Research Article

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Near Field Asymptotic Behavior for the Porous Medium Equation on the Half-Line

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Abstract: Kamin and Vázquez [11] proved in 1991 that solutions to the Cauchy–Dirichlet problem for the porous medium equation $u_t = (u^m)_{xx}$, $m > 1$, on the half-line with zero boundary data and nonnegative compactly supported integrable initial data behave for large times as a dipole-type solution to the equation having the same first moment as the initial data, with an error which is $o(t^{-1/m})$. However, on sets of the form $0 < x < g(t)$, with $g(t) = o(t^{1/(2m)})$ as $t \to \infty$, in the so-called near field, a scale which includes the particular case of compact sets, the dipole solution is $o(t^{-1/m})$, and their result gives neither the right rate of decay of the solution nor a nontrivial asymptotic profile. In this paper, we will improve the estimate for the error, showing that it is $o(t^{-(2m+1)/(2m^2)}(1+x)^{1/m})$. This allows in particular to obtain a nontrivial asymptotic profile in the near field limit, which is a multiple of $x^{1/m}$, thus improving in this scale the results of Kamin and Vázquez.

Keywords: Porous Medium Equation on the Half-Line, Asymptotic Behavior, Matched Asymptotics

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Communicated by: Antonio Ambrosetti and David Arcoya
Dedicated to our dear friend Ireneo Peral on the occasion of his 70th birthday

1 Introduction

This paper is concerned with the large time behavior of solutions to the porous medium equation (PME in what follows) on the half-line with zero boundary data,

$$u_t = (u^m)_{xx} \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \qquad u(0,t) = 0, \quad t \in \mathbb{R}_+, \qquad u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \tag{1.1}$$

with $m > 1$, and nonnegative and compactly supported integrable initial data. This problem, which models the flow of a fluid in a porous medium, has a unique weak solution; see [12]. The asymptotic behavior was first studied by Kamin and Vázquez in [11], and depends heavily on the fact that solutions to (1.1) preserve the first moment along the evolution, $\int_0^\infty xu(x,t)\,dx =$ constant for all $t > 0$. Indeed, the behavior is given in terms of the so-called *dipole solution* of the PME with first moment $M = \int_0^\infty xu_0(x)\,dx$, that is,

$$D_M(x,t) = t^{-\alpha} F_M(\xi), \qquad \xi = \frac{x}{t^\beta},\ \alpha = \frac{1}{m},\ \beta = \frac{1}{2m}, \tag{1.2}$$

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for a certain profile function F_M. Note that this solution has a self-similar structure, and that, due to the choice of the similarity exponents α and β, its first moment is constant in time. The precise result in [11] states that

$$\lim_{t\to\infty} t^{\alpha} \sup_{x\in\mathbb{R}_+} |u(x,t) - D_M(x,t)| = 0, \qquad M = \int_0^{\infty} x u_0(x)\,dx. \tag{1.3}$$

In order for D_M to be a weak solution of the equation on the half-line, with $D_M(0,t) = 0$, the profile F_M has to solve, in a weak sense,

$$(F_M^m)''(\xi) + \beta\xi F_M'(\xi) + \alpha F_M(\xi) = 0, \quad \xi \in \mathbb{R}_+, \qquad F_M(0) = 0, \tag{1.4}$$

while the condition on the value of the first moment imposes $\int_0^\infty \xi F_M(\xi)\,d\xi = M$. A simple scaling argument shows that

$$F_M(\xi) = M^{\frac{1}{m}} F_1(\xi / M^{\frac{m-1}{2m}}). \tag{1.5}$$

It turns out that there is a unique bounded profile corresponding to $M = 1$, namely

$$F_1(\xi) = \xi^{\frac{1}{m}} (C_m - \kappa_m \xi^{\frac{m+1}{m}})_+^{\frac{1}{m-1}}, \tag{1.6}$$

with constants C_m and κ_m given by

$$\kappa_m = \frac{m-1}{2m(m+1)}, \qquad C_m = \left(\frac{\kappa_m^{\frac{2m+1}{m+1}}}{\int_0^1 s^{\frac{m+1}{m}} (1 - s^{\frac{m+1}{m}})^{\frac{1}{m-1}}\,ds} \right)^{\frac{m^2-1}{2m^2}}; \tag{1.7}$$

see [2, 8, 9]. Note that F_1 has compact support $[0, \xi_1]$, where $\xi_1 = (C_m/\kappa_m)^{m/(m+1)}$. Thus, F_M has compact support $[0, \xi_M]$, with

$$\xi_M = \xi_1 M^{\frac{m-1}{2m}}.$$

Let us remark that $\lim_{t\to 0^+} \int_0^\infty D_M(x,t)\varphi(x)\,dx = M\varphi'(0)$. In other words, the antisymmetric extension $\overline{D}_M$ of D_M satisfies $\overline{D}_M(\cdot, t) \to -2M\delta'$, where δ' is the distributional derivative of the delta function. In physics, this is called an elementary dipole. Hence the name dipole solution of the PME for D_M.

