A LOWER BOUND FOR FAITHFUL REPRESENTATIONS OF NILPOTENT LIE ALGEBRAS

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ABSTRACT. In this paper we present a lower bound for the minimal dimension $\mu(\mathfrak{n})$ of a faithful representation of a finite dimensional *p*-step nilpotent Lie algebra \mathfrak{n} over a field of characteristic zero. Our bound is given as the minimum of a quadratically constrained linear optimization problem, it works for arbitrary p and takes into account a given filtration of \mathfrak{n} . We present some estimates of this minimum which leads to a very explicit lower bound for $\mu(\mathfrak{n})$ that involves the dimensions of \mathfrak{n} and its center. This bound allows us to obtain $\mu(\mathfrak{n})$ for some families of nilpotent Lie algebras.

1. INTRODUCTION AND MAIN RESULTS

In this paper all Lie algebras and representations are finite dimensional over field k of characteristic zero. Given a representation (π, V) of a nilpotent Lie algebra \mathfrak{n} , we say that (π, V) is a *nilrepresentation* if $\pi(X)$ is a nilpotent for all $X \in \mathfrak{n}$.

Ado's Theorem states that any Lie algebra has a faithful representation (see [J, p. 202]). Nevertheless, given a Lie algebra \mathfrak{n} , the invariants

$$\mu(\mathfrak{n}) = \min\{\dim V : (\pi, V) \text{ is a faithful representation of } \mathfrak{n}\},\$$

 $\mu_{nil}(\mathfrak{n}) = \min\{\dim V : (\pi, V) \text{ is a faithful nilrepresentation of } \mathfrak{n}\}.$

are, in general, very difficult to compute or even to estimate. Apart from its intrinsic interest, the map μ is not only important in computational mathematics, but it is also connected to the theory of compact affine manifolds and crystallographic groups (see for instance [Be, B, K, Mi, Se2]) and to the theory of polycyclic groups (see for instance [Se1][Ch. 5,6], [GSe][§3.2]).

The value of $\mu(\mathfrak{n})$ has been obtained only for very few families of Lie algebras \mathfrak{n} (see, for instance [Be, B, BM1, CRo, Ro, S]).

Obtaining general results about μ , in particular new bounds, is very hard. On the one hand, there is a number of papers investigating new methods for constructing faithful representations of small dimension for a given class of (nilpotent) Lie algebras (see for instance [BM2, BEdG, dG, dGN, Ne]) and thus obtaining upper bounds for μ . In this direction, an ambitious goal is to find out whether there is a fixed polynomial p such that $\mu(\mathfrak{n}) \leq p(\dim \mathfrak{n})$ for all Lie algebras \mathfrak{n} (at least inside a wide class).

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On the other hand, general lower bounds are crucial for proving that a given faithful representation of a Lie algebra is actually of minimal dimension. They are also important for their applications to other problems. For instance, the counterexample obtained by Benoist [Be] to Milnor's conjecture [Mi] is based on a family of Lie algebras satisfying $\mu(\mathfrak{n}) > \dim \mathfrak{n} + 1$. On the group theory side, lower bounds for faithful representations of finite groups have been used to obtain a lower bound for the smallest non-trivial eigenvalue of the Laplace-Beltrami operator on certain manifolds [SX], or to answer questions of Lubotzky about the uniform expansion bounds for the Cayley graphs of $SL_2(\mathbb{F}_p)$ [BoG].

In this paper we obtain the following lower bound of μ_{nil} for nilpotent Lie algebras.

Theorem 1.1. Let \mathfrak{n} be a Lie algebra and let $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_1 = \mathfrak{n}$ be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} . Then

$$\mu_{nil}(\mathfrak{n}) \ge r_0^{min}$$

where r_0^{\min} is the minimum value of

$$a_0 = a_0 + a_1 + \dots + a_p, \quad a_0, a_1, \dots, a_p \in \mathbb{Z},$$

 $r_0 = a_0 + a_1 + \dots + a_p, \quad a_0, a_1, \dots, a_p \in \mathbb{Z},$ subject to the following restrictions: (a) $a_0, a_p \ge 1$ and $a_k \ge 0,$ for $k = 1, \dots, p-1;$ (b) $\sum_{i=0}^{p_0-k} a_i(a_{k+i}+\dots+a_p) \ge \dim \mathfrak{n}_k, \text{ for } k=1,\dots,p_0;$

(c)
$$a_0(a_k + \dots + a_p) \ge \dim \mathfrak{n}_k$$
, for $k = p_0, \dots, p$.

The quadratically constrained linear optimization problem involved in the above theorem seems to be difficult. In this paper we present some quick, but not trivial, estimations of r_0^{\min} and the lower bounds obtained are already interesting. We are confident that future research on r_0^{\min} will provide very good lower bounds for $\mu_{nil}(\mathfrak{n})$. As a consequence of our estimates, we obtain the following theorem.

Theorem 1.2. Let \mathfrak{n} be a p-step nilpotent Lie algebra, p > 1, and let \mathfrak{z} be the center of \mathfrak{n} .

(1) If dim $\mathfrak{n} \ge ((p-1)^2 + p^2) \dim \mathfrak{z}$ then

$$\mu_{nil}(\mathfrak{n}) \geq \sqrt{\frac{2p}{p-1}} (\dim \mathfrak{n} - \dim \mathfrak{z}).$$

(2) If dim $\mathfrak{n} \leq ((p-1)^2 + p^2)$ dim \mathfrak{z} then

$$\mu_{nil}(\mathfrak{n}) \ge \sqrt{\frac{2(p-1)}{p-2}} \dim \mathfrak{n} + \frac{2p(p-1)}{(p-2)^2} \dim \mathfrak{z} - \frac{2}{p-2} \sqrt{\dim \mathfrak{z}},$$

if
$$p \neq 2$$
, and $\mu_{nil}(\mathfrak{n}) \geq \frac{\dim \mathfrak{n} + 3 \dim \mathfrak{j}}{2\sqrt{\dim \mathfrak{j}}}$ if $p = 2$.

