ f \times g\right)\left(\nu^{\prime}\right) .
$$

But

$$
e_{1} \circ(f \times g)=f \circ \pi_{S}
$$

on $W$, so that

$$
\nu=\left(\mathbb{S}(f) \circ \mathbb{S}\left(\pi_{S}\right)\right)\left(\nu^{\prime}\right)=(\mathbb{S} f)(\mu) .
$$

Hence

$$
\mu=\Psi\left(\mathbb{S}(f) \circ \mathbb{S}\left(\pi_{S}\right)\right)\left(\nu^{\prime}\right)=\left(\mathbb{S} \pi_{S}\right)\left(\nu^{\prime}\right)
$$

Constructing the Mediator. Now define $W$ as in (14) and endow it with $(f \times g)^{-1}\left[\mathcal{B}\left(\Delta_{U}\right)\right]$ as a $\sigma$-algebra. We know from Proposition 5.6 that $\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] \subseteq Z$ iff $\left(\mathbb{S} \pi_{T}\right)^{-1}[\mathcal{L}(t)] \subseteq Z$ for $f(s)=g(t)$. This suggests the following definition for the dynamics for $\langle s, t\rangle \in W$.

$$
\begin{equation*}
\tau(s, t):=\left\lfloor\left(\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)]\right)\right\rfloor\left(=\left\lfloor\left(\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{L}(t)]\right)\right\rfloor\right) \tag{15}
\end{equation*}
$$

Hence $\tau: W \rightarrow \mathbb{V}(W)$ renders this diagram commutative:


In fact, if $\langle s, t\rangle \in W$, we have

$$
\begin{aligned}
\left(\mathbb{S} \pi_{S}\right)^{-1}[Z] \in \tau(s, t) & \Leftrightarrow\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] \subseteq\left(\mathbb{S} \pi_{S}\right)^{-1}[Z] \\
& \Leftrightarrow \mathcal{K}(s) \subseteq Z
\end{aligned}
$$

The last conclusion follows from the observation that since $\pi_{S}$ is onto and continuous, $\mathbb{S} \pi_{S}$ is onto as well, so that $\left(\mathbb{S} \pi_{S}\right)^{-1}$ is injective as a set valued map.

Hence we see in a similar way, for $\langle s, t\rangle \in W$

$$
\begin{aligned}
& P_{f}(s)=\left\{Z \in \mathcal{B}(\mathbb{S} W) \mid\left(\mathbb{S} \pi_{S}\right)^{-1}[Z] \in \tau(s, t)\right\} \\
& Q_{g}(t)=\left\{Z \in \mathcal{B}(\mathbb{S} W) \mid\left(\mathbb{S} \pi_{S}\right)^{-1}[Z] \in \tau(s, t)\right\} .
\end{aligned}
$$

Proposition 5.8. $\tau$ is $t$-measurable.
Before we are in a position to establish Proposition 5.8, we need an auxiliary statement.
Lemma 5.9. Define $e_{i}: \Delta_{U} \rightarrow U$ as the $i$-th projection. If $H \in w(W)$, then there exists $H^{\prime} \in w(U)$ such that $H=\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[H^{\prime}\right]$.

Proof. Let $\mathcal{H}$ be the set of all $H \in w(W)$ for which the assertion is true. Then $\mathcal{H}$ is a $\sigma$-algebra, and $\boldsymbol{\beta}_{W}(D, \bowtie q) \in \mathcal{H}$, whenever $D \in \Sigma_{f \times g} \cap W$. In fact, $D$ can be written as $D=(f \times g)^{-1}\left[D_{1}\right]$ for some $D_{1} \in \mathscr{S}\left(\Delta_{U}\right)$, and $D_{1}=e_{1}^{-1}\left[D_{0}\right]$ for some $D_{0} \in \mathcal{B}(U)$. Hence $D=\left(e_{1} \circ f \times g\right)^{-1}\left[D_{0}\right]$, from which

$$
\boldsymbol{\beta}_{W}(D, \bowtie q)=\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[\beta_{U}\left(D_{0}, \bowtie q\right)\right]
$$

follows. Thus $\mathcal{H}$ contains the generator for $w(W)$, from which the assertion follows.
This yields as an immediate consequence the following observation.

