

Double field theory: a pedagogical review

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TOPICAL REVIEW

Double field theory: a pedagogical review

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Online at stacks.iop.org/CQG/30/163001**Abstract**

Double field theory (DFT) is a proposal to incorporate T-duality, a distinctive symmetry of string theory, as a symmetry of a field theory defined on a double configuration space. The aim of this review is to provide a pedagogical presentation of DFT and its applications. We first introduce some basic ideas on T-duality and supergravity in order to proceed to the construction of generalized diffeomorphisms and an invariant action on the double space. Steps towards the construction of a geometry on the double space are discussed. We then address generalized Scherk–Schwarz compactifications of DFT and their connection to gauged supergravity and flux compactifications. We also discuss U-duality extensions and present a brief parcours on worldsheet approaches to DFT. Finally, we provide a summary of other developments and applications that are not discussed in detail in the review.

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(Some figures may appear in colour only in the online journal)

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1. Introduction

Double field theory (DFT) is a proposal to incorporate T-duality, a distinctive symmetry of string (or M-)theory, as a symmetry of a field theory [1, 2, 3]. At first sight, such attempt could appear to lead to a *blind alley* since the very presence of T-duality requires extended objects like strings which, unlike field theory particles, are able to wrap non-contractible cycles. It is the very existence of winding modes (associated with these wrappings) and momentum modes that underlies T-duality, which manifests itself by connecting the physics of strings defined on geometrically very different backgrounds. Then, a T-duality symmetric field theory must take information about windings into account.

A way to incorporate such information is suggested by compactification of strings on a torus. In string toroidal compactifications, there are compact momentum modes, dual to compact coordinates y^m , $m = 1, \dots, n$, as well as string winding modes. Therefore, it appears that a new set of coordinates \tilde{y}_m , dual to windings, should be considered for the compactified sector in the field theory description. It is in this sense that DFT is a ‘doubled’ theory: it doubles the coordinates of the compact space. Formally, the non-compact directions x^μ , $\mu = 1, \dots, d$, are also assigned duals \tilde{x}_μ for completion, although this is merely aesthetical since nothing really depends on them. The DFT proposal is that, for a D -dimensional space with d non-compact spacetime dimensions and n compact dimensions, i.e. $D = n + d$, the fields depend on

coordinates $X^M = (\tilde{x}_\mu, \tilde{y}_m, x^\mu, y^m)$, where x^μ are spacetime coordinates, \tilde{x}_μ are there simply for decoration and $\mathbb{Y}^A = (\tilde{y}_m, y^m)$ are $2n$ compact coordinates, with $A = 1, \dots, 2n$.

When the compactification scale is much bigger than the string size, it is hard for strings to wrap cycles and winding modes are ineffective at low energies. In the DFT framework, this corresponds to the usual situation where there is no dependence on dual coordinates. Oppositely, in the T-dual description, if the compactification scale is small, then the momentum (winding) modes are heavy (light), and DFT only depends on dual coordinates. Either way, these (de)compactification limits typically amount on the DFT side to constrain the theory to depend only on a subset of coordinates. In particular, when all the coordinates are non-compact, one finds complete correspondence with supergravity in $D = 10$ dimensions.

The T-duality group associated with string toroidal compactifications on T^n is $O(n, n)$. The doubled internal coordinates \mathbb{Y}^A mix (span a vector representation) under the action of this group. However, it proves useful to formulate the theory in a double space with a full duality group $O(D, D)$, where all D coordinates are doubled, mimicking a string theory where all dimensions are compact⁵.

The next step in the construction of DFT is to choose the defining fields. In the simplest formulation of DFT, the field content involves the D -dimensional metric g_{ij} , a 2-form field b_{ij} and a scalar dilaton field ϕ . From a string perspective, they correspond to the universal gravitational massless bosonic sector, present in the bosonic, heterotic and Type II string theories as well as in the closed sector of Type I strings, in which case b_{ij} would be a Ramond–Ramond (R-R) field. However, since we are looking for an $O(D, D)$ invariant theory, the fundamental fields should be $O(D, D)$ tensors with $2D$ -dimensional indices. In fact, in DFT the g_{ij} and b_{ij} fields are *unified* in a single object: a generalized $O(D, D)$ symmetric metric \mathcal{H}_{MN} , with $M, N = 1, \dots, 2D$, defined in the double space. Then, based on *symmetries*, DFT *unifies* through *geometrization*, since it incorporates the 2-form into a generalized geometric picture. There is also a field d , which is a T-scalar combining the dilaton ϕ and the determinant of the metric g . The first part of this review will be dedicated to discuss the consistent construction of a DFT action as a functional of these generalized fields on a doubled configuration space.

In the decompactification limit (taking for example $D = 10$ so as to make contact with string theory), when the dual coordinates are projected out, the DFT action reproduces the action of the universal massless bosonic sector of supergravity

$$S = \int dx \sqrt{g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right),$$

where $H_{ijk} = 3\partial_{[i}b_{jk]}$ is the field strength of the 2-form. This limit action is invariant under the usual diffeomorphisms of General Relativity and gauge transformations of the 2-form. Following with the unification route, we then expect to combine these transformations into ‘generalized diffeomorphisms’ under which the DFT action should be invariant. They should then reduce to standard general coordinate and gauge transformations in the decompactification limit. In section 3, we will define these transformations and discuss constraint equations required by the gauge consistency of the generalized diffeomorphisms. Generically, these constraints restrict the space of configurations for which DFT is a consistent theory, i.e. DFT is a *restricted theory*.

The constraints of the theory are solved in particular when a *section condition* or *strong constraint* is imposed. This restriction was proposed in the original formulations of DFT, inspired by string field theory constraints. It implies that the fields of the theory only depend on a slice of the double space parameterized by half of the coordinates, such that there always

⁵ Time is treated here at the same level of other space coordinates for simplicity, but it can be restored by a standard Wick rotation.

exists a frame in which, locally, the configurations do not depend on the dual coordinates. Since the strong constraint is covariant under the global symmetries, the theory can still be covariantly formulated, but it is actually not truly doubled after it is solved.

Nevertheless, one can also find other solutions to the constraints that violate the strong constraint. In particular, Scherk–Schwarz (SS)-dimensional reductions of DFT, where the spacetime fields are twisted by functions of the internal coordinates, have proven to be interesting scenarios where consistent strong-constraint violating configurations are allowed. Interestingly enough, the SS reduction of (bosonic) DFT on the doubled space leads to an action that can be identified with (part of) the action of the bosonic sector of four-dimensional half-maximal gauged supergravities. Recall the fact that gauged supergravities are deformations of ordinary Abelian supergravity theories, in which the deformation parameters (gaugings) are encoded in the embedding tensor. DFT provides a higher dimensional interpretation of these gaugings in terms of SS double T-duality twists. Moreover, the quadratic constraints on gaugings are in one to one correspondence with the closure constraints of the generalized diffeomorphisms.

Gauged supergravities describe superstring compactifications with fluxes, where the gaugings correspond to the quantized fluxes. Therefore it is instructive to look at the connection between SS reductions of DFT and string flux compactifications. This connection is subtle. It is known that orientifold compactifications of $D = 10$ effective supergravity actions, corresponding to the low-energy limit of string theories, lead to four-dimensional superpotentials in which the coefficients are the fluxes. However, by looking at flux compactifications of string theories, expected to be T-duality related (for instance, type IIA and type IIB theories), the effective superpotentials turn out not to be T-dual. Namely, these compactifications are gauged supergravities but with different orbits of gaugings turned on, not connected by T-duality. By invoking symmetry arguments, it has been suggested that new fluxes should be included in order for the full superpotentials to be T-duals, so as to repair the mismatch. Similarly, more fluxes are required by invoking type IIB S-duality, M-theory or heterotic/type I S-duality, etc. Then, by imposing duality invariance at the level of the four-dimensional effective theory, the full (orientifold truncated) supergravity theory is obtained with all allowed gaugings.

Hence, we can conclude that four-dimensional gauged supergravity incorporates stringy information that, generically, is not present in the reduction of a ten-dimensional effective supergravity action. Compactification of DFT contains this stringy information from the start and provides a *geometric* interpretation for fluxes, even for those that are non-geometric from a supergravity point of view.

There have also been different proposals to extend DFT ideas to incorporate the full stringy U-duality symmetry group. Take $E_{7(7)}$ as an example, which includes T-dualities and strong–weak duality. The symmetrization now requires an extended geometry on which one can define an extended field theory (EFT). Interestingly enough, from a string theory perspective such formulation automatically incorporates information on NS-NS and R-R fields. While in DFT with $O(n, n)$ symmetry a doubled $2n$ compactified space is needed, in EFT coordinates span a mega-space with more dimensions, where SS compactifications lead to four-dimensional gauged maximal supergravity.

Closely related to DFT (or EFT) is the framework of generalized geometry (GG) (or exceptional generalized geometry), a program that also incorporates duality as a building block. In GG, the tangent space, where the vectors generating diffeomorphisms live, is enlarged to include the 1-forms corresponding to gauge transformations of the 2-form. The internal space is not extended, but the notion of geometry is still modified. DFT and GG are related when the section condition (which un-doubles the double space) is imposed.

To summarize, DFT is a T-duality invariant reformulation of supergravity which appears to offer a way to go beyond the supergravity limits of string theory by introducing some stringy features into particle physics. DFT is all about T-duality *symmetries, unification and geometry*. It is a rather young theory, still under development, but it has already produced plenty of new perspectives and results. There are still many things to understand, and the number of applications is increasing. Here we intend to review this beautiful theory and some of its applications, in as much a pedagogical fashion as we can.

2. Some references and a guide to the review

In this review we intend to provide a self-contained pedagogical introduction to DFT. We will introduce the basics of the theory in lecture-like fashion, mostly intended for non-experts who are willing to know more about this fascinating theory. We will mainly review the recent literature on the formulation and applications of the theory. The field is undergoing a quick expansion, and many exciting results are still to appear. Given the huge amount of material in this active area of research, we are forced to leave out many developments that are as important and stimulating as those that we consider here. With the purpose of reducing the impact of this restriction, we provide an updated list of references, where the reader can find more specific information. We apologize if, unintentionally, we have omitted important references.

Let us first start with a brief list of books on string theory [4]. There are already some very good and complete reviews and lectures on this and related topics that we strongly suggest. In [5, 6], the reader will find a complete exposition on T-duality. Flux compactifications are nicely reviewed in [7]. Comprehensive reports on non-geometric fluxes and their relation to gauged supergravities are those in [8] and [9], respectively. DFT has also been reviewed in [10] and GG in [11]. A review on duality symmetric string and M-theory will be found in [12].

Historically, the idea of implementing T-duality as a manifest symmetry goes back to Duff [13] and Tseytlin [1], where many of the building blocks of DFT were introduced. In [13] one can identify already the double coordinates and the generalized metric among other things, and in [1] the idea of DFT was essentially present. Soon after, Siegel contributed his pioneer work [2], in which a full duality symmetric action for the low-energy superstring was built in superspace formalism. More recently, Hull and Zwiebach combined their expertise on double geometry [14] and string field theory [15] to build DFT [3]. Later, together with Hohm, they constructed a background independent [16] and generalized metric [17] formulation of the theory. The relation of their work to Siegel's was analyzed in [18]. Closely related to DFT is the GG introduced by Hitchin and Gualtieri [19] and related to string theory in the works by Graña *et al* [20].

The inclusion of heterotic vector fields in the theory was discussed in [21] (see also [22]). R-R fields and a unification of Type II theories were included in [23–25], while the massive Type II theory was treated in [26]. The inclusion of fermions and supersymmetrization was performed in [27, 24]. There are many works devoted to explore the geometry of DFT [24, 28–30]. A fully covariant supersymmetric Type II formulation was constructed by Jeon *et al* in [31]. The gauge symmetries and equations of motion were analyzed by Kwak [32], and the gauge algebra and constraints of the theory were discussed in [33, 34]. The connection with duality symmetric nonlinear sigma models was established by Berman *et al* in [35–37]. Many of these studies were inspired by Siegel's construction [2].

Covariant frameworks extending T-duality to the full U-duality group were built as well. These include works by Hull [38], Pacheco and Waldram [39], Berman and Perry [40], the E_{11} programme by West *et al* [41] and [42, 43]. More recent DFT-related developments can

be found in [44–51]. Also, in this direction, but more related to non-geometry and gauged supergravities we have [52–55].

The ideas introduced in [56–58] led to the development of non-geometry, and T-dual non-geometric fluxes were named as such in [59] (see also [60]). Later, S-dual fluxes were introduced in [61], and finally the full U-dual set of fluxes was completed in [52]. Fluxes were considered from a generalized geometrical point of view in [20], and also from a double geometrical point of view in [14, 62, 63]. The relation between DFT, non-geometry and gauged supergravities was explored in [64–69]. Different worldsheet perspectives for fluxes were addressed in [70–72].

Some other developments on DFT and related works can be found in [73]. In the final section, we include more references, further developments and applications of DFT.

The present review covers the following topics.

- Section 3 provides a general introduction to DFT. Starting with some basics on T-duality as a motivation, double space and generalized fields are then defined. A generalized Lie derivative encoding usual diffeomorphisms and 2-form gauge transformations is introduced, together with its consistency constraints. We then present the DFT action, its symmetries and equations of motion.
- Section 4 reviews the construction of an underlying double geometry for DFT. Generalized connections, torsion and curvatures are discussed, and their similarities and differences with ordinary Riemannian geometry are examined.
- Section 5 is devoted to a discussion of dimensional reductions of DFT. After a brief introduction of usual SS compactifications, the procedure is applied to deal with generalized SS compactifications of DFT. The notions of geometric and non-geometric fluxes are addressed and the connection with gauged supergravity is established.
- Section 6 considers the U-duality extension of DFT, extended geometries, EFTs and their relation to maximal gauged supergravity.
- Section 7 reviews the various attempts to construct $O(D, D)$ invariant nonlinear sigma models and their relation to DFT.
- Section 8 provides a brief summary of different developments related to DFT (and guiding references), together with open problems, that are not discussed in detail in the review.

3. Double field theory

Strings feature many amazing properties that particles lack, and this manifests in the fact that string theory has many stringy symmetries that are absent in field theories like supergravity. Field theories usually describe the dynamics of particles, which have no dimension. Since the string is one dimensional, closed strings can wind around non-contractible cycles if the space is compact. So clearly, if we aimed at describing the dynamics of strings with a field theory, the particles should be assigned more degrees of freedom, to account for their limitations to reproduce stringy dynamics like winding. DFT is an attempt to incorporate some stringy features into a field theory in which the new degrees of freedom are introduced by doubling the space of coordinates.

DFT can be thought of as a T-duality invariant formulation of the ‘low-energy’ sector of string theory on a compact space. The reason why *low-energy* is quoted here is, although it is $O(D, D)$ symmetric, DFT keeps the levels that would be massless in the decompactification limit of the string spectrum. In some sense, DFT can be thought of as a T-duality symmetrization of supergravity. Our route will begin with the NS-NS sector, and later we will see how these ideas can be extended to the other sectors. As a starting point, we will briefly introduce the

basic notions of T-duality and supergravity, mostly in an ‘informal’ way, with the only purpose of introducing the fundamental concepts that will then be applied and extended for DFT. A better and more complete exposition of these topics can be found in the many books on string theory [4].

3.1. T-duality basics

T-duality is a symmetry of string theory that relates winding modes in a given compact space with momentum modes in another (dual) compact space. Here we summarize the basic ingredients of T-duality. For a complete and comprehensive review see [5].

Consider the mass spectrum of a closed string on a circle of radius R ,

$$M^2 = (N + \tilde{N} - 2) + p^2 \frac{l_s^2}{R^2} + \tilde{p}^2 \frac{l_s^2}{\tilde{R}^2}, \quad (3.1)$$

where l_s is the string length scale and $\tilde{R} = \frac{l_s^2}{R}$, the dual radius. The first terms contain the infinite mass levels of the string spectrum, and the last two terms are proportional to their quantized momentum p and winding \tilde{p} . The modes are constrained to satisfy the level matching condition (LMC)

$$N - \tilde{N} = p\tilde{p}, \quad (3.2)$$

reflecting the fact that there are no special points in a closed string.

If we take the decompactification limit $R \gg l_s$, the winding modes become heavy, and the mass spectrum for the momentum modes becomes a continuum. On the other hand, if we take the opposite limit $R \ll l_s$, the winding modes become light and the momentum modes heavy. These behaviors are very reasonable: if the compact space is large, it would demand a lot of energy to stretch a closed string around a large circle, so that it can wind, but very little if the space were small.

Note that for any level, the mass spectrum is invariant under the following exchange:

$$\frac{R}{l_s} \leftrightarrow \frac{\tilde{R}}{l_s} = \frac{l_s}{R}, \quad p \leftrightarrow \tilde{p}, \quad (3.3)$$

so if we could only measure masses, we would never be able to distinguish between a closed string moving with a given momentum k on a circle of radius R , and a closed string winding k times on a circle of radius l_s^2/R . This symmetry not only holds for the mass spectrum, but it is actually a symmetry of any observable one can imagine in the full theory!

DFT is currently restricted to the modes of the string that are massless in the decompactified limit, i.e. with $N + \tilde{N} = 2$, but it considers them on a compact space (actually, some or all of these dimensions can be taken to be non-compact). These modes correspond to the levels⁶ $N = \tilde{N} = 1$ (note that the LMC forbids the possibilities $(N, \tilde{N}) = (2, 0)$ and $(0, 2)$ when $p\tilde{p} = 0$) corresponding to a symmetric metric g_{ij} , an antisymmetric 2-form b_{ij} and a dilaton ϕ .

The T-duality symmetry of circle compactifications is generalized to $O(D, D, \mathbb{Z})$ in toroidal compactifications with a constant background metric and an antisymmetric field. The elements of the infinite discrete group $O(D, D)$ (we will drop the \mathbb{Z} in this review because it is irrelevant for our purposes of introducing DFT at the classical level) can be defined as the set of $2D \times 2D$ matrices h_{MN} that preserve the $O(D, D)$ invariant metric η_{MN} :

$$h_M^P \eta_{PQ} h_N^Q = \eta_{MN}, \quad (3.4)$$

⁶ For the cases $(N, \tilde{N}) = (1, 0)$ and $(0, 1)$ there is a particular enhancement of the massless degrees of freedom at $R = l_s$, which has not been contemplated in DFT so far.

where

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}, \quad \eta^{MN} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}, \quad \eta^{MP} \eta_{PN} = \delta_N^M, \quad (3.5)$$

raises and lowers all the $O(D, D)$ indices $M, N = 1, \dots, 2D$.

The momentum and winding modes are now D -dimensional objects p^i and \tilde{p}_i respectively. They can be arranged into a larger object (a generalized momentum)

$$\mathcal{P}^M = \begin{pmatrix} \tilde{p}_i \\ p^j \end{pmatrix}, \quad (3.6)$$

in terms of which the mass operator becomes

$$M^2 = (N + \tilde{N} - 2) + \mathcal{P}^P \mathcal{H}_{PQ} \mathcal{P}^Q, \quad (3.7)$$

where

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix} \quad (3.8)$$

is called the *generalized metric* [74, 75]. The LMC now takes the form

$$N - \tilde{N} = \frac{1}{2} \mathcal{P}^M \mathcal{P}_M \quad (3.9)$$

and implies that, for the DFT states $N = \tilde{N} = 1$, the generalized momenta must be orthogonal with respect to the $O(D, D)$ metric $p^i \tilde{p}_i = 0$.

Any element of $O(D, D)$ can be decomposed as successive products of the following transformations:

$$\text{Diffeomorphisms : } h_M^N = \begin{pmatrix} E_j^i & 0 \\ 0 & E_i^j \end{pmatrix}, \quad E \in GL(D)$$

$$\text{Shifts : } h_M^N = \begin{pmatrix} \delta_j^i & 0 \\ B_{ij} & \delta_i^j \end{pmatrix}, \quad B_{ij} = -B_{ji} \quad (3.10)$$

$$\text{T-dualities : } h^{(k)}_M^N = \begin{pmatrix} \delta_j^i - t^i_j & t^{ij} \\ t_{ij} & \delta_i^j - t_i^j \end{pmatrix}, \quad t = \text{diag}(0 \dots 0 \ 1 \ 0 \dots 0).$$

If the antisymmetric $D \times D$ matrix B_{ij} in the shifts were written in the North–East block, the resulting transformation is usually called β -transformation, for reasons that will become clear later. The diffeomorphisms correspond to basis changes of the lattice underlying the torus, and the factorized T-dualities generalize the $\frac{R}{l_s} \leftrightarrow \frac{\tilde{R}}{l_s}$ symmetry discussed above. The 1 in the $D \times D$ matrix t is in the k th position. It is therefore common to find statements about T-duality being performed on a given k -direction, in which case the resulting transformations for the metric g_{ij} and 2-form b_{ij} are named *Buscher rules*

$$\begin{aligned} g_{kk} &\rightarrow \frac{1}{g_{kk}}, & g_{ki} &\rightarrow \frac{b_{ki}}{g_{kk}}, & g_{ij} &\rightarrow g_{ij} - \frac{g_{ki}g_{kj} - b_{ki}b_{kj}}{g_{kk}}, \\ b_{ki} &\rightarrow \frac{g_{ki}}{g_{kk}}, & b_{ij} &\rightarrow b_{ij} - \frac{g_{ki}b_{kj} - b_{ki}g_{kj}}{g_{kk}}. \end{aligned} \quad (3.11)$$

These transformation rules were first derived by Buscher from a worldsheet perspective in [76, 77], and they rely on the fact that the T-duality is performed in an isometric direction (i.e., a direction in which the fields are constant). Note that g and g^{-1} get exchanged in the k th direction, just as it happens in the circle with the inversion R/l_s and l_s/R . Also note that the metric (3.5) corresponds to a product of n successive T-dualities, and for this reason this matrix is usually called the inversion metric (as we will see, it inverts the full generalized metric (3.8)).

