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Maximal Inequalities in Orlicz Spaces[1]

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Abstract. Given non negative measurable real valued functions f and g, we get inequalities of the type $\int_\Omega \Psi(f)\, d\mu \le K \int_\Omega \Psi(\frac{g}{c})\, d\mu$, assuming weak type inequalities $\mu(\{f > a\}) \le K \int_{\{f>a\}} \varphi(\frac{g}{a})\, d\mu$ where φ, $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are nondecreasing functions related by $\prec_N$ and where Ψ is a Young function given by $\Psi(x) = \int_0^x \psi(t)\, dt$. We apply these results to best approximation operators and sub additive operators.

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1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{A}, \mu)$ be the set of all $\mathcal{A}$-measurable real valued functions.

By Φ we denote the set of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are nonnegative, even, nondecreasing on $[0, \infty)$, such that $\varphi(x) > 0$ for all $x > 0$, $\varphi(0+) = 0$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Let $\mathbb{R}_0^+ = [0, \infty)$. We say that a nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies Δ_2-condition, symbolically $\varphi \in \Delta_2$, if there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \le \Lambda\varphi(a)$ for all $a \ge 0$.

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A nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies ∇_2-condition, symbolically $\varphi \in \nabla_2$, if there exists a constant $\lambda = \lambda_\varphi > 2$ such that $\varphi(2a) \geq \lambda\varphi(a)$ for all $a \geq 0$.

We claim that a nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies Δ'-condition, symbolically $\varphi \in \Delta'$, if there exists a constant $K_1 > 0$ such that $\varphi(xy) \leq K_1\varphi(x)\varphi(y)$ for all $x, y \geq x_0 \geq 0$.
If $x_0 = 0$ then we say that φ satisfies the Δ'-condition globally.

Let Φ be a Young function, that is, an even and convex function $\Phi : \mathbb{R} \to \mathbb{R}_0^+$ such that $\Phi(a) = 0$ iff $a = 0$.
Unless it makes a different statement, the Young function Φ is the one given by $\Phi(x) = \int_0^x \varphi(t)\,dt$, where $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the right-continuous derivative of Φ.

If $\varphi \in \Phi$, we define

$$L^\varphi(\Omega, \mathcal{A}, \mu) = \Big\{ f \in \mathcal{M} : \int_\Omega \varphi(tf)\,d\mu < \infty \ \text{ for some } t > 0 \Big\}.$$

If φ is a Young function, then $L^\varphi(\Omega, \mathcal{A}, \mu)$ is an Orlicz Space (see [7]).

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\varphi(0) = 0$ and let $f, g : \Omega \to \mathbb{R}_0^+$ be two fixed measurable functions.
Mazzone and Zó have proved in [6] that the Weak Type Inequality

$$\mu(\{f > a\}) \leq \frac{C_w}{\varphi(a)} \int_{\{f>a\}} \varphi(g)\,d\mu \quad \text{for all } a > 0 \tag{1.1}$$

implies, *under some conditions*, the inequality

$$\mu(\{f > a\}) \leq \frac{C_w}{\varphi(a)} \int_{\{g>c.a\}} \varphi(g)\,d\mu \tag{1.2}$$

for all $a > 0$ and some $c \in (0, 1)$;
then, from (1.2), they reach the Strong Type Inequality

$$\int_\Omega \Psi(f)\,d\mu \leq 2C_w\rho \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\,d\mu \tag{1.3}$$

for a class of Young functions $\Psi \in C^1([0, \infty))$ whose derivative ψ is related, in some way, to φ.

We wish to develop a similar scheme leaving from a different weak type inequality, that is

$$\mu(\{f > a\}) \leq C_w \int_{\{f>a\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu, \quad \text{for all } a > 0; \tag{1.4}$$

moving on to other weak type inequality, different to (1.2), like

$$\mu(\{f > a\}) \leq C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu, \quad \text{for all } a > 0 \tag{1.5}$$

and some $c > 0$; and finally to obtain a strong type inequality like

$$\int_\Omega \Psi(f)\,d\mu \le C_w K \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\,d\mu \tag{1.6}$$

where K is a positive constant depending only on c and ρ.

2. Weak Type Inequalities

First, we state conditions to reach the Weak Type Inequality (1.5) from (1.4), as it has done in [6] to get (1.1) from (1.2).

Lemma 2.1. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\varphi(0) = 0$.*
Suppose that f and g are nonnegative measurable functions satisfying (1.4).
If $\varphi(0+) = 0$, then there exists a constant $c > 0$ such that

$$\mu(\{f > a\}) \le 2C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu, \quad \textit{for all } a > 0.$$

Proof. From the hypothesis, we choose $c > 0$ such that $1 - C_w\varphi(c) > \frac{1}{2}$.
We write $\{f > a\} = (\{g \le ca\} \cap \{f > a\}) \cup (\{g > ca\} \cap \{f > a\})$, we split the integral in the right hand side of (1.4) on the sets $\{g \le ca\} \cap \{f > a\}$ and $\{g > ca\} \cap \{f > a\}$ and we employ the fact that φ is a nondecreasing function to obtain

$$\mu(\{f > a\}) \le C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu + C_w\varphi(c)\mu(\{f > a\} \cap \{g \le ca\}).$$

Owing to $\mu(\{f > a\}) \le \mu(\{f > a\} \cap \{g \le ca\})$, we have

$$\mu(\{f > a\}) \le C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu + C_w\varphi(c)\mu(\{f > a\}),$$

and consequently $\quad [1 - C_w\varphi(c)]\mu(\{f > a\}) \le C_w \displaystyle\int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu.$

Since $1 - C_w\varphi(c) > \frac{1}{2}$, we get $\frac{C_w}{1 - C_w\varphi(c)} < 2C_w$ and eventually

$$\mu(\{f > a\}) \le 2C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu \quad \forall a > 0.$$

□

Remark 1. If $c \ge 1$ in (1.5), then there exists $k \in (0, 1)$ such that

$$\mu(\{f > a\}) \le 2C_w \int_{\{g>ka\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu \quad \forall a > 0.$$

Remark 2. In Lemma 2.1 we **only** demand $\varphi(0+) = 0$ regardless of the condition $\varphi(rx) \le \frac{1}{2}\varphi(x)$ for a constant $r \in (0, 1)$ and for all $x > 0$ which is essential to prove Lemma 2.2 in [6].

Next, we exhibit measurable functions f, $g : \Omega \to \mathbb{R}_0^+$, and a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\varphi(0+) = 0$ verifying (1.5), that is,

$$\mu(\{f > a\}) \leq K_1 \int_{\{g > c_1 a\}} \varphi\Big(\frac{g}{a}\Big)\, d\mu$$

for all $a > 0$ and for a pair of constants $K_1 > 0$ and $c_1 > 0$; while, (1.2) does not hold, i.e, the following inequality

$$\mu(\{f > a_1\}) \leq \frac{C}{\varphi(a_1)} \int_{\{g > c a_1\}} \varphi(g)\, d\mu$$

is false for some $a_1 > 0$ and for any pair of positive constants C and c.

Let $\Omega = \mathbb{R}_0^+$, $\varphi(x) = e^x - 1$ and $g(x) = \frac{1}{2}\chi[0,1]$ where $\mu = |.|$ is the Lebesgue measure. For a fixed number $c > 0$, we have

$$\int_{\{g > ca\}} \varphi\Big(\frac{g}{a}\Big)\, dx = \begin{cases} \varphi(\frac{1}{2a}) & \text{if } a < \frac{1}{2c} \\ 0 & \text{if } a \geq \frac{1}{2c}. \end{cases}$$

The function $F(a) = \varphi(\frac{1}{2a})$ is decreasing, continuous and it also satisfies $\lim_{a\to\infty} F(a) = 0$ y $F(0+) = \infty$.