Remark 1.1. (a) The result in [11] states that solutions to the signed PME in the whole real line

$$u_t = (|u|^{m-1}u)_{xx}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.8}$$

with an integrable and compactly supported initial data having zero mass and a nontrivial first moment $\int_{\mathbb{R}} x u(x,0)\,dx = P$ converge to $\overline{D}_{P/2}$, with an error which is $o(t^{-\alpha})$. Solutions to (1.1) clearly fall within this framework, since they coincide with the restriction to the half-line of the solution to (1.8) having as initial datum the antisymmetric extension of u_0.

(b) The proof in [11] uses that $v(x,t) = \int_{-\infty}^x u(y,t)\,dy$ is a solution to the p-Laplacian evolution equation with $p = m + 1$, that is,

$$v_t = (|v_x|^{m-1} v_x)_x, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.9}$$

The convergence for u is deduced from the convergence of v and its derivatives. In particular, $D_M = \partial_x B_{-2M}$, where B_K denotes the source-type solution to (1.9) that has $K\delta$ as initial datum.

The limit (1.3) gives the first nontrivial term in the asymptotic expansion of u for $x = O(t^\beta)$, in the so-called *far field limit*. However, since $F_M(0) = 0$, in the *near field*, $x = o(t^\beta)$, it only says that u is $o(t^{-\alpha})$. The aim of this paper is to improve the result of Kamin and Vázquez in the near field by giving a sharp decay rate, which is faster than that in the far field, and a nontrivial asymptotic profile, which turns out to be a multiple of $x^{1/m}$. The precise result reads as follows.

Theorem 1.2. *Let u be the unique weak solution to* (1.1). *Then*

$$\lim_{t\to\infty} t^{\alpha + \frac{\beta}{m}} \sup_{x\in\mathbb{R}_+} \frac{|u(x,t) - D_M(x,t)|}{(1+x)^{\frac{1}{m}}} = 0, \tag{1.10}$$

where D_M is the unique dipole solution to the PME with first moment $M = \int_0^\infty x u_0(x)\,dx$.

In particular, on compact sets $x \in [0, K]$, u is $O(t^{-(\alpha+\beta/m)})$, and $t^{\alpha+\beta/m}u(x,t)$ converges to $M^{2/(m+1)}C_m^{1/(m-1)}x^{1/m}$, while

$$\sup_{0\le x\le g(t)} u(x,t) = O\bigg(\frac{t^{\alpha+\frac{\beta}{m}}}{g(t)^{\frac{1}{m}}}\bigg)$$

if $g(t) \to \infty$, $g(t) = o(t^\beta)$.

We already know that the result is true in the far-field scale, $\xi_1 \le x/t^\beta \le \xi_2$, $0 < \xi_1 \le \xi_2 < \infty$; see (1.3). If $x \ge g(t)t^\beta$, with $g(t) \to \infty$ as $t \to \infty$ (the very far field scale), formula (1.3) yields a better result. Nevertheless, since $D_M(x,t) = 0$ for $x \ge \xi_M t^\beta$, it only says that the solution is $o(t^{-\alpha})$ there. However, Kamin and Vázquez proved, by using some asymptotic formulas from [6], that $s(t) = \sup\{x : u(x,t) > 0\}$ satisfies

$$s(t) = \xi_M t^\beta + o(1), \quad s(t) \ge \xi_M t^\beta,$$

which gives a complete characterization of the asymptotic behavior for $x > \xi_M t^\beta$. Hence, it only remains to check what happens in the near-field. This is done through a matching argument with the outer behavior, which is based on a clever choice of sub- and supersolutions. Comparison is performed in sets of the form $0 < x < \delta t^\beta$ for some small δ. The ordering in the outer boundary comes from the outer behavior which was already known from the analysis in [11]. We devote Section 2 to obtain the upper limit and Section 3 to get the lower one.

A similar analysis for the linear heat equation has recently been performed in [4]. However, in that case linearity made things easier since a representation formula for the solution in terms of the initial datum was available. That paper also considers a nonlocal version of the heat equation.

Problem (1.1) admits at least two generalizations to higher dimensions. The first one consists in considering the problem posed on a half-space, with zero boundary conditions at the boundary, which is a hyperplane. In this case, we also have that the asymptotic behavior of solutions with bounded and compactly supported initial data is given by a dipole-type self-similar solution; see [10]. But, as in the one-dimensional case, the result does not describe properly the large time behavior in the near field scale. Due to the distinguished role of one of the directions, the study of the near field limit will require new ideas, and will be considered elsewhere.

The second generalization consists in considering the problem in outer domains, which are the complement of open bounded sets, once more with zero boundary data. In dimension one a hole disconnects the domain in several components, and one can reduce the study of the unbounded ones to the case of the half-line.

For large dimensions, larger than or equal to three, a full description of the large time behavior in outer domains, including both the near and the far field limit, was given in [3]. One of the main differences to the one-dimensional case is that in large dimensions the rate of decay of solutions does not depend on the scale, which makes the analysis easier. Moreover, there is a nontrivial asymptotic mass, and the far field limit is given not by a dipole-type solution but by an instantaneous point-source solution with this residual mass. In the critical two-dimensional case mass decays to zero, but very slowly. The far field behavior is still given by an instantaneous point source solution, which in this case has a variable mass that decays to zero; see [7]. The near field limit, which is quite involved, is studied in [5]. As in the one-dimensional case, in this critical dimension there are also different decay rates in different scales.