In both cases, the given bound is bigger than $\sqrt{\frac{2(p+1)}{p}} \dim \mathfrak{n}$.

From this theorem, μ_{nil} is obtained for the following families.

(i) Given $p, a \in \mathbb{N}$, let

$$\mathfrak{n}_{a,p} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} \dots & A_{1p+1} \\ 0 & A_{23} \dots & A_{2p+1} \\ & \ddots & \vdots \\ 0 & & A_{pp+1} \\ 0 \end{pmatrix} : A_{ij} \in M_a(\mathbf{k}) \text{ para } 1 \le i < j \le p+1 \right\},$$

then $\mu(\mathfrak{n}_{a,p}) = (p+1)a.$

(ii) Given
$$a, b, c \in \mathbb{N}$$
 let

$$\mathfrak{n}_{a,b,c} = \left\{ \begin{pmatrix} 0 & A_{ab} & A_{ac} \\ 0 & A_{bc} \\ 0 & 0 \end{pmatrix} : A_{ab} \in M_{a,b}(\mathbf{k}), A_{ac} \in M_{a,c}(\mathbf{k}), A_{bc}(\mathbf{k}) \in M_{b,c} \right\}.$$
Then, if either $h = a + c$ or $a = c$ and $b \in 2a$

Then, if either b = a + c, or a = c and $b \le 2a$,

$$\mu(\mathfrak{n}_{a,b,c}) = a + b + c.$$

The above two families are nilradicals of parabolic subalgebras of simple Lie algebras of type A. The above result shows that their defining representation is faithful of minimal dimension. However this is not true for all nilradicals of type A. For instance if a = b = 1, then the Lie algebra $\mathfrak{n}_{1,1,c}$ given in (ii) satisfies $\mu(\mathfrak{n}_{1,1,c}) = \lfloor 2\sqrt{2c} \rfloor < 2 + c$ for all $c \in \mathbb{N}$, as shown in [ARo].

The paper is organized as follows. In §2 we prove Theorem 2.3 which is a key result. It allows us to obtain certain special bases for faithful representations of nilpotent Lie algebras that eventually lead, in §3, to the optimization problem of Theorem 1.1. In this section, an open question is posed. In §4 we compute μ_{nil} for the families (i) and (ii). In §5 we obtain estimates for the minimum of our optimization problem and prove Theorem 1.2.

2. Linearly independent subsets associated to chains of endomorphisms

In this section we describe an algorithm that, given a faithful n-module V, will provide a basis of V with certain special properties that will allow us to estimate dim V.

First, we recall the following standard lemma.

Lemma 2.1. Let V be a vector space and let $\mathcal{T}_1, \ldots, \mathcal{T}_p$ be vector subspaces of End(V). If $r_i = \max\{\dim \mathcal{T}_i v : v \in V\}$ and $W_i = \{w \in V : \dim \mathcal{T}_i w = r_i\}$, then $\bigcap_{i=1}^p W_i$ is a non-empty open dense subset of V. In particular, there exists $v \in V$ such that $\dim \mathcal{T}_i v = r_i$ for all $i = 1, \ldots, p$.

Proof. Since the intersection of open dense subsets is a non-empty open dense subset, it suffices to prove that W_i is open and dense for all *i*. Let us fix $i = 1, \ldots, p$, and let $w \in W_i$ and let $\{T_1, \ldots, T_{r_i}\} \subseteq \mathcal{T}_i$ be such that

$$\{T_1(w),\ldots,T_{r_i}(w)\}$$

is a basis of $\mathcal{T}_i w$. For any $v \in V$, A(v) denote the matrix whose columns are the coordinates of $T_1(v), \ldots, T_{r_i}(v)$ in a given basis B of V. Since $\{T_1(w), \ldots, T_r(w)\}$ is a linearly independent set, the matrix A(v) has an $(r \times r)$ -minor a(v) such that det $a(w) \neq 0$. Therefore, the open set $U = \{v \in V : \det a(v) \neq 0\}$ contains w, is contained in W_i and, since k is an infinite field, it is dense. \Box **Definition 2.2.** Given a vector space V and a sequence of vector subspaces $\mathcal{T}_1, \ldots, \mathcal{T}_p$ of End(V), we say that $v \in V$ is *rank-vector* for the sequence $\mathcal{T}_1, \ldots, \mathcal{T}_p$, if

$$\dim \mathcal{T}_i v = \max\{\dim \mathcal{T}_i w : w \in V\}$$

for all $i = 1, \ldots, p$.

Let $\mathcal{T}_p \subset \cdots \subset \mathcal{T}_1 = \mathcal{T}$ be a chain of vector subspaces of $\operatorname{End}(V)$ and let $\{v_1, v_2, v_3, \ldots\}$ the sequence (which eventually will be finite) obtained by applying the following procedure:

- (1) choose a rank-vector v_1 for the chain \mathcal{T} ,
- (2) choose a (special) linear complement \mathcal{T}' of the annihilator of v_1 in \mathcal{T} ,
- (3) choose a rank-vector v_2 for the chain \mathcal{T}' ,

and so on. More precisely, the procedure is given by the following algorithm.