Corollary 5.10. If $H \in \mathscr{S}(\mathbb{S}(W) \otimes[0,1])$, then there exists $H^{\prime} \in \mathscr{S}(\mathbb{S}(U) \otimes[0,1])$ such that $H=\left(\mathbb{S}\left(e_{1} \circ f \times g\right) \times i d_{[0,1]}\right)^{-1}\left[H^{\prime}\right]$.

This Corollary will permit this proof strategy: if we want to establish a property for some measurable $H \subseteq \mathbb{S}(W) \otimes[0,1]$, we investigate $H^{\prime}$ instead, establish a suitably modified property for $H^{\prime}$ and prove things for $H^{\prime}$. But let us have a look at the proof.

Proof. Again, the set for which the assertion is true is a $\sigma$-algebra, which contains by Lemma 5.9 all measurable rectangles $H_{1} \times V$ with $H_{1} \in w(W)$ and $V \in \mathcal{B}([0,1])$.

Proof. (of Proposition [5.8) 1. Since $\tau(s, t)=\left\lfloor\left(\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)]\right)\right\rfloor$ for $\langle s, t\rangle \in W$, we have to show that

$$
T_{H}:=\left\{\langle s, t, q\rangle \mid H_{q} \in \tau(s, t)\right\}
$$

is a member of $\mathscr{S}(W \otimes[0,1])$, whenever $H \subseteq \mathbb{S}(W) \times[0,1]$ is measurable. Thus we have to show that $T_{H}=\left\{\langle s, t, q\rangle \in W \times[0,1] \mid\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] \subseteq H_{q}\right\} \in \mathcal{B}(W \otimes[0,1])$.
2. Now Corollary 5.10 kicks in. For $H$ there exists $G \in \mathscr{S}(\mathbb{S}(U) \otimes[0,1])$ such that $H_{q}=$ $\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[G_{q}\right]$.
We claim

$$
\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] \subseteq H_{q} \Leftrightarrow \mathcal{K}(s) \subseteq\left(\mathbb{S}(f) \times i d_{[0,1]}\right)^{-1}[G]_{q}
$$

" $\Leftarrow$ ": Note that $f \circ \pi_{S}=e_{1} \circ f \times g$ on $W$, hence

$$
\begin{aligned}
\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] & \subseteq\left(\mathbb{S} \pi_{S}\right)^{-1}\left[\left(\mathbb{S}(f) \times i d_{[0,1]}\right)^{-1}[G]_{q}\right] \\
& =\left(\mathbb{S} \pi_{S}\right)^{-1}\left[(\mathbb{S} f)^{-1}\left[G_{q}\right]\right] \\
& =\left(\mathbb{S}\left(f \circ \pi_{S}\right)\right)^{-1}\left[G_{q}\right] \\
& =\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[G_{q}\right] \\
& =H_{q}
\end{aligned}
$$

" $\Rightarrow$ ": Because $\mathbb{S} \pi_{s}$ is onto by Lemma 5.7, we infer from $\left(\mathbb{S} \pi_{S}\right)^{-1}[\mathcal{K}(s)] \subseteq H_{q}$ that $\mathcal{K}(s) \subseteq$ $\left(\mathbb{S} \pi_{S}\right)\left[H_{q}\right]$. Now let $\mu \in\left(\mathbb{S} \pi_{S}\right)\left[H_{q}\right]=\left(\mathbb{S} \pi_{S}\right)\left[\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[G_{q}\right]\right]$, then there exists $\nu \in$ $\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)^{-1}\left[G_{q}\right]$ with $\mu=\left(\mathbb{S} \pi_{S}\right)(\nu)$, hence

$$
(\mathbb{S} f)(\mu)=\left(\mathbb{S} f \circ \pi_{S}\right)(\mu)=\left(\mathbb{S}\left(e_{1} \circ f \times g\right)\right)(\nu) \in G_{q}
$$

3. Hence we have $T_{H}=\left\{\langle s, t, q\rangle \in W \times[0,1] \mid \mathcal{K}(s) \subseteq\left(\mathbb{S}(f) \times i d_{[0,1]}\right)^{-1}[G]_{q}\right\}$, which is a measurable subset of $W \times[0,1]$.