Summarizing, the T-duality symmetry of the circle compactification is generalized in toroidal compactifications to $O(n, n)$ acting as

$$\mathcal{H}_{MN} \leftrightarrow h_M^P \mathcal{H}_{PQ} h_N^Q, \quad \mathcal{P}^M \leftrightarrow h^M_N \mathcal{P}^N, \quad h \in O(n, n), \quad (3.12)$$

on constant backgrounds. More generally, T-duality in DFT is allowed in non-isometric directions, as we will see.

Let us now consider the dilaton, on which T-duality acts non-trivially. The closed string coupling in D -dimensions, $g_s^{(D)} = e^{-2\phi}$, is related to the $(D-1)$ -dimensional coupling when one dimension is compactified on a circle as $g_s^{(D-1)} = \sqrt{R/l_s} g_s^{(D)}$. Given that the scattering amplitudes for the dilaton states are invariant under T-duality, so must be the $(D-1)$ -dimensional coupling. Therefore, the dilaton of two theories compactified on circles of dual radii R and l_s^2/R must be related. When the compact space is n -dimensional, the T-duality invariant d is given by the following combination:

$$e^{-2d} = \sqrt{g} e^{-2\phi}. \quad (3.13)$$

This intriguing symmetry of string theory is not inherited by the fully decompactified low-energy effective theory (supergravity), because all the winding modes are infinitely heavy and play no role in the low-energy dynamics. Therefore, decompactified supergravity describes the ‘particle limit’ of the massless modes of the string. However, it is likely that a fully compactified supergravity in D -dimensions (i.e. where all dimensions are compact, and then $D = n$) can be rewritten in a T-duality, or more generally $O(D, D)$ covariant way, such that the symmetry becomes manifest at the level of the field theory. Then, DFT can be thought of as a T-duality invariant formulation of supergravity with compact dimensions. Actually, as we will see, DFT is more general than just a compactification of a fully decompactified theory (where the winding modes have been integrated out). The generalization relies on the fact that the winding dynamics is kept from the beginning, and at low-energy winding modes only decouple when the corresponding directions of the fully compactified theory are decompactified.

In order to begin with the construction of DFT, it is instructive to first introduce supergravity in D decompactified dimensions.

3.2. Supergravity basics

Before trying to assemble the NS-NS sector of supergravity in a T-duality invariant formulation, let us briefly review the bosonic sector of the theory that we will then try to covariantize. The degrees of freedom are contained in a D -dimensional metric (of course, we always keep in mind that the relevant dimension is $D = 10$) $g_{ij} = g_{(ij)}$, with $i, j, \dots = 1, \dots, D$, a D -dimensional 2-form $b_{ij} = b_{[ij]}$ (also known as the b -field or the Kalb–Ramond field) and a dilaton ϕ . All these fields depend on the D coordinates of spacetime x^i .

There is a pair of local gauge transformations under which the physics does not change.

- Diffeomorphisms, or change of coordinates, parameterized by infinitesimal vectors λ^i :

$$\begin{aligned} g_{ij} &\rightarrow g_{ij} + L_\lambda g_{ij}, & L_\lambda g_{ij} &= \lambda^k \partial_k g_{ij} + g_{kj} \partial_i \lambda^k + g_{ik} \partial_j \lambda^k, \\ b_{ij} &\rightarrow b_{ij} + L_\lambda b_{ij}, & L_\lambda b_{ij} &= \lambda^k \partial_k b_{ij} + b_{kj} \partial_i \lambda^k + b_{ik} \partial_j \lambda^k, \\ \phi &\rightarrow \phi + L_\lambda \phi, & L_\lambda \phi &= \lambda^i \partial_i \phi. \end{aligned} \quad (3.14)$$

Here, L_λ is the Lie derivative, defined as follows for arbitrary vectors V^i :

$$L_\lambda V^i = \lambda^j \partial_j V^i - V^j \partial_j \lambda^i = [\lambda, V]^i. \quad (3.15)$$

In the last equality we have defined the Lie Bracket, which is antisymmetric and satisfies the Jacobi identity. It is very important to keep the Lie derivative in mind, because it will be

generalized later, and the resulting generalized Lie derivative is one of the building blocks of DFT. The action of the Lie derivative amounts to diffeomorphic transformations, and the invariance of the action signals the fact that the physics remains unchanged under a change of coordinates.

- Gauge transformations of the 2-form, parameterized by infinitesimal 1-forms $\tilde{\lambda}_i$:

$$b_{ij} \rightarrow b_{ij} + \partial_i \tilde{\lambda}_j - \partial_j \tilde{\lambda}_i. \quad (3.16)$$

The supergravity action takes the following form:

$$S = \int d^D x \sqrt{g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} \right], \quad (3.17)$$

where we have defined the following 3-form with the corresponding Bianchi identity (BI):

$$H_{ijk} = 3\partial_{[i} b_{jk]}, \quad \partial_{[i} H_{jkl]} = 0, \quad (3.18)$$

and R is the Ricci scalar constructed from g_{ij} in the usual Riemannian sense. It is an instructive warm-up exercise to show that this action is invariant under diffeomorphisms (3.14) and the 2-form gauge transformations (3.16).

The equations of motion derived from the supergravity action take the form

$$R_{ij} - \frac{1}{4} H_i{}^{pq} H_{jpq} + 2\nabla_i \nabla_j \phi = 0, \quad (3.19)$$

$$\frac{1}{2} \nabla^p H_{pij} - H_{pij} \nabla^p \phi = 0, \quad (3.20)$$

$$R + 4(\nabla^i \nabla_i \phi - (\partial\phi)^2) - \frac{1}{12} H^2 = 0. \quad (3.21)$$

From the string theory point of view, they imply the Weyl invariance of the theory at the one-loop quantum level.

We have described in this section the bosonic NS-NS sector of supergravity. This sector is interesting on its own because it determines the moduli space of the theory. Given that the fermions are charged with respect to the Lorentz group, for any given configuration the vacuum expectation value (VEV) of a fermion would break the Lorentz invariance. In order to preserve this celebrated symmetry, one considers vacua in which the fermions have vanishing VEV. For this reason, and also for simplicity, in this review we will restrict ourselves to bosonic degrees of freedom.

3.3. Double space and generalized fields

So far, we have introduced the basic field-theoretical notions of supergravity and explained the importance of T-duality in string theory. It is now time to start exploring how the supergravity degrees of freedom can be rearranged in a T-duality invariant formulation of DFT [3]. For this to occur, we must put everything in T-duality representations, i.e., in objects that have well-defined transformation properties under T-duality.

Let us begin with the fields. As mentioned, we consider on the one hand a metric g_{ij} and a 2-form b_{ij} which can be combined into a symmetric generalized metric \mathcal{H}_{MN} given by

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix}. \quad (3.22)$$

Note that this metric has the same form as the one defined in (3.8), but here the fields are non-constant. This is an $O(D, D)$ element, and its inverse is obtained by raising the indices with the $O(D, D)$ metric η^{MP} introduced in (3.5):

$$\mathcal{H} \in O(D, D), \quad \mathcal{H}^{MN} = \eta^{MP} \mathcal{H}_{PQ} \eta^{QN}, \quad \mathcal{H}_{MP} \mathcal{H}^{PN} = \delta_M^N. \quad (3.23)$$

Actually, all the indices in DFT are raised and lowered with the $O(D, D)$ invariant metric (3.5). On the other hand, the dilaton ϕ is combined with the determinant of the metric g in an $O(D, D)$ scalar d :

$$e^{-2d} = \sqrt{g} e^{-2\phi}. \tag{3.24}$$

Before showing how these objects transform under local and global symmetries, let us mention where these generalized fields are defined. Since everything must be organized in T-duality representations, the coordinates cannot be an exception. Paradoxically, we only have D of them: x^i , while the lowest dimensional representation of $O(D, D)$ is the fundamental, which has dimension $2D$. We therefore face the question of what should we combine the supergravity coordinates with, in order to complete the fundamental representation. It turns out that there are no such objects in supergravity, so we must *introduce* new coordinates \tilde{x}_i . We can now define a generalized notion of coordinates,

$$X^M = (\tilde{x}_i, x^i), \tag{3.25}$$

and demand that the generalized fields depend on this double set of coordinates:

$$\mathcal{H}_{MN}(X), \quad d(X). \tag{3.26}$$

From the point of view of compactifications on tori, these coordinates correspond to the Fourier duals to the generalized momenta \mathcal{P}^M (3.6). However, here we will consider more generally a background-independent formulation [16] in which the generalized metric [17] can be defined on more general backgrounds.

It is important to recall that here the coordinates can either parameterize compact or non-compact directions indistinctively. Even if non-compact, one can still formulate a full $O(D, D)$ covariant theory. In this case, the duals to the non-compact directions are just ineffective, and one can simply assume that nothing depends on them. This will become clear later, when we consider DFT in the context of four-dimensional effective theories. For the moment, this distinction is irrelevant.

Being in the fundamental representation, the coordinates rotate under $O(D, D)$ as follows:

$$X^M \rightarrow h^M_N X^N, \quad h \in O(D, D), \tag{3.27}$$

so they mix under these global transformations. Given that x^i and \tilde{x}_i are related by T-duality, the latter are usually referred to as *dual coordinates*. Under $O(D, D)$ transformations, the fields change as follows:

$$\mathcal{H}_{MN}(X) \rightarrow h_M^P h_N^Q \mathcal{H}_{PQ}(h X), \quad d(X) \rightarrow d(h X). \tag{3.28}$$

In the particular case in which h corresponds to T-dualities (3.10) in isometric directions (i.e. in directions in which the fields have no coordinate dependence), these transformations reproduce the Buscher rules (3.11) and (3.13) for g_{ij} , b_{ij} and ϕ . It can be shown that the different components of (3.28) are equivalent to (3.11), which were derived assuming the fact that T-duality is performed along an isometry. More generally, the transformation rules (3.28) admit the possibility of performing T-duality in non-isometric directions [14], the reason being that DFT is defined on a double space, so, contrary to what happens in supergravity, if a T-duality hits a non-isometric direction, the result is simply that the resulting configuration will depend on the T-dual coordinate.

The reader might be quite confused at this point, wondering what these dual coordinates correspond to in the supergravity picture. Well, they simply have no meaning from a supergravity point of view. Then, there must be some mechanism to constrain the coordinate dependence, and moreover since we want a T-duality invariant formulation, such constraint must be duality invariant. The constraint in question goes under many names in the literature,

the most common ones being *strong constraint* or *section condition*. This restriction consists of a differential equation

$$\eta^{MN}\partial_M\partial_N(\dots) = 0, \quad (3.29)$$

where η_{MN} is the $O(D, D)$ invariant metric introduced in (3.5). For later convenience, we recast it as

$$Y^M{}_{P^N}{}_Q\partial_M\partial_N(\dots) = 0, \quad (3.30)$$

where we have introduced the tensor

$$Y^M{}_{P^N}{}_Q = \eta^{MN}\eta_{PQ} \quad (3.31)$$

following the notation in [45], which is very useful to explore generalizations of DFT to more general U-duality groups, as we will see in section 6. The dots in (3.30) represent any field or gauge parameter and also products of them. Note that since the tensor Y is an $O(D, D)$ invariant, so is the constraint. This means that if a given configuration solves the strong constraint, any T-duality transformation of it will also do. When written in components, the constraint takes the form

$$\tilde{\partial}^i\partial_i(\dots) = 0, \quad (3.32)$$

so a possible solution is $\tilde{\partial}^i(\dots) = 0$, or any $O(D, D)$ rotation of this. Actually, it can be proven that this is the only solution. Therefore, even if formally in this formulation the fields depend on the double set of coordinates, when the strong constraint is imposed the only possible configurations allowed by it depend on a D -dimensional section of the space. When this section corresponds to the x^i coordinates of supergravity (i.e., when all fields and gauge parameters are annihilated by $\tilde{\partial}^i$), we will say that the strong constraint is solved in the *supergravity frame*.

When DFT is evaluated on tori, a weaker version of the strong constraint can be related to the LMC (3.9). In this case, the generalized fields must be expanded in the modes of the double torus $\exp(iX^M\mathcal{P}_M)$, such that when the derivatives hit the mode expansion, the LMC contraction $\mathcal{P}^M\mathcal{P}_M$ makes its appearance. Here we will pursue background independence, and moreover we will later deal with twisted double-tori only up to the zero mode, so the LMC should not be identified with the strong constraint (or any weaker version) in this review. We will be more specific on this point in section 5.

Throughout this review we will not necessarily impose the strong constraint, and in many occasions we will explicitly write the terms that would vanish when it is imposed. The reader can choose whether he/she wants to impose it or not. Only when we intend to compare with supergravity in D dimensions, we will explicitly impose the strong constraint and choose the supergravity frame (in these cases we will mention this explicitly). The relevance of dealing with configurations that violate the strong constraint will become apparent when we get to the point of analyzing dimensional reductions of DFT, and the risks of going beyond supergravity will be properly explained and emphasized. Let us emphasize that DFT is a restricted theory though, so one cannot just relax it and consider generic configurations: the consistency constraints of the theory are imposed by demanding closure of the gauge transformations, as we will discuss later. These closure constraints are solved in particular by the solutions to the strong constraint, but other solutions exist, and then it is convenient to stay as general as possible.

3.4. Generalized Lie derivative

We have seen that the D -dimensional metric and 2-form field transform under diffeomorphisms (3.14) and that the 2-form also enjoys a gauge symmetry (3.16). These fields have been unified

into a single object called the generalized metric (3.22), and then one wonders whether there are generalized diffeomorphisms unifying the usual diffeomorphisms (3.14) and gauge transformations (3.16). Since the former are parameterized by a D -dimensional vector, and the latter by a D -dimensional 1-form, one can think of considering a generalized gauge parameter

$$\xi^M = (\tilde{\lambda}_i, \lambda^i). \tag{3.33}$$

Then, the generalized diffeomorphisms and gauge transformations of the 2-form can be unified as

$$\mathcal{L}_\xi e^{-2d} = \partial_M(\xi^M e^{-2d}), \tag{3.34}$$

$$\mathcal{L}_\xi \mathcal{H}_{MN} = L_\xi \mathcal{H}_{MN} + Y^R{}_{M^P}{}_Q \partial^Q \xi_P \mathcal{H}_{RN} + Y^R{}_{N^P}{}_Q \partial^Q \xi_P \mathcal{H}_{MR}, \tag{3.35}$$

where L_ξ is the Lie derivative (3.15) in $2D$ dimensions, and Y , already defined in (3.31), measures the departure from the conventional Riemannian geometry. We see here that e^{-2d} transforms as a density, and as such it will correspond to the integration measure when we deal with the action. When the generalized metric is parameterized as in (3.22) in terms of g_{ij} and b_{ij} , and the strong constraint is imposed in the supergravity frame (i.e., when $\tilde{\partial}^i = 0$), the different components of (3.35) yield

$$\mathcal{L}_\xi g_{ij} = L_\lambda g_{ij}, \tag{3.36}$$

$$\mathcal{L}_\xi b_{ij} = L_\lambda b_{ij} + 2 \partial_{[i} \tilde{\lambda}_{j]}, \tag{3.37}$$

and then the local transformations of supergravity (3.14)–(3.16) are recovered. The generalization of the usual Lie derivative with the addition of the term with Y is not only essential in order to recover the standard transformations of the bosonic NS-NS sector of supergravity, but also to preserve the $O(D, D)$ metric

$$\mathcal{L}_\xi \eta_{MN} = 0. \tag{3.38}$$

To end this discussion, we present the general form of the generalized Lie derivative with respect to a vector ξ acting on a tensorial density V^M with weight $\omega(V)$, which is given by the following gauge transformation:

$$\mathcal{L}_\xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P + \omega(V) \partial_P \xi^P V^M. \tag{3.39}$$

This expression is trivially extended to other tensors with a different index structure. In particular, when this is applied to e^{-2d} with $\omega(e^{-2d}) = 1$ and \mathcal{H}_{MN} with $\omega(\mathcal{H}) = 0$, the transformations (3.34) and (3.35) are respectively recovered. As we will discuss in the following section, the closure of these generalized diffeomorphisms imposes differential constraints on the theory.

Let us finally highlight that the action of these generalized diffeomorphisms has been defined when transforming tensorial quantities. Note however that, for example, the derivative of a vector $\partial_M V^N$ is non-tensorial. It is then instructive to denote its transformation as

$$\delta_\xi (\partial_M V^N) = \partial_M (\delta_\xi V^N) = \partial_M (\mathcal{L}_\xi V^N), \tag{3.40}$$

where we have used the fact that the transformation of a vector is dictated by the generalized Lie derivative (3.39). One can however extend the definition of the generalized diffeomorphisms \mathcal{L}_ξ to act on non-tensorial quantities as if they were actually tensorial. Since δ_ξ represents the actual transformation, one can define the failure of any object to transform covariantly as

$$\Delta_\xi \equiv \delta_\xi - \mathcal{L}_\xi, \tag{3.41}$$

such that when acting on tensors, say V^M , one finds

$$\Delta_\xi V^M = \delta_\xi V^M - \mathcal{L}_\xi V^M = 0, \tag{3.42}$$

or equivalently, for any non-tensorial quantity, say W^M , we have

$$\delta_\xi W^M = \mathcal{L}_\xi W^M + \Delta_\xi W^M. \quad (3.43)$$

This notation is very useful for the analysis of the consistency constraints of the theory, to which we now move.

3.5. Consistency constraints

Given the structure of generalized diffeomorphisms (3.35), one must check that they actually define a closed group [33]. This requires, in particular, that two successive gauge transformations parameterized by ξ_1 and ξ_2 , acting on a given field ξ_3 , must reproduce a new gauge transformation parameterized by some given $\xi_{12}(\xi_1, \xi_2)$ acting on the same vector

$$\Delta_{123}^M = -\Delta_{\xi_1}(\mathcal{L}_{\xi_2}\xi_3^M) = ([\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] - \mathcal{L}_{\xi_{12}})\xi_3^M = 0, \quad (3.44)$$

where we have defined Δ_ξ as in (3.41). In other words, the generalized Lie derivative must send tensors into tensors. The resulting parameter is given by

$$\xi_{12} = \mathcal{L}_{\xi_1}\xi_2, \quad (3.45)$$

provided the following constraint holds:

$$\Delta_{123}^M = Y^P{}_{RQ}{}^S (2\partial_P\xi_{[1}^R \partial_Q\xi_2^M \xi_3^S - \partial_P\xi_1^R \xi_2^S \partial_Q\xi_3^M) = 0. \quad (3.46)$$

This was written here for vectors with vanishing weight, for simplicity. The parameter ξ_{12} goes under the name of the D-bracket, and its antisymmetric part is named the C-bracket:

$$\xi_{[12]}^M = [[\xi_1, \xi_2]]^M = \frac{1}{2}(\mathcal{L}_{\xi_1}\xi_2^M - \mathcal{L}_{\xi_2}\xi_1^M) = [\xi_1, \xi_2]^M + Y^M{}_{N^P}{}^Q \xi_{[1}^N \partial_P\xi_2^Q]. \quad (3.47)$$

It corresponds to an extension of the Lie bracket (3.15), since it contains a correction proportional to the invariant Y , which in turn corrects the Lie derivative. The D- and C-brackets, respectively, reduce to the Dorfman and Courant brackets [78] when the strong constraint is imposed in the supergravity frame. Under the constraint (3.46), the following relation holds:

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[[\xi_1, \xi_2]]}. \quad (3.48)$$

Note also that symmetrizing (3.44), we find the so-called Leibniz rule (which arises here as a constraint)

$$\mathcal{L}_{((\xi_1, \xi_2))} = 0, \quad ((\xi_1, \xi_2)) = \xi_{(12)}. \quad (3.49)$$

The D-bracket, which satisfies the Jacobi identity, then contains a symmetric piece that must generate trivial gauge transformations. This fact is important because, on the other hand, the C-bracket, which is antisymmetric, has a non-vanishing Jacobiator

$$J(\xi_1, \xi_2, \xi_3) = [[[[\xi_1, \xi_2]], \xi_3]] + \text{cyclic}. \quad (3.50)$$

However, using (3.48) and (3.49), one can rapidly show that [45]

$$J(\xi_1, \xi_2, \xi_3) = \frac{1}{3}([[[\xi_1, \xi_2]], \xi_3]) + \text{cyclic}, \quad (3.51)$$

and then the Jacobiator generates trivial gauge transformations by virtue of (3.49).