- (i) For all k = 1, ..., p, let $s_k := 0$ and $\mathcal{R}_k := \mathcal{T}_k$. Let i := 0, q := p.
- (ii) Increase *i* by 1.
- (iii) Let v_i be a rank-vector associated to $\mathcal{R}_q \subset \cdots \subset \mathcal{R}_1.$
- (iv) For all $k = 1, \ldots, q$, let

 $\tilde{\mathcal{R}}_k = \operatorname{Ann}_{\mathcal{R}_k}(v_i) = \{ T \in \mathcal{R}_k : T(v_i) = 0 \},\$

If $ilde{\mathcal{R}}_k
eq \mathcal{R}_k$, increase s_k by 1 and let $\mathcal{T}_{k,i}$ be such that

 $\mathcal{R}_k = \mathcal{T}_{k,i} \oplus \tilde{\mathcal{R}}_k$ and $\mathcal{T}_{k,i} \supseteq \mathcal{T}_{k+1,i}$ (assume $\mathcal{T}_{q+1,i} = 0$).

(v) If $\tilde{\mathcal{R}}_1 \neq 0$, let q be the largest j such that $\tilde{\mathcal{R}}_j \neq 0$ and let

$$\mathcal{R}_k := \tilde{\mathcal{R}}_k, \qquad k = 1, \ldots, q$$

(we have $\mathcal{R}_q \subset \cdots \subset \mathcal{R}_1$). Go to (ii).

(vi) End.

As a result we obtain:

- (a) A partition $s_1 \ge s_2 \ge \cdots \ge s_p > 0$, $(s_1 \text{ is the final value of } i)$.
- (b) A set $\{v_1, v_2, \ldots, v_{s_1}\}$.
- (c) A family of subspaces $\mathcal{T}_{k,j} \subset \operatorname{End}(V)$, $1 \leq j \leq s_k$ and $1 \leq k \leq p$.

The following theorem summarizes some of the main properties of the set $\{v_1, v_2, \ldots, v_{s_1}\}$ and the family of subspaces $\mathcal{T}_{k,j} \subset \operatorname{End}(V)$.

Theorem 2.3. Let V be a vector space and let $\mathcal{T}_p \subset \cdots \subset \mathcal{T}_1$ be a chain of subspaces in End(V). Then there exist a partition $s_1 \geq s_2 \geq \cdots \geq s_p > 0$, a linearly independent set $\{v_1, \ldots, v_{s_1}\} \subset V$ and a family of subspaces $\mathcal{T}_{k,j} \subset End(V)$, $1 \leq j \leq s_k$ and $1 \leq k \leq p$, such that:

We notice that this display resembles the Young diagram of the partition $s_1 \ge s_2 \ge \cdots \ge s_p$.

(2) dim $\mathcal{T}_{k,j} = \dim \mathcal{T}_{k,j}v_j$ for $j = 1, \ldots, s_k$ and $k = 1, \ldots, p$. (3) $\mathcal{T}_{k,j}v_i = 0$ for $1 \le i < j \le s_k$ and $k = 1, \ldots, p$. (4) $\mathcal{T}_{k,j}V \subseteq \mathcal{T}_{k,i}v_i$ for $1 \le i < j \le s_k$ and $k = 1, \ldots, p$. Moreover, if \mathcal{T}_1 consists of nilpotent operators and $[\mathcal{T}_1, \mathcal{T}_{p_0}] = 0, 1 \le p_0 \le p$, then $\mathcal{T}_{1,1}v_1 \cap \operatorname{span}_k\{v_1, \ldots, v_{s_{p_0}}\} = 0$.

Proof. By construction, it is clear that properties (1), (2) and (3) hold.

We first prove that $\{v_1, v_2, \ldots, v_{s_1}\}$ is linearly independent. By construction, we may assume, as an induction hypothesis, that $\{v_2, \ldots, v_{s_1}\}$ is linearly independent. Thus we must show that $v_1 \notin \operatorname{span}_k\{v_2, \ldots, v_{s_1}\}$. If

(2.1)
$$v_1 = \sum_{j=2}^{s_1} a_j v_j$$

let $j_0 = \max\{j : a_j \neq 0\} \ge 2$ and let $T \in \mathcal{T}_{1,j_0} \subset \mathcal{T}_1, T \ne 0$. We now apply T to both sides of (2.1). Property (3) implies that the left-hand side is zero and the right hand side is $a_{j_0}T(v_{j_0})$. On the other hand, property (2) says that $T(v_{j_0}) \ne 0$, which is a contradiction.

We now prove (4). If $s_1 = 1$ then $\mathcal{T}_{k,1} = \mathcal{T}_k$ for all $k = 1, \ldots, p$ and condition (4) is empty. As we did earlier, we may assume by induction that $\mathcal{T}_{k,j}V \subseteq \mathcal{T}_{k,i}v_i$ for $2 \leq i < j \leq s_k$ and $k = 1, \ldots, p$. Thus, we only need to prove (4) when i = 1. This is equivalent to prove that $T(v) \in \mathcal{T}_{k,1}v_1$ for all $v \in V$ and all $T \in \mathcal{T}_{k,2} \oplus \cdots \oplus \mathcal{T}_{k,s_k}$, $k = 1, \ldots, p$. If $r_k = \dim \mathcal{T}_{k,1}v_1$ and $\{T_1, \ldots, T_{r_k}\}$ is a basis of $\mathcal{T}_{k,1}$, we must show that

$$T(v), T_1(v_1), \ldots, T_{r_k}(v_1)$$

is linearly dependent for all $T \in \mathcal{T}_{k,2} \oplus \cdots \oplus \mathcal{T}_{k,s_k}$, $k = 1, \ldots, p$, and all $v \in V$. Let us fix such T, v and k. By the definition of v_1 , the set

$$\{T(v_1+tv), T_1(v_1+tv), \dots, T_{r_k}(v_1+tv)\}$$

is linearly dependent for all $t \in k$. Since $T(v_1) = 0$, we obtain that $\{T(v), T_1(v_1+tv), \ldots, T_{r_k}(v_1+tv)\}$ is linearly dependent for all $t \neq 0$. Since k infinite, we conclude that this last set is linearly dependent for t = 0. This completes the proof of (4).