2 Control from Above

The purpose of this section is to prove the "upper" part of equation (1.10). To this end we will construct a supersolution V approaching D_M with the right rate as t goes to infinity. We only need the function V to be a supersolution in sets of the form

$$A_{\delta,T} = \{(x,t) : t \ge T,\ 0 < x < \delta t^\beta\}$$

for $T > 0$ big and $\delta > 0$ small. Our candidate is

$$V(x,t) = k(t)t^{-\alpha}F_M\Big(\frac{x+a}{t^\beta}\Big), \quad a > 0, \tag{2.1}$$

for some function k satisfying $k(t) \searrow 1$ as $t \to \infty$. It will turn out that a good choice for k is given by the solution to

$$tk'(t) = -\alpha(k^m(t) - k(t)), \quad t > T, \qquad k(T) = k_0 > 1. \tag{2.2}$$

Note that $k(t)$ is well defined and that it is a monotone decreasing function of time.

We start by proving that V is a supersolution to the PME in $A_{\delta,T}$ if δ is small and T is big.

Lemma 2.1. *Let $m > 1$ and $M > 0$. There exist values $\bar\delta > 0$ and $\overline{T} > 0$ depending only on M and m such that for all $a \in (0,1)$, $T \geq \overline{T}$ and $k_0 > 1$ the function V given by* (2.1)–(2.2) *satisfies*

$$V_t - (V^m)_{xx} \geq 0 \quad \text{in } A_{\delta,T} \text{ for all } \delta \in (0,\bar\delta).$$

Proof. Let $\xi = (x+a)/t^\beta$. Since $m\alpha + 2\beta = \alpha + 1$, a straightforward computation combined with (1.4) shows that

$$(V_t - (V^m)_{xx})(x,t) = t^{-\alpha-1}\big(tk'(t)F_M(\xi) + (k(t) - k^m(t))(F_M^m)''(\xi)\big).$$

Thus, if we choose k satisfying (2.2), we get

$$(V_t - (V^m)_{xx})(x,t) = t^{-\alpha-1}(k^m(t) - k(t))\big(-\alpha F_M(\xi) - (F_M^m)''(\xi)\big), \quad x \in \mathbb{R}_+,\ t \geq T.$$

Now we observe that there is a value $\bar\xi \in (0,\xi_1)$ such that $F_1'(\xi) > 0$ for $\xi \in (0,\bar\xi)$. Therefore,

$$F_M'(\xi) > 0, \quad \xi \in \big(0, \bar\xi M^{\frac{m-1}{2m}}\big). \tag{2.3}$$

On the other hand, if we take $\bar\delta < \bar\xi M^{(m-1)/(2m)}/2$ and then $\overline{T} = (1/\bar\delta)^{1/\beta}$, for any $\delta \in (0,\bar\delta)$ and $T \geq \overline{T}$ we get

$$0 < \xi = \frac{x+a}{t^\beta} \leq 2\bar\delta < \bar\xi M^{\frac{m-1}{2m}}, \quad (x,t) \in A_{\delta,T}.$$

Hence, $-(F_M^m)''(\xi) - \alpha F_M(\xi) = \beta\xi F_M'(\xi) > 0$ if $(x,t) \in A_{\delta,T}$, and the desired result follows since $k(t) > 1$ for all times. □

We now arrive at the matching part of the result where, using the behavior in the far field scale, we obtain an upper bound in sets of the form $A_{\delta,T}$ for δ small and T large.

Lemma 2.2. *Let u be the unique weak solution to* (1.1), *$M = \int_0^\infty xu_0(x)\,dx$ and $\bar\delta$ and $\overline{T}$ as in Lemma 2.1. For every $\varepsilon > 0$ there exists a value $T_\varepsilon \geq \overline{T}$ such that for all $a \in (0,1)$ and $T \geq T_\varepsilon$ there is a value $k_0 \geq 1$ such that the function V given by* (2.1)–(2.2) *satisfies*

$$u(x,t) \leq (1 + C_\delta\varepsilon)V(x,t), \quad (x,t) \in A_{\delta,T}, \qquad C_\delta = \frac{1}{F_M(\delta)}, \quad \delta \in (0,\bar\delta). \tag{2.4}$$

Proof. Formula (1.3) implies that for every $\varepsilon > 0$ there exists T_ε, which may be assumed to be larger than $\overline{T}$, such that

$$t^\alpha|u(x,t) - D_M(x,t)| \leq \varepsilon, \quad x \in \mathbb{R}_+,\ t \geq T_\varepsilon. \tag{2.5}$$

For any given $T \geq T_\varepsilon$ and some big enough $k_0 > 1$ to be determined below, we define V by (2.1)–(2.2).