Condition (3.44) poses severe consistency constraints on the generalized diffeomorphisms (i.e. their possible generalized gauge parameters). Therefore, DFT is a *constrained* or *restricted* theory. The generalized gauge parameters cannot be generic, but must be constrained to solve (3.44). Supergravity is safe from this problem, because the usual D -dimensional diffeomorphisms and 2-form gauge transformations do form a group. It is then to be expected that under the imposition of the strong constraint, (3.44) is automatically satisfied. This is trivial from (3.46) because all of its terms are of the form (3.30), but more generally these equations leave room for strong constraint-violating configurations [65, 66, 34], as we will see later.

3.6. The action

The NS-NS sector of DFT has an action from which one can derive equations of motion. Before showing its explicit form, let us introduce some objects that will be useful later. The generalized metric can be decomposed as

$$\mathcal{H}_{MN} = E^{\bar{A}}_M S_{\bar{A}\bar{B}} E^{\bar{B}}_N, \quad (3.52)$$

with an $O(D, D)$ generalized frame

$$\eta_{MN} = E^{\bar{A}}_M \eta_{\bar{A}\bar{B}} E^{\bar{B}}_N, \quad (3.53)$$

where $\eta_{\bar{A}\bar{B}}$ raises and lowers flat indices and takes the same form as η_{MN} (3.5). Under generalized diffeomorphisms, the generalized frame $E^{\bar{A}}_M$ transforms as follows:

$$\mathcal{L}_{\xi} E^{\bar{A}}_M = \xi^P \partial_P E^{\bar{A}}_M + (\partial_M \xi^P - \partial^P \xi_M) E^{\bar{A}}_P, \quad (3.54)$$

and can be parameterized in terms of the vielbein of the D -dimensional metric $g_{ij} = e^{\bar{a}}_i s_{\bar{a}\bar{b}} e^{\bar{b}}_j$, where $s_{\bar{a}\bar{b}} = \text{diag}(- + \cdots +)$ is the D -dimensional Minkowski metric, as

$$E^{\bar{A}}_M = \begin{pmatrix} e^{\bar{a}i} & e^{\bar{a}j} b_{ji} \\ 0 & e^{\bar{a}}_i \end{pmatrix}, \quad S_{\bar{A}\bar{B}} = \begin{pmatrix} s_{\bar{a}\bar{b}} & 0 \\ 0 & s_{\bar{a}\bar{b}} \end{pmatrix}. \quad (3.55)$$

Since the Minkowski metric is invariant under Lorentz transformations $O(1, D-1)$, the metric $S_{\bar{A}\bar{B}}$ is invariant under double Lorentz transformations

$$H = O(1, D-1) \times O(1, D-1) \quad (3.56)$$

which correspond to the maximal (pseudo-)compact subgroup of $G = O(D, D)$. Therefore, the generalized metric is invariant under local double Lorentz transformations, and thus it parameterizes the coset G/H . The dimension of the coset is D^2 , and this allows us to accommodate a symmetric D -dimensional metric g_{ij} and an antisymmetric D -dimensional 2-form b_{ij} , as we have seen. Technically, the triangular parametrization of the generalized frame would break down under a T-duality, and then one has to restore the triangular gauge through an H-transformation. From the generalized frame $E^{\bar{A}}_M$ and dilaton d , one can build the *generalized fluxes*

$$\mathcal{F}_{\bar{A}\bar{B}\bar{C}} = E_{\bar{C}M} \mathcal{L}_{E_{\bar{A}}} E_{\bar{B}}^M = 3\Omega_{[\bar{A}\bar{B}\bar{C}]}, \quad (3.57)$$

$$\mathcal{F}_{\bar{A}} = -e^{2d} \mathcal{L}_{E_{\bar{A}}} e^{-2d} = \Omega^{\bar{B}}_{\bar{B}\bar{A}} + 2E_{\bar{A}}^M \partial_M d, \quad (3.58)$$

out of the following object:

$$\Omega_{\bar{A}\bar{B}\bar{C}} = E_{\bar{A}}^M \partial_M E_{\bar{B}}^N E_{\bar{C}N} = -\Omega_{\bar{A}\bar{C}\bar{B}}, \quad (3.59)$$

that will be referred to as the generalized *Weitzenböck connection*.

Since all these objects are written in planar indices, they are manifestly $O(D, D)$ invariant, so any combination of them will also be. The generalized fluxes (3.57) and (3.58) depend on the fields and are therefore dynamical. Later, when we will analyze compactifications of the theory, they will play an important role, as they will be related to the covariant quantities in the effective action, and will moreover reduce to the usual constant fluxes, or gaugings in the lower dimensional theory, hence the name *generalized fluxes*.

The generalized frame and dilaton enter in the action of DFT only through the dynamical fluxes (3.57) and (3.58). Indeed, up to total derivatives, the action takes the form

$$S = \int dX e^{-2d} \mathcal{R}, \quad (3.60)$$

with

$$\mathcal{R} = \mathcal{F}_{\bar{A}\bar{B}\bar{C}} \mathcal{F}_{\bar{D}\bar{E}\bar{F}} \left[\frac{1}{4} S^{\bar{A}\bar{D}} \eta^{\bar{B}\bar{E}} \eta^{\bar{C}\bar{F}} - \frac{1}{12} S^{\bar{A}\bar{D}} S^{\bar{B}\bar{E}} S^{\bar{C}\bar{F}} - \frac{1}{6} \eta^{\bar{A}\bar{D}} \eta^{\bar{B}\bar{E}} \eta^{\bar{C}\bar{F}} \right] + \mathcal{F}_{\bar{A}} \mathcal{F}_{\bar{B}} [\eta^{\bar{A}\bar{B}} - S^{\bar{A}\bar{B}}]. \quad (3.61)$$

In this formulation, it takes the same form as the scalar potential of half-maximal supergravity in four dimensions. We will be more specific about this later, but for the readers who are familiar with gauged supergravities, note that identifying here the dynamical fluxes with gaugings and the $S_{\bar{A}\bar{B}}$ matrix with the moduli scalar matrix, this action resembles the form of the scalar potential of [79]. This frame formulation was introduced in [2], later related to other formulations in [18], and also discussed in [68].

Written in this form, the $O(D, D)$ invariance is manifest. However, some local symmetries are hidden and the invariance of the action must be explicitly verified. Under generalized diffeomorphisms, the dynamical fluxes transform as

$$\begin{aligned} \delta_{\xi} \mathcal{F}_{\bar{A}\bar{B}\bar{C}} &= \xi^{\bar{D}} \partial_{\bar{D}} \mathcal{F}_{\bar{A}\bar{B}\bar{C}} + \Delta_{\xi \bar{A}\bar{B}\bar{C}}, \\ \delta_{\xi} \mathcal{F}_{\bar{A}} &= \xi^{\bar{D}} \partial_{\bar{D}} \mathcal{F}_{\bar{A}} + \Delta_{\xi \bar{A}}, \end{aligned} \quad (3.62)$$

where

$$\begin{aligned} \Delta_{\xi \bar{A}\bar{B}\bar{C}} &= 4 \mathcal{Z}_{\bar{A}\bar{B}\bar{C}\bar{D}} \xi^{\bar{D}} + 3 \partial_{\bar{D}} \xi_{[\bar{A}} \Omega^{\bar{D}}{}_{\bar{B}\bar{C}]}, \\ \Delta_{\xi \bar{A}} &= \mathcal{Z}_{\bar{A}\bar{B}} \xi^{\bar{B}} + \mathcal{F}^{\bar{B}} \partial_{\bar{B}} \xi_{\bar{A}} - \partial^{\bar{B}} \partial_{\bar{B}} \xi_{\bar{A}} + \Omega^{\bar{C}}{}_{\bar{A}\bar{B}} \partial_{\bar{C}} \xi^{\bar{B}}, \end{aligned} \quad (3.63)$$

and we have defined

$$\begin{aligned} \mathcal{Z}_{\bar{A}\bar{B}\bar{C}\bar{D}} &= \partial_{[\bar{A}} \mathcal{F}_{\bar{B}\bar{C}\bar{D}]} - \frac{3}{4} \mathcal{F}_{[\bar{A}\bar{B}}{}^{\bar{E}} \mathcal{F}_{\bar{C}\bar{D}]\bar{E}} = -\frac{3}{4} \Omega_{\bar{E}[\bar{A}\bar{B}} \Omega^{\bar{E}}{}_{\bar{C}\bar{D}]}, \\ \mathcal{Z}_{\bar{A}\bar{B}} &= \partial^{\bar{C}} \mathcal{F}_{\bar{C}\bar{A}\bar{B}} + 2 \partial_{[\bar{A}} \mathcal{F}_{\bar{B}]} - \mathcal{F}^{\bar{C}} \mathcal{F}_{\bar{C}\bar{A}\bar{B}} = (\partial^M \partial_M E_{[\bar{A}}{}^N) E_{\bar{B}]N} - 2 \Omega^{\bar{C}}{}_{\bar{A}\bar{B}} \partial_{\bar{C}} d. \end{aligned} \quad (3.64)$$

The vanishing of (3.63) follows from the closure conditions (3.44), precisely because the dynamical fluxes are defined through generalized diffeomorphisms (3.57) and (3.58)

$$\begin{aligned} \Delta_{\xi} \mathcal{F}_{\bar{A}\bar{B}\bar{C}} &= \Delta_{\xi \bar{A}\bar{B}\bar{C}} = E_{\bar{C}M} \Delta_{\xi} (\mathcal{L}_{E_{\bar{A}}} E_{\bar{B}}{}^M) = 0, \\ \Delta_{\xi} \mathcal{F}_{\bar{A}} &= \Delta_{\xi \bar{A}} = -e^{2d} \Delta_{\xi} (\mathcal{L}_{E_{\bar{A}}} e^{-2d}) = 0. \end{aligned} \quad (3.65)$$

Therefore, the dynamical fluxes in flat indices transform as scalars under generalized diffeomorphisms.

Let us now argue that due to the closure constraints (3.65), the action of DFT is invariant under generalized diffeomorphisms. In fact, since e^{-2d} transforms as a density (recall (3.34))

$$\delta_{\xi} e^{-2d} = \partial_P (\xi^P e^{-2d}), \quad (3.66)$$

for the action to be invariant under generalized diffeomorphisms, \mathcal{R} must transform as a scalar. Using the gauge transformation rules for the generalized fluxes (3.62) together with (3.65), one arrives at the following result:

$$\delta_{\xi} \mathcal{R} = \mathcal{L}_{\xi} \mathcal{R} = \xi^P \partial_P \mathcal{R}. \quad (3.67)$$

Combining (3.66) with (3.67), it can be checked that the Lagrangian density $e^{-2d} \mathcal{R}$ transforms as a total derivative, and then the action (3.60) is invariant.

We have seen that in addition to generalized diffeomorphisms, the theory must be invariant under local double Lorentz transformations (3.56) parameterized by an infinitesimal $\Lambda_{\bar{A}}{}^{\bar{B}}$. This parameter must be antisymmetric $\Lambda_{\bar{A}\bar{B}} = -\Lambda_{\bar{B}\bar{A}}$ to guarantee the invariance of $\eta_{\bar{A}\bar{B}}$ and it must also satisfy $S_{\bar{A}}{}^{\bar{C}} \Lambda_{\bar{C}\bar{B}} = \Lambda_{\bar{A}\bar{C}} S^{\bar{C}}{}_{\bar{B}}$ to guarantee the invariance of $S_{\bar{A}\bar{B}}$. The frame transforms as

$$\delta_{\Lambda} E_{\bar{A}}{}^M = \Lambda_{\bar{A}}{}^{\bar{B}} E_{\bar{B}}{}^M, \quad (3.68)$$

and this guarantees that the generalized metric is invariant. The invariance of the action is, however, less clear, and a short computation shows that

$$\delta_\Lambda S = \int dX e^{-2d} \mathcal{Z}_{\bar{A}\bar{C}} \Lambda_{\bar{B}}^{\bar{C}} (\eta^{\bar{A}\bar{B}} - S^{\bar{A}\bar{B}}), \quad (3.69)$$

with $\mathcal{Z}_{\bar{A}\bar{B}}$ defined in (3.64). Then, the invariance of the action (3.60) under double Lorentz transformations (3.68) is also guaranteed from closure, since

$$\mathcal{Z}_{\bar{A}\bar{B}} = \Delta_{E_{\bar{A}}} \mathcal{F}_{\bar{B}} = 0. \quad (3.70)$$

As happens with all the constraints in DFT, which follow from (3.44), they are solved by the strong constraint but admit more general solutions (this can be seen especially in (3.46) where cancelations could occur without demanding each contribution to vanish independently).

This flux formulation of DFT is a small extension of the generalized metric formulation introduced in [17]. It incorporates terms that would vanish under the imposition of the strong constraint in a covariant way. After some algebra, it can be shown that the action (3.60) can be recast in the form

$$S = \int dX e^{-2d} \left(4\mathcal{H}^{MN} \partial_M \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} - 4\mathcal{H}^{MN} \partial_M d \partial_N d + 4\partial_M \mathcal{H}^{MN} \partial_N d \right. \\ \left. + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} + \Delta_{(\text{SC})} \mathcal{R} \right), \quad (3.71)$$

up to total derivatives. Here, we have separated all terms in (3.60) that vanish under the imposition of the strong constraint $\Delta_{(\text{SC})} \mathcal{R}$ to facilitate the comparison with the generalized metric formulation [17].

To conclude this section, we recall that in order to recover the supergravity action (3.17), the strong constraint must be imposed in the supergravity frame. Then, when $\tilde{\delta}^i = 0$ is imposed on (3.71), and the generalized metric is parameterized in terms of the D -dimensional metric and 2-form as in (3.22), the DFT action (3.71) reproduces (3.17) exactly.

3.7. Equations of motion

The equations of motion in DFT were extensively discussed in [32] for different formulations of the theory. For the flux formulation we have just presented, the variation of the action with respect to $E_{\bar{A}}^{\bar{M}}$ and to d takes the form

$$\delta_E S = \int dX e^{-2d} \mathcal{G}^{\bar{A}\bar{B}} \delta E_{\bar{A}\bar{B}}, \quad (3.72)$$

$$\delta_d S = \int dX e^{-2d} \mathcal{G} \delta d, \quad (3.73)$$

where

$$\delta E_{\bar{A}\bar{B}} = \delta E_{\bar{A}}^{\bar{M}} E_{\bar{B}\bar{M}} = -\delta E_{\bar{B}\bar{A}}, \quad (3.74)$$

to incorporate the fact that the generalized bein preserves the $O(D, D)$ metric (3.5). It can easily be checked that the variations of the generalized fluxes are given by

$$\delta_E \mathcal{F}_{\bar{A}\bar{B}\bar{C}} = 3(\partial_{[\bar{A}} \delta E_{\bar{B}\bar{C}]} + \delta E_{[\bar{A}}^{\bar{D}} \mathcal{F}_{\bar{B}\bar{C}]\bar{D}}), \quad (3.75)$$

$$\delta_E \mathcal{F}_{\bar{A}} = \partial^{\bar{B}} \delta E_{\bar{B}\bar{A}} + \delta E_{\bar{A}}^{\bar{B}} \mathcal{F}_{\bar{B}}, \quad (3.76)$$

$$\delta_d \mathcal{F}_{\bar{A}} = 2\partial_{\bar{A}} \delta d. \quad (3.77)$$

We then obtain

$$\mathcal{G}^{[\bar{A}\bar{B}]} = 2(S^{\bar{D}[\bar{A}} - \eta^{\bar{D}[\bar{A}})\partial^{\bar{B}]}\mathcal{F}_{\bar{D}} + (\mathcal{F}_{\bar{D}} - \partial_{\bar{D}})\check{\mathcal{F}}^{\bar{D}[\bar{A}\bar{B}]} + \check{\mathcal{F}}^{\bar{C}\bar{D}[\bar{A}}\mathcal{F}_{\bar{C}\bar{D}}^{\bar{B}]}, \quad (3.78)$$

$$\mathcal{G} = -2\mathcal{R}, \quad (3.79)$$

where

$$\check{\mathcal{F}}^{\bar{A}\bar{B}\bar{C}} = \frac{3}{2}\mathcal{F}_{\bar{D}}^{\bar{B}\bar{C}}S^{\bar{A}\bar{D}} - \frac{1}{2}\mathcal{F}_{\bar{D}\bar{E}\bar{F}}S^{\bar{A}\bar{D}}S^{\bar{B}\bar{E}}S^{\bar{C}\bar{F}} - \mathcal{F}^{\bar{A}\bar{B}\bar{C}}. \quad (3.80)$$

The equations of motion are then

$$\mathcal{G}^{[\bar{A}\bar{B}]} = 0, \quad \mathcal{G} = 0. \quad (3.81)$$

Upon decomposing these equations in components, and standing in the supergravity frame of the strong constraint, one recovers the equations of motion of supergravity (3.19)–(3.21), provided the generalized frame is parameterized as in (3.55).

For completeness, let us also mention that had we varied the action in the generalized metric formulation (3.71) with respect to the generalized metric (and setting to zero the strong-constraint-like terms), we would have found [17, 32]

$$\delta_{\mathcal{H}}S = \int dX e^{-2d}\delta\mathcal{H}^{MN}\mathcal{K}_{MN}, \quad (3.82)$$

with

$$\begin{aligned} \mathcal{K}_{MN} = & \frac{1}{8}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{4}(\partial_L - 2(\partial_L d))(\mathcal{H}^{LK}\partial_K\mathcal{H}_{MN}) + 2\partial_M\partial_N d \\ & - \frac{1}{2}\partial_{(M}\mathcal{H}^{KL}\partial_L\mathcal{H}_{|N)K} + \frac{1}{2}(\partial_L - 2(\partial_L d))(\mathcal{H}^{KL}\partial_{(M}\mathcal{H}_{N)K} + \mathcal{H}^K{}_{(M}\partial_K\mathcal{H}^L{}_{|N)}). \end{aligned} \quad (3.83)$$

Note, however, that the variations $\delta\mathcal{H}_{MN}$ are not generic, but must be subjected to constraints inherited from (3.23). This implies that only some projections of \mathcal{K}_{MN} give the equations of motion, through a generalized Ricci flatness equation:

$$\widehat{\mathcal{R}}_{MN} = \widehat{P}_{(M}{}^P\check{P}_{N)}{}^Q\mathcal{K}_{PQ} = 0, \quad (3.84)$$

where we introduced some projectors that will be useful in the following section:

$$\widehat{P}_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}_{MN}), \quad \check{P}_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}_{MN}). \quad (3.85)$$

Finally, imposing the strong constraint to (3.81), they can be taken to the form (3.84).

These equations of motion will be revisited in the following section from a geometrical point of view.

4. Double geometry

We have explored the basics of the bosonic NS-NS sector of DFT, starting from its degrees of freedom, the double space on which it is defined, its consistency constraints, the action and equations of motion, etc. In particular, the action was tendentiously written in terms of a generalized Ricci scalar and the equations of motion were cast in a generalized Ricci flatness form. But, is there some underlying geometry? Can DFT be formulated in a more fundamental (generalized) geometrical way? It turns out that there is such a formulation, but it differs from the Riemannian geometry out of which General Relativity is constructed. We find it instructive to begin this section with a basic review of the notions of the Riemannian geometry that will then be generalized for DFT.