Let $f(x) = \begin{cases} F^{-1}(x) & \text{if } x > \varphi(c) \\ \frac{1}{2c} & \text{if } 0 < x \leq \varphi(c) \end{cases}$, then $|\{f > a\}| = \begin{cases} F(a) & \text{if } a < \frac{1}{2c} \\ 0 & \text{if } a \geq \frac{1}{2c} \end{cases}$.

Consequently, (1.5) is true with $c = 2C_w = 1$.

On the other hand, if $a < \frac{1}{2c}$ then $\displaystyle\int_{\{g > \tilde{c}a\}} \frac{\varphi(g)}{\varphi(a)}\, dx = \frac{\varphi(\frac{1}{2})}{\varphi(a)}$.

Therefore, for every pair of positive constants K and c there exists $a :\ 0 < a < \min\{\frac{1}{2\tilde{c}}; \frac{1}{2c}\}$ such that

$$K \int_{\{g > \tilde{c}.a\}} \frac{\varphi(g)}{\varphi(a)}\, dx < \int_{\{g > \tilde{c}a\}} \varphi\Big(\frac{g}{a}\Big)\, dx,$$

since $\quad \dfrac{\varphi(\frac{1}{2a}).\varphi(a)}{\varphi(\frac{1}{2})} \to \infty \quad$ as $a \to 0$. Hence, (1.2) is not verified.

We also reach, in some cases, inequalities (1.5) and (1.4) from inequalities (1.2) and (1.1) respectively.

Proposition 2.2. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\varphi(0) = 0$ and assume $\varphi \in \Delta'$. Suppose f and g are nonnegative measurable functions. Then,* (1.2) *implies* (1.5) *and* (1.1) *implies* (1.4).

Proof. It follows straightforward from $\varphi \in \Delta'$ globally and $\varphi(a) > 0$ for any $a > 0$. ∎

3. Strong Type Inequality

Let us recall a concept introduced in [6]

Definition 3.1. *A function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is quasi-increasing iff there exists a constant $\rho > 0$ such that* $\frac{1}{x}\int_0^x \eta(t)\,dt \le \rho\eta(x)$ *for all $x \in \mathbb{R}^+$.* We will call ρ the q.i constant.

From the previous definition Mazzone and Zó, in [6], established

Definition 3.2. *Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.*
$\varphi \prec \psi$ iff $\frac{\psi}{\varphi}$ is a quasi-increasing function; that is, iff there exists a constant $\rho > 0$ such that

$$\frac{1}{x}\int_0^x \frac{\psi(t)}{\varphi(t)}\,dt \le \rho\frac{\psi(x)}{\varphi(x)} \quad \text{for all } x \in \mathbb{R}^+.$$

In Theorem 2.4 in [6], the authors employed relation $\prec$ to get a strong type inequality like (1.6). Consequently, with the aim of following an analogous pattern, we define

Definition 3.3. *Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.*
$\varphi \prec_N \psi$ iff $\{\psi(x)\varphi(\frac{\alpha}{x})\}_{\alpha\in\mathbb{R}^+}$ is a collection of quasi-increasing functions with the same q.i constant; namely, iff there exists a constant $\rho > 0$ such that

$$\frac{1}{x}\int_0^x \psi(t)\varphi\Big(\frac{\alpha}{t}\Big)\,dt \le \rho\psi(x)\varphi\Big(\frac{\alpha}{x}\Big) \quad \text{for all } x \in \mathbb{R}^+ \text{ and for all } \alpha \in \mathbb{R}^+.$$

First, we notice that $\prec$ is always a reflexive relation while $\prec_N$ is not.
In fact, for any $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ there exists $\rho \ge 1 > 0$ such that

$$\frac{1}{x}\int_0^x \frac{\varphi(t)}{\varphi(t)}\,dt \le \rho\frac{\varphi(x)}{\varphi(x)} \text{ for all } x \in \mathbb{R}^+; \text{ that is to say, } \varphi \prec \varphi.$$

However, if $\varphi(x) = x(x+1)$ there does not exist $\rho > 0$ such that

$$\frac{1}{x}\int_0^x t(t+1)\frac{\alpha}{t}\Big(\frac{\alpha}{t}+1\Big)\,dt \le \rho x(x+1)\frac{\alpha}{x}\Big(\frac{\alpha}{x}+1\Big)$$

for all $\alpha \in \mathbb{R}^+$ and for all $x \in \mathbb{R}^+$. Hence, $\varphi \not\prec_N \varphi$.

Next, we set sufficient conditions to assure the relation $\prec_N$.

Proposition 3.4. *Let $\varphi,\ \psi : \mathbb{R}^+ \to \mathbb{R}^+$.*
If $H(x) = \psi(x)\varphi(\frac{\alpha}{x})$ is a nondecreasing function from $\mathbb{R}^+$ into itself for all $\alpha > 0$, then $\varphi \prec_N \psi$.

Proof. It follows straightforward from $0 < H(t) \le H(x)$ $\forall t \in (0, x)$ due to $H(x)$ is a nondecreasing function on $(0, \infty)$. □

The following result follows straightforward from the definitions of $\prec$ and $\prec_N$.

Proposition 3.5. *Let φ, ψ, $M : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing functions.*
a) *If $\varphi \prec_N \psi$, then $\varphi \prec_N M\psi$.*
b) *If $\varphi \prec \psi$, then $\varphi \prec M\psi$.*

Proposition 3.4 claims that every nondecreasing function is a quasi-increasing one; in addition, a nonincreasing function may be a quasi-increasing one because Lemma 3.1 in [6] establishes

Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function.
If η satisfies $\eta(\frac{x}{2}) \leq K\eta(x)$ with $K < 2$, then η is quasi-increasing.

Thus, from this last result, we obtain

Proposition 3.6. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function.*
If $\varphi \in \Delta_2$ with $\Lambda_\varphi < 2$, then
a) *$\{\varphi(\frac{\alpha}{x})\}_{\{\alpha \in \mathbb{R}^+\}}$ is a collection of quasi-increasing functions with the same q.i constant.*
b) *$\frac{1}{\varphi(x)}$ is a quasi-increasing function.*

Example 3.7. *$\{\ln(\sqrt[3]{\frac{\alpha}{x}}+1)\}_{\alpha \in \mathbb{R}^+}$ is a collection of quasi-increasing functions with the same q.i constant and $\frac{1}{\ln(\sqrt[3]{x}+1)}$ is quasi-increasing.*

Remark 3. Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing functions.
a) If $\{\varphi(\frac{\alpha}{x})\}_{\{\alpha \in \mathbb{R}^+\}}$ is a collection of quasi-increasing functions with the same q.i constant, then $\varphi \prec_N \psi$.
b) If $\frac{1}{\varphi(x)}$ is a quasi-increasing function on $\mathbb{R}^+$, then $\varphi \prec \psi$.

Proposition 3.8. *Let $\Phi(x) = \int_0^x \varphi(t)\, dt$ and $\Psi(x) = \int_0^x \psi(t)\, dt$.*
Let $\Psi(x)\Phi(\frac{\alpha}{x})$ be a nonincreasing function for all $\alpha \in \mathbb{R}^+$, $\Phi \in \Delta_2$ and $\Psi \in \nabla_2$.
If $\lambda_\Psi^{-1}\Lambda_\Phi < 2$, then we have $\Phi \prec_N \Phi$.