Since $k(t) > 1$, inequality (2.3) implies that $D_M(x,t) \leq V(x,t)$ in $A_{\delta,\overline{T}}$, and hence in $A_{\delta,T}$. Therefore,

$$t^\alpha(u(x,t) - V(x,t)) \leq t^\alpha|u(x,t) - D_M(x,t)| + t^\alpha(D_M(x,t) - V(x,t)) \leq \varepsilon, \quad (x,t) \in A_{\delta,T}.$$

On the other hand, for $x = \delta t^\beta$ and $t \geq T$ we have

$$t^\alpha V(x,t) \geq t^\alpha D_M(x,t) = F_M\Big(\frac{x}{t^\beta}\Big) = F_M(\delta),$$

and we conclude that

$$t^{\alpha}(u(x,t)-V(x,t)) \le t^{\alpha} C_{\delta}\varepsilon V(x,t).$$

Thus,

$$u(x,t) \le \underbrace{(1+C_{\delta}\varepsilon)V(x,t)}_{W(x,t)} \quad \text{if } x=\delta t^{\beta},\ t\ge T.$$

Since solutions to (1.8) with integrable initial data are bounded for $t \ge \tau > 0$, see [12], formula (1.3) implies that there is a constant $C_0 > 0$ such that $u(x,t) \le C_0 t^{-\alpha}$ for $t \ge \overline{T}$. Thus, using the monotonicity property (2.3), we get

$$V(x,T) = k(T)T^{-\alpha}F_M\Big(\frac{x+a}{T^{\beta}}\Big) \ge k_0 F_M\Big(\frac{a}{T^{\beta}}\Big)T^{-\alpha} \ge C_0 T^{-\alpha} \ge u(x,T) \quad \text{for } 0<x<\delta T^{\beta}$$

if $k_0 > \max\{C_0/F_M(a/T^{\beta}), 1\}$. Therefore, with that choice of k_0, we have

$$u(x,T) \le V(x,T) \le W(x,T) \quad \text{if } 0<x<\delta T^{\beta}.$$

We now observe that W is a supersolution to the PME in $A_{\delta,T}$. Indeed, in that set

$$(V^m)_{xx}(x,t) = -k(t)t^{-(\alpha+1)}(\alpha F_M(\xi)+\beta\xi F_M'(\xi)) < 0,$$

and hence Lemma 2.1 implies that

$$\begin{aligned} W_t-(W^m)_{xx} &= (1+C_{\delta}\varepsilon)V_t-(1+C_{\delta}\varepsilon)^m(V^m)_{xx} \\ &= (1+C_{\delta}\varepsilon)(V_t-(V^m)_{xx}) - ((1+C_{\delta}\varepsilon)^m-(1+C_{\delta}\varepsilon))(V^m)_{xx} \ge 0. \end{aligned}$$

We finally notice that $W(0,t) > 0$ for all $t > T$. Therefore, comparison yields (2.4). □

The third ingredient, that we prove next, is that V and D_M are ε-close in sets of the form $A_{\delta,T}$ for large times, even when the difference is multiplied by

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}},$$

if the parameter a in the definition of V is $O(\varepsilon^m)$.

Lemma 2.3. *Let $m>1$, $M>0$, $\varepsilon>0$, and let $\bar\delta$ and $\overline{T}$ be as in Lemma 2.1. There exist values $\hat\delta \in (0,\bar\delta)$, $\widehat{T} \ge \overline{T}$ independent of ε, and $a_\varepsilon \in (0,1]$, such that for all $\delta\in(0,\hat\delta)$, $T \ge \widehat{T}$ and $a \in (0,a_\varepsilon)$ the function V given by* (2.1)–(2.2) *satisfies*

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}|V(x,t)-D_M(x,t)| < \varepsilon \quad \textit{in } A_{\delta,\widehat{T}_\varepsilon} \textit{ for some } \widehat{T}_\varepsilon \ge T. \tag{2.6}$$

Proof. There holds that

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}|D_M(x,t)-V(x,t)| = \underbrace{\frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\Big|F_M\Big(\frac{x+a}{t^{\beta}}\Big)-F_M\Big(\frac{x}{t^{\beta}}\Big)\Big|}_{\mathrm{I}} + \underbrace{\frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}F_M\Big(\frac{x+a}{t^{\beta}}\Big)|k(t)-1|}_{\mathrm{II}}.$$

In order to estimate I we notice that there exist constants $\hat\xi \in (0,\bar\xi)$ and $K>0$ such that $\xi F_1'(\xi) \le K\xi^{1/m}$ for $\xi \in (0,\hat\xi)$. Thus, if we take $\hat\delta < \hat\xi M^{(m-1)/(2m)}/2$, and then $\widehat{T} = (1/\hat\delta)^{1/\beta}$, for any $\delta\in(0,\hat\delta)$ and $T \ge \widehat{T}$, we get

$$\begin{aligned} \mathrm{I} &= \frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\int_0^1 F_M'\Big(\frac{x+sa}{t^{\beta}}\Big)\frac{a}{t^{\beta}}\,ds = \frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\int_0^1 F_M'\Big(\frac{x+sa}{t^{\beta}}\Big)\frac{x+sa}{t^{\beta}}\frac{a}{x+sa}\,ds \\ &\le K\int_0^1 (x+sa)^{\frac{1}{m}-1}a\,ds \le mKa^{\frac{1}{m}} \quad \text{in } A_{\delta,T}. \end{aligned}$$

Therefore, $\mathrm{I} < \frac{\varepsilon}{2}$ if $a < a_\varepsilon := \min\{(\frac{\varepsilon}{2mK})^m, 1\}$.