We now prove the 'moreover' part of the theorem. We must show that

$$\mathcal{T}_{1,1}v_1 \cap \operatorname{span}_{\mathbf{k}}\{v_1, \dots, v_{s_{p_0}}\} = 0.$$

Suppose, on the contrary, that there exist $T \in \mathcal{T}_{1,1}$, $T \neq 0$, and $a_1, \ldots, a_{s_{p_0}} \in k$ such that

(2.2)
$$T(v_1) = \sum_{j=0}^{s_{p_0}} a_j v_j.$$

Since $T \in \mathcal{T}_{1,1}$ and $T \neq 0$, it follows that $T(v_1) \neq 0$ and thus $a_j \neq 0$ for some j. Let $j_0 = \max\{j : a_j \neq 0\}$. Since T is nilpotent, its only eigenvalue is zero and thus $1 < j_0 \leq s_{p_0}$.

Let $T' \in \mathcal{T}_{p_0,j_0}, T' \neq 0$, and let us apply T' to both sides of (2.2). Since $T' \in \mathcal{T}_{p_0,j_0}$ and $j_0 > 1$ we obtain on the left hand side $T'T(v_1) = TT'(v_1) = 0$. On the other hand, it follows from properties (2) and (3) that the right hand side is $a_{j_0}T'(v_{j_0}) \neq 0$, which is a contradiction.

3. An optimization problem leading to a lower bound for μ_{nil}

Let V be a vector space and let \mathfrak{n} be a Lie subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms. Let

$$\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_2 \subset \mathfrak{n}_1 = \mathfrak{n}$$

be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} for some $1 \leq p_0 \leq p$.

Applying Theorem 2.3 to the filtration $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_2 \subset \mathfrak{n}_1 = \mathfrak{n}$ we obtain a partition $s_1 \geq s_2 \geq \cdots \geq s_p > 0$, a linearly independent set $\{v_1, \ldots, v_{s_1}\} \subset V$, and a decomposition

 $\begin{array}{rclcrcl} \mathfrak{n}_{1} & = & \mathfrak{n}_{1,1} & \oplus \cdots \oplus & \mathfrak{n}_{1,s_{p}} & \oplus \cdots \oplus & \mathfrak{n}_{1,s_{p-1}} & \oplus \cdots \oplus & \mathfrak{n}_{1,s_{2}} & \oplus \cdots \oplus & \mathfrak{n}_{1,s_{1}} \\ \cup & \cup & & \cup & & \cup & & \cup \\ \mathfrak{n}_{2} & = & \mathfrak{n}_{2,1} & \oplus \cdots \oplus & \mathfrak{n}_{2,s_{p}} & \oplus \cdots \oplus & \mathfrak{n}_{2,s_{p-1}} & \oplus \cdots \oplus & \mathfrak{n}_{2,s_{2}} \\ \cup & \cup & & \cup & & \cup & & \cup \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \cup & \cup & & \cup & & \cup & & \cup \\ \mathfrak{n}_{p-1} & = & \mathfrak{n}_{p-1,1} & \oplus \cdots \oplus & \mathfrak{n}_{p-1,s_{p}} & \oplus \cdots \oplus & \mathfrak{n}_{p-1,s_{p-1}} \\ \cup & & \cup & & \cup & & \cup \\ \mathfrak{n}_{p} & = & \mathfrak{n}_{p,1} & \oplus \cdots \oplus & \mathfrak{n}_{p,s_{p}} \end{array}$ such that

$$\mathfrak{n}_{1,1}v_1 \cap \operatorname{span}_{\mathbf{k}}\{v_1,\ldots,v_{s_{n_0}}\} = 0$$

Let $r_k = \dim \mathfrak{n}_{k,1}$. Since $\mathfrak{n}_{k,1} \subseteq \mathfrak{n}_{k-1,1}$, there exists a basis $\{X_1, \ldots, X_{r_1}\}$ of $\mathfrak{n}_{1,1}$ such that $\{X_1, \ldots, X_{r_k}\}$ is a basis of $\mathfrak{n}_{k,1}$, $k = 1, \ldots, p$. It follows from Theorem 2.2 that

It follows from Theorem 2.3 that

(3.2)
$$\{X_1(v_1), \dots, X_{r_k}(v_1)\}$$

is a basis of $\mathfrak{n}_{k,1}v_1$, $k = 1, \ldots, p$. We now fix an ordered basis

(3.3)
$$B = \{X_1(v_1), \dots, X_{r_1}(v_1), w_1, \dots, w_q, v_1, \dots, v_{s_{p_0}}\}$$

of V and let

$$W = \operatorname{span}_{k} \{ w_{1}, \dots, w_{q} \},$$
$$V_{0} = \operatorname{span}_{k} \{ v_{1}, \dots, v_{s_{p_{0}}} \}.$$

We now consider the matrix of a given $X \in \mathfrak{n}$ with respect to the basis B

$$[X]_{B} = \begin{bmatrix} \overbrace{A_{1,1}(X)}^{r_{1}} & \overbrace{A_{1,2}(X)}^{q} & \overbrace{A_{1,3}(X)}^{s_{p_{0}}} \\ A_{2,1}(X) & A_{2,2}(X) & A_{2,3}(X) \\ A_{3,1}(X) & A_{3,2}(X) & A_{3,3}(X) \end{bmatrix} \right\} r_{1}$$

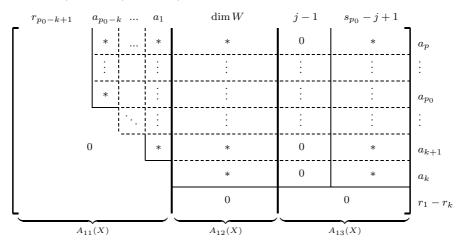
where the row and columns correspond to the decomposition

$$V = \mathfrak{n}_{1,1}v_1 \oplus W \oplus V_0.$$

The following proposition describe the main properties of $[X]_B$.