4.1. Riemannian geometry basics

Even though General Relativity follows from an action of the form

$$S = \int dx \sqrt{g} R, \quad (4.1)$$

where R is the Ricci scalar and g is the determinant of the metric, we know that there exists an underlying geometry out of which this theory can be obtained. The starting point can be taken to be the Lie derivative (3.15)

$$L_\xi V^i = \xi^k \partial_k V^i - \partial_k \xi^i V^k. \quad (4.2)$$

The derivative of a vector is non-tensorial under the diffeomorphisms (4.2), so one starts by introducing a covariant derivative

$$\nabla_i V^j = \partial_i V^j + \Gamma_{ik}^j V^k, \quad (4.3)$$

defined in terms of a Christoffel connection Γ , whose purpose is to compensate the failure of the derivative to transform as a tensor. Therefore, the failure to transform as a tensor under diffeomorphisms parameterized by ξ , denoted by Δ_ξ , is given by

$$\Delta_\xi \Gamma_{ij}^k = \partial_i \partial_j \xi^k. \quad (4.4)$$

The torsion can be defined through

$$T_{ij}^k \xi^i V^j = (L_\xi^\nabla - L_\xi) V^k = 2\Gamma_{[ij]}^k \xi^i V^j. \quad (4.5)$$

The superscript ∇ is just notation to indicate that, in the Lie derivative, the partial derivatives should be replaced by covariant derivatives. A condition to be satisfied in the Riemannian geometry is covariant constancy of the metric g_{ij} . It receives the name of metric compatibility

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} = 0. \quad (4.6)$$

This fixes the symmetric part of the connection

$$\Gamma_{(ij)}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) - g_{m(i} T_{j)l}^m g^{lk}. \quad (4.7)$$

When the connection is torsionless

$$T_{ij}^k = 2\Gamma_{[ij]}^k = 0, \quad (4.8)$$

it is named Levi-Civita. Note that the Levi-Civita connection is symmetric and completely fixed by metric compatibility (4.7) in terms of the degrees of freedom of General Relativity, namely the metric g_{ij} :

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \quad (4.9)$$

Let us note that the Levi-Civita connection satisfies the partial integration rule in the presence of the measure \sqrt{g} :

$$\int dx \sqrt{g} U \nabla_i V^i = - \int dx \sqrt{g} V^i \nabla_i U, \quad (4.10)$$

given that its trace satisfies

$$\Gamma_{ki}^k = \frac{1}{\sqrt{g}} \partial_i \sqrt{g}. \quad (4.11)$$

In a vielbein formulation, one also introduces a spin connection $W_{i\bar{a}}^{\bar{b}}$, so that

$$\nabla_i e_{\bar{a}}^j = \partial_i e_{\bar{a}}^j + \Gamma_{ik}^j e_{\bar{a}}^k - W_{i\bar{a}}^{\bar{b}} e_{\bar{b}}^j, \quad (4.12)$$

and compatibility with the vielbein, $\nabla_i e_{\bar{a}}^j = 0$, relates the Christoffel connection to the Weitzenböck connection

$$\Omega_{\bar{a}\bar{b}}^{\bar{c}} = e_{\bar{a}}^i \partial_i e_{\bar{b}}^j e^{\bar{c}}_j, \quad (4.13)$$

through

$$W_{i\bar{a}}^{\bar{b}} = \Omega_{\bar{c}\bar{a}}^{\bar{b}} e^{\bar{c}}_i + \Gamma_{ij}^k e_{\bar{a}}^j e^{\bar{b}}_k. \quad (4.14)$$

For future reference, we also introduce the notion of dynamical Scherk–Schwarz flux, defined by the Lie derivative as

$$e^{\bar{c}}_i L_{e_{\bar{a}}} e_{\bar{b}}^i = f_{\bar{a}\bar{b}}^{\bar{c}} = 2\Omega_{[\bar{a}\bar{b}]}^{\bar{c}}. \quad (4.15)$$

Notice the analogy with the generalized fluxes (3.57) defined in terms of the generalized Lie derivative (3.39). Then, the projection of the torsionless spin connection to the space of fluxes (i.e., its antisymmetrization in the first two indices) is proportional to the fluxes, given that the projection of the Levi-Civita connection to this space vanishes. In fact, it can be shown that, in general,

$$e_{\bar{a}}^i W_{i\bar{b}}^{\bar{c}} = \frac{1}{2} (f_{\bar{a}\bar{b}}^{\bar{c}} + s_{\bar{a}\bar{d}} s^{\bar{c}\bar{e}} f_{\bar{e}\bar{b}}^{\bar{d}} + s_{\bar{b}\bar{d}} s^{\bar{c}\bar{e}} f_{\bar{e}\bar{a}}^{\bar{d}}). \quad (4.16)$$

Then, the spin connection is fully expressible in terms of dynamical Scherk–Schwarz fluxes.

Having introduced the connections and their properties, we now turn to curvatures. The commutator of two covariant derivatives reads

$$[\nabla_i, \nabla_j] V^k = R_{ijl}^k V^l - T_{ij}^l \nabla_l V^k, \quad (4.17)$$

with

$$R_{ijl}^k = \partial_i \Gamma_{jl}^k - \partial_j \Gamma_{il}^k + \Gamma_{im}^k \Gamma_{jl}^m - \Gamma_{jm}^k \Gamma_{il}^m \quad (4.18)$$

the Riemann tensor, which is covariant under Lie derivatives. It takes the same form when it is written in terms of the spin connection

$$R_{i\bar{j}\bar{a}}^{\bar{b}} = R_{ijk}^l e_{\bar{a}}^k e^{\bar{b}}_l = \partial_i W_{j\bar{a}}^{\bar{b}} - \partial_j W_{i\bar{a}}^{\bar{b}} + W_{i\bar{c}}^{\bar{b}} W_{j\bar{a}}^{\bar{c}} - W_{j\bar{c}}^{\bar{b}} W_{i\bar{a}}^{\bar{c}}, \quad (4.19)$$

and it has the following properties in the absence of torsion:

$$R_{ijlk} = R_{ijl}^m g_{mk} = R_{(ij)[lk]}, \quad R_{[ijl]}^k = 0, \quad (4.20)$$

the latter known as BI. The Riemann tensor is a very powerful object in the sense that it dictates how tensors are parallel-transported, and for this reason it is also known as the curvature tensor. Tracing the Riemann tensor, one obtains the (symmetric) Ricci tensor

$$R_{ij} = R_{ikj}^k = R_{ji}, \quad (4.21)$$

and tracing further leads to the Ricci scalar

$$R = g^{ij} R_{ij}. \quad (4.22)$$

The later defines the object out of which the action of General Relativity (4.1) is built, while the vanishing of the former gives the equations of motion

$$R_{ij} = 0. \quad (4.23)$$

This equation is known as *Ricci flatness*, and the solutions to these equations are said to be *Ricci flat*. Note that the Riemann and Ricci tensors and the Ricci scalar are completely defined for a torsionless and metric compatible connection in terms of the metric.

Before turning to the generalizations of these objects needed for DFT, let us mention that combining the above results, the action of General Relativity can be written purely in terms of dynamical Scherk–Schwarz fluxes as

$$S = \frac{1}{4} \int dx \sqrt{g} f_{\bar{a}\bar{b}}^{\bar{c}} f_{\bar{d}\bar{e}}^{\bar{f}} [4\delta_{\bar{c}}^{\bar{a}} \delta_{\bar{f}}^{\bar{d}} s^{\bar{b}\bar{e}} - 2\delta_{\bar{f}}^{\bar{b}} \delta_{\bar{c}}^{\bar{e}} s^{\bar{a}\bar{d}} - s^{\bar{a}\bar{d}} s^{\bar{b}\bar{e}} s_{\bar{c}\bar{f}}]. \quad (4.24)$$

This is also analogue to the situation in DFT (3.61).

4.2. Generalized connections and torsion

Some of the ingredients discussed in the last subsection already found their generalized analogues in previous sections. For example, the Lie derivative (4.2) has already been extended to its generalized version in the double geometry in (3.39). The Weitzenböck connection (4.13) has also been generalized in (3.59), and out of it, so have the fluxes (3.57) been extended to (4.15). Moreover, the actions (4.24) and (3.61) were both shown to be expressible in terms of fluxes. So, how far can we go? The aim of this section is to continue with the comparison, in order to find similarities and differences between the usual Riemannian geometry and double geometry. This is mostly based on [2, 18, 28, 24, 29].

Having defined a generalized Lie derivative, it is natural to seek a covariant derivative. We consider one of the forms

$$\nabla_M V_A^N = \partial_M V_A^N + \Gamma_{MP}^N V_A^P - W_{MA}^{\bar{B}} V_B^N, \quad (4.25)$$

with trivial extension to tensors with more indices. Here we have introduced a Christoffel connection Γ and a spin connection W whose transformation properties must compensate the failure of the partial derivative of a tensor to transform covariantly both under generalized diffeomorphisms and double Lorentz transformations.

We can now demand some properties on the connections, as we did in the Riemannian geometry construction. Let us analyze the implications of the following conditions.

- Compatibility with the generalized frame

$$\nabla_M E_A^N = 0. \quad (4.26)$$

As in the conventional Riemannian geometry, this simply relates the Christoffel connection to the spin connection through

$$W_{MA}^{\bar{B}} = E_{\bar{C}A}^{\bar{B}} \Omega_{\bar{C}}^{\bar{B}} + \Gamma_{MN}^P E_A^N E^{\bar{B}}_P, \quad (4.27)$$

where we have written the Weitzenböck connection defined in (3.59), which is totally determined by the generalized frame. Then, this condition simply says that if some components of the spin (Christoffel) connection were determined, the corresponding components of the Christoffel (spin) connection would also be.

- Compatibility with the $O(D, D)$ invariant metric

$$\nabla_M \eta^{PQ} = 2\Gamma_M^{(PQ)} = 0. \quad (4.28)$$

This simply states that the Christoffel connection must be antisymmetric in its two last indices

$$\Gamma_{MNP} = -\Gamma_{MPN}. \quad (4.29)$$

Note that, since we have seen in (3.59) that the Weitzenböck connection satisfies this property as well, due to (4.27), so does the spin connection

$$W_{MA\bar{B}} = -W_{M\bar{B}A}. \quad (4.30)$$

- Compatibility with the generalized metric

$$\nabla_M \mathcal{H}^{PQ} = \partial_M \mathcal{H}^{PQ} + 2\Gamma_{MR}^{(P} \mathcal{H}^{Q)R} = 0. \quad (4.31)$$

Its planar variant $\nabla_M S^{\bar{A}\bar{B}} = 0$ is then automatically guaranteed if compatibility with the generalized frame (4.26) is imposed.

The implications of the combined $O(D, D)$ and generalized metric compatibilities are better understood through the introduction of the following two projectors (3.85):

$$\hat{P}_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}_{MN}), \quad \check{P}_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}_{MN}), \quad (4.32)$$

which satisfy the properties

$$\hat{P}_M^Q \hat{P}_Q^N = \hat{P}_M^N, \quad \check{P}_M^Q \check{P}_Q^N = \check{P}_M^N, \quad \hat{P}_M^N + \check{P}_M^N = \delta_M^N. \quad (4.33)$$

Compatibility with both metrics then equals compatibility with these projectors

$$\nabla_M \hat{P}_N^Q = 0, \quad \nabla_M \check{P}_N^Q = 0, \quad (4.34)$$

which in turn implies

$$\check{P}_N^R \hat{P}_S^Q \Gamma_{MR}^S = \hat{P}_R^Q \partial_M \check{P}_N^R. \quad (4.35)$$

Then, compatibility with the generalized metric and $O(D, D)$ metric combined implies that only these projections of the connection are determined.

- Partial integration in the presence of the generalized density e^{-2d} (3.66):

$$\int e^{-2d} U \nabla_M V^M = - \int e^{-2d} V^M \nabla_M U \quad \Rightarrow \quad \Gamma_{PM}^P = -2\partial_M d. \quad (4.36)$$

Note that if the generalized frame were compatible, this would imply in turn that

$$E_C^N W_{NA}^{\bar{C}} = -\mathcal{F}_{\bar{A}}. \quad (4.37)$$

This requirement can also be considered as compatibility with the measure e^{-2d} , provided a trace part in the covariant derivative is added when acting on tensorial densities.

- Vanishing torsion. The Riemannian definition of torsion (4.5) is tensorial with respect to the Lie derivative, but not under generalized diffeomorphisms. In order to define a covariant notion of torsion, one can mimic its definition in terms of the Lie derivative, and replace it with the covariant derivative [24]

$$(\mathcal{L}_\xi^\nabla - \mathcal{L}_\xi)V^M = \mathcal{T}_{PQ}^M \xi^P V^Q, \quad \mathcal{T}_{PQ}^M = 2\Gamma_{[PQ]}^M + Y^M_Q{}^R \Gamma_{RP}^S. \quad (4.38)$$

This defines a covariant *generalized torsion*, which corrects the usual Riemannian definition through the invariant Y defined in (3.31), which in turn corrects the Lie derivative. Vanishing generalized torsion has the following consequence on the Christoffel connection:

$$2\Gamma_{[PQ]}^M + \Gamma^M{}_{PQ} = 0. \quad (4.39)$$

If this is additionally supplemented with the $O(D, D)$ metric compatibility (4.28), one gets that the totally antisymmetric part of the Christoffel connection vanishes

$$\Gamma_{[MNP]} = 0 \quad \Leftrightarrow \quad 3W_{[\bar{A}\bar{B}\bar{C}]} = \mathcal{F}_{\bar{A}\bar{B}\bar{C}}, \quad (4.40)$$

where the implication assumes generalized frame compatibility.

- Connections determined in terms of physical degrees of freedom. Typically, under the imposition of the above constraints on the connections, only some of their components get determined in terms of the physical fields. In [28], the connections were further demanded to live in the kernel of some projectors, allowing for a full determination of the connection. The price to pay is that under these projections, the derivative is ‘semi-covariant’, i.e. only some projections of it behave covariantly under transformations.

As we reviewed in the previous section, compatibility with the $O(D, D)$ metric is absent in the Riemannian geometry. There, metric compatibility and vanishing torsion determine the connection completely, and moreover guarantee partial integration in the presence of the measure \sqrt{g} . Here, the measure contains a dilaton-dependent part, and then one has to demand in addition, compatibility with the generalized dilaton. Agreement between the Riemannian geometry and double geometry is that vanishing (generalized) torsion implies that the projection of the spin connection to the space of fluxes is proportional to the fluxes (4.15) and (4.40).

Table 1. A list of conditions is given for objects in the Riemannian and double geometries, with their corresponding implications on the connections. Every line assumes that the previous ones hold.

	Riemannian geometry	Double geometry
Frame compatibility	$W = \Omega + \Gamma$	$W = \Omega + \Gamma$
$O(D, D)$ compatibility	–	$\Gamma_{MNP} = -\Gamma_{MPN}$ $W_{M\bar{A}\bar{B}} = -W_{M\bar{B}\bar{A}}$
Metric compatibility	$\partial_i g_{jk} = 2\Gamma_{i(j}{}^l g_{k)l}$	$\partial_M \mathcal{H}_{PQ} = 2\Gamma_{M(P}{}^N \mathcal{H}_{Q)N}$
Vanishing torsion	$\Gamma_{[ij]}{}^k = 0$ $W_{[\bar{a}\bar{b}]}{}^{\bar{c}} = 2f_{\bar{a}\bar{b}}{}^{\bar{c}}$	$\Gamma_{[MNP]} = 0$ $W_{[\bar{A}\bar{B}\bar{C}]} = 3\mathcal{F}_{\bar{A}\bar{B}\bar{C}}$
Measure compatibility	$\Gamma_{ki}{}^k = \frac{1}{\sqrt{g}} \partial_i \sqrt{g}$ $W_{\bar{b}\bar{a}}{}^{\bar{b}} = f_{\bar{b}\bar{a}}{}^{\bar{b}}$	$\Gamma_{PM}{}^P = e^{2d} \partial_M e^{-2d}$ $W_{\bar{B}\bar{A}}{}^{\bar{B}} = -\mathcal{F}_{\bar{A}}$
Determined part	Totally fixed $\Gamma_{ij}{}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$	Only some projections
Covariance failure	$\Delta_\xi \Gamma_{ij}{}^k = \partial_i \partial_j \xi^k$	$\Delta_\xi \Gamma_{MNP} = 2\partial_M \partial_{[N} \xi_{P]} + \Omega_{RNP} \Omega^R{}_{MS} \xi^S$

Despite many coincidences between the Riemannian geometry and double geometry, there is a striking difference. While in the former demanding metric compatibility and vanishing torsion determines the connection completely, in the latter these requirements turn out to leave undetermined components of the connection. Only some projections of the connections are determined, such as the trace (4.37) and its full antisymmetrization (4.40), among others.

To highlight the differences and similarities between the Riemannian and double geometries, we list in table 1 some of the quantities appearing in both frameworks.

4.3. Generalized curvature

In this section, we will assume that the generalized Christoffel and spin connections satisfy all the conditions listed in table 1. We would now like to seek a generalized curvature. The first natural guess would be to consider the conventional definition of the Riemann tensor (4.18) and extend it straightforwardly to the double space, namely

$$R_{MNP}{}^Q = 2\partial_{[M} \Gamma_{N]P}{}^Q + 2\Gamma_{[M|R}{}^Q \Gamma_{|N]P}{}^R. \tag{4.41}$$

However, this does not work because this expression is non-covariant under generalized diffeomorphisms:

$$\Delta_\xi R_{MNP}{}^Q = 2\Delta_\xi \Gamma_{[MN]}{}^R \Gamma_{RP}{}^Q + \text{strong constraint}. \tag{4.42}$$

In the Riemannian geometry, this would be proportional to the failure of the torsion to be covariant, which is zero. Here however $\Gamma_{[MN]}{}^P$ is not the torsion, because as we have seen, it is non-covariant. This in turn translates into the non-covariance of the Riemann tensor. As explained above, one has to resort to a generalized version of torsion (4.38)

$$\mathcal{T}_{PQ}{}^M = 2\Gamma_{[PQ]}{}^M + Y^M{}_Q{}^R \Gamma_{RP}{}^S = 0. \tag{4.43}$$

In addition, even if the first term in (4.42) were zero, we would have to deal with the other terms taking the form of the strong constraint if we were not imposing it from the beginning. For the moment, let us ignore them, and we will come back to them later. Note that vanishing torsion (4.43) implies

$$\Delta_\xi R_{MNPQ} = -\Delta_\xi \Gamma_{RMN} \Gamma_{PQ}{}^R + \text{strong constraint}, \tag{4.44}$$

and then it is trivial to check that the following combination

$$\mathcal{R}_{MNPQ} = R_{MNPQ} + R_{PQMN} + \Gamma_{RMN}\Gamma^R_{PQ} + \text{strong constraint} \quad (4.45)$$

is tensorial up to terms taking the form of the strong constraint:

$$\Delta_\xi \mathcal{R}_{MNPQ} = \text{strong constraint}. \quad (4.46)$$

Taking the strong constraint-like terms into account, the full generalized Riemann tensor is given by

$$\mathcal{R}_{MNPQ} = R_{MNPQ} + R_{PQMN} + \frac{1}{2D} Y^R_L{}^{SL} (\Gamma_{RMN}\Gamma_{SPQ} - \Omega_{RMN}\Omega_{SPQ}), \quad (4.47)$$

and is now covariant up to the consistency constraints of the theory discussed in section 3.5.

Since the connection has undetermined components, so does this generalized Riemann tensor. This combination of connections and derivatives does not project the connections to their determined part, so we are left with an undetermined Riemann tensor. The projections of the Riemann tensor with the projectors (4.32) turn out to be either vanishing or unprojected as well. This situation marks a striking difference with the Riemannian geometry.

We can now wonder whether some traces (and further projections) of this generalized Riemann tensor lead to sensible quantities, such as some generalized Ricci tensor related to the equations of motion of DFT (3.81) or some generalized Ricci scalar related to (3.61). For this to occur, the traces must necessarily project the connections in the Riemann tensor in such a way that only their determined part survives.