Proof. As $\Phi \in \Delta_2$, $\exists\ \Lambda_\Phi > 0$ such that $\Phi(2x) \leq \Lambda_\Phi \Phi(x)\ \ \forall x > 0$; and due to $\Psi \in \nabla_2$, $\exists\ \lambda_\Psi > 0$ such that $\Psi(2x) \geq \lambda_\Psi \Psi(x)\ \ \forall x > 0$. Consequently, we have

$$\Psi\Big(\frac{x}{2}\Big)\Phi\Big(\frac{2\alpha}{x}\Big) \leq\ \lambda_\Psi^{-1}\Lambda_\Phi \Psi(x)\Phi\Big(\frac{\alpha}{x}\Big)\ \ \forall \alpha \in \mathbb{R}^+ \text{ and } \forall x > 0.$$

By hypothesis $\Psi(x)\Phi(\frac{\alpha}{x})$ is a nonincreasing function $\forall \alpha \in \mathbb{R}^+$ then, by application of Lemma 3.1 in [6], $\{\Psi(x)\Phi(\frac{\alpha}{x})\}_{\{\alpha \in \mathbb{R}^+\}}$ is a collection of quasi-increasing functions with the same q.i constant iff $\Phi \prec_N \Psi$. □

Right afterwards, we state conditions under which relations $\prec$ and $\prec_N$ are simultaneously valid.

Proposition 3.9. *Let Φ_1 and Φ_2 be two Young functions restricted to $\mathbb{R}^+$ and let φ_{1+}, φ_{2+} be their right derivatives.*
If Φ_1, $\Phi_2 \in \Delta_2$, we have $\Phi_1 \prec \Phi_2$ iff $\varphi_{1+} \prec \varphi_{2+}$ and $\Phi_1 \prec_N \Phi_2$ iff $\varphi_{1+} \prec_N \varphi_{2+}$
.

Proof. To begin with, we obtain some inequalities which will be employed later. As Φ_1 and Φ_2 are Young functions restricted to $\mathbb{R}^+$ and φ_{1+} and φ_{2+} are their right derivatives, we get

$$\frac{x}{K_2}\varphi_{2+}(x) \le \Phi_2(x) \le x\varphi_{2+}(x) \quad \forall x \in \mathbb{R}^+ \tag{3.1}$$

and

$$\frac{\alpha}{K_1 x}\varphi_{1+}\Big(\frac{\alpha}{x}\Big) \le \Phi_1\Big(\frac{\alpha}{x}\Big) \le \frac{\alpha}{x}\varphi_{1+}\Big(\frac{\alpha}{x}\Big) \quad \forall x \in \mathbb{R}^+ \text{ and } \forall \alpha \in \mathbb{R}^+. \tag{3.2}$$

$\Rightarrow$) If $\Phi_1 \prec_N \Phi_2$, then $\exists\, \rho_1 > 0$ such that

$$\frac{1}{x}\int_0^x \Phi_2(t)\Phi_1\Big(\frac{\alpha}{t}\Big)\,dt \le \rho_1\Phi_2(x)\Phi_1\Big(\frac{\alpha}{x}\Big) \quad \forall x \in \mathbb{R}^+ \text{ and } \forall \alpha \in \mathbb{R}^+.$$

From (3.1), (3.2) and the hypothesis, $\exists R_1 = K_1K_2\rho_1 > 0$ such that

$$\frac{1}{x}\int_0^x \varphi_{2+}(t).\varphi_{1+}\Big(\frac{\alpha}{t}\Big)\,dt \le R_1\varphi_{2+}(x)\varphi_{1+}\Big(\frac{\alpha}{x}\Big) \quad \forall x \in \mathbb{R} \text{ and } \alpha \in \mathbb{R}^+$$

Therefore, $\varphi_{1+} \prec_N \varphi_{2+}$.

$\Leftarrow$) If $\varphi_{1+} \prec_N \varphi_{2+}$, then $\exists\, \rho_2 > 0$ such that

$$\frac{1}{x}\int_0^x \varphi_{2+}(t)\varphi_{1+}\Big(\frac{\alpha}{t}\Big)\,dt \le \rho_2\varphi_{2+}(x)\varphi_{1+}\Big(\frac{\alpha}{x}\Big)\forall x \in \mathbb{R}^+ \text{ and } \forall \alpha \in \mathbb{R}^+.$$

From (3.1), (3.2) and the hypothesis, $\exists R_2 = K_1K_2\rho_2 > 0$ such that

$$\frac{1}{x}\int_0^x \Phi_2(t)\Phi_1\Big(\frac{\alpha}{t}\Big)\,dt \le R_2\Phi_2(x)\Phi_1\Big(\frac{\alpha}{x}\Big) \quad \forall x \in \mathbb{R}^+ \text{ and } \forall \alpha \in \mathbb{R}^+.$$

Therefore, $\Phi_1 \prec_N \Phi_2$. □

The following result follows straightforward from the definitions

Proposition 3.10. *Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.*
Let $p \in \mathbb{R}$. If $\varphi(x) = x^p$, then $\varphi \prec \psi$ iff $\varphi \prec_N \psi$.

The following result is an immediate consequence of Proposition 3.6 and Remark 3.

Proposition 3.11. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function.*
If $\varphi \in \Delta_2$ with $\Lambda_\varphi < 2$, then $\varphi \prec \psi$ and $\varphi \prec_N \psi$.

Proposition 3.12. *Let $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.*
If there exist constants $0 < K_1 \le K_2$ such that $K_1 \le \varphi(x)\varphi(\frac{\alpha}{x}) \le K_2$ for all $\alpha > 0$ and for all $x > 0$, then $\varphi \prec \psi$ iff $\varphi \prec_N \psi$.

Proof. From the hypothesis, there exist $0 < K_1 \le K_2$ such that

$$\frac{1}{K_1}\varphi\Big(\frac{\alpha}{x}\Big) \ge \frac{1}{\varphi(x)} \text{ and } \varphi\Big(\frac{\alpha}{x}\Big) \le \frac{K_2}{\varphi(x)} \quad \forall \alpha > 0 \text{ and } \forall x > 0. \tag{3.3}$$

$\Rightarrow$) Due to $\varphi \prec \psi$, $\exists \rho > 0$ such that

$$\frac{1}{x}\int_0^x \frac{\psi(t)}{\varphi(t)}\,dt \le \rho \frac{\psi(x)}{\varphi(x)} \;\; \forall x > 0$$

and therefore, by (3.3), $\exists K_3 = \frac{K_2}{K_1}\rho > 0$ such that

$$\frac{1}{x}\int_0^x \psi(t)\varphi\Big(\frac{\alpha}{t}\Big)\,dt \le K_3\psi(x)\varphi\Big(\frac{\alpha}{x}\Big) \;\; \forall \alpha > 0 \text{ and } \forall x > 0 \text{ iff } \varphi \prec_N \psi.$$

$\Leftarrow$) Due to $\varphi \prec_N \psi$, $\exists \rho > 0$ such that

$$\frac{1}{x}\int_0^x \psi(t)\varphi\Big(\frac{\alpha}{t}\Big)\,dt \le \rho\psi(x)\varphi\Big(\frac{\alpha}{x}\Big) \;\; \forall \alpha > 0 \text{ and } \forall x > 0$$

and then, by (3.3), $\exists K_3 = \frac{K_2}{K_1}\rho > 0$ such that

$$\frac{1}{x}\int_0^x \frac{\psi(t)}{\varphi(t)}\,dt \le K_3 \frac{\psi(x)}{\varphi(x)} \quad \forall x > 0 \text{ iff } \varphi \prec \psi.$$

□

Example 3.13. *Let* $\varphi(t) = \begin{cases} \frac{1}{2}\sin t + \frac{1}{2} & \text{for } 0 < t < \frac{\pi}{2} \\ 1 & \text{for } t \ge \frac{\pi}{2} \end{cases}$

then $\varphi\Big(\frac{\alpha}{t}\Big) = \begin{cases} \frac{1}{2}\sin(\frac{\alpha}{t}) + \frac{1}{2} & \text{for } t > \frac{2\alpha}{\pi} \\ 1 & \text{for } \frac{2\alpha}{\pi} \ge t \ge 0 \end{cases}$

and consequently $0 < \dfrac{1}{4} \le \varphi(t)\varphi\Big(\dfrac{\alpha}{t}\Big) \le 1 \;\; \forall \alpha > 0$ *and* $\forall t > 0$; *thus* $\varphi \prec_N \varphi$ *owing to* $\varphi \prec \varphi$.