Proposition 3.1. Let $X \in \mathfrak{n}_{k,j}$ for some k = 1, ..., p and $j = 1, ..., s_k$. (1) If j = 1 then $(A_{1,3}(X))_{h,1} = 0$ for all $h = r_k + 1, ..., r_1$. In addition, $(A_{1,3}(X))_{h,1} = 0$ for all $h = 1, ..., r_1$ if and only if X = 0.

(2) If $j \ge 2$ then $A_{m,n}(X) = 0$ for m = 2, 3, n = 1, 2, 3. On the other hand, the row $A_{1,1}(X) A_{1,2}(X) A_{1,3}(X)$ has the following structure



where $a_h = r_h - r_{h+1}$, h = 1, ..., p-1 and $a_p = r_p$. In particular, if $k \ge p_0$, then $A_{1,1}(X) = 0$.

Proof. Part (1) is a consequence of Theorem 2.3(1) and (4).

If $j \geq 2$, it follows from Theorem 2.3(4) that $X(v) \in \mathfrak{n}_{k,1}v_1$ for all $v \in V$. This proves that $(A_{1,*}(X))_{h,*} = 0$ for all $h \geq k$. It follows from Theorem 2.3(3) that $(A_{1,3}(X))_{*h} = 0$ for all $h \leq j - 1$.

Finally, let us prove that $A_{1,1}(X)$ has the staircase-shape stated above. If $i = 1, \ldots, r_1$, then i^{th} element of B is $X_i(v_1)$. If additionally $i \leq r_h$, for some $h = 1, \ldots, p$, then $X_i \in \mathfrak{n}_{h,1}$ and since $X \in \mathfrak{n}_{k,j}$ we obtain $[X, X_i] \in \mathfrak{n}_{k+h}$. Thus

$$XX_{i}(v_{1}) = X_{i}X(v_{1}) + [X, X_{i}](v_{1})$$

= [X, X_{i}](v_{1}) \epsilon \mathbf{n}_{k+h}v_{1} (since j \ge 2).

This completes the proof.

Question. Since $\mathfrak{n} \subset \mathfrak{gl}(V)$ consists of nilpotent endomorphisms, it would be very interesting to obtain a basis B such that $[X]_B$ is upper triangular for all $X \in \mathfrak{n}$, in addition to the properties stated in Proposition 3.1 (or similar ones). This would transform Proposition 3.1 into a detailed version of Lie's Theorem that takes into account a given filtration of the Lie algebra \mathfrak{n} . As stated, Proposition 3.1 is enough to obtain the lower bounds that we are looking for.

Theorem 3.2. Let \mathfrak{n} be a Lie subalgebra of nilpotent operators of $\mathfrak{gl}(V)$ and let $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_1 = \mathfrak{n}$ be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} . Then there exists integers $a_k \geq 0$, $k = 0, \ldots, p$, with $a_0, a_p \geq 1$, such that:

(1) dim
$$\mathfrak{n}_k \leq \sum_{i=0}^{p_0-\kappa} a_i (a_{k+i} + \dots + a_p)$$
 for $k = 1, \dots, p_0$.

(2) dim
$$\mathfrak{n}_k \leq a_0 (a_k + \cdots + a_p)$$
 for $k = p_0, \ldots, p$.

(3) dim $V = a_0 + a_1 + \dots + a_p$.

Proof. Let B be the basis of V as in (3.3) and let $T : \mathfrak{n} \to \mathfrak{gl}(V) \oplus V$ be defined by

$$T(X) = \begin{cases} X(v_1) \in V, & \text{if } X \in \mathfrak{n}_{1,1}; \\ X \in \mathfrak{gl}(V), & \text{if } X \in \mathfrak{n}_{1,j}, j \ge 2 \end{cases}$$

It follows from Theorem 2.3 that T is injective. We apply Proposition 3.1 to obtain a bound for dim $T(\mathfrak{n}_k)$. On the one hand, we know from Theorem 2.3(2) that

$$\dim T(\mathfrak{n}_{k,1}) = r_k.$$

On the other hand, from Proposition 3.1(2), when $j \ge 2$, we know the shape of the matrices $[T(\mathfrak{n}_{k,j})]_B$. Taking into account that the first column of $A_{1,3}(T(\mathfrak{n}_{k,j}))$ is zero if $j \ge 2$, we obtain

$$\dim T\left(\bigoplus_{j=2}^{o_k} n_{k,j}\right) \leq \underbrace{(\dim W + \mathfrak{s}_{p_0})r_k - r_k}_{\text{size of } A_{1,2} \text{ and } A_{1,3}} + \underbrace{a_1r_{k+1} + a_2r_{k+2} + \dots + a_{p_0-k}r_{p_0}}_{\text{size of the staircase in } A_{1,1}, \text{ it appears only if } k < p_0}$$

where $a_h = r_h - r_{h+1} \ge 0, \ h = 1, ..., p - 1$. Therefore

$$\dim T(n_k) \leq \begin{cases} (\dim W + \mathfrak{s}_{p_0})r_k + \sum_{i=1}^{p_0-k} a_i r_{k+i}, & \text{if } k < p_0; \\ (\dim W + \mathfrak{s}_{p_0})r_k, & \text{if } k \ge p_0. \end{cases}$$