Concluding, we have shown
Proposition 5.11. Let $P: S \rightarrow S$ and $Q: T \rightarrow T$ be finitely supported and behaviorally equivalent stochastic effectivity functions, and assume that $S, T$ and the mediator's state space are standard Borel spaces. Then there exist subsystems $\mathcal{C}_{P}$ and $\mathcal{C}_{Q}$ of $P$ resp. $Q$ and effectivity functions $P_{s}: S \rightarrow\left(S, \mathcal{C}_{P}\right)$ and $Q_{s}: T \rightarrow\left(T, \mathcal{C}_{Q}\right)$ such that $P_{s}$ and $Q_{s}$ are bisimilar.

Let us briefly look back and see what we did, and how we did it. We identified subsystems for the given morphisms. On the equalizer $W$ of these morphisms, we constructed a Borel structure, and from this measurable space we obtained a stochastic effectivity function. So far the proof resembles the classic proof for the existence of bisimulations for set based systems. The complications arise when having to establish that we have here a span of morphisms, in part because the measurable structure on the subprobabilities of $W$ is given by appealing to the measurable structure on the diagonal of the target space for the co-span, from which we started, through the given morphisms and through projections. Investigating this structure, which arises through these delegations (as an object oriented programmer would say) and which requires the maps and their images under $\mathbb{S}$ as handles, renders the proof somewhat involved.

One might be tempted now to capitalize on this result for establishing the existence of a bisimulation for stochastic relations. This approach would rest on the observation that each stochastic relation $K: S \rightsquigarrow T$ for measurable spaces $S$ and $T$ generates an effectivity function $P_{K}$ upon setting $P_{K}(s):=\{A \in w(S) \mid K(s) \in A\}$. Hence $P_{K}$ is clearly finitely supported. Since each morphism for stochastic relations yields a morphism for the associated effectivity function, the result above applies to stochastic relations $K: S \rightsquigarrow S$ and $L: T \rightsquigarrow T$, and one finds a mediating effectivity function for $P_{K, f}$ and $Q_{L, g}$. This effectivity function is finitely supported, but not by the associated Borel sets. It is rather defined on a sub- $\sigma$-algebra of the Borel sets of $W$, the space we have identified here. It cannot be concluded right away, however, that the stochastic relation which underlies the mediating function is also a stochastic relation for the Borel sets of the underlying Polish or analytic spaces; looking at the proofs establishing the existence of a semi-pullback [15, 8], it turns out that it is this property which is the crucial one for stochastic relations.
5.2. Event Bisimulations Revisited. Since both event bisimulations and subsystems are given by sub- $\sigma$-algebras, it is natural to inquire about the connections between these two notions, which adapt the remarks made above concerning the relationships between subsystems and event bisimulations for stochastic relations.

In first place, we need to make some detailed observations concerning the definition of event bisimulations. Fix a nondeterministic kernel $\kappa: S \rightarrow w(S)$. We have defined an event bisimulation to be any sub- $\sigma$-algebra $\mathcal{C}$ of $\mathscr{S}(S)$ such that $\kappa:(S, \mathcal{C}) \rightarrow(w(S), \mathscr{H}(w(\mathcal{C})))$ is measurable. This means that for all $G \in w(\mathcal{C})$, the preimage

$$
\kappa^{-1}\left[\mathrm{H}_{G}\right]=\{s \in S \mid \kappa(s) \cap G \neq \varnothing\}
$$