If we soften the hypothesis in the preceding proposition, we achieve

Proposition 3.14. *Let* $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.
If $\varphi \prec_N \psi$ *and there exist constants* $0 < K_1 \le K_2$ *such that* $K_1 \le \varphi(x)\varphi(\frac{1}{x}) \le K_2$ *for all* $x > 0$, *then* $\varphi \prec \psi$.

Proof. From the hypothesis, there exist $0 < K_1 \le K_2$ such that

$$\frac{1}{K_1}\varphi\Big(\frac{1}{x}\Big) \ge \frac{1}{\varphi(x)} \;\text{ and }\; \varphi\Big(\frac{1}{x}\Big) \le \frac{K_2}{\varphi(x)} \;\; \forall x > 0. \tag{3.4}$$

Due to $\varphi \prec_N \psi$, $\exists \rho > 0$ such that

$$\frac{1}{x}\int_0^x \psi(t)\varphi\Big(\frac{\alpha}{t}\Big)\,dt \le \rho\psi(x)\varphi\Big(\frac{\alpha}{x}\Big) \;\; \forall \alpha > 0 \text{ and } \forall x > 0;$$

now we choose $\alpha = 1 > 0$, we get

$$\frac{1}{x}\int_0^x \psi(t)\varphi\Big(\frac{1}{t}\Big)\,dt \le \rho\psi(x)\varphi\Big(\frac{1}{x}\Big) \;\; \forall x > 0$$

and then, employing (3.4), $\exists K_3 = \frac{K_2}{K_1}\rho > 0$ such that

$$\frac{1}{x}\int_0^x \frac{\psi(t)}{\varphi(t)}\,dt \leq K_3\frac{\psi(x)}{\varphi(x)} \quad \forall x > 0 \text{ iff } \varphi \prec \psi.$$

□

Example 3.15. $x + \ln(x+1) \prec_N x$ *and* $x + \ln(x+1) \prec x$.
It is remarkable that functions of this example **belong to Φ.**

Proposition 3.16. *Let* $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$.
If $\frac{\psi}{\varphi}$ *and* $\varphi(\frac{\alpha}{x})\psi(x)$ *are two nonincreasing functions for all* $\alpha > 0$, $\varphi \in \Delta_2$, $\psi \in \nabla_2$ *and* $\frac{\Lambda_\varphi}{\lambda_\psi} < 2$; *then* $\varphi \prec \psi$ *and* $\varphi \prec_N \psi$.

Proof. It follows in the same way as Proposition 3.8. □

Remark 4. The advantage of this statement resides in the fact that φ and ψ could be any nondecreasing functions.

Now, we reach a strong type inequality from a weak type one and provided that the involved functions are related by $\prec_N$.

Theorem 3.17. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nondecreasing function such that* $\varphi(0) = 0$.
Let f *and* g *be nonnegative measurable functions satisfying*

$$\mu(\{f > a\}) \leq 2C_w \int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu \quad \text{for all } a > 0 \text{ and some } c > 0.$$

Let Ψ *be a* $C^1([0,\infty))$ *Young function and let* $\psi = \Psi'$; *and, assume that* $\varphi \prec_N \psi$. *Then*

$$\int_\Omega \Psi(f)\,d\mu \leq 2C_wC_q \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\,d\mu \tag{3.5}$$

where C_q *is a constant that depends only on* ρ *and* c.

Proof. It follows the same pattern of the proof of Theorem 2.4 in [6].
First, we write $\int_\Omega \Psi(f)\,d\mu$ using the distribution function of f; then, we apply the Weak Type Inequality of the hypothesis and Fubini's Theorem, obtaining the next chain of inequalities

$$\int_\Omega \Psi(f)\,d\mu = \int_0^\infty \psi(a)\mu(\{f>a\})\,da \leq$$

$$2C_w\int_0^\infty \psi(a)\left(\int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu\right)da = 2C_w\int_\Omega\left(\int_0^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\,da\right)d\mu.$$

As $\varphi \prec_N \psi$ we get

$$\int_0^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\,da \leq \rho.\varphi(c)(c^{-1}g)\psi(c^{-1}g) = C_q(c^{-1}g)\psi(c^{-1}g)$$

being $C_q = \rho\varphi(c)$, and consequently

$$2C_w \int_\Omega \left(\int_0^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\, da \right) d\mu \le 2C_w \int_\Omega C_q(c^{-1}g)\psi(c^{-1}g)\, d\mu.$$

Due to $\Psi(x) = \int_0^x \psi(t)\, dt$ where $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is nondecreasing, we have $\Psi(x) \ge \int_{\frac{x}{2}}^x \psi(t)\, dt \ge \frac{x}{2}\psi(\frac{x}{2})$, so

$$2C_w \int_\Omega C_q(c^{-1}g)\psi(c^{-1}g)\, d\mu \le 2C_w C_q \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\, d\mu.$$

Finally $\displaystyle\int_\Omega \Psi(f)\, d\mu \le 2.C_w C_q \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\, d\mu.$ □

Now we can obtain new versions of Corollaries 2.6, 2.7 and 2.8 in [6] as follows.

Corollary 3.18. *Let f and g be nonnegative measurable functions. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$, let Ψ be a $C^1([0,\infty)) \cap \Delta_2$ Young function and let $\psi = \Psi'$; then $\displaystyle\int_\Omega \Psi(f)\, d\mu \le C \int_\Omega \Psi(g)\, d\mu$ where the positive constant C is independent of f and g if*

1. *$\varphi(0+) = 0$, (1.4) and $\varphi \prec_N \psi$; or*
2. *(1.5), $\Phi \in \Delta_2$ such that $\varphi = \Phi'_+$ and $\Phi \prec_N \Psi$; or*
3. *(1.4), $\Phi \in \Delta_2$ such that $\varphi = \Phi'_+$ and $\Phi \prec_N \Psi$.*

Proof. (1) From Lemma 2.1 and the fact that $\Psi \in \Delta_2$.

(2) By Proposition 3.9, Theorem 3.17 and the fact that $\Psi \in \Delta_2$.

(3) By application of Lemma 2.1 and point (2). □

Remark 5. In points (1) and (3) we did not require $\Phi \in \nabla_2$ which is an indispensable condition to prove Corollaries 2.6 and 2.8 in [6].

In the following theorem, we also obtain a strong type inequality although (1.4) or (1.5) do not hold for all $a > 0$ and provided that we consider a Finite Measure Space.

Theorem 3.19. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\varphi(0) = 0$. Let f and g be nonnegative measurable functions satisfying*

$$\mu(\{f > a\}) \le K_1 \int_{\{g > ca\}} \varphi\Big(\frac{g}{a}\Big)\, d\mu,$$

for all $a \ge a_0 > 0$ and some $K_1 > 0$ and some $c > 0$. Let Ψ be a $C^1([0,\infty))$ Young function such that $\psi = \Psi'$ and assume that $\varphi \prec_N \psi$, then

$$\int_\Omega \Psi(f)\, d\mu \le K_2 + K_3 \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\, d\mu$$

where $0 < K_3$ and $0 < K_2 = \psi(a_0)\mu(\Omega)$ are independent of f and g.