If $a_p = r_p \ge 1$ and $a_0 = \dim W + \mathfrak{s}_{p_0} \ge 1$, then rewriting the above inequality in terms of $a_k's$, we obtain

$$\dim T(n_k) \le \begin{cases} a_0(a_k + \dots + a_p) + \sum_{i=1}^{p_0 - k} a_i(a_{k+i} + \dots + a_p), & \text{if } k < p_0; \\ a_0(a_k + \dots + a_p), & \text{if } k \ge p_0. \end{cases}$$

This shows (1) and (2).

S1-

Finally $a_0 + a_1 + \dots + a_p = \dim W + \mathfrak{s}_{p_0} + r_1 = \dim V.$

Theorem 3.2 leads us to consider the following optimization problem.

Problem 3.3. Given integer numbers $p \ge p_0 \ge 1$ and n_1, \ldots, n_p , let

$$r_k = a_k + a_{k+1} + \dots + a_p, \quad a_0, a_1, \dots, a_p \in \mathbb{Z},$$

for $k = 0, \ldots, p$. Find the minimum value r_0^{\min} of

$$r_0 = a_0 + a_1 + \dots + a_p,$$

subject to the following restrictions

(a) $a_0, a_p \ge 1$ and $a_k \ge 0$, for k = 1, ..., p - 1; (b) $\sum_{i=0}^{p_0-k} a_i r_{k+i} \ge n_k$, for $k = 1, ..., p_0$; (c) $a_0 r_k \ge n_k$, for $k = p_0, ..., p$.

The solution to this problem gives us a lower bound for μ_{nil} .

Corollary 3.4. Let \mathfrak{n} be a Lie algebra and let $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_1 = \mathfrak{n}$ be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} . Then

$$\mu_{nil}(\mathfrak{n}) \ge r_0^{min}$$

where r_0^{\min} is the minimum value of Problem 3.3 associated to $p \ge p_0 \ge 1$ and $n_k = \dim \mathfrak{n}_k$ with $k = 1, \ldots, p$.

The optimization problem above seems to be difficult and, in this paper, we will just give a pair of quick, but not trivial, estimates of its solution. In the last section we discuss two simplifications of Problem 3.3 that lead respectively to Theorem 3.5 and Theorem 3.6 below. We think that it is worth studying Problem 3.3 in more detail in the future to obtain more accurate results than the following two theorems.

Theorem 3.5 (First simplification). Let \mathfrak{n} be a nilpotent Lie algebra and let $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_1 = \mathfrak{n}$ be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} for some $p_0 = 1, \ldots, p$. Then

$$\mu_{nil}(\mathfrak{n}) \ge \sqrt{\frac{2(p_0+1)}{p_0}} \dim \mathfrak{n}.$$

In particular, if \mathfrak{n} is p-step nilpotent Lie algebra then $\mu_{nil}(\mathfrak{n}) \geq \sqrt{\frac{2(p+1)}{p} \dim \mathfrak{n}}$.

Theorem 3.6 (Second simplification). Let \mathfrak{n} be a nilpotent Lie algebra and let $\mathfrak{n}_p \subset \cdots \subset \mathfrak{n}_1 = \mathfrak{n}$ (p > 1) be a filtration of \mathfrak{n} such that \mathfrak{n}_{p_0} is contained in the center of \mathfrak{n} for some $p_0 = 2, \ldots, p$ and let $n_i = \dim \mathfrak{n}_i, i = 1, \ldots, p$.

(1) If $n_1 \ge ((p_0 - 1)^2 + p_0^2)n_{p_0}$ then

$$\mu_{nil}(\mathfrak{n}) \ge \sqrt{\frac{2p_0}{p_0 - 1}(n_1 - n_{p_0})}.$$

(2) If $n_1 \leq ((p_0 - 1)^2 + p_0^2)n_{p_0}$ then

$$\mu_{nil}(\mathfrak{n}) \ge \sqrt{\frac{2(p_0-1)}{p_0-2}}n_1 + \frac{2p_0(p_0-1)}{(p_0-2)^2}n_{p_0} - \frac{2}{p_0-2}\sqrt{n_{p_0}},$$

if $p_0 \neq 2$, and $\mu_{nil}(\mathfrak{n}) \geq \frac{n_1+3n_2}{2\sqrt{n_2}}$, if $p_0 = 2$. In both cases, the given bound is bigger than $\sqrt{\frac{2(p_0+1)}{p_0}n_1}$.

Both results are proved in §5. Although Theorem 1.2 is an immediate corollary of Theorem 3.6 they will be treated separately since it is much easier to obtain directly Theorem 1.2 from Corollary 3.4. This will also show some of the difficulties involved in Problem 3.3.