should belong to $\mathcal{C}$. But we have to exercise some care here: although any measure $\mu$ in $\mathbb{S}(S)=\mathbb{S}(S, \mathscr{S}(S))$ assigns also values to elements of $\mathcal{C}$, it is actually not a member of the space $\mathbb{S}(S, \mathcal{C})$. We should restrict its domain by using the map $\mathbb{S} \iota_{\mathcal{C}}: \mathbb{S}(S, \mathscr{S}(S)) \rightarrow \mathbb{S}(S, \mathcal{C})$. So we finally obtain the following more precise formulation of event bisimulation on $\kappa: S \rightarrow w(S)$ : any $\mathcal{C}$ such that for all $G \in w(\mathcal{C})$,

$$
\kappa^{-1}\left[\mathrm{H}_{(\mathbb{S} \iota \mathcal{C})^{-1}[G]}\right]=\left\{s \in S \mid \kappa(s) \cap(\mathbb{S} \iota \mathcal{C})^{-1}[G] \neq \varnothing\right\} \in \mathcal{C} .
$$

Finally, by considering Lemma 2.2, this is equivalent to having

$$
\begin{equation*}
\left\{s \in S \mid \kappa(s) \subseteq\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[G]\right\} \in \mathcal{C} \tag{16}
\end{equation*}
$$

for all $G \in w(\mathcal{C})$.

As for stochastic relations, subsystems induce event bisimulations. This is fairly straightforward.

Proposition 5.12. Let $\mathcal{C}$ be a subsystem of $\lfloor\kappa\rfloor: S \rightarrow w(S)$. Then $\mathcal{C}$ is an event bisimulation of $\kappa$.

Proof. The hypothesis tells us that for all $H \in \mathscr{S}(\mathbb{S}(S, \mathcal{C}) \otimes[0,1])$, we have

$$
\left\{\langle s, q\rangle \mid H_{q} \in(\lfloor\kappa\rfloor)_{\mathcal{C}}(s)\right\} \in \mathscr{S}((S, \mathcal{C}) \otimes[0,1])
$$

that is,

$$
\begin{equation*}
\left\{\langle s, q\rangle \mid\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}\left[H_{q}\right] \in\lfloor\kappa(s)\rfloor\right\} \in \mathscr{S}((S, \mathcal{C}) \otimes[0,1]) \tag{17}
\end{equation*}
$$

We aim at showing that (16) holds for all $G \in w(\mathcal{C})$.
In order to do this, take $H:=G \times[0,1]$ in (17) and calculate:

$$
\begin{aligned}
\left\{\langle s, q\rangle \mid\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}\left[H_{q}\right] \in\lfloor\kappa(s)\rfloor\right\} & =\left\{\langle s, q\rangle \mid\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[G] \in\lfloor\kappa(s)\rfloor\right\} \\
& =\left\{\langle s, q\rangle \mid \kappa(s) \subseteq\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[G]\right\} \\
& =\left\{s \mid \kappa(s) \subseteq\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[G]\right\} \times[0,1]
\end{aligned}
$$

Now the set $\left\{s \mid \kappa(s) \subseteq\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[G]\right\}$ is a cut of a set in $\mathscr{S}((S, \mathcal{C}) \otimes[0,1])=\mathcal{C} \otimes \mathcal{B}([0,1])$, and hence it belongs to $\mathcal{C}$.

We will need for the converse to strengthen our hypothesis by following the ideas put forward in Proposition 4.8, making the parameter from $[0,1]$ explicit. It works like this: Let $\kappa$ : $S \rightarrow w(S)$ be a generating kernel. We will show that parametrized event bisimulations, appropriately defined for the extended system with base space $S \times[0,1]$, are exactly the subsystems of the effectivity function based on $\kappa$.