Proof. Let $a_0 > 0$. We write $\Omega = \{f > a_0\} \cup \{f \leq a_0\}$, then

$$\int_\Omega \Psi(f)\,d\mu = \int_{\{f>a_0\}} \Psi(f)\,d\mu + \int_{\{f\leq a_0\}} \Psi(f)\,d\mu.$$

First we suppose $f > a_0$, we rewrite the $\int_\Omega \Psi(f)\,d\mu$ and, due to the Weak Type Inequality in the hypothesis is valid $\forall a \geq a_0$, we have

$$\int_\Omega \Psi(f)\,d\mu = \int_\Omega \left(\int_0^{a_0} \psi(a)\,da \right) d\mu + \int_\Omega \left(\int_{a_0}^{f} \psi(a)\,da \right) d\mu.$$

We recall $\Psi(f) = \int_0^f \psi(a)\,da$ and we take $K_2 = \mu(\Omega)\psi(a_0)$ to obtain

$$\int_\Omega \left(\int_0^{f} \psi(a)\,da \right) d\mu \leq K_2 + \int_\Omega \left(\int_{a_0}^{f} \psi(a)\,da \right) d\mu.$$

Now, we apply Fubini's Theorem and the hypothesis to produce

$$\int_\Omega \left(\int_{a_0}^{f} \psi(a)\,da \right) d\mu = \int_{a_0}^{\infty} \psi(a) \left(\int_{\{f>a\}} d\mu \right) da =$$

$$\int_{a_0}^{\infty} \psi(a)\mu(\{f > a\})\,da \leq K_1 \int_{a_0}^{\infty} \psi(a) \left(\int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu \right) da.$$

Again, we employ Fubini's Theorem to express

$$K_1 \int_{a_0}^{\infty} \psi(a) \left(\int_{\{g>ca\}} \varphi\Big(\frac{g}{a}\Big)\,d\mu \right) da = K_1 \int_\Omega \left(\int_{a_0}^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\,da \right) d\mu;$$

because of $\psi(a)\varphi(\frac{g}{a})$ being a nonnegative function on $[0,\infty)$, we get

$$K_1 \int_\Omega \left(\int_{a_0}^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\,da \right) d\mu \leq K_1 \int_\Omega \left(\int_{0}^{c^{-1}g} \psi(a)\varphi\Big(\frac{g}{a}\Big)\,da \right) d\mu.$$

From here the proof is similar to the one developed in Theorem 3.17; and eventually,

$$\int_\Omega \Psi(f)\,d\mu \leq K_2 + K_3 \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\,d\mu \ \text{ where } \ K_3 = K_1 C_q.$$

If $f \leq a_0$,

$$\int_\Omega \Psi(f)\,d\mu \leq \int_\Omega \left(\int_0^{a_0} \psi(a)\,da \right) d\mu = \mu(\Omega)\Psi(a_0) = K_2$$

because $\Psi(f) = \int_0^f \psi(t)\,dt$ with $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a nondecreasing function. Combining the two previous results, we obtain

$$\int_\Omega \Psi(f)\,d\mu \leq 2K_2 + K_3 \int_\Omega \Psi\Big(\frac{2}{c}g\Big)\,d\mu$$

where $K_3 = K_1 C_q$ y $K_2 = \mu(\Omega)\Psi(a_0)$. $\square$

4. Inequalities for Best Approximation Operators

A subset $\mathcal{L} \subset \mathcal{A}$ is a σ-lattice iff $\emptyset$, $\Omega \in \mathcal{L}$ and $\mathcal{L}$ is closed under countable unions and intersections.
Set $L^\Phi(\mathcal{L})$ for the set of $\mathcal{L}$-measurable functions in $L^\Phi(\Omega)$.

A function $g \in L^\Phi(\mathcal{L})$ is called a best Φ-approximation to $f \in L^\Phi$ iff

$$\int_\Omega \Phi(f-g)\,d\mu = \min_{h \in L^\Phi(\mathcal{L})} \int_\Omega \Phi(f-h)\,d\mu.$$

We denote by $\mu(f, \mathcal{L})$ the set of all the best Φ-approximants to f.
It is well known that for every $f \in L^\Phi$, $\mu(f,\mathcal{L}) \neq \emptyset$, see [5].

Recall that a Young function Φ such that $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$ is called an N-function.
Let Φ be a derivable N-function and let $\varphi = \Phi'$, then $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a right continuous, nondecreasing function that satisfies $0 < \varphi(x) < \infty$ for all $x \in (0,\infty)$, $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$ ([4] and [7]).

If $f \in L^\Phi$, we will write $\overline{f}$ for the best Φ-approximation to $f \in L^\Phi$.
Given two functions f and g, we denote $f \vee g$ ($f \wedge g$) the pointwise maximum (minimum) of the functions.
Assume that $\Phi \in C^1 \cap \Delta_2$ is strictly convex, then the function $\Phi' = \varphi$ also fulfills the Δ_2-condition.
Let $f \in L^\varphi$ and let n be a fixed positive number.
Thus, we define $\overline{(-n \vee f)}$ as the increasing limit of $\overline{((-n \vee f) \wedge m)}$ as $m \to \infty$.
And, the decreasing limit of $\overline{(-n \vee f)}$ as $n \to \infty$ will be, by definition, the Extended Best Approximation Operator of f from L^Φ to L^φ, which will denote $\overline{f_e}$.

In [2] Favier and Zó obtained

Theorem 4.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.
Let $\Phi \in C^1 \cap \Delta_2$ be a strictly convex N-function and assume $\varphi = \Phi'$.
Let $f \in L^\varphi$ such that $f \geq 0$ and let $\overline{f_e}$ be the Extension of the Best Approximation Operator to L^φ.
If there exists a constant $c > 0$ such that $\varphi(x+y) \leq c[\varphi(x) + \varphi(y)]$ for all $x, y \geq 0$, then*

$$\mu(\{\overline{f_e} > a\}) \leq \frac{c+1}{\varphi(a)} \int_{\{\overline{f_e} > a\}} \varphi(f)\,d\mu \quad \textit{for all } a > 0.$$

We begin proving an equivalence similar to Lemma 4.1 in [6] and Lemma 2.5 in [2].

Proposition 4.2. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function.
$\varphi \in \Delta_2$ iff there exists $c > 0$ such that $\varphi(x+y) \leq c[\varphi(x)+\varphi(y)]$ for all $x, y \geq 0$.*

Proof. $\Rightarrow$) If $\varphi \in \Delta_2$, then $\exists\, c > 0$ such that $\varphi(2x) \leq c\varphi(x)$ $\forall x \geq 0$.
Let $x, y \geq 0$, then $x \leq y$ ó $x \geq y$.
Without any loss of generality, let us assume $x \geq y$; then,
$\varphi(x+y) \leq \varphi(2x) \leq c\varphi(x) \leq c\varphi(x) + c\varphi(y) = c[\varphi(x) + \varphi(y)]$.
$\Leftarrow$) If $\exists\, c > 0$ such that $\varphi(x+y) \leq c[\varphi(x) + \varphi(y)]$ $\forall x, y \geq 0$; for $x = y \geq 0$ we have $\varphi(2x) \leq 2c\varphi(x)$ $\forall x \geq 0$, i.e, $\varphi \in \Delta_2$. □

Due to every Δ'-function is a Δ_2-function (see [7]), it is also true $\varphi \in \Delta'$(globally) implies the existence of a constant $c > 0$ such that $\varphi(x+y) \leq c[\varphi(x) + \varphi(y)]$ for all $x, y \geq 0$.
In consequence, if we demand $\varphi \in \Delta'$ globally, by Theorem 4.1 and Proposition 2.2, we obtain the Weak Type Inequality (1.4) with f and g replaced by $\overline{f_e}$ and $f \in L^{\varphi}$ respectively.
Moreover, if φ is a continuous function such that $\varphi(0) = 0$, by Lemma 2.1, we reach (1.5) for $\overline{f_e}$. That is,