4. Some applications

(1) Given $p, a \in \mathbb{N}$, let

$$\mathfrak{n}_{a,p} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} & \dots & A_{1p+1} \\ 0 & A_{23} & \dots & A_{2p+1} \\ & \ddots & & \vdots \\ 0 & & & A_{pp+1} \\ 0 & & & 0 \end{pmatrix} : A_{ij} \in M_a(\mathbf{k}) \text{ para } 1 \le i < j \le p+1 \right\}.$$

It is clear that $\mathfrak{n}_{a,p}$ is a *p*-step nilpotent Lie subalgebra of $\mathfrak{sl}((p+1)a, \mathbf{k})$ and dim $\mathfrak{n}_{a,p} = \frac{(p+1)p}{2}a^2$. Its defining representation has dimension (p+1)a. Since Theorem 3.5 states that

$$\mu(\mathfrak{n}_{a,p}) \ge \sqrt{\frac{2(p+1)}{p}} \dim \mathfrak{n}_{a,p} = (p+1)a,$$

we obtain $\mu(\mathfrak{n}_{a,p}) = (p+1)a$. (2) Given $a, b, c \in \mathbb{N}$ let

$$\mathfrak{n}_{a,b,c} = \left\{ \begin{pmatrix} 0 & A_{ab} & A_{ac} \\ 0 & A_{bc} \\ 0 \end{pmatrix} : A_{ab} \in M_{a,b}(\mathbf{k}), A_{ac} \in M_{a,c}(\mathbf{k}), A_{bc}(\mathbf{k}) \in M_{b,c} \right\}.$$

Now $\mathfrak{n}_{a,b,c}$ is a 2-step nilpotent Lie subalgebra of $\mathfrak{sl}(a+b+c,\mathbf{k})$ and $\dim \mathfrak{n}_{a,p} = ab + bc + ac$. The center of $\mathfrak{n}_{a,b,c}$ is the dimension ac.

If b = a + c then we are under the conditions stated in part (1), Theorem 1.2 and the given lower bound for $\mu(\mathbf{n}_{a,b,c})$ coincides with the dimension of the defining representation of $\mathbf{n}_{a,b,c}$.

If a = c and $b \leq 2a$ then we are under the conditions stated in (2), Theorem 1.2 and the given lower bound for $\mu(\mathbf{n}_{a,b,c})$ coincides with the dimension of the defining representation of $\mathbf{n}_{a,b,c}$.

Thus, if either b = a + c, or a = c and $b \le 2a$, we have

$$\mu(\mathfrak{n}_{a,b,c}) = a + b + c.$$

We point out that in some cases $\mu(\mathfrak{n}_{a,b,c}) < a+b+c$. For instance, it is shown in [ARo] that

$$\mu(\mathfrak{n}_{1,1,c}) = \left\lceil 2\sqrt{2c} \right\rceil < 1 + 1 + c.$$

for all $c \in \mathbb{N}$.

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5. Estimates for the solution of Problem 3.3

In this section we will show some bounds for r_0^{\min} resulting from considering Problem 3.3 with real (instead of integer) variables. Since r_0 is linear, it is clear that, in this case, r_0^{\min} will be reached in a boundary point of the restriction set.

5.1. A first simplification. It is clar that r_0^{\min} is greater than or equal to the minimum of

$$r_0 = a_0 + \dots + a_p$$

subject to

(a') $a_0, a_p > 0 \text{ and } a_k \in \mathbb{R}_{\geq 0}, \text{ for } k = 0, \dots, p;$ (b') $\sum_{i=0}^{p_0-1} a_i r_{1+i} \geq n_1;$

We will think r_{p_0} as an independent variable in this problem and thus we can reformulate it looking for a minimum of

$$r_0 = a_0 + \dots + a_{p_0 - 1} + r_{p_0}$$

subject to

(a')
$$a_0, r_{p_0} > 0$$
 and $a_k \in \mathbb{R}_{\geq 0}$, for $k = 0, \dots, p_0 - 1$;
(b') $\sum_{i=0}^{p_0-1} a_i r_{1+i} \ge n_1$; (here $r_i = a_i + \dots + a_{p_0-1} + r_{p_0}$)

Let us call this problem as Problem (a'b') for p_0 . We notice that, if we consider Problem (a'b') for p_0 with the additional restriction $a_k = 0$ for some $k = 1, \ldots, p_0 - 1$, then the problem becomes Problem (a'b') for $p_0 - 1$. Therefore, in order to find the minimum value of r_0 we may consider $a_k > 0$ for all $k = 0, \ldots, p_0 - 1$. Moreover, since the minimum will be reached at the boundary, we can reformulate Problem (a'b') for p_0 as: find the minimum of r_0 subject to

(a')
$$a_k, r_{p_0} > 0$$
, for $k = 0, \dots, p_0 - 1$;
(b') $\sum_{i=0}^{p_0-1} a_i r_{1+i} = n_1$.

We now will find the minimum value of r_0 in this problem. We use (b') in order to eliminate the variable a_0 . Thus we will think r_0 as a function of $a_1, \ldots, a_{p_0-1}, r_{p_0}$, and we will find the critical values of r_0 , and next its minimum.

It is not difficult to see that the only critical value of r_0 is

$$(a_1, \ldots, a_{p_0-1}, r_{p_0}) = (a_0, a_0, \ldots, a_0, a_0)$$

with $n_1 = \frac{p_0(p_0+1)}{2} a_0^2$. Also, it is not difficult to see that this is a local minimum and it yields

$$r_0 = \sqrt{\frac{2(p_0+1)}{p_0}n_1}.$$

This is in fact a global minimum. Indeed, since $\sqrt{\frac{2(p_0+1)}{p_0}n_1}$ is decreasing as a function of p_0 , taking into account the remark explained above, we can not obtain smaller values of r_0 by allowing $a_k = 0$ for some k.