Formally, a parametrized event bisimulation on $\kappa: S \rightarrow w(S)$ is a sub- $\sigma$-algebra of $\mathscr{S}(S)$ such that $\kappa \times i d_{[0,1]}: S \times[0,1] \rightarrow w(S) \times[0,1]$ is $(\mathcal{C} \otimes \mathcal{B}([0,1]))-\mathscr{H}(w(\mathcal{C}) \otimes \mathcal{B}([0,1]))$ measurable: for all $G \in w(S, \mathcal{C}) \otimes \mathcal{B}([0,1])$,

$$
\left\{\langle s, q\rangle \in S \times[0,1] \mid\left(\kappa \times i d_{[0,1]}\right)(s, q) \subseteq\left(\mathbb{S}_{\mathcal{C}} \times i d_{[0,1]}\right)^{-1}[G]\right\} \in \mathcal{C} \otimes \mathcal{B}([0,1])
$$

compare this with (16).
Proposition 5.13. For every generating kernel $\kappa: S \rightarrow w(S)$, the parametrized event bisimulations on $\kappa$ are exactly the subsystems of $\lfloor\kappa\rfloor$.

Proof. Let $\mathcal{C}$ be a parametrized event bisimulation. By recalling (17) in the proof of the previous proposition, we would like to show that

$$
\left\{\langle s, q\rangle \mid\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}\left[G_{q}\right] \in\lfloor\kappa(s)\rfloor\right\} \in \mathscr{S}((S, \mathcal{C}) \otimes[0,1]) .
$$

for all $G \in \mathscr{S}(\mathbb{S}(S, \mathcal{C}) \otimes[0,1])$. By definition of $\lfloor\cdot\rfloor$, we may write this set as

$$
\left\{\langle s, q\rangle \mid \kappa(s) \subseteq\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}\left[G_{q}\right]\right\}
$$

Next make explicit the cut component $q$ :

$$
\left\{\langle s, q\rangle \mid \kappa(s) \times\{q\} \subseteq\left(\mathbb{S}_{\mathcal{C}} \times i d_{[0,1]}\right)^{-1}[G]\right\}
$$

That is, we need

$$
\left\{\langle s, q\rangle \mid\left(\kappa \times i d_{[0,1]}\right)(s, q) \subseteq\left(\mathbb{S}_{\mathcal{C}} \times i d_{[0,1]}\right)^{-1}[G]\right\} \in \mathcal{C} \otimes \mathcal{B}([0,1])
$$

But this last condition is exactly the definition of parametrized event bisimulation, and therefore the two notions are equivalent.

To finish this section, we will show that in a finitary setting, the relational approach to bisimilarity of effectivity functions is compatible to that of subsystems, by showing that the relation of state bisimilarity induces a subsystem in a natural way. We actually obtain a stronger result: every smooth bisimulation induces a subsystem, the result on finitary effectivity functions follows as a corollary.
A technical lemma is required first.
Lemma 5.14. Let $P: S \rightarrow S$ an effectivity function, $R$ a state bisimulation on $P$, and $\mathcal{C}:=\Sigma_{R}$. Then for given $H \in \mathscr{S}(\mathbb{S}(S, \mathcal{C}) \otimes[0,1])$, assume $H_{q} \in P_{\mathcal{C}}(s)$, and let $s R$. Then $H_{q} \in P_{\mathcal{C}}(t)$.

Proof. Since $R$ is a bisimulation, if $s R t$ we have

$$
\forall G \in P(s) \exists E \in P(t): \forall \nu \in E \exists \mu \in G(\mu \bar{R} \nu) .
$$

By definition of $\iota_{\mathcal{C}}$,

$$
\forall G \in P(s) \exists E \in P(t): \mathbb{S} \iota_{\mathcal{C}}[E] \subseteq \mathbb{S}_{\mathcal{C}}[G]
$$

Now we can take $E^{\prime}:=E \cup G$, and this set belongs to $P(t)$ by upper-closedness, and we obtain

$$
\begin{equation*}
\forall G \in P(s) \exists E^{\prime} \in P(t): \mathbb{S}_{\mathcal{C}}\left[E^{\prime}\right]=\mathbb{S}_{\mathcal{C}}[G] . \tag{18}
\end{equation*}
$$