Theorem 4.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite meaure space.*
Let $\Phi \in C^1 \cap \Delta_2$ a strictly convex N-function and assume $\varphi = \Phi'$.
Suppose $f \in L^{\varphi}$ and $f \geq 0$; and, let $\overline{f_e}$ be the Extension of the Best Approximation Operator to L^{φ}.
If $\varphi \in \Delta'$ globally, then

$$\mu(\{\overline{f_e} > a\}) \leq K \int_{\{\overline{f_e} > a\}} \varphi\Big(\frac{f}{a}\Big)\, d\mu \;\; \textit{for all } a > 0;$$

and, there also exists a constant $c > 0$ such that

$$\mu(\{\overline{f_e} > a\}) \leq 2K \int_{\{f > ca\}} \varphi\Big(\frac{f}{a}\Big)\, d\mu \;\; \textit{for all } a > 0.$$

Now, if $\varphi \in \Delta'$ globally, from Theorems 4.3 and 3.17, we get

Theorem 4.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.*
Let $\Phi \in C^1 \cap \Delta_2$ a strictly convex N-function and let $\varphi = \Phi'$.
Supose $f \in L^{\varphi}$ and $f \geq 0$; and, let $\overline{f_e}$ be the Extension of the Best Approximation Operator to L^{φ}.
Let $\Psi \in C^1([0, \infty))$ be a Young function and let $\psi = \Psi'$.
If $\varphi \in \Delta'$ globally and $\varphi \prec_N \psi$, then

$$\int_{\Omega} \Psi(\overline{f_e})\, d\mu \leq K_2 \int_{\Omega} \Psi\Big(\frac{2}{c} f\Big)\, d\mu \tag{4.1}$$

where $K_2 > 0$ is independent of the function f.

Remark 6. In [2], Favier and Zó obtained strong type inequalities like (1.6) for $\overline{f_e}$ with Ψ belonging to specific classes of functions. However, the Strong Type Inequality for $\overline{f_e}$ is not characterized.

Now, we consider another maximal operator related with the best approximation operator.
Suppose that $\mathcal{L}_n$ is an increasing sequence of σ-lattices, i.e $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ for all $n \in \mathbb{N}$.
Let $f \in L^\Phi$ such that $f \geq 0$ and let f_n be any selection of functions in $\mu(f, \mathcal{L}_n)$.
In [6] it is defined the maximal function $f^* = \sup\limits_n f_n$.

Let Φ be a Young function such that $\hat{\Phi} \in \Delta_2 \cap \nabla_2$ being

$$\hat{\Phi}(x) = \int_0^x \hat{\varphi}(t)\,dt \quad \text{with} \quad \hat{\varphi}(x) = \varphi_+(x) - \varphi_+(0)\text{sign}(x)$$

and φ_+ the right-continuous derivative of Φ.
In Theorem 1.1 in [6], Mazzone and Zó proved that f^* satisfies

$$\mu(\{f^* > a\}) \leq \frac{C}{\varphi_+(a)} \int_{\{f>ca\}} \varphi_+(f)\,d\mu \tag{4.2}$$

for all $a > 0$ and some $C > 0$; they also stated that if $\varphi_+(0) = 0$, then

$$\mu(\{f^* > a\}) \leq \frac{C}{\varphi_+(a)} \int_{\{f^*>a\}} \varphi_+(f)\,d\mu \ \text{ for all } a > 0. \tag{4.3}$$

In the proof of the case $\varphi_+(0) = 0$, the authors did not employ the fact that $\hat{\Phi} \in \nabla_2$; nevertheless, this condition became essential to get (4.2).
For this reason, we assume $\varphi_+(0) = 0$ and $\varphi_+ \in \Delta'$ globally and then we get a weak type inequality like (1.4) where $f = f^*$ and $g = f$.
Moreover, if φ_+ is a right continuous function, we apply Lemma 2.1 and we obtain (1.5) for $f = f^*$ and $g = f$. That is,

Theorem 4.5. *Let Φ be a Young function such that φ_+ is the right continuous derivative of Φ.*
If $\varphi_+(0) = 0$ and $\varphi_+ \in \Delta'$ globally, then

$$\mu(\{f^* > a\}) \leq K \int_{\{f^*>a\}} \varphi_+\Big(\frac{f}{a}\Big)\,d\mu \ \text{ for all } a > 0; \tag{4.4}$$

and, it is also true

$$\mu(\{f^* > a\}) \leq K \int_{\{f>ca\}} \varphi_+\Big(\frac{f}{a}\Big)\,d\mu \tag{4.5}$$

for all $a > 0$ and some $c > 0$.

In consequence, if $\varphi_+ \in \Delta'$ globally and $\varphi_+(0) = 0$, by Theorem 4.5, f^*satisfies a weak type inequality like (1.4) and, from Theorem 3.17, we get

Theorem 4.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space.*
Let Φ be a Young function such that φ_+ is its right-continuous derivative; suppose $\varphi_+ \in \Delta'$ globally and $\varphi_+(0) = 0$.
Let $\mathcal{L}_n$ be an increasing sequence of σ-lattices and consider $f \in L^\Phi$ such that $f \geq 0$ and let f_n be any selection of functions in $\mu(f, \mathcal{L}_n)$ and $f^* = \sup\limits_n f_n$.

Let $\Psi \in C^1([0,\infty))$ *be a Young function with* $\psi = \Psi'$.
If $\varphi \prec_N \psi$, *then*

$$\int_\Omega \Psi(f^*)\, d\mu \le K_2 \int_\Omega \Psi\Big(\frac{2}{c} f\Big)\, d\mu \tag{4.6}$$

where K_2 *is a positive constant independent of the function* f.

We point out that the Young functions Φ whose right-continuous derivatives $\varphi_+ \in \Delta'$ globally and satisfy $\varphi_+(0) = 0$ can be considered for the above Theorem while, for Theorem 1.1 in [6], we need $\hat{\Phi}(x) \in \nabla_2$.

5. Inequalities for Sub Additive Operators

The following result follows by a standard procedure as Remark (1), page 38, in [1], and Lemma 3.1 in [2].

Proposition 5.1. *Let* $T : L^1(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n)$ *be a subaddtive operator,* $f \in L^1(\mathbb{R}^n)$ *and* $\varphi \in \Phi$ *such that* $\varphi(0) = 0$. *Assume*

$$|\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}| \le C \int_{\mathbb{R}^n} \varphi\Big(\frac{Cf(x)}{\lambda}\Big)\, dx \tag{5.1}$$

for all $\lambda > 0$ *and some* $C > 0$ *independent of* f; *and, suppose*

$$\|Tf\|_\infty \le \|f\|_\infty. \tag{5.2}$$

Then

$$|\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}| \le C \int_{\{x : |f(x)| > \frac{\lambda}{2}\}} \varphi\Big(\frac{2Cf(x)}{\lambda}\Big)\, dx$$

for all $\lambda > 0$; *and, being the constant* C *independent of the function* f.