As for the other term, we will use that

$$F_M(\xi) \le C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} \xi^{\frac{1}{m}}, \qquad \xi \in \mathbb{R}_+; \tag{2.7}$$

see formulas (1.5)–(1.6). Therefore, taking into account that $a < 1$, we obtain

$$\mathrm{II} \le C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} \Big(\frac{x+a}{x+1}\Big)^{\frac{1}{m}} |k(t)-1| \le C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} |k(t)-1| < \frac{\varepsilon}{2}$$

if $t \ge \widehat{T}_\varepsilon$ for some $\widehat{T}_\varepsilon \ge T$ since $k(t) \to 1$ as $t \to \infty$. □

We finally arrive at the main result of this section, the upper limit.

Proposition 2.4. *Let u be the unique weak solution to* (1.1) *and let D_M be the unique dipole solution to the PME with first moment $M = \int_0^\infty x u_0(x)\,dx$. If $\hat{\delta}$ is the constant given by Lemma 2.3, then*

$$\limsup_{t\to\infty} t^{\alpha+\frac{\beta}{m}} \sup_{0<x<\delta t^\beta} \frac{(u(x,t) - D_M(x,t))}{(1+x)^{\frac{1}{m}}} \le 0$$

for all $\delta \in (0, \hat{\delta})$.

Proof. Given $\varepsilon > 0$, let T_ε be as in Lemma 2.2, and $\widehat{T}$ and a_ε as in Lemma 2.3. We take $T \ge \max\{T_\varepsilon, \widehat{T}\}$ and $a \in (0, a_\varepsilon)$, and then $k_0 > 1$ large so that the function V defined by (2.1)–(2.2) satisfies (2.4) and (2.6) for any given $\delta \in (0, \hat{\delta})$ for some large $\widehat{T}_\varepsilon \ge T$.

On the other hand, since $k(t) \to 1$ as $t \to \infty$ and $a \in (0, 1)$, using (2.7), we get

$$\frac{t^{\alpha+\frac{\beta}{m}} V(x,t)}{(1+x)^{\frac{1}{m}}} = \frac{k(t) t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}} F_M\Big(\frac{x+a}{t^\beta}\Big) \le 2C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} \Big(\frac{x+a}{x+1}\Big)^{\frac{1}{m}} \le 2C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}}$$

for all large enough times.

Combining all the estimates mentioned above we finally get, for $0 < x < \delta t^\beta$ and all large enough times,

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}} (u(x,t) - D_M(x,t)) \le C_\delta \varepsilon \frac{t^{\alpha+\frac{\beta}{m}} V(x,t)}{(1+x)^{\frac{1}{m}}} + \varepsilon \le (C_\delta 2 C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} + 1)\varepsilon.$$ □

3 Control from Below

We will now deal with the "lower" part of (1.10). The proof is quite similar to that of the "upper" part. However, in this case, subsolutions are only obtained in sets of the form

$$A_{a,\delta,T} = \{(x,t) : a < x < \delta t^\beta,\ t \ge T\},$$

and the points $x \in (0, a)$ have to be treated separately.

The subsolution approaching D_M with the right rate as t goes to infinity will have the form

$$v(x,t) = c(t) t^{-\alpha} F_M\Big(\frac{x-a}{t^\beta}\Big), \qquad a > 0, \tag{3.1}$$

where c is the solution to the initial value problem

$$tc'(t) = \alpha(c(t) - c^m(t)), \quad t > T, \qquad c(T) = c_0 \in (0,1). \tag{3.2}$$

The function c is well defined for $t \ge T$. It is monotone increasing and $c(t) \nearrow 1$ as $t \to \infty$, as desired.

We start by proving that v is a subsolution to the PME in $A_{a,\delta,T}$ if δ is small and T is big, no matter the value of $a \in (0, 1)$.

Lemma 3.1. *Let $m > 1$ and $M > 0$, and let $\bar{\delta} > 0$ be as in Lemma 2.1. For all $a \in (0, 1)$, $T > 0$ and $c_0 \in (0, 1)$ the function v given by* (3.1)–(3.2) *satisfies*

$$v_t - (v^m)_{xx} \leq 0 \quad \text{in } A_{a,\delta,T} \text{ for all } \delta \in (0, \bar{\delta}).$$

Proof. Let $\xi = (x - a)/t^\beta < x/t^\beta$. A computation analogous to the one we did in the proof of Lemma 2.1 shows that

$$(v_t - (v^m)_{xx})(x, t) = -t^{-\alpha-1}(c(t) - c^m(t))\beta\xi F_M'(\xi);$$

but

$$0 < \xi = \frac{x - a}{t^\beta} < \frac{x}{t^\beta} < \delta < \bar{\delta} < \bar{\xi} M^{\frac{m-1}{2m}}, \quad (x, t) \in A_{a,\delta,T},$$

and hence the result follows from (2.3) since $c(t) < 1$ for all times. □

The matching with the outer behavior will require to know that u is positive in some set $A_{\delta,T}$. This is what we prove next.