This yields

$$
\begin{array}{rlr}
A & :=\left\{\langle s, q\rangle \mid H_{q} \in P_{\mathcal{C}}(s)\right\} & \quad \text { definition of } P_{\mathcal{C}} \\
& =\left\{\langle s, q\rangle \mid\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}\left[H_{q}\right] \in P(s)\right\} & \\
& =\left\{\langle s, q\rangle \mid \exists D \in P(s):\left(\mathbb{S} \iota_{\mathcal{C}}\right)^{-1}[D] \subseteq H_{q} \in P(s)\right\} &
\end{array}
$$

If $\langle s, q\rangle \in A$ there exists a $D$ as in the previous line. And if $s R t$, there must exist some $D^{\prime} \in P(t)$ such that $\mathbb{S} \iota_{\mathcal{C}}\left[D^{\prime}\right]=\mathbb{S}_{\mathcal{C}}[D]$ by (18). Hence $\langle t, q\rangle \in A$.

Now all preparations are finally in place for showing that bisimulation equivalences induce subsystems, provided they are smooth. By using our logical characterization for finitary effectivity functions we may also prove that state bisimilarity induces a subsystem.

Theorem 5.15. Let $S$ be Polish, $P: S \rightarrow \mathbb{V}(S)$ an effectivity function, and $R$ a smooth state bisimulation equivalence on $P$. Then $\Sigma_{R}$ is a subsystem. In particular, if $P$ is finitary, then $\Sigma\left(\sim_{\mathrm{s}}, \mathscr{S}(S)\right)$ is a subsystem.

Proof. 1. We have to prove that $P_{\mathcal{C}}$ is t-measurable. That means, for all $H \in \mathscr{S}(\mathbb{S}(S, \mathcal{C}) \otimes$ $[0,1])$, the set $A$ as defined for $H$ in the proof for Lemma 5.14 should belong to $\Sigma_{R}(\mathcal{B}(S)) \otimes$ $\mathcal{B}([0,1])$. By Lemma 5.14, we have $A \in \Sigma_{R \times \Delta_{[p, 1]}}(\mathcal{B}(S) \otimes \mathcal{B}([0,1]))$. But since $[0,1]$ is a Polish space, and since $R$ is smooth, we obtain from [3, Lemma 6.4.2] via [13, Proposition 2.12] that

$$
\Sigma_{R}(\mathcal{B}(S)) \otimes \mathcal{B}([0,1])=\Sigma_{R \times \Delta_{[0,1]}}(\mathcal{B}(S) \otimes \mathcal{B}([0,1]))
$$

2. By Corollary 4.23 the relation $\sim_{\mathrm{s}}$ of state bisimilarity on $P$ is smooth, hence Theorem 5.15 applies.

## 6. Conclusions \& Further Work

We shed some light into the relationship of nondeterministic kernels and stochastic effectivity functions, forging links between the respective concepts of morphism and bisimulation. While nondeterministic kernels appear as a natural generalization of purely probabilistic systems, the properties proposed for effectivity functions are useful for interpreting game logics. Both are descendants of stochastic relations, and this common origin is one reason for having a tighter bond between the finitely-generated versions of each of them. Two of the main contributions of this work concern the finitary setting, namely: the fact that image-finite kernels always give rise to effectivity functions via the filter construction, and that state bisimilarity on finitary effectivity functions is characterized by the two-level modal logic $2 \mathcal{L}$.
We also started the coalgebraic study of nondeterministic kernels, by translating the notion of morphism for effectivity functions to them, thus obtaining a category of NLMPs. There is still much work to be done in this direction, since we still only have some preliminary approximations for the candidate endofunctor for the coalgebraic structure. One pertinent observation is that if we obtain a successful definition of this functor, it would most probably be a contravariant one. This is not yet encompassed in the standard theory of coalgebras, so a systematic study of contravariant coalgebras might be a natural step to follow.