Next, from Proposition 5.1 and Theorem 3.17, we get

Theorem 5.2. *Let* $T : L^1_{loc}(\mathbb{R}^n) \to \mathcal{M}_{ed}(\mathbb{R}^n)$ *be a subadditive operator,* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $\varphi \in \Phi$ *such that* $\varphi(0) = 0$. *Suppose*

$$|\{x \in \mathbb{R}^n : |T(f)(x)| > \lambda\}| \le C \int_{\mathbb{R}^n} \varphi\Big(\frac{Cf(x)}{\lambda}\Big)\, dx \tag{5.3}$$

for all $\lambda > 0$ *and some* $C > 0$ *independent of the function* f; *and, assume* $\|Tf\|_\infty \le \|f\|_\infty$.
Let Ψ *be a* $C^1([0,\infty))$ *Young function and let* $\psi = \Psi'$.
If $\varphi \prec_N \psi$, *then* $\displaystyle\int_{\mathbb{R}^n} \Psi(|T(f)|)\, dx \le K \int_{\mathbb{R}^n} \Psi(4f)\, dx.$

Hereafter, we consider the Hardy Littlewood Maximal Operator M defined over cubes $Q \subset \mathbb{R}^n$ and given by the formula

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q f(t)\, dt \quad \text{for } f \in L^1_{loc}(\mathbb{R}^n).$$

Kokilashvili and Krbec, in [3], introduced the following concept

Definition 5.3. *A function $\varphi : [0,\infty) \to \mathbb{R}$ is said to be quasiconvex on $[0,\infty)$ if there exist a convex function ω and a constant $c > 0$ such that $\omega(t) \le \varphi(t) \le c\omega(ct)$ for all $t \in [0,\infty)$.*

Next, we employ previous definition to establish sufficient conditions for the validity of a weak type inequality for M.

Theorem 5.4. *Let $\varphi \in \Phi$.*
If φ is quasiconvex on $[0,\infty)$, there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le C \int_{\mathbb{R}^n} \varphi\Big(\frac{Cf(x)}{\lambda}\Big)\, dx \tag{5.4}$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and for all $\lambda > 0$.

Proof. By Lemma 1.2.4 in [3], $\exists\, c > 0$ such that

$$|\{x \in \mathbb{R}^n : M(g)(x) > \lambda\}| \le \frac{c}{\varphi(\lambda)} \int_{\mathbb{R}^n} \varphi(cg(x))\, dx$$

$\forall g \in L^1_{loc}(\mathbb{R}^n)$ and $\forall \lambda > 0$ iff φ is a quasiconvex function.
Choosing $\lambda = 1$ and next $g = \frac{f}{\lambda}$, with $f \in L^1_{loc}(\mathbb{R}^n)$ and $\lambda > 0$, we have

$$\Big|\Big\{x \in \mathbb{R}^n : M\Big(\frac{f}{\lambda}\Big)(x) > 1\Big\}\Big| \le \frac{c}{\varphi(1)} \int_{\mathbb{R}^n} \varphi\Big(\frac{cf(x)}{\lambda}\Big)\, dx.$$

As M is a homogeneous operator,

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le \frac{c}{\varphi(1)} \int_{\mathbb{R}^n} \varphi\Big(\frac{cf(x)}{\lambda}\Big)\, dx;$$

and, since $\varphi \in \Phi$, $\exists C = \max\{c; \frac{c}{\varphi(1)}\} > 0$ such that

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le C \int_{\mathbb{R}^n} \varphi\Big(\frac{C}{\lambda} f(x)\Big)\, dx$$

$\forall f \in L^1_{loc}(\mathbb{R}^n)$ and $\forall \lambda > 0$. □

However, the quasiconvexity is not a necessary condition to hold (5.4).
Let $\psi(x) = \begin{cases} x^p & \text{if } x \ge 0 \\ (-x)^p & \text{if } x < 0 \end{cases}$ for $p \ge 1$,
then $\psi \in \Phi$ and ψ is quasiconvex on $[0,\infty)$.
By Theorem 5.4, there exists a constant $c > 0$ such that

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and for all $\lambda > 0$.

Now, we consider the function $\varphi(x) = \begin{cases} x^p & \text{if } x \ge 1 \\ x^{\frac{1}{p}} & \text{if } 0 \le x < 1 \\ (-x)^p & \text{if } x \le -1 \\ (-x)^{\frac{1}{p}} & \text{if } -1 < x < 0 \end{cases}$

for $p > 1$, which begins to Φ and $0 \le \psi(x) \le \varphi(x)$ for all $x \in \mathbb{R}$. So

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le c \int_{\mathbb{R}^n} \varphi\Big(\frac{cf(x)}{\lambda}\Big)\, dx$$

$\forall \lambda > 0$ and $\forall f \in L^1_{loc}(\mathbb{R}^n)$, but φ is not a quasiconvex function.
Hence, the converse of Theorem 5.4 is not true.

With the aim of relaxing the hypothesis of quasiconvexity in Theorem 5.4, we derive some properties of functions belonging to Φ.

Let $a > 0$. We denote Φ_a the set of functions $\varphi : (-a, a) \to \mathbb{R}$ which are nonnegative, even, nondecreasing on $[0, a)$, such that $\varphi(x) > 0$ for all $x > 0$ and $\varphi(0+) = 0$.

The following two lemmas can be easily proved from the hypothesis.

Lemma 5.5. *Let $\psi \in \Phi$. If there exists $x_v : 0 < x_v \le x_0$ such that ψ is convex on $(0, x_v)$, then there exist a convex function $\omega \in \Phi_{x_0}$ such that $\omega(x) \le \psi(x)$ on $(0, x_0)$.*

Lemma 5.6. *Let $\psi \in \Phi$ and $x_0 > 0$. Suppose $\psi(x) \ge c_1 x$ for all $x \in [x_0, \infty)$ and there exists a subinterval $(x_1, x_2) \subseteq (0, x_0)$, with x_1 not necessarily zero, such that $\psi(x) \le c_1 x$ for all $x \in (x_1, x_2)$.*
If ψ is a concave function on $(0, x_v) \subseteq (0, x_0)$, then $\psi(x_v-) \le c_1 x_0$.

Lemma 5.7. *Let $\psi \in \Phi$. If there exist constants $c_1 > 0$ and $x_0 \ge 0$ such that $\psi(x) \ge c_1 x$ for all $x \in [x_0, \infty)$, and there exists a subinterval $(0, x_v) \subseteq (0, x_0)$ where ψ is either a convex or a concave function; then, there exists a convex function $\varphi \in \Phi$ that verifies $\varphi(x) \le \psi(x)$ for all $x > 0$.*

Proof. According with the behavior of ψ on the interval $(0, x_0)$, we build a convex function $\varphi \in \Phi$ such that $\varphi(x) \le \psi(x)\ \forall x > 0$.

A) If $x_0 = 0$, then $\varphi(x) = c_1.x$.

B) If ψ is a concave function on $(0, x_v) \subseteq (0, x_0)$, by Lemma 5.5, there exists a convex function $\omega \in \Phi_{x_0}$ such that $0 < \omega(x) \le \psi(x)\ \forall x \in (0, x_0)$.

- If $0 < \omega'_+(0) \le c_1$, then $r_0(x) = \omega'_+(0).x$ is the tangent line to $\omega(x)$ on $(0, 0)$ and it verifies $r_0(x) \le \omega(x) \le \psi(x)\ \ \forall x \in (0, x_0),\ \ r_0(x) \le c_1.x$ $\forall x \in [x_0, \infty)$ and $r_0 \in \Phi$. Thus, $\varphi(x) = r_0(x)$.
- If $\omega'_+(0) = 0$, then $\exists\, x_w \in (0, x_0)$ such that $0 < \omega'_-(x_w) \le c_1$. Let r_1 be the tangent line to $\omega(x)$ on $(x_w, \omega(x_w))$, in consequence $r_1(x) \le \psi(x)$ $\forall x \in [x_w, x_0)$; and, we also have $r_1(x) \le c_1.x\ \forall x \in [x_0, \infty)$. Therefore, the convex function
$$\varphi(x) = \begin{cases} \omega(x) & \text{if } x \in (0, x_w) \\ r_1(x) & \text{if } x \in [x_w, \infty) \end{cases}$$
belongs to Φ and verifies $\varphi(x) \le \psi(x)\ \forall x > 0$.
- If $\nexists\, x_w \in (0, x_0)$ such that $0 < \omega'_-(x_w) \le c_1$ and $\omega'_+(0) > c_1$, then $r_2(x) = \omega'_+(0).x$ is the tangent line to $\omega(x)$ on $(0, 0)$ that satisfies

$c_1.x < r_2(x) \leq \omega(x) \leq \psi(x)$ $\forall x \in (0, x_0)$; we also have $\psi(x) \geq c_1.x$ $\forall x \geq x_0$ and $r_2 \in \Phi$. Therefore, $\varphi(x) = c_1.x$.