5.2. A second simplification. In this case, we can do an analysis similar to what we did in the first simplification to conclude that r_0^{\min} is greater than or equal to the minimum of

(5.1)
$$r_0 = a_0 + \dots + a_{p_0-1} + r_{p_0}$$

subject to

(a')
$$a_k, r_{p_0} > 0$$
, for $k = 0, \dots, p_0 - 1$;
(b') $\sum_{i=0}^{p_0-1} a_i r_{1+i} = n_1$;
(c') $a_0 r_{p_0} = n_{p_0}$,

In this case we will use (b') and (c') in order to eliminate the variables r_{p_0} and a_{p_0-1} . Thus we will think r_0 as a function of a_0, \ldots, a_{p_0-2} , we will find its critical values, and its minimum.

It follows from (c') that

$$\frac{\partial r_{p_0}}{\partial a_j} = \begin{cases} -\frac{r_{p_0}}{a_0}, & j = 0; \\ 0, & 1 \le j \le p_0 - 2. \end{cases}, \quad \frac{\partial^2 r_{p_0}}{\partial a_i a_j} = \begin{cases} \frac{2r_{p_0}}{a_0^2}, & i = j = 0; \\ 0, & 1 \le i, j \le p_0 - 2. \end{cases}$$

It follows from (b') that

(5.3)
$$n_1 = (r_0 - r_{p_0-1})(a_{p_0-1} + r_{p_0}) + a_{p_0-1}r_{p_0} + \sum_{i=0}^{p_0-3} a_i(r_{i+1} - r_{p_0-1}),$$

and we obtain from (5.3)

(5.4)
$$\frac{\partial a_{p_0-1}}{\partial a_j} = \begin{cases} -\frac{(r_0 - a_0 - r_{p_0})(a_0 - r_{p_0})}{(r_0 - a_{p_0-1})a_0}, & j = 0; \\ -\frac{r_0 - a_j}{r_0 - a_{p_0-1}}, & 1 \le j \le p_0 - 2. \end{cases}$$

and (5.5)

$$\frac{\partial^2 a_{p_0-1}}{\partial a_i a_j} = \begin{cases} \frac{2(r_0 - a_0 - r_{p_0})(a_0^2 - r_{p_0}(2a_0 + r_0 - r_{p_0-1}))}{(r_0 - a_{p_0-1})^2 a_0^2}, & i = j = 0; \\ \frac{(r_0 + a_{p_0-1} - a_0 - a_i - r_{p_0})(a_0 - r_{p_0})}{(r_0 - a_{p_0-1})^2 a_0}, & 0 = j < i \le p_0 - 2; \\ \frac{(r_0 + a_{p_0-1} - a_j - a_i)}{(r_0 - a_{p_0-1})^2}, & 0 < j < i \le p_0 - 2; \\ \frac{2(r_0 - a_j)}{(r_0 - a_{p_0-1})^2}, & 0 < j = i \le p_0 - 2; \end{cases}$$

Therefore, it follows from (5.1), (5.2) and (5.4) that

$$\frac{\partial r_0}{\partial a_j} = \begin{cases} \frac{(r_{p_0} + a_0 - a_{p_0-1})(a_0 - r_{p_0})}{(r_0 - a_{p_0-1})a_0}, & j = 0; \\ \frac{a_j - a_{p_0-1}}{r_0 - a_{p_0-1}}, & 1 \le j \le p_0 - 2. \end{cases}$$

The (possible) critical values of r_0 are two. First

$$(a_0, a_1, a_2, \dots, a_{p_0-1}, r_{p_0}) = (a_0, a_1, a_1, \dots, a_1, a_0)$$

with

$$n_1 = \frac{(p_0 - 1)(p_0 - 2)}{2} a_1^2 + 2(p_0 - 1) a_0 a_1 + a_0^2,$$

$$n_{p_0} = a_0^2;$$

whose positive solutions are $a_0 = \sqrt{n_{p_0}}$ and

$$a_1 = \frac{\sqrt{2(p_0 - 1)((p_0 - 2)n_1 + p_0 n_{p_0})} - 2(p_0 - 1)\sqrt{n_{p_0}}}{(p_0 - 1)(p_0 - 2)}.$$

This yields

(5.6)
$$r_0 = \sqrt{\frac{2(p_0 - 1)}{p_0 - 2}} n_1 + \frac{2p_0(p_0 - 1)}{(p_0 - 2)^2} n_{p_0} - \frac{2}{p_0 - 2} \sqrt{n_{p_0}}$$

This critical value always exists. The second case is

$$(a_0, a_1, a_2, \dots, a_{p_0-1}, r_{p_0}) = (a_0, a_1, a_1, \dots, a_1, a_1 - a_0)$$

with

$$n_1 = \frac{p_0(p_0 - 1)}{2} a_1^2 + a_0 a_1 - a_0^2;$$

$$n_{p_0} = n_1 - \frac{p_0(p_0 - 1)}{2} a_1^2;$$

whose positive solutions are $a_1 = \sqrt{\frac{2(n_1 - n_{p_0})}{p_0(p_0 - 1)}}$ and

$$a_0 = \sqrt{\frac{n_1 - n_{p_0}}{2p_0(p_0 - 1)}} \pm \sqrt{\frac{n_1 - n_{p_0}}{2p_0(p_0 - 1)}} - n_{p_0}.$$

These \pm critical values exist if and only if

(5.7)
$$n_1 \ge \left((p_0 - 1)^2 + p_0^2 \right) n_{p_0}$$

and either of them yields

(5.8)
$$r_0 = \sqrt{\frac{2p_0}{p_0 - 1}(n_1 - n_{p_0})}$$

If condition (5.7) holds, then the value of (5.8) is a local minimum (and the value of (5.6) is a local maximum). If condition (5.7) does not hold, then the value of (5.6) is a local minimum.

Arguing as we did with in first simplification we conclude that the value of (5.8), if (5.7) holds, and the value of (5.6), if (5.7) does not hold, is a global minimum.

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