Tracing the generalized Riemann tensor with the projector \hat{P} (4.32), one can define a generalized notion of Ricci tensor

$$\mathcal{R}_{MN} = \hat{P}^{PQ} \mathcal{R}_{MPNQ}, \quad (4.48)$$

from which the action of DFT and its equations of motion can be obtained from traces and projections. Taking another trace one recovers the (already defined) generalized Ricci scalar

$$\mathcal{R} = \frac{1}{4} \hat{P}^{MN} \mathcal{R}_{MN} \quad (4.49)$$

that defines the action of DFT (it actually gives this tensor up to terms that constitute total derivatives when introduced in the action (3.60)). On the other hand, the following projections of this new generalized Ricci tensor contain the information on the equations of motion (3.81)

$$\mathcal{G}_{[MN]} = \hat{P}_{[M}{}^P \check{P}_{N]}{}^Q \mathcal{R}_{PQ} = 0. \quad (4.50)$$

It might be quite confusing that the projections of the generalized Ricci tensor yielding the equations of motion correspond to the vanishing of an antisymmetric tensor. However, there is a remarkable property of matrices of the form (4.50)

$$\begin{array}{ccccc} \hat{P}_M{}^R \check{P}_N{}^S \mathcal{R}_{RS} = 0 & \Rightarrow & \hat{P}_{[M}{}^R \check{P}_{N]}{}^S \mathcal{R}_{RS} = 0 & \Rightarrow & \hat{P}_Q{}^M \hat{P}_{[M}{}^R \check{P}_{N]}{}^S \mathcal{R}_{RS} = 0 \\ & & \updownarrow & & \\ \hat{P}_Q{}^M \hat{P}_{(M}{}^R \check{P}_{N)}{}^S \mathcal{R}_{RS} = 0 & \Leftarrow & \hat{P}_{(M}{}^R \check{P}_{N)}{}^S \mathcal{R}_{RS} = 0 & \Leftarrow & \hat{P}_M{}^R \check{P}_N{}^S \mathcal{R}_{RS} = 0 \end{array} \quad (4.51)$$

Therefore, the vanishing of the antisymmetric part of $\hat{P}_M{}^R \check{P}_N{}^S \mathcal{R}_{RS}$ contains the same information as the vanishing of the symmetric part. Then, one can alternatively define a symmetric generalized Ricci tensor whose vanishing yields the equations of motion as well:

$$\widehat{\mathcal{R}}_{MN} = \hat{P}_{(M}{}^R \check{P}_{N)}{}^S \mathcal{R}_{RS} = 0. \quad (4.52)$$

We summarize some differences between the geometric quantities in the Riemannian and double geometries in table 2.

An alternative to this approach was considered in [30], where only the Weitzenböck connection is non-vanishing and the spin connection is set to zero. The Weitzenböck connection

Table 2. A list of definitions of curvatures is given for Riemannian and double geometry.

	Riemannian geometry	Double geometry
Torsion	$T_{ij}{}^k = 2\Gamma_{[ij]}{}^k$	$\mathcal{T}_{MN}{}^P = 2\Gamma_{[MN]}{}^P + \Gamma^P{}_{MN}$
Riemann tensor	Determined $R_{ijl}{}^k = 2\partial_{[i}\Gamma_{j]l}{}^k + 2\Gamma_{[im}{}^k\Gamma_{j]l}{}^m$	Undetermined $\mathcal{R}_{MNPQ} = R_{MNPQ} + R_{PQMN} + \Gamma_{RMN}\Gamma^R{}_{PQ} - \Omega_{RMN}\Omega^R{}_{PQ}$
Ricci tensor	Determined $R_{ij} = R_{ikj}{}^k$	Undetermined $\mathcal{R}_{MN} = \hat{P}_P{}^Q\mathcal{R}_{MQN}{}^P$
EOM	$R_{ij} = 0$	$\hat{P}_{(M}{}^R\check{P}_{N)}{}^S\mathcal{R}_{RS} = 0$
Ricci Scalar	$R = g^{ij}R_{ij}$	$\mathcal{R} = \frac{1}{4}\hat{P}^{MN}\mathcal{R}_{MN}$

is torsionful, and the torsion coincides with the generalized fluxes (3.57). This connection is flat, and then the Riemann tensor vanishes, but the dynamics is encoded in the torsion and one can still build the DFT action and equations of motion from it, by demanding H -invariance (3.56). Since the connection and torsion are fully determined, this approach has the advantage of the absence of unphysical degrees of freedom. This also has a General Relativity analogue with its corresponding similarities and differences.

4.4. Generalized Bianchi identities

The generalized Riemann tensor satisfies the same symmetry properties as in the Riemannian geometry (4.20):

$$R_{MNPQ} = R_{(MN)[PQ]}, \quad (4.53)$$

plus a set of generalized BI

$$\mathcal{R}_{[\bar{A}\bar{B}\bar{C}\bar{D}]} = \mathcal{Z}_{\bar{A}\bar{B}\bar{C}\bar{D}} = \partial_{[\bar{A}}\mathcal{F}_{\bar{B}\bar{C}\bar{D}]} - \frac{3}{4}\mathcal{F}_{[\bar{A}\bar{B}}{}^{\bar{E}}\mathcal{F}_{\bar{C}\bar{D}]\bar{E}}, \quad (4.54)$$

which under the strong constraint in the supergravity frame simply become the BI of supergravity (3.18) and (4.20), as we will see later. Note that due to the consistency constraints (3.44) this vanishes as in the usual Riemannian case

$$\mathcal{Z}_{\bar{A}\bar{B}\bar{C}\bar{D}} = \Delta_{E_{\bar{A}}}\mathcal{F}_{\bar{B}\bar{C}\bar{D}} = E_{\bar{D}M}\Delta_{\bar{A}}\mathcal{L}_{E_{\bar{B}}}E_{\bar{C}}{}^M = 0. \quad (4.55)$$

BI in DFT were extensively discussed in [29].

5. Dimensional reductions

In order to make contact with four-dimensional physics, we have to address the dimensional reduction of DFT. Strictly speaking, we were already assuming that some directions were compact, but now make the distinction between compact and non-compact directions precise, and evaluate the dynamics around particular backgrounds. We begin this section with a brief review of Scherk–Schwarz (SS) compactifications of supergravity [80], and then extend these ideas to dimensionally reduce DFT to four dimensions, following [65, 66]. We show that the resulting effective action corresponds to the electric bosonic sector of half-maximal gauged supergravity [79] containing all duality orbits of electric fluxes, including the non-geometric ones [67].

5.1. Scherk–Schwarz compactifications

Let us briefly recall how geometric fluxes emerge in SS compactifications of supergravity, along the lines of [81].

Consider the NS-NS sector of supergravity containing a D -dimensional metric $g_{ij} = e_i^{\bar{a}} s_{\bar{a}\bar{b}} e_j^{\bar{b}}$, a 2-form field b_{ij} and a dilaton ϕ , all depending on D spacetime coordinates x^i (we are thinking of $D = 10$). We will refer to the D -dimensional theory as the *parent theory*. When dimensionally reduced to $d = D - n$ dimensions, the resulting lower dimensional theory will be referred to as the *effective theory*.

SS reductions can be introduced as the following set of steps to be performed in order to obtain the effective theory.

- Split coordinates

$$x^i = (x^\mu, y^m). \quad (5.1)$$

The coordinates y^m , $m = 1, \dots, n$ correspond to the compact space directions, while x^μ , $\mu = 1, \dots, d$ are the spacetime directions of the effective theory. The former (latter) are called *internal* (*external*).

- Split indices in fields and parameters. The original D -dimensional theory enjoys a set of symmetries and the fields belong to representations of these symmetries. Upon compactification, the parent symmetry groups will be broken to those of the effective theory. The fields then must be decomposed into the representations of the symmetry group in the lower dimensional theory:

$$g_{ij} = \begin{pmatrix} g_{\mu\nu} + g_{pq} A^p_\mu A^q_\nu & A^p_\mu g_{pn} \\ g_{mp} A^p_\nu & g_{mn} \end{pmatrix}, \quad (5.2)$$

$$b_{ij} = \begin{pmatrix} b_{\mu\nu} - \frac{1}{2}(A^p_\mu V_{p\nu} - A^p_\nu V_{p\mu}) + A^p_\mu A^q_\nu b_{pq} & V_{n\mu} - b_{np} A^p_\mu \\ -V_{m\nu} + b_{mp} A^p_\nu & b_{mn} \end{pmatrix}, \quad (5.3)$$

i.e. into internal, external and mixed components. Note that here there is an abuse of notation in that $g_{\mu\nu}$ are not the $\mu\nu$ components of g_{ij} .

Also, the parameters of gauge transformations must be split

$$\lambda^i = (\epsilon^\mu, \Lambda^m), \quad \tilde{\lambda}_i = (\epsilon_\mu, \Lambda_m). \quad (5.4)$$

- Provide a reduction ansatz. The particular dependence of the fields on the external and internal coordinates is of the form

$$\begin{aligned} g_{\mu\nu} &= \widehat{g}_{\mu\nu}(x), & b_{\mu\nu} &= \widehat{b}_{\mu\nu}(x), \\ A^m_\mu &= u^m_a(y) \widehat{A}^a_\mu(x), & V_{m\mu} &= u^a_m(y) \widehat{V}_{a\mu}(x), \\ g_{mn} &= u^a_m(y) u^b_n(y) \widehat{g}_{ab}(x), & b_{mn} &= u^a_m(y) u^b_n(y) \widehat{b}_{ab}(x) + v_{mn}(y), \end{aligned} \quad (5.5)$$

and similarly for the dilaton $\phi = \widehat{\phi}(x)$. The procedure even tells you what form this ansatz should have. If there is a global symmetry in the theory, such as a shift in the 2-form $b \rightarrow b + v$, then one simply ‘gauges’ the global symmetry by making it depend on the internal coordinates $v \rightarrow v(y)$. The y -dependent elements $u(y)$ and $v(y)$ are called twists. Once the procedure is over, the dependence on internal coordinates will disappear, but the information on the twists will remain in the form of structure-like constants that will parameterize the possible deformations of the effective action. For this reason, the twist matrices are taken to be constant in the external directions, because otherwise Lorentz invariance would be explicitly broken by these constants in the effective action. The hatted fields on the other hand depend only on the external coordinates, and will therefore

correspond to the dynamical degrees of freedom in the effective action. These are a d -dimensional metric $\widehat{g}_{\mu\nu}$ and a 2-form $\widehat{b}_{\mu\nu}$, plus $2n$ vectors $(\widehat{A}^a_\mu, \widehat{V}_{a\mu})$, plus $n^2 + 1$ scalars $(\widehat{g}_{ab}, \widehat{b}_{ab}, \widehat{\phi})$.

The gauge parameters must be twisted as well

$$\lambda^i = (\widehat{\epsilon}^\mu(x), u_a^m(y)\widehat{\Lambda}^a(x)), \quad \tilde{\lambda}_i = (\widehat{\epsilon}_\mu(x), u_m^a(y)\widehat{\Lambda}_a(x)). \quad (5.6)$$

- Identify residual gauge transformations. The gauge transformations of the parent supergravity theory are given by the Lie derivatives (3.14)

$$L_\lambda V^i = \lambda^j \partial_j V^i - V^j \partial_j \lambda^i, \quad (5.7)$$

plus gauge transformations of the 2-form (3.16). Plugging the fields and gauge parameters with the SS form into these, one obtains the resulting gauge transformations of the effective theory. For example, taking $V^i = (\widehat{v}^\mu(x), u_a^m(y)\widehat{v}^a(x))$, one obtains

$$L_\lambda V^\mu = \widehat{\epsilon}^\nu \partial_\nu \widehat{v}^\mu - \widehat{v}^\nu \partial_\nu \widehat{\epsilon}^\mu \equiv \widehat{L}_{\widehat{\epsilon}} \widehat{v}^\mu, \quad (5.8)$$

and then the d -dimensional Lie derivative of the effective action is obtained. Similarly,

$$L_\lambda V^m = u_a^m \widehat{L}_{\widehat{\lambda}} \widehat{V}^a, \quad (5.9)$$

where the resulting transformation is *gauged*

$$\widehat{L}_{\widehat{\lambda}} \widehat{V}^a = L_{\widehat{\lambda}} \widehat{V}^a + f_{bc}^a \widehat{\Lambda}^b \widehat{v}^c, \quad (5.10)$$

since it receives the contribution from the following combination of twist matrices:

$$f_{ab}^c = u_a^m \partial_m u_b^n u_n^c - u_b^m \partial_m u_a^n u_n^c, \quad (5.11)$$

which takes the same form as the SS flux (4.15). Even if these objects are defined in terms of the twist u^a_m , which is y -dependent, given that they appear in the residual transformations and we look for a y -independent theory, one must impose the constraint that they are constant. In the literature, these constants are known as *metric fluxes*, since they correspond to the background fluxes of the metric (notice that the twist $u^a_m(y)$ corresponds to the internal coordinate dependence of the metric (5.5)).

Pursuing this procedure with all the components of all the gauge transformations, we find the gauge transformations for all the fields in the effective action. To render the result readable, let us rearrange things in a compact language. The gauge parameters are taken to be of the form

$$\widehat{\xi} = (\widehat{\epsilon}_\mu, \widehat{\epsilon}^\mu, \widehat{\Lambda}^A), \quad \widehat{\Lambda}^A = (\widehat{\lambda}_a, \widehat{\lambda}^a), \quad (5.12)$$

and similarly the vector fields

$$\widehat{A}^A_\mu = (\widehat{V}_{a\mu}, \widehat{A}^a_\mu) \quad (5.13)$$

and the scalars

$$\widehat{\mathcal{M}}_{AB} = \begin{pmatrix} \widehat{g}^{ab} & -\widehat{g}^{ac} \widehat{b}_{cb} \\ \widehat{b}_{ac} \widehat{g}^{cb} & \widehat{g}_{ab} - \widehat{b}_{ac} \widehat{g}^{cd} \widehat{b}_{db} \end{pmatrix}. \quad (5.14)$$

Then, the different gauge transformations, parameterized by the different components of $\widehat{\xi}$ are inherited from the parent gauge transformations, and take the form

$$\delta_{\widehat{\xi}} \widehat{g}_{\mu\nu} = L_{\widehat{\epsilon}} \widehat{g}_{\mu\nu}, \quad (5.15)$$

$$\delta_{\widehat{\xi}} \widehat{b}_{\mu\nu} = L_{\widehat{\epsilon}} \widehat{b}_{\mu\nu} + (\partial_\mu \widehat{\epsilon}_\nu - \partial_\nu \widehat{\epsilon}_\mu), \quad (5.16)$$

$$\delta_{\widehat{\xi}} \widehat{A}^A_\mu = L_{\widehat{\epsilon}} \widehat{A}^A_\mu - \partial_\mu \widehat{\Lambda}^A + f_{BC}^A \widehat{\Lambda}^B \widehat{A}^C_\mu, \quad (5.17)$$

$$\delta_{\widehat{\xi}} \widehat{\mathcal{M}}_{AB} = L_{\widehat{\epsilon}} \widehat{\mathcal{M}}_{AB} + f_{AC}^D \widehat{\Lambda}^C \widehat{\mathcal{M}}_{DB} + f_{BC}^D \widehat{\Lambda}^C \widehat{\mathcal{M}}_{AD}. \quad (5.18)$$

Hence, we can readily identify the role of the different components of $\widehat{\xi}$: $\widehat{\epsilon}^\mu$ are the diffeomorphism parameters, $\widehat{\epsilon}_\mu$ generate gauge transformations of the 2-form and $\widehat{\Lambda}^A$ are the parameters of the gauge transformations for vectors. While here we have made a great effort to unify all these transformations, in DFT this unification is there from the beginning, as we will see later.

Here, we have introduced the ‘gaugings’ or ‘fluxes’ f_{AB}^C , which have the following non-vanishing components:

$$\begin{aligned} f_{abc} &= 3(\partial_{[a}v_{bc]} + f_{[ab}{}^d v_{c]d}), \\ f_{ab}{}^c &= u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n, \end{aligned} \quad (5.19)$$

while the rest of them vanish

$$f_a{}^{bc} = 0, \quad f^{abc} = 0. \quad (5.20)$$

This compact way of writing the results assumes that indices are raised and lowered with an $O(n, n)$ metric

$$\eta_{AB} = \begin{pmatrix} 0 & \delta_a{}^b \\ \delta_a{}^b & 0 \end{pmatrix}. \quad (5.21)$$

When written in the form of $f_{ABC} = f_{AB}{}^D \eta_{DC}$, they are totally antisymmetric $f_{ABC} = f_{[ABC]}$.

- Obtain the d -dimensional effective action. When the SS ansatz is plugged in the supergravity action, the result is

$$\begin{aligned} S = \int dx \sqrt{\widehat{g}} e^{-2\widehat{\phi}} & \left(\mathbf{R} + 4\partial^\mu \widehat{\phi} \partial_\mu \widehat{\phi} - \frac{1}{4} \widehat{\mathcal{M}}_{AB} \mathcal{F}^{A\mu\nu} \mathcal{F}^B{}_{\mu\nu} \right. \\ & \left. - \frac{1}{12} \mathcal{G}_{\mu\nu\rho} \mathcal{G}^{\mu\nu\rho} + \frac{1}{8} D_\mu \widehat{\mathcal{M}}_{AB} D^\mu \widehat{\mathcal{M}}^{AB} + V \right). \end{aligned} \quad (5.22)$$

Here \mathbf{R} is the d -dimensional Ricci scalar, and we have defined the field strengths as

$$\begin{aligned} \mathcal{F}^A{}_{\mu\nu} &= \partial_\mu \widehat{A}^A{}_\nu - \partial_\nu \widehat{A}^A{}_\mu - f_{BC}{}^A \widehat{A}^B{}_\mu \widehat{A}^C{}_\nu, \\ \mathcal{G}_{\mu\rho\lambda} &= 3\partial_{[\mu} \widehat{b}_{\rho\lambda]} - f_{ABC} \widehat{A}^A{}_\mu \widehat{A}^B{}_\rho \widehat{A}^C{}_\lambda + 3\partial_{[\mu} \widehat{A}^A{}_\rho \widehat{A}_{\lambda]A}, \end{aligned} \quad (5.23)$$

and a covariant derivative for scalars as

$$D_\mu \widehat{\mathcal{M}}_{AB} = \partial_\mu \widehat{\mathcal{M}}_{AB} - f_{AD}{}^C \widehat{A}^D{}_\mu \widehat{\mathcal{M}}_{CB} - f_{BD}{}^C \widehat{A}^D{}_\mu \widehat{\mathcal{M}}_{AC}. \quad (5.24)$$

Also, due to the gaugings, a scalar potential arose

$$V = -\frac{1}{4} f_{DA}{}^C f_{CB}{}^D \widehat{\mathcal{M}}^{AB} - \frac{1}{12} f_{AC}{}^E f_{BD}{}^F \widehat{\mathcal{M}}^{AB} \widehat{\mathcal{M}}^{CD} \widehat{\mathcal{M}}_{EF} - \frac{1}{6} f_{ABC} f^{ABC}, \quad (5.25)$$

which strongly resembles the form of the DFT action (3.61). Let us mention that we have actually considered a simplified ansatz. Lorentz invariance is also preserved if a warp factor is included in the reductions ansatz, which would turn on additional flux backgrounds of the form f_A , in which case the effective action would exactly coincide with the DFT action.

This concludes the introduction to the basic notions of SS compactifications of supergravity. We should say that there exist different related Scherk–Schwarz compactifications, and their distinction goes beyond the scope of this review. Also, the consistency of these reductions is subtle and by no means automatic, and we refer to the literature for detailed discussions on these points (see for example [9, 82]).

5.2. Geometric fluxes

In the SS reduction defined in (5.5), we restricted ourselves to the zero modes and truncated all the states of the infinite tower of Kaluza–Klein (KK) modes. Had we conserved them, the effective action would have been more involved and would have had towers of KK degrees of freedom. Typically, these modes are neglected because their masses scale proportionally to the order of each mode. If they were kept, other stringy states with comparable masses should be kept as well, and the effective theory would have to be completed with the corresponding contributions.

This can be more clearly seen in a toroidal compactification. Indeed, notice that a compactification on a torus with vanishing background of the 2-form corresponds to taking $u^a_m = \delta^a_m$ and $v_{ab} = 0$ in the SS procedure. In this case, (5.19) would give $f_{ABC} = 0$, i.e. we obtain an ungauged theory. Recalling the mass spectrum of closed strings on tori (3.1) and the fact that the winding modes decouple in the field theory limit, we see that the zero mode of such a compactification is massless for the fields considered in supergravity (with $N = \tilde{N} = 1$). Had we kept states with $p \neq 0$ to be consistent, we should have also taken into account other string excitations with comparable masses. Since all the fluxes vanish in this case, no masses can be generated in the effective theory.

These compactifications on tori with vanishing form fluxes (i.e., configurations with $f_{ABC} = 0$) present many phenomenological problems.

- The scalar potential vanishes, so *any* configuration of scalars corresponds to a possible minimum of the theory. The moduli space is then fully degenerate, and all scalars are massless. This poses a problem because, on the one hand, there are no massless scalars in nature, and, on the other hand, the theory loses all predictability since one has the freedom to choose any vacuum of the effective theory.
- Since the scalar potential vanishes, there is no way to generate a cosmological constant in the lower dimensional theory. This is contrary to experimental evidence, which indicates that our universe has a tiny positive cosmological constant, i.e. it is a de Sitter (dS) universe.
- The gravitinos of the supersymmetric completion of the theory are massless as well. If we start with $\mathcal{N} = 1$ theory in $D = 10$, we would end with $\mathcal{N} = 4$ theory in $d = 4$. This is too much supersymmetry and we have no possibility of breaking it.
- Since the fluxes play the role of structure constants, their vanishing implies that the gauge symmetries are Abelian. Then, Standard-Model-like interactions are not possible.