C) Assume $\exists x_v \in (0, x_0)$ such that $\psi(x)$ is a concave function on $(0, x_v) \subseteq (0, x_0)$.

- If $\psi(x) \geq c_1.x$ $\forall x \in (0, x_0)$ and due to $\psi(x) \geq c_1.x$ $\forall x \geq x_0$, then $\psi(x) \geq c_1.x$ $\forall x > 0$. Therefore, $\varphi(x) = c_1.x$.
- If ψ is concave on $(0, x_v) \subseteq (0, x_0)$ and $\psi(x) \leq c_1.x$ $\forall x \in (x_1, x_2) \subseteq (0, x_0)$ where x_1 is not necessarily 0; then, by Lemma 5.6 , $\psi(x_v-) \leq c_1.x_0$.
- If $x_v = x_0$, we have $\psi(x_0-) \leq c_1.x_0$ and let $r_3(x) = \frac{\psi(x_0-)}{x_0}.x$ be the chord between $(0, 0)$ and $(x_0, \psi(x_0-))$ then $r_3(x) \leq \psi(x) \leq c_1.x$ $\forall x \in (0, x_0)$; it is also true that $r_3(x) \leq c_1.x$ $\forall x \geq x_0$. Thus $\varphi(x) = r_3(x)$.
- If $x_v < x_0$ and $\psi(x)$ concave on $(0, x_v)$, we define
$$\psi_c(x) = \begin{cases} \psi(x) & \text{if } x \in (0, x_v) \\ \psi(x_v-) & \text{if } x \in [x_v, x_0) \end{cases}$$
that satisfies $\psi_c(x) \leq \psi(x)$ $\forall x \in (0, x_0)$.
Moreover $\psi_c(x)$ is concave on $(0, x_0)$, then $r_4(x) = \frac{\psi(x_v-)}{x_0}.x$ is the chord between $(0, 0)$ and $(x_0, \psi_c(x_0-))$ and verifies $r_4(x) \leq \psi_c(x)$ $\forall x \in (0, x_0)$; we also have $r_4(x) \leq c_1.x$ $\forall x \geq x_0$. Thus, $\varphi(x) = r_4(x)$.

□

In consequence, we achieve another way to obtain (5.4); namely,

Theorem 5.8. *Let $\psi \in \Phi$. Suppose there exist constants $c_1 > 0$ and $x_0 \geq 0$ such that $\psi(x) \geq c_1 x$ for all $x > x_0$, and there exists a subinterval $(0, x_v) \subseteq (0, x_0)$ where ψ is either a convex or a concave function. Then, there exists a constant $c > 0$ such that*

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \leq c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx \tag{5.5}$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and for all $\lambda > 0$.

Proof. Let $\psi \in \Phi$. Assume $\exists\, x_0 \geq 0$, $c_1 > 0$ such that $\psi(x) \geq c_1 x$ $\forall x > x_0$; and, there exists a subinterval $(0, x_v) \subseteq (0, x_0)$ where ψ is either a convex or a concave function.
Then, by Lemma 5.7, there exists a convex function $\varphi \in \Phi$ such that $\varphi(x) \leq \psi(x)$ $\forall x > 0$; therefore,

$$c \int_{\mathbb{R}^n} \varphi\Big(\frac{cf(x)}{\lambda}\Big)\, dx \leq c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx. \tag{5.6}$$

Due to any convex function is a quasiconvex one (see Lemma 1.1.1 in [3]), we apply Theorem 5.4 to φ and we get

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \leq c \int_{\mathbb{R}^n} \varphi\Big(\frac{cf(x)}{\lambda}\Big)\, dx. \tag{5.7}$$

Eventually, from (5.6) and (5.7), we have

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx.$$

□

Moreover, we also find a necessary condition for the validity of the Weak Type Inequality (5.5).

Theorem 5.9. *Let $\psi \in \Phi$. If there exists a constant $c > 0$ such that*

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and for all $\lambda > 0$, then there exist $c_1 > 0$ and $x_0 \ge 0$ such that $\psi(x) \ge c_1 x$ for all $x > x_0$.

Proof. We follow the idea of [3] to prove Lemma 1.2.4.
Let $0 < t_1 < t_2$, $I = \Big\{x = (x_1, ..., x_n) \in \mathbb{R}^n : 0 < x_i < \Big(\frac{t_1}{t_2}\Big)^{\frac{1}{n}}, i = 1, ..., n\Big\}$ then $I \subset (0,1)^n$ and $|I| = \frac{t_1}{t_2} < 1$; and, put $f(x) = t_2 \chi_I(x)$.
For any $x \in (0,1)^n$, we have $M(f)(x) > t_1$ and thus

$$|\{x \in \mathbb{R}^n : M(f)(x) > t_1\}| \ge 1. \tag{5.8}$$

From the hypothesis, $\exists c > 0$ such that

$$|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}| \le c \int_{\mathbb{R}^n} \psi\Big(\frac{cf(x)}{\lambda}\Big)\, dx$$

$\forall f \in L^1_{loc}(\mathbb{R}^n)$ and $\forall \lambda > 0$; so, choosing I, f and λ as in the beginning and in (5.8), $\exists c > 0$ such that

$$1 \le |\{x \in \mathbb{R}^n : M(f)(x) > t_1\}| \le c \int_{\psi} \Big(c\frac{t_2}{t_1}\Big)\, dx = c\psi\Big(c\frac{t_2}{t_1}\Big)\Big(\frac{t_1}{t_2}\Big)$$

and hence $\frac{t_2}{t_1} \le c\psi(c\frac{t_2}{t_1})$.
Due to $t_1 < t_2$ and naming $x = c\frac{t_2}{t_1}$, we get $\frac{x}{c^2} \le \psi(x)\ \forall x > c$; therefore, $\exists c_1 = c^{-2} > 0$ and $x_0 = c > 0$ such that $c_1 x \le \psi(x)\ \forall x > x_0$. □

References

[1] M. de Guzmán (1975): *Differentiation of Integrals in* $\mathbb{R}^n$. Springer-Verlag, Berlin-Heidelberg-New York.

[2] S. Favier, F. Zó (2001): *Extension of the best approximation operator in Orlicz spaces and weak-type inequalities.* Abstract and Applied Analysis, **6**: 101-114.

[3] V. Kokilashvili, M. Krbec (1991): *Weighted Inequalities in Lorentz and Orlicz Spaces.*World Scientific, Singapore.

[4] M. A. Krasnosel'skii and Ya. B. Rutickii (1961): *Convex Functions and Orlicz Spaces.* P. Noordhoff Ltd. Groningen The Netherlands

[5] D. Landers, L. Rogge (1980): *Best approximants in* L_Φ*-spaces.* Z. Wahrsch. Verw. Gabiete, **51**: 215-237.

[6] F.D. Mazzone, F. Zó (2009): *On Maximal Inequalities Arising in Best Approximation*. J. Inequal. Pure and Appl. Math., 10(2), Art. 58, 10 pp. issn: 1443-5756, Victoria University, Australia.

[7] M. Rao, Z. Ren (1991): *Theory of Orlicz Spaces*. Marcel Dekker, Inc. New York.

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