In the opposite way, the study of the translation from nondeterministic kernels to effectivity functions made apparent the need of different notions of morphism for the latter. The plain definition of effectivity function morphism corresponds to bounded morphisms for models of monotonic non-normal modal logic, as studied by Hansen [18]. Now, both our finitary effectivity functions and those generated by a kernel are analogous to core-complete models, meaning that their value for each state is a union of principal filters. In this context, strong morphisms correspond to bounded core morphisms. Somehow related to this, it is important to note that the working hypothesis used to prove a Hennessy-Milner theorem for monotonic models (i.e., a logical characterization of bisimilarity [18, Definition 4.30]) is the same as ours of having a finitary effectivity function. Further connections between the standard theory of monotonic models and the present stochastic version should be studied in the future.

We would like to see an interpretation of game logic without the fairly strong condition of t measurability, which, however, has turned out to be technically necessary. One point of attack may be given by the observation that certain crucial real functions that appear in the study of composability of effectivity functions [13, Section 3.2] are monotonically decreasing and hence Borel measurable. If it could be verified that t-measurability can be deduced from some more lightweight properties of nondeterministic kernels, the boundaries between NLMPs and effectivity functions could be lowered, permitting an easier exchange of properties. It seems necessary to start by investigating the composition of two nondeterministic kernels; there are natural definitions of this composition, stemming from the discrete case, but measurability obstacles preclude a direct generalization.

Another topic of interest is the relationship of the two-level logic $2 \mathcal{L}$ and game logic. The interpretation of the "upper level" of $2 \mathcal{L}$ in terms of measures would open up a lot of interesting
perspectives on game logic, not least for the exploration of bisimulations for probabilistic neighborhood models, about we do not know too much even under the assumption of Polish or analytic state spaces. Also, we will give an account for the relation between $2 \mathcal{L}$ and the logic for NLMPs somewhere else. This may help in completing the picture in this stochastic landscape.

Finally a third area of further work includes a deeper study of coalgebraic matters, in particular of expressivity including bisimilarity. We believe that (at least under the assumption of finitary effectivity functions) stronger results may be obtained.