It is then clear that a torus compactification is not interesting from a phenomenological point of view. The situation changes when the twists $u^a_m(y)$ and $v_{ab}(y)$ are such that $f_{ABC} \neq 0$. We have seen that they allow us to turn on metric fluxes f_{ab}^c (through u^a_m) and 2-form fluxes f_{abc} (through v_{ab}) in (5.19). The appearance of these fluxes now generates a scalar potential (5.25) that classically lifts the moduli space. This in turn generates masses for scalars and gravitinos, renders the gauge symmetries non-Abelian and allows for the possibility of a cosmological constant. However, although the phenomenological perspectives improved, it turns out that geometric fluxes seem not to be enough for moduli stabilization and dS vacua, and then one has to go beyond them. There are a number of no-go theorems and evidence [83] pointing in this direction.

In the literature, the 2-form and metric fluxes both go under the name of *geometric fluxes*, and are denoted

$$\begin{aligned} H_{abc} &\equiv f_{abc} = 3(\partial_{[a}v_{bc]} + f_{[ab}{}^d v_{c]d}), \\ \omega_{ab}{}^c &\equiv f_{ab}{}^c = u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n, \end{aligned} \tag{5.26}$$

respectively. Since T-dualities exchange metric and 2-form components (3.11), they exchange these fluxes as well

$$H_{abc} \xleftrightarrow{h^{(c)}} \omega_{ab}{}^c. \tag{5.27}$$

Let us now devote a few lines to give an interpretation of the SS procedure in terms of a compactification. The SS ansatz (5.5) can be interpreted as follows. The twists $u^a{}_m(y)$ correspond to the metric background in the compact space and \widehat{g}_{ab} amount to perturbations. The full internal metric reads

$$g_{mn} = u^a{}_m(y)\widehat{g}_{ab}(x)u^b{}_n(y). \tag{5.28}$$

When plugging this in the supergravity action, one obtains an effective theory for the perturbations \widehat{g}_{mn} , which is deformed by parameters that only depend on the background. Then, freezing the perturbations as

$$\widehat{g}_{ab}(x) = \delta_{ab} \quad \Rightarrow \quad g_{mn} = u^a{}_m(y)\delta_{ab}u^b{}_n(y) \tag{5.29}$$

gives the background on which one compactifies, and the effective action dictates the dynamics of the perturbations around the background. Similarly, the perturbations of the 2-form are given by \widehat{b}_{ab} , and freezing them gives the corresponding 2-form background

$$\widehat{b}_{ab}(x) = 0 \quad \Rightarrow \quad b_{mn} = v_{mn}(y). \tag{5.30}$$

The twist matrices $u^a{}_m$ and v_{mn} can then be interpreted as the backgrounds associated with the vielbein and the 2-form of the compact space. From now on, when referring to backgrounds we shall assume that the perturbations are frozen.

Let us now explore a very simple setting that gives rise to a flux for the 2-form, H_{abc} (later, we will consider all its T-duals). This is the canonical example in the literature on (non-)geometric fluxes, and it is nicely discussed in [8]. Most of the terminology related to (non-)geometric fluxes is taken from this example, so we find it instructive to revisit it here. For simplicity, we consider a three-dimensional internal space, which can be embedded in the full internal six-dimensional space. Consider a compactification on a 3-torus with a non-trivial 2-form:

$$g_{mn} = \delta_{mn}, \quad b_{23} = Ny^1 \quad \Leftrightarrow \quad u_m{}^a = \delta_m^a, \quad v_{23} = Ny^1. \tag{5.31}$$

Plugging this into (5.26), we obtain

$$H_{123} = N, \quad \omega_{12}{}^3 = \omega_{23}{}^1 = \omega_{31}{}^2 = 0, \tag{5.32}$$

so a compactification on a torus with a non-trivial 2-form field turns on a H -flux in the effective action.

Since this background has isometries in the directions y^2, y^3 , we can perform a T-duality in one of these directions, let us say $h^{(3)}$, through the Buscher rules (3.11). Then, we obtain the background

$$ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + (dy^2)^2 + (dy^3 + Ny^1 dy^2)^2, \quad b_{mn} = 0. \tag{5.33}$$

This corresponds to

$$u^a{}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & Ny^1 & 1 \end{pmatrix}, \quad v_{mn} = 0. \tag{5.34}$$

Plugging this into (5.26), we find that the fluxes turned on in the effective action are now

$$H_{123} = \omega_{23}{}^1 = \omega_{31}{}^2 = 0, \quad \omega_{12}{}^3 = N, \tag{5.35}$$

in agreement with the T-duality chain (5.27). The background (5.33) is called *twisted torus*, and it generates metric fluxes $\omega_{ab}{}^c$ upon compactifications. In more general backgrounds, SS compactifications allow us to turn on form and metric fluxes simultaneously, provided the compactification is done on a twisted torus with a non-trivial 2-form background. Examples of different Scherk–Schwarz compactifications in different scenarios, and their relation to gauged supergravity can be found in [84].

5.3. Gauged supergravities and duality orbits

The effective action (5.22), obtained by means of an SS compactification, is a particular gauged supergravity. For a review of gauged supergravity see [9]. These kinds of dimensional reductions preserve all the supersymmetries of the parent theory and are therefore highly constrained. When the starting point is $D = 10$ supergravity with $\mathcal{N} = 1$ supersymmetries (16 supercharges), the $d = 4$ effective theory preserves all the supercharges and has therefore $\mathcal{N} = 4$ supersymmetries. This corresponds to the half of the maximal allowed supersymmetries, and so they are called *half-maximal gauged supergravities*. These theories have been widely studied irrespective of their stringy higher dimensional origin, and the full set of possible deformations have been classified in [79]. Let us here review the basics of the bosonic sector of $d = 4$ half-maximal gauged supergravity, so that we can then identify particular gaugings as specific reductions in different backgrounds.

The bosonic field content of half-maximal gauged supergravity in four dimensions consists of a metric $\widehat{g}_{\mu\nu}$, 12 vector fields $\widehat{A}^A{}_\mu$ and 38 scalars, arranged in two objects: a complex parameter $\tau = e^{-2\widehat{\phi}} + i\widehat{B}_0$ and a scalar matrix $\widehat{\mathcal{M}}_{AB}$ with 36 independent components parameterizing the coset $O(6, 6)/O(6) \times O(6)$.

There is an additional freedom to couple an arbitrary number N of vector multiplets but, for simplicity, we will not consider this possibility (otherwise the global symmetry group would have to be extended to $O(D, D + N)$ [65]). Also, the global symmetry group contains an $SL(2)$ factor as well, related to S-duality, which mixes the electric and magnetic sectors. This is not captured by DFT (unless the global symmetry group is further extended to include S-duality), and then one can only obtain the electric sector.

The ungauged theory is invariant under an $O(6, 6)$ global symmetry group, and by ‘ungauged’ we mean that the gauge group is the Abelian $U(1)^{12}$. This group can however be rendered non-Abelian by gauging a subgroup of $O(6, 6)$. Given the $O(6, 6)$ generators $(t_\alpha)_A{}^B$, with $\alpha = 1, \dots, 66$; $A = 1, \dots, 12$, there is a powerful object named embedding tensor $\Theta_A{}^\alpha$ that dictates the possible gaugings of the theory. The gauge group generators are given by $\Theta_A{}^\alpha (t_\alpha)_B{}^C$, so $\Theta_A{}^\alpha$ establishes how the gauge group is embedded in the global symmetry group. The $\mathbf{12} \otimes \mathbf{66}$ components of $\Theta_A{}^\alpha$ are restricted by a linear constraint that leaves only $\mathbf{12} + \mathbf{220}$ components, parameterized by

$$\xi_A, \quad f_{ABC} = f_{[ABC]}, \quad (5.36)$$

and these are further restricted by quadratic constraints

$$\xi_A \xi^A = 0, \quad (5.37)$$

$$\xi^C f_{ABC} = 0 \quad (5.38)$$

$$f_{E[AB} f^E{}_{CD]} = \frac{1}{3} f_{[ABC} \xi_{D]}, \quad (5.39)$$

necessary for the gauge invariance of the embedding tensor (and closure of the algebra). The $O(6, 6)$ indices are raised and lowered with the invariant metric (5.21).

In four dimensions, 2-forms are dual to scalars. Dualizing the scalar $\widehat{B}_0 \rightarrow \widehat{b}_{\mu\nu}$, the action of the electric bosonic sector of half-maximal gauged supergravity takes the form (5.22) when $\xi_A = 0$. We then see that the SS compactification of $D = 10$ supergravity on a twisted torus with 2-form flux leads to a particular half-maximal gauged supergravity in $d = 4$. The only possible deformations in that theory are given by (5.26). From now on, we will restrict ourselves to the gaugings f_{ABC} and set the rest of them to zero, i.e. $\xi_A = 0$, for simplicity.

The global symmetries of the ungauged theory amount to $O(6, 6)$ transformations

$$\widehat{A}^A_\mu \rightarrow h_B^A \widehat{A}^B_\mu, \quad \widehat{\mathcal{M}}_{AB} \rightarrow h_A^C \widehat{\mathcal{M}}_{CD} h_B^D, \quad (5.40)$$

where the elements $h \in O(6, 6)$ were introduced in section 3.1. When the gaugings are turned on, the global symmetry group is broken by them. However, $O(6, 6)$ transformations do not change the physics. In fact, given a configuration of gaugings f_{ABC} with their corresponding action (5.22), any $O(6, 6)$ rotation of them,

$$f_{ABC} \rightarrow h_A^D h_B^E h_C^F f_{DEF}, \quad (5.41)$$

would yield a different configuration with a corresponding different action. However, through a field redefinition of the form (5.40), this action can be taken to the original form. In other words, we have the relation

$$S[h_A^D h_B^E h_C^F f_{DEF}, \widehat{A}^A_\mu, \widehat{\mathcal{M}}_{AB}] = S[f_{ABC}, h_B^A \widehat{A}^B_\mu, h_A^C \widehat{\mathcal{M}}_{CD} h_B^D], \quad (5.42)$$

and so an $O(6, 6)$ transformation of the gaugings just amounts to a field redefinition. Then, it corresponds to the same theory. For this reason, it is not convenient to talk about configurations of gaugings, but rather of *orbits* of gaugings. An orbit is a set of configurations related by duality transformations, so that different theories correspond to different duality orbits of gaugings.

An intriguing feature of gauged supergravities is that they admit more deformations than those that can be reached by means of geometric compactifications on twisted tori with 2-form flux. In fact, for generic configurations, the embedding tensor has components

$$Q_a^{bc} = f_a^{bc}, \quad R^{abc} = f^{abc}, \quad (5.43)$$

that cannot be turned on through the canonical SS compactification (5.20). Since the other set of gaugings f_{abc} , f^a_{bc} were identified with the geometrical fluxes, these are said to be *non-geometric* gaugings. Here we have named them Q and R to match the standard parlance in the literature of flux compactifications. One then wonders to what kind of backgrounds or compactifications these gaugings would correspond to. As we will see, T-duality has a very concrete answer to this question.

Before moving to a discussion on non-geometric fluxes, let us briefly review the arguments of [59, 61] to invoke non-geometric fluxes from a string theory perspective. In [59, 61], all supergravities in $D = 10$ and 11 dimensions are compactified in a geometric sense to four dimensions. These higher dimensional supergravities are the low-energy limit of duality-related string theories, like for instance Types IIA and Type IIB strings. Each compactification gives rise to a fluxed effective action containing only geometric fluxes.

When duality transformations are applied at the level of the four-dimensional effective action, one finds that, although the parent theories are connected by dualities, the effective theories are not [59, 61]. Thus, new non-geometric fluxes have to be invoked, so that the theories match. In this process, gaugings (or fluxes) that look geometric in one picture (duality frame) are non-geometric in others, and all of them should be included in string compactifications in order to preserve all the *stringy* information at the level of the effective action. Moreover, when all the gaugings are considered together in the effective action, the resulting (super-)potential includes *all* the possible deformations (gaugings) of gauged supergravity. All

the T-dual deformations are captured by generalized geometric compactifications of DFT, as we will see.

We have seen in section 5.2 that starting with a toroidal background with a 2-form flux H_{123} (5.31), an $h^{(3)}$ T-duality can be performed in the direction y^3 leading to a twisted torus with the metric flux ω_{12}^3 (5.33). The latter still has an isometry in the direction y^2 , so nothing prevents us from doing a new T-duality, namely $h^{(2)}$. At the level of fluxes, the chain would go as

$$H_{abc} \xleftrightarrow{h^{(c)}} \omega_{ab}{}^c \xleftrightarrow{h^{(b)}} Q_a{}^{bc}, \quad (5.44)$$

and so a compactification on the resulting background would turn on a $Q_1{}^{23}$ flux in the effective action. Instead of using the Buscher rules, we find it more instructive to T-dualize via the construction of a generalized metric. For the twisted torus (5.33), it takes the form

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{mn} & -g^{mp}b_{pn} \\ b_{mp}g^{pn} & g_{mn} - b_{mp}g^{pq}b_{qn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -Ny^1 & 0 & 0 & 0 \\ 0 & -Ny^1 & 1 + (Ny^1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + (Ny^1)^2 & Ny^1 \\ 0 & 0 & 0 & 0 & Ny^1 & 1 \end{pmatrix}.$$

Now acting on this twisted torus background with a T-duality in the direction y^2 :

$$\mathcal{H}_{MN} \rightarrow h^{(2)}{}_M{}^P h^{(2)}{}_N{}^Q \mathcal{H}_{PQ}, \quad h^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.45)$$

we obtain

$$h^{(2)}{}_M{}^P h^{(2)}{}_N{}^Q \mathcal{H}_{PQ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 + (Ny^1)^2 & 0 & 0 & 0 & Ny^1 \\ 0 & 0 & 1 + (Ny^1)^2 & 0 & -Ny^1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -Ny^1 & 0 & 1 & 0 \\ 0 & Ny^1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5.46)$$

and from here we can obtain the background metric

$$ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + \frac{1}{1 + (Ny^1)^2} [(dy^2)^2 + (dy^3)^2] \quad (5.47)$$

and the 2-form

$$b_{23} = -\frac{Ny^1}{1 + (Ny^1)^2} \quad (5.48)$$

associated with the $Q_1{}^{23}$ flux. This background only depends on y^1 in the directions orthogonal to y^1 , so this corresponds to a base coordinate. When undergoing a monodromy $y^1 \rightarrow y^1 + 1$, the solution does not come back to itself, but rather to an $O(2, 2) \in O(3, 3)$ rotation of it. Since this duality element mixes the metric and the 2-form in a non-trivial way, it is called a T-fold [14]. From a supergravity point of view, these backgrounds are globally ill-defined because the T-duality element needed to ‘glue’ the two different coordinate patches is not an element of the geometric (i.e. diffeos + shifts (3.10)) subgroup of $O(3, 3)$. This background is then said to correspond to a *globally non-geometric flux* $Q_a{}^{bc}$. Note however that from the

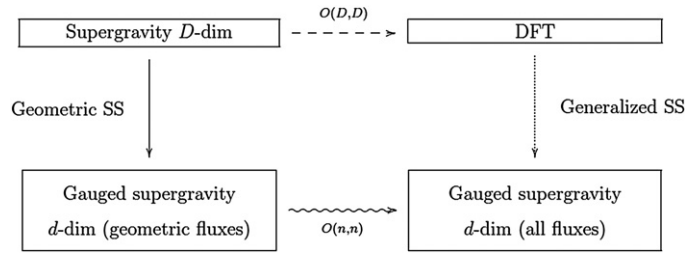


Figure 1. We picture the logic of DFT compactifications [65]. While standard SS reductions from supergravity in $D = d + n$ dimensions (solid line) give rise to gauged supergravity involving only geometric fluxes in d dimensions, invoking duality arguments at the level of the effective action one can conjecture the need for dual fluxes [59, 61] to complete all the deformations of gauged supergravity (waved line). More fundamentally, DFT is the $O(D, D)$ -covariantization of supergravity (dashed line), and generalized SS compactifications of DFT give rise to gauged supergravities with all possible deformations (dotted line).

double-space point of view, there is no such global issue provided one allows transitions with the full $O(3, 3)$ symmetry group (including T-dualities (3.10)). In this case, the identifications between the coordinates under monodromies involve the dual ones, and the generalized bein is globally well defined on the double space.

If we intended to do a further T-duality [59], say in the direction y^1 ,

$$H_{abc} \xleftrightarrow{h^{(c)}} \omega_{ab}{}^c \xleftrightarrow{h^{(b)}} Q_a{}^{bc} \xleftrightarrow{h^{(a)}} R^{abc}, \tag{5.49}$$

we would face the problem that we ran out of isometries. Therefore the resulting background would have to depend on a ‘dual’ coordinate and we would lose any notion of locality in terms of the usual coordinates on which supergravity is defined. For this reason, the fluxes R^{abc} are usually named *locally non-geometric*. Clearly, again, this form of non-geometry is not a problem in the double space either.

Note that the chain (5.49) connects different configurations of gaugings via T-duality. By definition, they all correspond to the same orbit, so the four-dimensional theory really does not distinguish between compactifications on tori with 2-form flux, twisted tori or T-folds, that are connected by T-dualities. In this sense, the orbit itself is basically geometric: if we were given an action with a single flux, either H , ω , Q or R , we would always find a geometric uplift and face its corresponding phenomenological problems. A different situation would be that of an action containing both geometric *and* non-geometric fluxes *simultaneously* turned on. T-duality would exchange geometric with non-geometric fluxes, and it would never be able to get rid of the non-geometric ones. These kinds of configurations are said to belong to a duality orbit of non-geometric fluxes [67], and they cannot be reached by means of a standard SS compactification of supergravity. They are actually the most interesting orbits since they circumvent all the no-go theorems preventing moduli fixing, dS vacua, etc [83].

As we will see, being T-duality invariant and defined on a double space, DFT is free from global and local issues. Generalized SS compactifications of DFT will be the topic of the forthcoming subsections. We will see that DFT provides a beautiful geometric uplift of all duality orbits, including the non-geometric ones. We anticipate the final picture in figure 1.

5.4. Generalized Scherk–Schwarz compactifications

Here, we generalize the SS procedure in a duality covariant way by applying it to DFT. This and the following sections are mostly based on [65, 66, 34]. Let us

then follow the steps introduced in section 5.1, although in a different order for convenience.

- In the double space, we have coordinates $X^M = (\tilde{x}_i, x^i)$, so we split them as follows: $\tilde{x}_i = (\tilde{x}_\mu, \tilde{y}_m)$ and $x^i = (x^\mu, y^m)$. As before, $m = 1, \dots, n$ are indices denoting internal directions and $\mu = 1, \dots, d$ are spacetime indices. Then, we have a *double* external space and a *double* internal one with coordinates $\mathbb{X} = (\tilde{x}_\mu, x^\mu)$ and $\mathbb{Y}^A = (\tilde{y}_m, y^m)$, respectively.
- Next, we propose a reduction ansatz for the fields and gauge parameters in the theory, inspired in the global symmetries of DFT. For the generalized bein and dilaton, we have

$$E^{\hat{A}}_M(X) = \widehat{E}^{\hat{A}}_I(\mathbb{X}) U^I_M(\mathbb{Y}), \quad d(X) = \widehat{d}(\mathbb{X}) + \lambda(\mathbb{Y}), \quad (5.50)$$

and for the gauge parameters, we have

$$\xi^M(X) = \widehat{\xi}^I(\mathbb{X}) U_I^M(\mathbb{Y}). \quad (5.51)$$

Here, $M, N = 1, \dots, 2D$ are curved indices in the parent theory and $I, J = 1, \dots, 2D$ are curved indices in the effective theory. Again, we use the notation that hatted objects are \mathbb{X} -dependent, and all the (double) internal \mathbb{Y} -dependence enters through the twists $U_I^M \in O(n, n)$ and λ .