Lemma 3.2. *Let u be the unique weak solution to* (1.1), $M = \int_0^\infty x u_0(x)\,dx$. *Given $\delta \in (0, \xi_M)$, there exists a time T_δ such that $u(x, t) > 0$ in A_{δ,T_δ}.*

Proof. Since $\delta < \xi_M$, the convergence result (1.3) implies that there is a time t_δ such that $u(x, t) \geq K t^{-\alpha}$ for some $K > 0$ if $x \in (\frac{\delta}{2} t^\beta, \delta t^\beta)$, $t \geq t_\delta$.

We now use that nonnegative solutions to (1.1) have the so-called retention property: if $u(x, \bar{t}) > 0$, then $u(x, t) > 0$ for all $t \geq \bar{t}$. This can be proved in several ways, for instance, by using that the application $t \mapsto t^{1/(m-1)} u(x, t)$ is non-decreasing. This monotonicity property follows easily from the estimate $u_t \geq -\frac{u}{(m-1)t}$ which is proved by using comparison arguments; see, for instance, [12]. Hence, we have $u(x, t) > 0$ for $x \in (\frac{\delta}{2} t_\delta^\beta, \delta t^\beta)$, $t \geq t_\delta$.

It only remains to prove that $u(x, T_\delta) > 0$ if $x \in (0, \frac{\delta}{2} T_\delta^\beta)$ for some large enough $T_\delta \geq t_\delta$ since the result will then follow from the retention property. The positivity in this fixed interval is achieved by comparison with a suitable translate of a source-type solution of the PME,

$$B(x, t; C) = t^{-\frac{1}{m+1}} (C - \kappa_m |\xi|^2)_+^{\frac{1}{m-1}}, \qquad \xi = \frac{x}{t^{\frac{1}{m+1}}},\ C > 0,$$

where the constant κ_m is as in formula (1.7). Such solutions are due to Zel'dovič and Kompaneets [13] and Barenblatt [1]. Indeed, take $x_0 \in (\frac{\delta}{2} t_\delta^\beta, \delta t_\delta^\beta)$. It is easy to check that if $C > 0$ is small enough, then

$$B(x - x_0, t_\delta; C) = 0 \quad \text{if } x \notin \Big(\frac{\delta}{2} t_\delta^\beta, \delta t_\delta^\beta\Big), \qquad \sup_{\frac{\delta}{2} t_\delta^\beta \leq x \leq \delta t_\delta^\beta} B(x - x_0, t_\delta; C) \leq K t_\delta^{-\alpha}.$$

Moreover, $B(x - x_0, t; C)$ is a solution to (1.1) until it touches the boundary $x = 0$. This will happen in a finite time $T_\delta \geq t_\delta$. Then comparison yields that $u(x, t) \geq B(x - x_0, t; C)$ for all $t \in [t_\delta, T_\delta]$, and hence the required positivity. □

We now perform the matching with the outer behavior in order to obtain the control from below.

Lemma 3.3. *Let u be the unique weak solution to* (1.1), $M = \int_0^\infty x u_0(x)\,dx$ *and $\bar{\delta}$ as in Lemma 2.1. Given $\varepsilon > 0$, $a \in (0, 1)$ and $\delta \in (0, \bar{\delta})$, there is a time $T_{\varepsilon,a,\delta} > 0$ such that for all $T \geq T_{\varepsilon,a,\delta}$ there is a value $c_0 \in (0, 1)$ such that the function v given by* (3.1)–(3.2) *satisfies*

$$u(x, t) \geq (1 - C_\delta \varepsilon) v(x, t), \quad (x, t) \in A_{a,\delta,T},\ C_\delta = \frac{1}{F_M(\delta)}. \tag{3.3}$$

Proof. Let $\delta \in (0, \bar{\delta})$ and $a \in (0, 1)$. Note that $\bar{\delta} < \xi_M$. The convergence result (1.3) implies that, given $\varepsilon > 0$, there exists a value $T_{a,\varepsilon,\delta} \geq \max\{T_{\bar{\delta}}, (\frac{a}{\delta})^{1/\beta}\}$ such that (2.5) holds with $T_\varepsilon = T_{\varepsilon,a,\delta}$.

Let $T \geq T_{\varepsilon,a,\delta}$. We know from Lemma 3.2 that there is a constant $\kappa = \kappa(a, \delta, T)$ such that $u(x, T) \geq \kappa$ if $a < x < \delta T^\beta$. Take now $c_0 \in (0, 1)$ small so that $c_0 T^{-\alpha} F_M(\delta) \leq \kappa$. With this choice of T and c_0, we define v by (3.1)–(3.2). Using the monotonicity property (2.3), we get

$$u(x, T) \geq c_0 T^{-\alpha} F_M\Big(\frac{x - a}{T}\Big) = v(x, T), \quad a < x < \delta T^\beta.$$

On the other hand, since $T \geq T_{\varepsilon,a,\delta}$, the convergence result (1.3) together with the self-similar form of D_M, cf. formula (1.2), yields

$$u(x,t) \geq -\varepsilon t^{-\alpha} + D_M(x,t) = (1 - F_M(\delta)^{-1}\varepsilon)D_M(x,t), \quad x = \delta t^\beta,\ t \geq T.$$

We notice now that $c(t) < 1$. Therefore, inequality (2.3) implies that $D_M \geq v$ in $A_{\delta,T}$, and we conclude that

$$u(x,t) \geq \underbrace{(1 - C_\delta\varepsilon)v(x,t)}_{w(x,t)}, \quad x = \delta t^\beta,\ t \geq T$$

for $\varepsilon < F_M(\delta)$ and $C_\delta = 1/F_M(\delta)$.