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## References

[1] W. Arveson. An Invitation to $C^{*}$-Algebra, volume 39 of Graduate Texts in Mathematics. Springer-Verlag, New York, Berlin, Heidelberg, 1976.
[2] P. Blackburn, M. de Rijke, and Y. Venema. Modal Logic. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, UK, 2001.
[3] V. I. Bogachev. Measure Theory. Springer-Verlag, 2007.
[4] P. R. D'Argenio, P. Sánchez Terraf, and N. Wolovick. Bisimulations for non-deterministic labelled Markov processes. Math. Struct. Comp. Science, 22(1):43-68, February 2012.
[5] P.R. D'Argenio, N. Wolovick, P. Sánchez Terraf, and P. Celayes. Nondeterministic labeled Markov processes: Bisimulations and logical characterization. In $Q E S T$, pages 11-20. IEEE Computer Society, 2009.
[6] J. Desharnais, A. Edalat, and P. Panangaden. Bisimulation of labelled Markov processes. Information and Computation, 179(2):163-193, 2002.
[7] Josée Desharnais. Labeled Markov Process. PhD thesis, Department of Computer Science, McGill University, Montreal, 1999.
[8] E.-E. Doberkat. Stochastic relations: congruences, bisimulations and the HennessyMilner theorem. SIAM J. Computing, 35(3):590 - 626, 2006.
[9] E.-E. Doberkat. Kleisli morphisms and randomized congruences for the Giry monad. J. Pure Appl. Alg., 211:638-664, 2007.
[10] E.-E. Doberkat. Stochastic Relations. Foundations for Markov Transition Systems. Chapman \& Hall/CRC Press, Boca Raton, New York, 2007.
[11] E.-E. Doberkat. A stochastic interpretation of propositional dynamic logic: Expressivity. J. Symb. Logic, 77(2):687-716, 2012.
[12] E.-E. Doberkat. A stochastic interpretation of game logic. Technical Report 193, Chair for Software Technology, Technische Universität Dortmund, December 2012, arXiv: 1403.7765.
[13] E.-E. Doberkat. Algebraic properties of stochastic effectivity functions. J. Logic and Algebraic Progr., 83:339 - 358 (arXiv: 1403.7745), 2014.
[14] E.-E. Doberkat. A Companion to Mathematics for Computer Science: Sets, Categories, Topologies, Measures. Springer-Verlag, New York, Berlin, Heidelberg, 2015.
[15] A. Edalat. Semi-pullbacks and bisimulations in categories of Markov processes. Math. Struct. Comp. Science, 9(5):523-543, 1999.
[16] M. Giry. A categorical approach to probability theory. In Categorical Aspects of Topology and Analysis, number 915 in Lect. Notes Math., pages 68 - 85, Berlin, 1981. SpringerVerlag.
[17] R. Goldblatt. Topological proofs of some Rasiowa-Sikorski lemmas. Studia Logica, 100:1 - 18, 2012.
[18] Helle Hvid Hansen. Monotonic modal logics. Master's thesis, Institute for Logic, Language and Computation (ILLC), University of Amsterdam, 2003.
[19] E. Hewitt and K. R. Stromberg. Real and Abstract Analysis. Springer-Verlag, Berlin, Heidelberg, New York, 1965.
[20] T. Jech. Set Theory. Springer-Verlag (The Third Millennium Edition), Berlin, Heidelberg, New York, 2006.
[21] A. S. Kechris. Classical Descriptive Set Theory. Graduate Texts in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, 1994.
[22] K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. Information and Computation, 94:1-28, 1991.
[23] E. Michael. Topologies on spaces of subsets. Trans. Am. Math. Soc., 71(2):152-182, 1951.
[24] P. Panangaden. Probabilistic relations. In C. Baier, M. Huth, M. Kwiatkowska, and M. Ryan, editors, Proc. PROBMIV, pages $59-74,1998$.
[25] R. Parikh. The logic of games and its applications. In M. Karpinski and J. van Leeuwen, editors, Topics in the Theory of Computation, volume 24, pages 111-140. Elsevier, 1985.
[26] A. Parma and R. Segala. Logical characterizations of bisimulations for discrete probabilistic systems. In Proc. FOSSACS'07, volume 4423 of Lect. Notes Comp. Sci., pages 287 - 301, Berlin, Heidelberg, New York, 2007. Springer Verlag.
[27] M. Pauly and R. Parikh. Game logic - an overview. Studia Logica, 75:165-182, 2003.
[28] Roberto Segala. A compositional trace-based semantics for probabilistic automata. In Insup Lee and Scott A. Smolka, editors, CONCUR, volume 962 of Lecture Notes in Computer Science, pages 234-248. Springer, 1995.
[29] Roberto Segala and Nancy A. Lynch. Probabilistic simulations for probabilistic processes. Nord. J. Comput., 2(2):250-273, 1995.
[30] S. M. Srivastava. A Course on Borel Sets. Number 180 in Graduate Texts in Mathematics. Springer-Verlag, Berlin, 1998.
[31] Mariëlle Stoelinga. An introduction to probabilistic automata. Bulletin of the European Association for Theoretical Computer Science, 78:176-198, 2002.
[32] Y. Venema. Algebras and co-algebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, Handbook of Modal Logic, volume 3 of Studies in Logic and Practical Reasoning, pages 331-426. Elsevier, Amsterdam, 2007.
[33] Nicolás Wolovick. Continuous Probability and Nondeterminism in Labeled Transition Systems. PhD thesis, Universidad Nacional de Córdoba, 2012.

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[^1]:    ${ }^{1}$ The notation of indicating the horizontal cut through an index conflicts with indexing, but it is customary, so we will be careful to make sure which meaning we have in mind.