- We plug this ansatz in the generalized fluxes, and obtain

$$\mathcal{F}_{\hat{A}\hat{B}\hat{C}} = \widehat{F}_{\hat{A}\hat{B}\hat{C}} + f_{IJK} \widehat{E}_A^I \widehat{E}_B^J \widehat{E}_C^K, \quad (5.52)$$

$$\mathcal{F}_{\hat{A}} = \widehat{F}_{\hat{A}} + f_I \widehat{E}_A^I, \quad (5.53)$$

where we have split the coordinate dependence in \mathbb{X} -dependent quantities

$$\widehat{F}_{\hat{A}\hat{B}\hat{C}} = 3\widehat{\Omega}_{[\hat{A}\hat{B}\hat{C}]}, \quad \widehat{\Omega}_{\hat{A}\hat{B}\hat{C}} = \widehat{E}_A^I \partial_I \widehat{E}_B^J \widehat{E}_{\hat{C}J}, \quad (5.54)$$

$$\widehat{F}_{\hat{A}} = \widehat{\Omega}^{\hat{B}}_{\hat{B}\hat{A}} + 2\widehat{E}_A^I \partial_I \widehat{d}, \quad (5.55)$$

and \mathbb{Y} -dependent ones

$$f_{IJK} = 3\tilde{\Omega}_{[IJK]}, \quad \tilde{\Omega}_{IJK} = U_I^M \partial_M U_J^N U_{KN}, \quad (5.56)$$

$$f_I = \tilde{\Omega}^J_{JI} + 2U_I^M \partial_M \lambda. \quad (5.57)$$

This splitting is possible provided one imposes the following constraint on the duality twist U_I^M :

$$U_I^M \partial_M \widehat{g} = \partial_I \widehat{g}, \quad \partial^M U_I^N \partial_M \widehat{g} = 0. \quad (5.58)$$

This restriction on the duality twist implies that it must be trivial in the \mathbb{X} -directions. There is a very important physical reason for this constraint to hold. The quantities f_{IJK} and f_I are named *gaugings*, and we take them to be constant

$$f_{IJK} = \text{constant}, \quad f_I = \text{constant}. \quad (5.59)$$

This is due to the fact that they appear in the action through the generalized fluxes $\mathcal{F}_{\hat{A}\hat{B}\hat{C}}$ and $\mathcal{F}_{\hat{A}}$, and since we look for a \mathbb{Y} -independent effective Lagrangian, they must be \mathbb{Y} -independent because their dependence comes only through the gaugings, which were requested to be constant. This in turn implies that the internal space is parallelizable, namely the twist must be globally defined. The constraint (5.58) can be recast as

$$f_{IJ}{}^K \partial_K \widehat{g} = 0, \quad f^I \partial_I \widehat{g} = 0. \quad (5.60)$$

Its role is to protect Lorentz invariance in the reduced theory. Note that $\partial_I \widehat{g}$ is only non-vanishing in the external directions, and then if the gaugings had legs in these directions,

they would explicitly break Lorentz symmetry. Therefore, f_{IJK} and f_I can only be non-vanishing along the double internal space. For simplicity, here we will only analyze the case $f_I = 0$, since consistency of the theory would otherwise require a slightly modified reduction ansatz. A discussion on how to turn the gaugings on in the usual supergravity picture can be found in [85], and in a duality covariant way in DFT in [66].

- We next plug (5.52) and (5.53) into the action of DFT (3.60) to obtain the effective theory

$$S_{\text{GDFT}} = v \int d\mathbb{X} e^{-2\hat{d}} \left[-\frac{1}{4} (\widehat{F}_{IK}{}^L + f_{IK}{}^L) (\widehat{F}_{JL}{}^K + f_{JL}{}^K) \widehat{\mathcal{H}}^{IJ} \right. \\ \left. - \frac{1}{12} (\widehat{F}_{IJ}{}^K + f_{IJ}{}^K) (\widehat{F}_{LH}{}^G + f_{LH}{}^G) \widehat{\mathcal{H}}^{IL} \widehat{\mathcal{H}}^{JH} \widehat{\mathcal{H}}_{KG} \right. \\ \left. - \frac{1}{6} (\widehat{F}_{IJK} + f_{IJK}) (\widehat{F}^{IJK} + f^{IJK}) + (\widehat{\mathcal{H}}^{IJ} - \eta^{IJ}) \widehat{F}_I \widehat{F}_J \right]. \quad (5.61)$$

The internal coordinate dependence factorizes and it just amounts to an overall constant factor

$$v = \int d\mathbb{Y} e^{-2\lambda}. \quad (5.62)$$

If the gaugings vanish $f_{IJK} = 0$, one recovers the usual DFT action (3.60) in less dimensions. This then corresponds to a *gauged* DFT (GDFT) [21, 34], which has been obtained through a generalized SS compactification of a higher dimensional *parent* DFT.

- The symmetries of the GDFT are inherited from those of the parent DFT. For instance, the generalized Lie derivative induces the gauge transformations in the effective action

$$\mathcal{L}_{\widehat{\xi}} V^M = U_I{}^M \widehat{\mathcal{L}}_{\widehat{\xi}} \widehat{V}^I, \quad (5.63)$$

namely

$$\widehat{\mathcal{L}}_{\widehat{\xi}} \widehat{V}^I = \mathcal{L}_{\widehat{\xi}} \widehat{V}^I - f^I{}_{JK} \widehat{\xi}^J \widehat{V}^K. \quad (5.64)$$

The first term is the usual generalized Lie derivative and the second one amounts to a deformation due to the gaugings. These induced transformations now close (in the sense of (3.46)) when the following quadratic constraints are imposed on the gaugings

$$f_{H[IJ} f_{KL]}{}^H = 0, \quad (5.65)$$

and the strong constraint holds in the external space

$$\partial_I \widehat{V} \partial^I \widehat{W} = 0 \quad (5.66)$$

for any hatted quantity, such as effective fields or gauged parameters. The action of GDFT (5.61) is invariant under (5.64) up to these constraints. Moreover, it can be checked that compactifying the constraints of the parent DFT gives the same result that one would obtain by directly computing the consistency conditions of the effective GDFT. These amount to even more relaxed versions of (5.65) and (5.66).

5.5. From gauged DFT to gauged supergravity

Now that we have built a covariant formulation of the effective theory, we can choose to solve the *effective* strong constraint (5.66) in the usual frame of a gauged supergravity $\partial_I \widehat{V} \partial^I \widehat{W} = 0 \rightarrow \tilde{\partial}^\mu \widehat{V} = 0$, i.e. we solve the strong constraint in the effective action by demanding that the effective fields and gauged parameters only depend on x^μ . Due to the coordinate splitting $X \rightarrow \mathbb{X}, \mathbb{Y}$, a convenient re-parameterization of the effective generalized

metric $\widehat{\mathcal{H}}^{IJ}$ is in order. The $O(D, D)$ group is now broken to $O(d, d) \times O(n, n)$, and then it is convenient to rotate the group metric to the form

$$\eta_{IJ} = \begin{pmatrix} & \delta^{\mu\nu} \\ \delta_{\mu\nu} & \\ & \eta_{AB} \end{pmatrix}, \quad (5.67)$$

where η_{AB} has been defined in (5.21). This amounts to a re-parameterization of the generalized bein

$$\widehat{E}^{\bar{A}}_I = \begin{pmatrix} \widehat{e}_a^\mu & -\widehat{e}_a^\rho \widehat{c}_{\rho\mu} & -\widehat{e}_a^\rho \widehat{A}_{A\rho} \\ 0 & \widehat{e}^{\bar{a}}_\mu & 0 \\ 0 & \widehat{\Phi}^{\bar{A}}_B \widehat{A}^B_\mu & \widehat{\Phi}^{\bar{A}}_A \end{pmatrix}, \quad (5.68)$$

which is now associated with the following generalized metric:

$$\widehat{\mathcal{H}}_{IJ} = \begin{pmatrix} \widehat{g}^{\mu\nu} & -\widehat{g}^{\mu\rho} \widehat{c}_{\rho\nu} & -\widehat{g}^{\mu\rho} \widehat{A}_{A\rho} \\ -\widehat{g}^{\nu\rho} \widehat{c}_{\rho\mu} & \widehat{g}_{\mu\nu} + \widehat{A}^C_\mu \widehat{\mathcal{M}}_{CD} \widehat{A}^D_\nu + \widehat{c}_{\rho\mu} \widehat{g}^{\rho\sigma} \widehat{c}_{\sigma\nu} & \widehat{\mathcal{M}}_{AC} \widehat{A}^C_\mu + \widehat{A}_{A\rho} \widehat{g}^{\rho\sigma} \widehat{c}_{\sigma\mu} \\ -\widehat{g}^{\nu\rho} \widehat{A}_{B\rho} & \widehat{\mathcal{M}}_{BC} \widehat{A}^C_\nu + \widehat{A}_{B\rho} \widehat{g}^{\rho\sigma} \widehat{c}_{\sigma\nu} & \widehat{\mathcal{M}}_{AB} + \widehat{A}_{A\rho} \widehat{g}^{\rho\sigma} \widehat{A}_{B\sigma} \end{pmatrix}. \quad (5.69)$$

Here, we have introduced the combination $\widehat{c}_{\mu\nu} = \widehat{b}_{\mu\nu} + \frac{1}{2} \widehat{A}^B_\mu \widehat{A}_{B\nu}$. Also, \widehat{A}^A_μ are the vectors (5.13) and $\widehat{\Phi}^{\bar{A}}_A$ is the scalar bein for the scalar metric $\widehat{\mathcal{M}}_{AB}$ defined in (5.14). Note that we have run out of indices, so we are denoting with the same letter \bar{A} the *full* flat index and the *internal* one, and the distinction should be clear from the context. Also, due to the splitting, now the gaugings are only non-vanishing in the internal components

$$f_{IJK} = \begin{cases} f_{ABC} & (I, J, K) = (A, B, C) \\ 0 & \text{otherwise} \end{cases}. \quad (5.70)$$

Then, plugging (5.68) and (5.70) into (5.54), and taking into account that the indices are now raised and lowered with (5.67), we can readily identify some of the components of the compactified generalized fluxes with covariant quantities in the effective action (5.23)–(5.24), namely

$$\mathcal{F}_{\bar{a}\bar{b}\bar{c}} = \widehat{e}_a^\mu \widehat{e}_b^\nu \widehat{e}_c^\rho \mathcal{G}_{\mu\nu\rho}, \quad (5.71)$$

$$\mathcal{F}_{\bar{a}\bar{b}}^{\bar{C}} = \widehat{e}_a^\mu \widehat{e}_b^\nu \widehat{\Phi}^{\bar{C}}_C F^C_{\mu\nu}, \quad (5.72)$$

$$\mathcal{F}_{\bar{a}\bar{B}}^{\bar{C}} = \widehat{e}_a^\mu \widehat{\Phi}^{\bar{C}}_C D_\mu \widehat{\Phi}^{\bar{C}}_B, \quad (5.73)$$

where

$$D_\mu \widehat{\Phi}^{\bar{C}}_B = \partial_\mu \widehat{\Phi}^{\bar{C}}_B - f_{AB}^C \widehat{A}^A_\mu \widehat{\Phi}^{\bar{C}}_B \quad (5.74)$$

is the covariant derivative of the scalar bein. Finally, plugging (5.68) and (5.70) in the action (5.61) of GDFT, one recovers the effective action of gauged supergravity (5.22). Therefore, gauged supergravities are particular examples of GDFT.

5.6. Duality orbits of non-geometric fluxes

Even if it looks like that the generalized SS procedure discussed in sections 5.4 and 5.5 leads to the same action (5.22) obtained from the usual geometric SS compactification of section 5.1, this is not correct. The difference resides in the gaugings. While the geometric SS reduction only allows to turn on the fluxes (5.19) and the others (5.20) vanish, the generalized SS reduction of DFT allows, in principle, to turn on *all* the gaugings simultaneously.

We have defined the gaugings or fluxes in (5.56) in terms of a duality-valued twist matrix $U(\mathbb{Y}) \in O(n, n)$. This generalizes the usual SS gaugings in two ways.

- *Global extension.* The geometric SS gaugings are generated through u^a_m and v_{mn} in (5.19), which respectively correspond to the metric and 2-form background. They can both be combined into the $O(n, n)$ duality twist matrix in the form

$$U^A_M = \begin{pmatrix} u_a^m & u_a^n v_{nm} \\ 0 & u^a_m \end{pmatrix}. \tag{5.75}$$

A T-duality transformation would break the triangular form of this matrix into a new element of $O(n, n)$ containing a South–West component. Therefore, the only backgrounds that are allowed in the usual SS compactification of supergravity are those that come back to themselves under monodromies, up to u and/or v -transformations only, i.e. the *geometric* subgroup of the full $O(n, n)$. In a generalized SS compactification, we now allow the duality twist to be a generic element of $O(n, n)$. This includes, in addition to the elements u^a_m and $v_{[mn]}$, a new component usually dubbed $\beta^{[mn]}$:

$$U^A_M = \begin{pmatrix} u_a^m & u_a^n v_{nm} \\ u^a_n \beta^{nm} & u^a_m + u^a_n \beta^{np} v_{pm} \end{pmatrix}. \tag{5.76}$$

The effect of this extension is now to allow for backgrounds that come back to themselves under monodromies, up to a generic $O(n, n)$ transformation. This is the case of the T-folds discussed before. Then, the generalized SS compactification allows for new backgrounds that are globally ill defined from the usual (geometric) supergravity point of view.

- *Local extension.* The fluxes (5.56) are now not only defined in terms of an extended duality twist, but also in terms of a generalized derivative with respect to all coordinates. This would allow for more richness in the space of gaugings, if the duality twist violated the strong constraint. In this case, the dual coordinate dependence would make no sense from a supergravity point of view, altering the standard notion of locality.

Let us now show that the quadratic constraints (5.65) are weaker than the strong constraint. For the duality twist, the strong constraint implies

$$\tilde{\Omega}_{EAB} \tilde{\Omega}^E_{CD} = 0, \tag{5.77}$$

where $\tilde{\Omega}_{ABC}$ has been defined in (5.56). On the other hand, similar to the BI (3.64), one can show that

$$\partial_{[A} f_{BCD]} - \frac{3}{4} f_{[AB}{}^E f_{CD]E} = -\frac{3}{4} \tilde{\Omega}_{E[AB} \tilde{\Omega}^E_{CD]}. \tag{5.78}$$

For constant gaugings, the first term drops out, and then we see that the quadratic constraints correspond to a relaxed version of the strong constraint (5.77), because they only require the totally antisymmetric part of (5.77) to vanish.

This is, however, not the end of the story. One has to show that there exist solutions to the quadratic constraints that violate the strong constraint. As we explained, the gauged supergravities we are dealing with are half-maximal. Half-maximal gauged supergravities split into two different groups: those that can be obtained by means of a truncation of a maximal supergravity, and those that cannot (in $d = 4$, see [86]). The former inherit the quadratic constraints of the maximal theory, which in the language of the electric half-maximal gaugings take the form (for simplicity we take $\lambda = 0$ in (5.50))

$$f_{ABC} f^{ABC} = 3 \tilde{\Omega}_{ABC} \tilde{\Omega}^{ABC} = 0. \tag{5.79}$$

Therefore, the genuine half-maximal theories, which violate the above constraint, must be necessarily generated through a truly doubled duality twist [67]. On the other hand, when the strong constraint holds, the only reachable theories are those that admit an uplift to a maximal supergravity. In figure 2, we have pictured the kind of orbits that one finds in half-maximal supergravities (the case $d = 4$ should be analyzed separately due to the extra $SL(2)$ factor

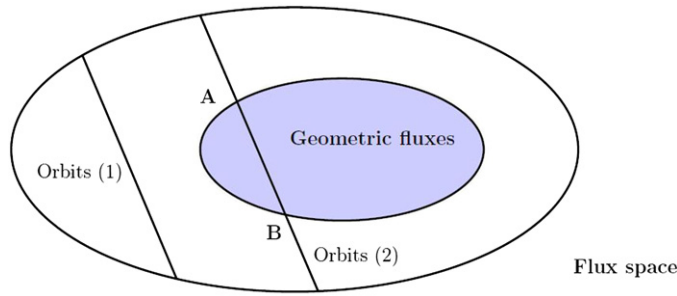


Figure 2. We picture the space of gaugings (or fluxes) in half-maximal supergravities in $d = 7, 8$ [67]. A point in this diagram corresponds to a given configuration. If two points lie in the same diagonal line (orbit), they are related by a duality transformation. Different theories are classified by orbits (lines) rather than configurations (points). The configuration space splits in a subgroup of geometric (i.e. only involving fluxes like H_{abc} and $\omega_{ab}{}^c$) and non-geometric (involving fluxes $Q_a{}^{bc}$ and R^{abc}) configurations. The space of orbits then splits into two: (1) non-geometric orbits (truly half-maximal) and (2) geometric orbits (basically maximal) that intersect the geometric space (between A and B).

[86]). Let us stress that the notion of non-geometry discussed in [67] is local, and then a duality orbit of non-geometric fluxes contains *all* fluxes simultaneously turned on H_{abc} , $\omega_{ab}{}^c$, $Q_a{}^{bc}$ and R^{abc} . However, one could also define a notion of globally non-geometric orbit, which could admit a representative without R^{abc} -flux.

The idea of combining geometric with non-geometric fluxes simultaneously is usually considered with some precaution. It is common to find objections against these configurations mostly based on scaling arguments. The non-geometric fluxes are sometimes associated with windings (since they are mostly generated through dual coordinate dependence), while the geometric ones are related to momentum. A quick look at the mass formula (3.1) shows that, for a given radius R , when momentum (winding) modes are heavy, the winding (momentum) modes are light. Considering both of them simultaneously then leads to unavoidable heavy modes in the spectrum. This enters in conflict with the fact that one is truncating the heavy mass levels of the string from the beginning. The conclusion of this argument is that one should then impose the strong constraint, so as to truncate the heavy part of the spectrum, and this in turn only permits geometric orbits.

Notice however that these arguments are purely based on the KK-mode expansions of fields on tori. The relation between (winding) momentum and (dual) coordinates, is given by the Fourier transforms of the KK-modes of the torus. Moreover, the mass formula (3.1) only holds for tori. In this section, we do not consider tori, and, moreover, we do not consider KK excitations. We only consider the zero-modes of the fields on twisted-double tori. The only connection between these two situations is when the duality twist is taken to be constant, in which case we would be dealing with the (massless) zero-modes on a torus, as discussed in section 5.2. When the twist matrix is non-constant, the effective theory becomes *massive*, but these masses are corrections to the *massless* modes on the torus through a twist. We are then correcting massless modes, through a procedure that truncates all the problematic (KK) modes. When dealing with moduli fixing, one has to make sure that for a given non-geometric orbit, the masses of the scalar fields (which are totally unrelated to (3.1)) in a given vacuum are small compared to the scales of the modes that we are neglecting.

A similar reasoning prevents us from relating the strong constraint (or a weaker version of it) to the LMC condition (3.9), as we explained in section 3.3. If we had considered tori

compactifications, and kept the tower of excited states, whenever the derivatives in strong-constraint like terms acted on the mode expansion they would form contractions like $\mathcal{P}^M \mathcal{P}_M$ related to the LMC. Here, we are considering the zero-modes, for which $\mathcal{P}^M \mathcal{P}_M = 0$ is trivially satisfied because $\mathcal{P}^M = 0$. Then, in SS compactifications in which the tower of KK modes is truncated, it does not seem to be correct to identify the LMC with strong-like constraints. Instead, the consistency constraints are given by the quadratic constraints for gaugings (5.65).

Combining fluxes and their derivatives, it is possible to construct three quantities that vanish upon imposition of the strong constraint [68]:

$$\partial_{[A} f_{BCD]} - \frac{3}{4} f_{[AB}{}^E f_{CD]E} = \mathcal{Z}_{ABCD}, \quad (5.80)$$

$$\partial^E f_{EAB} + 2\partial_{[A} f_{B]} - f^E f_{EAB} = \mathcal{Z}_{AB}, \quad (5.81)$$

$$\partial^E f_E - \frac{1}{2} f^E f_E + \frac{1}{12} f_{ABC} f^{ABC} = \mathcal{Z}. \quad (5.82)$$

They correspond to duality orbits of *generalized BI* for all the dual fluxes. The first two, (5.80) and (5.81), are related to the constraints of the theory, and are obtained from compactifications of (3.64). The last one (5.82) was associated with the embedding of the theory into a maximal theory (5.79).