We now observe that w is a subsolution to the PME in $A_{a,\delta,T}$. Indeed, in that set we have

$$(v^m)_{xx}(x,t) = -c(t)t^{-(\alpha+1)}(\alpha F_M(\xi) + \beta\xi F_M'(\xi)) < 0,$$

and hence Lemma 3.1 implies that

$$w_t - (w^m)_{xx} = (1 - C_\delta\varepsilon)(v_t - (v^m)_{xx}) - ((1 - C_\delta\varepsilon)^m - (1 - C_\delta\varepsilon))(v^m)_{xx} \leq 0 \quad \text{in } A_{a,\delta,T}.$$

We finally notice that $w(a,t) = 0$ for all $t > T$. Therefore, a comparison argument allows to conclude that (3.3) holds. □

The next step is to control the difference between v and D_M for large times.

Lemma 3.4. *Given $m > 1$, $M > 0$ and $\varepsilon > 0$, let $\hat{\delta}$ and $a_\varepsilon \in (0,1]$ be as in Lemma 2.3. Then for all $\delta \in (0,\hat{\delta})$, $T > 0$ and $a \in (0,a_\varepsilon)$ the function v given by* (3.1)–(3.2) *satisfies*

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}|D_M(x,t) - v(x,t)| < \varepsilon \quad \text{in } A_{a,\delta,\widetilde{T}_\varepsilon} \text{ for some } \widetilde{T}_\varepsilon \geq T. \tag{3.4}$$

Proof. Let $x \in (a, \delta t^\beta)$ with $\delta < \hat{\delta}$. Arguing as in the proof of Lemma 2.3, we get

$$\begin{aligned}\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\Big|t^{-\alpha}F_M\Big(\frac{x-a}{t^\beta}\Big) - D_M(x,t)\Big| &\leq \frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\int_0^1 F_M'\Big(\frac{x-sa}{t^\beta}\Big)\frac{x-sa}{t^\beta}\,\frac{a}{x-sa}\,ds\\ &\leq K\int_0^1 (x-sa)^{\frac{1}{m}-1}a\,ds\\ &\leq mKa^{\frac{1}{m}} < \frac{\varepsilon}{2}\end{aligned}$$

if $a < a_\varepsilon := \min\{(\frac{\varepsilon}{2mK})^m, 1\}$.

On the other hand, using (2.7), we obtain

$$\begin{aligned}\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}\Big|t^{-\alpha}F_M\Big(\frac{x-a}{t^\beta}\Big) - v(x,t)\Big| &= \frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}F_M\Big(\frac{x-a}{t^\beta}\Big)|1-c(t)|\\ &\leq C_m^{\frac{1}{m-1}}M^{\frac{m+1}{2m^2}}\Big(\frac{x-a}{1+x}\Big)^{\frac{1}{m}}|1-c(t)|\\ &\leq C_m^{\frac{1}{m-1}}M^{\frac{m+1}{2m^2}}|1-c(t)| < \frac{\varepsilon}{2}\end{aligned}$$

if $t \geq \widetilde{T}_\varepsilon$ for some large enough $\widetilde{T}_\varepsilon \geq T$ since $c(t) \to 1$ as $t \to \infty$.

The combination of the above two estimates yields the result. □

We now have all tools we need to prove the lower limit. Here a difference arises with respect to the upper limit: we will have to treat separately the limit in sets of the form $(0,a)$ with a small. This is done by using that both u and D_M are small in this set for large times.

Proposition 3.5. *Let u be the unique weak solution to* (1.1) *and let D_M be the unique dipole solution to the PME with first moment $M = \int_0^\infty xu_0(x)\,dx$. If $\hat{\delta}$ is the constant given by Lemma 2.3, then*

$$\liminf_{t\to\infty} t^{\alpha+\frac{\beta}{m}} \sup_{0<x<\delta t^\beta} \frac{(u(x,t)-D_M(x,t))}{(1+x)^{\frac{1}{m}}} \geq 0$$

for all $\delta \in (0,\hat{\delta})$.

Proof. Given $\delta \in (0,\hat{\delta})$ and $\varepsilon \in (0, F_M(\delta))$, we choose a small value $a \in (0, a_\varepsilon)$ with $a_\varepsilon \in (0,1]$ as in Lemma 2.3. We will specify how small it has to be later on. We take $T \geq \max\{T_\varepsilon, T_{\varepsilon,a,\delta}\}$, with T_ε as in Lemma 2.2 and $T_{\varepsilon,a,\delta}$ as in Lemma 3.3, and then $c_0 \in (0,1)$ small enough so that the function v defined by (3.1)–(3.2) satisfies (3.3) and (3.4).

By Lemma 2.2, we know that there is a value $k_0 \geq 1$ such that the function V defined by (2.1)–(2.2) satisfies (2.4). Besides, since $k(t) \to 1$ as $t \to \infty$, there exists a time $\check{T} \geq T$ such that $k(t) \leq 2$ for all $t \geq \check{T}$. Therefore, since $C_\delta \varepsilon < 1$, using (2.7), we get

$$\begin{aligned}\frac{t^{\alpha+\frac{\beta}{m}} u(x,t)}{(1+x)^{\frac{1}{m}}} &\leq (1+C_\delta\varepsilon)\frac{t^{\alpha+\frac{\beta}{m}} V(x,t)}{(1+x)^{\frac{1}{m}}}\\ &\leq 2(1+C_\delta\varepsilon)\frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}} F_M\Big(\frac{x+a}{t^\beta}\Big)\\ &\leq 4C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} (2a)^{\frac{1}{m}} < \frac{\varepsilon}{2}\end{aligned}$$

if a is small enough. On the other hand, using again (2.7), we get

$$\frac{t^{\alpha+\frac{\beta}{m}} D_M(x,t)}{(1+x)^{\frac{1}{m}}} = \frac{t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}} F_M\Big(\frac{x}{t^\beta}\Big) \leq C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} a^{\frac{1}{m}} < \frac{\varepsilon}{2}.$$

We conclude that

$$t^{\alpha+\frac{\beta}{m}} \sup_{0<x<a} \frac{|u(x,t)-D_M(x,t)|}{(1+x)^{\frac{1}{m}}} \leq \varepsilon$$

if t is large enough.

We now consider the set $a < x < \delta t^\beta$. Since $c(t) \to 1$ as $t \to \infty$, using (2.7), we get

$$\frac{t^{\alpha+\frac{\beta}{m}} v(x,t)}{(1+x)^{\frac{1}{m}}} = \frac{c(t) t^{\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}} F_M\Big(\frac{x-a}{t^\beta}\Big) \leq 2C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} \Big(\frac{x-a}{x+1}\Big)^{\frac{1}{m}} \leq 2C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}}$$

for all large enough times. Combining this estimate with (3.3) and (3.4), we finally get

$$\frac{t^{\alpha+\frac{\beta}{m}}}{(1+x)^{\frac{1}{m}}}(u(x,t)-D_M(x,t)) \geq -C_\delta\varepsilon \frac{t^{\alpha+\frac{\beta}{m}} v(x,t)}{(1+x)^{\frac{1}{m}}} - \varepsilon \geq -\big(C_\delta 2C_m^{\frac{1}{m-1}} M^{\frac{m+1}{2m^2}} + 1\big)\varepsilon$$

for $a < x < \delta t^\beta$ and all large enough times. □

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References

[1] G. I. Barenblatt, On some unsteady motions of a liquid and gas in a porous medium (in Russian), *Akad. Nauk SSSR. Prikl. Mat. Meh.* **16** (1952), no. 1, 67–78.

[2] G. I. Barenblatt and Y. B. Zel'dovich, On dipole solutions in problems of non-stationary filtration of gas under polytropic regime (in Russian), *Prikl. Mat. Mekh.* **21** (1957), no. 5, 718–720.

[3] C. Brändle, F. Quirós and J. L. Vázquez, Asymptotic behaviour of the porous media equation in domains with holes, *Interfaces Free Bound.* **9** (2007), no. 2, 211–232.

[4] C. Cortázar, M. Elgueta, F. Quirós and N. Wolanski, Asymptotic behavior for a nonlocal diffusion equation on the half line, *Discrete Contin. Dyn. Syst.* **35** (2015), no. 4, 1391–1407.

[5] C. Cortázar, F. Quirós and N. Wolanski, Near field asymptotics for the porous medium equation in exterior domains. The critical two-dimensional case, preprint (2016), https://arxiv.org/abs/1610.04772.

[6] J. R. Esteban and J. L. Vázquez, Homogeneous diffusion in $\mathbb{R}$ with power-like nonlinear diffusivity, *Arch. Ration. Mech. Anal.* **103** (1988), no. 1, 39–80.

[7] B. H. Gilding and J. Goncerzewicz, Large-time behaviour of solutions of the exterior-domain Cauchy–Dirichlet problem for the porous media equation with homogeneous boundary data, *Monatsh. Math.* **150** (2007), no. 1, 11–39.

[8] B. H. Gilding and L. A. Peletier, On a class of similarity solutions of the porous media equation, *J. Math. Anal. Appl.* **55** (1976), no. 2, 351–364.

[9] B. H. Gilding and L. A. Peletier, On a class of similarity solutions of the porous media equation. II, *J. Math. Anal. Appl.* **57** (1977), no. 3, 522–538.

[10] J. Hulshof and J. L. Vázquez, The dipole solution for the porous medium equation in several space dimensions, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **20** (1993), no. 2, 193–217.

[11] S. Kamin and J. L. Vázquez, Asymptotic behaviour of solutions of the porous medium equation with changing sign, *SIAM J. Math. Anal.* **22** (1991), no. 1, 34–45.

[12] J. L. Vázquez, *The Porous Medium Equation. Mathematical Theory*, Oxford Math. Monogr., Oxford University Press, Oxford, 2007.

[13] Y. B. Zel'dovich and A. S. Kompaneets, On the theory of propagation of heat with the heat conductivity depending upon the temperature (in Russian), in: *Collection in Honor of the Seventieth Birthday of Academician A. F. Ioffe*, Izdat. Akad. Nauk SSSR, Moscow (1950), 61–71.