The fluxes f_{ABC} encode the standard T-dual fluxes, as we reviewed. This can be seen by splitting the indices as

$$f_{abc} = H_{abc}, \quad f^a{}_{bc} = \omega_{bc}{}^a, \quad f^{ab}{}_c = Q_c{}^{ab}, \quad f^{abc} = R^{abc}. \quad (5.83)$$

Through T-dualities they are related according to the chain (5.49). Recall that we are only dealing here with the case $f_A = 0$. Splitting in components equation (5.80), we find

$$\begin{aligned} \partial_{[a} H_{bcd]} - \frac{3}{2} H_{e[ab} \omega_{cd]}{}^e &= \mathcal{Z}_{abcd}, \\ 3\partial_{[a} \omega_{bc]}{}^d - \partial^d H_{abc} + 3\omega_{[ab}{}^e \omega_{c]e}{}^d - 3Q_{[a}{}^{de} H_{bc]e} &= \mathcal{Z}_{abc}{}^d, \\ 2\partial_{[a} Q_{b]}{}^{cd} + 2\partial^{[c} \omega_{ab]}{}^{d]} - \omega_{ab}{}^e Q_e{}^{cd} - H_{abe} R^{ecd} + 4Q_{[a}{}^{e[c} \omega_{b]e}{}^{d]} &= \mathcal{Z}_{ab}{}^{cd}, \\ 3\partial^{[a} Q_d{}^{bc]} - \partial_d R^{abc} + 3Q_e{}^{[ab} Q_d{}^{c]e} - 3\omega_{de}{}^{[a} R^{bc]e} &= \mathcal{Z}^{abc}{}_d, \\ \partial^{[a} R^{bcd]} - \frac{3}{2} R^{e[ab} Q_e{}^{cd]} &= \mathcal{Z}^{abcd}. \end{aligned} \quad (5.84)$$

These reduce to those of [59] for constant fluxes under the strong constraint, and to those of [87] for non-constant fluxes. Equation (5.82), on the other hand, reads in components

$$\frac{1}{6} H_{abc} R^{abc} + \frac{1}{2} \omega_{ab}{}^c Q_c{}^{ab} = \mathcal{Z}. \quad (5.85)$$

This corresponds to an orthogonality condition between geometric ($H_{abc}, \omega_{ab}{}^c$) and non-geometric ($Q_a{}^{bc}, R^{abc}$) fluxes, as expected. And this is the reason why the failure of this equation to vanish *requires* non-geometric fluxes. Therefore, (5.85) can be used to classify duality orbits of non-geometric fluxes.

Using the extended parameterization of the twist matrix (5.76)

$$U^A{}_M = \begin{pmatrix} u_a{}^m & u_a{}^n v_{nm} \\ u^a{}_n \beta^{nm} & u^a{}_m + u^a{}_n \beta^{np} v_{pm} \end{pmatrix}, \quad (5.86)$$

and inserting this into the definition of the fluxes (5.56) and (5.70)

$$f_{ABC} = 3 U_{[A]}{}^M \partial_M U_{|B}{}^N U_{C]N}, \quad (5.87)$$

we find in components

$$\begin{aligned} H_{abc} &= 3[\nabla_{[a} v_{bc]} - v_{d[a} \tilde{\nabla}^d v_{bc]}], \\ \omega_{ab}{}^c &= 2\Gamma_{[ab]}{}^c + \tilde{\nabla}^c v_{ab} + 2\Gamma^{mc}{}_{[a} v_{b]m} + \beta^{cm} H_{mab}, \\ Q_c{}^{ab} &= 2\Gamma^{[ab]}{}_c + \partial_c \beta^{ab} + v_{cm} \tilde{\nabla}^m \beta^{ab} + 2\omega_{mc}{}^{[a} \beta^{b]m} - H_{mnc} \beta^{ma} \beta^{nb}, \\ R^{abc} &= 3[\beta^{[am} \nabla_m \beta^{bc]} + \tilde{\nabla}^c \beta^{bc}] + v_{mn} \tilde{\nabla}^n \beta^{[ab} \beta^{c]m} + \beta^{[am} \beta^{bn} \tilde{\nabla}^c] v_{mn}] + \beta^{am} \beta^{bn} \beta^{cl} H_{mnl}, \end{aligned} \quad (5.88)$$

where we have used the following relations and definitions:

$$u_a^m u_n^a = \delta_n^m, \quad u_a^m u_b^m = \delta_a^b, \quad v_{ab} = u_a^m u_b^n v_{mn}, \quad \beta^{ab} = u_a^m u_b^n \beta^{mn},$$

$$\begin{aligned} \partial_a &= u_a^m \partial_m, & \tilde{\partial}^a &= u_a^m \tilde{\partial}^m, \\ \nabla_a v_{bc} &= \partial_a v_{bc} - \Gamma_{ab}^d v_{dc} - \Gamma_{ac}^d v_{bd}, & \tilde{\nabla}^a v_{bc} &= \tilde{\partial}^a v_{bc} + \Gamma_{bc}^{ad} v_{dc} + \Gamma_{ac}^{ad} v_{bd}, \\ \nabla_a \beta^{bc} &= \partial_a \beta^{bc} + \Gamma_{ad}^b \beta^{dc} + \Gamma_{ad}^c \beta^{bd}, & \tilde{\nabla}^a \beta^{bc} &= \tilde{\partial}^a \beta^{bc} - \Gamma_{ab}^d \beta^{dc} - \Gamma_{ad}^c \beta^{bd}, \end{aligned}$$

and

$$\Gamma_{ab}^c = u_a^m \partial_m u_b^n e_n^c, \quad \Gamma^{ab}_c = u_a^m \tilde{\partial}^m u_b^n u_c^n. \quad (5.89)$$

Expressions (5.88) are very useful to explore the uplifting of fluxes to higher dimensional theories. Note that while H_{abc} and ω_{ab}^c can be generated through geometric twists u_a^m and v_{mn} , the non-geometric fluxes Q_a^{bc} and/or R^{abc} require β -twists and/or dual coordinate dependence. This also serves to show that the distinction between ‘globally’ and ‘locally’ non-geometric fluxes is just a terminology inherited from the toy example discussed before, since Q_a^{bc} can arise from dual coordinate dependence, and R^{abc} can arise from globally non-geometric compactifications with non-trivial β -twist. Setting $\beta^{ij} = 0$ and $\tilde{\partial}^i = 0$, expressions (5.88) reduce to (5.26).

6. U-duality and extended geometry

The compactification of $D = 11$ supergravity and M-theory on an n -dimensional torus enjoys a U-duality symmetry $E_{n(n)}$ (see for example [88–90]). The idea of extending the spacetime and/or the tangent space so as to accommodate such symmetries was introduced in [38, 39] and more recently considered in [40, 44, 46]. In this section, we review some of the approaches to replace the T-duality group by the U-duality group, in order to incorporate all the extra fields (like R-R in Type II theories or the three-form of M-theory) in a duality covariant manner, much under the same philosophy as that of DFT.

6.1. Generalized diffeomorphisms and the section condition

We have seen that the generalized diffeomorphisms of DFT (3.39) discussed in the previous sections enjoy the following properties.

- They preserve the duality group invariant, in that case the $O(D, D)$ metric η_{MN} .
- They are defined in terms of an invariant Y -tensor related to the definition of the strong constraint.
- They reproduce the gauge transformations of the D -dimensional metric and 2-form, upon application of the strong constraint.
- When ‘twisted’, they give rise to fluxes or gaugings in the representations allowed by supersymmetry.
- Their closure imposes a set of constraints that, on the one hand are solved by the strong constraint, and on the other, reproduce the quadratic constraints of the supergravity gaugings upon ‘twisting’.

Clearly, the inclusion of other fields, like R-R sector in Type II string theory or the more general 3-form of M-theory, requires an enlargement and further generalization of the already generalized Lie derivative, to be compatible now with U-duality. All the transformation properties of gravitational and tensorial degrees of freedom, which mix under U-duality, must now be accommodated (and unified) in a new generalized Lie derivative. We will see

Table 3. Some relevant representations of U-duality groups [45].

	R_1	R_2
$E_{4(4)} = SL(5)$	10	$\bar{\mathbf{5}}$
$E_{5(5)} = SO(5, 5)$	16	10
$E_{6(6)}$	27	$\bar{\mathbf{27}}$
$E_{7(7)}$	56	133

that such a generalized transformation enjoys the U-duality extension of the properties listed above.

This generalization is a little more involved since the U-duality group jumps with dimension. For the n internal dimensions of M-theory, it corresponds to exceptional groups $E_{n(n)}$, and in $n > 8$ one encounters the complication of infinite-dimensional Kac–Moody-type algebras. Given the disconnected structure of the groups for different dimensions, it is convenient to work case by case. As in DFT, where the space is doubled to account for the winding degrees of freedom of the string, here the space is further enlarged to account for the wrapping states of M -branes. The internal space is then replaced by an *extended* mega-space with extended dimensions, and here for simplicity we neglect the external spacetime. Relevant representations for the different U-duality groups are given in table 3. The mega-space associated with each of them is $\dim(R_1)$ dimensional.

The generalized diffeomorphisms [47, 44] formally preserve the structure of those analyzed before for DFT (we are using the notation of [45])

$$\mathcal{L}_\xi V^M = L_\xi V^M + Y^M{}_{N^P}{}_Q \partial_P \xi^Q V^N, \tag{6.1}$$

where Y is a U-duality invariant tensor ‘measuring’ the departure from the usual Lie derivative. It can be generically decomposed as

$$Y^M{}_{N^P}{}_Q = \delta_Q^M \delta_N^P - \alpha P_{(adj)}{}^M{}_{N^P}{}_Q + \beta \delta_N^M \delta_Q^P, \tag{6.2}$$

where $P_{(adj)}$ is a projector to the adjoint representation of the U-duality group, α is a group-theoretical quantity that depends on the dimension and β is a weight for tensorial densities that also depends on the group. Indices M and N are in the R_1 reps of table 3, and $P_{(adj)}$ corresponds to the adjoint projection contained in the tensor product $R_1 \otimes \bar{R}_1$. It can be checked that these generalized Lie derivatives preserve the invariants of each group. The appearance of the last β -term is due to the fact that in the U-duality case one usually considers $E_{n(n)} \times \mathbb{R}^+$ tensorial densities rather than just tensors (we will be more specific later).

Before showing the general results, to warm up let us first see how the DFT $O(n, n)$ case fits in this language. The projector to the adjoint representation is given by

$$O(n, n): P_{(adj)}{}^M{}_{N^P}{}_Q = \frac{1}{2}(\delta_Q^M \delta_N^P - \eta^{MP} \eta_{NQ}), \tag{6.3}$$

and then, introducing this in (6.2) and comparing with (3.31), we find that the correct value of the proportionality constants is given by $(\alpha, \beta) = (2, 0)$ for unweighted tensors. More generally, the expression for Y in the different duality groups is given in table 4.

In the U-duality case, there is also an analogue of the strong constraint, also known as the section condition [47, 44]:

$$P_{(R_2)MN}{}^{PQ} \partial_P \partial_Q (\dots) = 0, \tag{6.4}$$

which again acts on any product of fields and gauge parameters. Generically, any solution to this condition picks out an n -dimensional subspace of the mega-space, which can be associated with the physical space in M-theory compactifications. Again, when analyzing the closure of

Table 4. Invariant Y -tensor and proportionality constants for different dimensions. Here η_{MN} is the $O(n, n)$ invariant metric, ϵ_{iMN} is the $SL(5)$ alternating tensor, $(\gamma^i)^{MN}$ are 16×16 MW representations of the $SO(5, 5)$ Clifford algebra, d^{MNR} and K^{MNPQ} are the symmetric invariant tensors of $E_{6(6)}$ and $E_{7(7)}$, respectively, and ϵ^{MN} is the symplectic invariant in $E_{7(7)}$. These results were taken from [45]; we refer to that paper for more details.

	$Y^M{}_Q{}^N{}_P$	α	β
$O(n, n)$	$\eta^{MN}\eta_{PQ}$	2	0
$E_{4(4)} = SL(5)$	$\epsilon^{iMN}\epsilon_{iPQ}$	3	$\frac{1}{5}$
$E_{5(5)} = SO(5, 5)$	$\frac{1}{2}(\gamma^i)^{MN}(\gamma_i)_{PQ}$	4	$\frac{1}{4}$
$E_{6(6)}$	$10d^{MNR}\bar{d}_{PQR}$	6	$\frac{1}{3}$
$E_{7(7)}$	$12K^{MN}{}_{PQ} + \delta_P^{(M}\delta_Q^{N)} + \frac{1}{2}\epsilon^{MN}\epsilon_{PQ}$	12	$\frac{1}{2}$

these generalized diffeomorphisms, one finds that the closure is achieved automatically when restricted to configurations satisfying the section condition.

Analogous to the DFT case, when a duality twist reduction of these generalized Lie derivatives is performed, they induce an effective gauge transformation giving rise to the embedding tensor components of the different maximal gauged supergravities for different dimensions. The analogies do not stop here, since it is also possible to construct an extended geometrical formalism, introducing generalized connections, torsion and covariant Ricci-like tensors for these generalized transformations. Let us now review how this works, specializing to the $E_{7(7)}$ case for a detailed exposition.

The cases $n \leq 7$ were studied in [44] and [45]. The cases $n \leq 2$ just reduce to the ordinary Riemannian geometry. The case $n = 4$ was studied in [53] in the context of gauged supergravities, and a geometry for it was considered in [49]. The cases $n = 5$ and 6 were related to SS compactifications in [54], and $n = 7$ in [55], where an extended geometry was also formulated. A unified geometric description for $n \leq 7$ was considered in [50]. The case $n = 8$ was explored in [51], and for $n > 8$ the groups are much more involved. An ambitious programme intended to encompass all formulations under E_{11} can be found in [41].

6.2. The $E_{7(7)}$ case and maximal gauged supergravity

$E_{7(7)}$ is the U-duality group of gauged maximal supergravity in four dimensions [91], the ungauged theory being obtained through compactifications of M-theory on a 7-torus [92]. The idea here is to replace the internal 7-space by a 56-dimensional mega-space, and accommodate the internal degrees of freedom in a generalized metric \mathcal{H}_{MN} defined on the mega-space. This idea was first considered in [62], and here we will present the results of [55]. Since we are ignoring the four-dimensional spacetime, the generalized metric should be identified with the scalar degrees of freedom of the gauged supergravity. The generalized metric transforms covariantly under $G = E_{7(7)} \times \mathbb{R}^+$ and is invariant under the maximal compact subgroup $H = SU(8)$. It can be written in terms of a generalized bein $E_{\bar{A}}{}^M$ taking values in the quotient G/H :

$$E_{\bar{A}}{}^M = e^{-\Delta} \tilde{E}_{\bar{A}}{}^M, \tag{6.5}$$

where we have separated a conformal factor Δ corresponding to the \mathbb{R}^+ components. Here the flat indices \bar{A}, \bar{B}, \dots take values in H and the curved ones M, N, \dots in G . Then, $\tilde{E}_{\bar{A}}{}^M$ lives in the quotient $E_{7(7)}/SU(8)$, and the tilde refers to the $E_{7(7)}$ part of G only. G has a (weighted) symplectic invariant ω_{MN} that raises and lowers indices, and a quartic invariant K_{MNPQ} which is totally symmetric. The fundamental representation of $E_{7(7)}$ is **56** and the adjoint is **133**.

Given a tensorial density V^M , the generalized Lie derivative (or equivalently the exceptional Dorfman bracket) reads

$$\mathcal{L}_\xi V^M = \xi^P \partial_P V^M - 12P_{(adj)}{}^M{}_{N^P}{}_Q \partial_P \xi^Q V^N - \frac{1}{2} \partial_P \xi^P V^M. \quad (6.6)$$

Here, $P_{(adj)((MN)(PQ))}$ is the projector to the adjoint representation, defined in terms of the $E_{7(7)}$ invariants:

$$P_{(adj)MNPQ} = (t_\alpha)_{NM} (t^\alpha)_{PQ} = \frac{1}{12} \omega_{M(P} \omega_{Q)N} + K_{MNPQ}. \quad (6.7)$$

As we did before in the DFT case, here we can compute the closure of these transformations

$$\Delta_{123}{}^M = -\Delta_{\xi_1} \mathcal{L}_{\xi_2} \xi_3^M = ([\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] - \mathcal{L}_{\mathcal{L}_{\xi_1} \xi_2}) \xi_3^M = 0 \quad (6.8)$$

and obtain

$$\begin{aligned} \Delta_{[12]3}{}^M &= Y^Q{}_L{}^O{}_I \partial_O \xi_{[2}^I \xi_{1]}^L \partial_Q \xi_3^M + A^M{}_N{}^J{}_L Y^Q{}_J{}^O{}_I \partial_Q \xi_{[2}^I \partial_O \xi_{1]}^L \xi_3^N + Q^M{}_N{}^{QO}{}_{LI} \partial_Q \partial_O \xi_{[2}^I \xi_{1]}^L \xi_3^N \\ &= 0, \\ \Delta_{(12)3}{}^M &= -Y^Q{}_L{}^O{}_I \partial_Q \xi_{(1}^I \xi_{2)}^L \partial_O \xi_3^M + Q^M{}_N{}^{QO}{}_{LI} \partial_Q (\xi_{(1}^L \partial_O \xi_{2)}^I) \xi_3^N + \frac{1}{4} \omega_{LI} \omega^{QO} \partial_Q \xi_1^L \partial_O \xi_2^I \xi_3^M \\ &= 0, \end{aligned} \quad (6.9)$$

where we have defined

$$Q^M{}_N{}^{QO}{}_{LI} = Y^Q{}_J{}^O{}_{(L} A^J{}_{I)}{}^M{}_N + \frac{1}{2} \omega_{IL} Y^{QMO}{}_N - \frac{1}{2} Y^Q{}_L{}^O{}_I \delta_N^M. \quad (6.10)$$

Note that all derivatives are contracted as

$$Y^M{}_P{}^N{}_Q \partial_M \partial_N = \left(\frac{1}{2} \omega^{MN} \omega_{PQ} - 12P_{(adj)PQ}{}^{MN} \right) \partial_M \partial_N. \quad (6.11)$$

As we mentioned in the previous section, when the so-called section condition (6.4)

$$P_{(adj)PQ}{}^{MN} \partial_M \partial_N (\dots) = 0 \quad (6.12)$$

is imposed, the closure condition $\Delta_{123}{}^M = 0$ is automatically satisfied. In fact, it can be seen that any solution to this condition selects a seven-dimensional subspace of the full 56-dimensional mega-space, permitting us to make contact with the physical internal compact directions. When (6.12) holds, it can also be proven that $\omega^{MN} \partial_M \partial_N (\dots) = 0$, and therefore also $Y^M{}_P{}^N{}_Q \partial_M \partial_N (\dots) = 0$, in analogy with DFT (3.30).

Following the DFT logic (3.57), we can now define a dynamical flux

$$\mathcal{L}_{E_{\bar{A}}} E_{\bar{B}} = F_{\bar{A}\bar{B}}{}^{\bar{C}} E_{\bar{C}}, \quad (6.13)$$

with

$$F_{\bar{A}\bar{B}}{}^{\bar{C}} = \Omega_{\bar{A}\bar{B}}{}^{\bar{C}} - 12P_{(adj)}{}^{\bar{C}}{}_{\bar{B}}{}^{\bar{D}}{}_{\bar{E}} \Omega_{\bar{D}\bar{A}}{}^{\bar{E}} + \frac{1}{2} \Omega_{\bar{D}\bar{A}}{}^{\bar{D}} \delta_{\bar{B}}{}^{\bar{C}}, \quad (6.14)$$

where

$$\Omega_{\bar{A}\bar{B}}{}^{\bar{C}} = E_{\bar{A}}{}^M \partial_M E_{\bar{B}}{}^N (E^{-1})_N{}^{\bar{C}} \quad (6.15)$$

is the G -generalized Weitzenböck connection. Rotating these expressions with the bein, we can define the fluxes with curved indices

$$F_{MN}{}^P = \Omega_{MN}{}^P - 12P_{(adj)}{}^P{}_{N^R}{}_S \Omega_{RM}{}^S + \frac{1}{2} \Omega_{RM}{}^R \delta_N^P, \quad (6.16)$$

and the corresponding Weitzenböck connection in curved indices takes values in the algebra of G :

$$\Omega_{MN}{}^P = -\partial_M \Delta \delta_N^P + \tilde{\Omega}_{MN}{}^P = \Omega_M^0 (t_0)_N{}^P + \tilde{\Omega}_M{}^\alpha (t_\alpha)_N{}^P. \quad (6.17)$$

Here, $(t_0)_N{}^P = -\delta_N^P$ is the generator of \mathbb{R}^+ . The 56×133 part

$$\tilde{\Omega}_{MN}{}^P = (\tilde{E}^{-1})_N{}^{\bar{B}} \partial_M \tilde{E}_{\bar{B}}{}^P \quad (6.18)$$

contains the irreducible representations $\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480}$. The projectors onto the first two representations in this product are given by [93]

$$\begin{aligned} P_{(\mathbf{56})M}{}^{\alpha, N}{}_{\beta} &= \frac{56}{133}(t^{\alpha}t_{\beta})_M{}^N, \\ P_{(\mathbf{912})M}{}^{\alpha, N}{}_{\beta} &= \frac{1}{7}\delta_{\beta}^{\alpha}\delta_M^N - \frac{12}{7}(t_{\beta}t^{\alpha})_M{}^N + \frac{4}{7}(t^{\alpha}t_{\beta})_M{}^N. \end{aligned} \quad (6.19)$$

Equations (6.16)–(6.19) imply that the fluxes are in the $\mathbf{912}$ and $\mathbf{56}$ representations only. More precisely,

$$F_{MN}{}^P = X_{MN}{}^P + D_{MN}{}^P, \quad (6.20)$$

with

$$X_{MN}{}^P = \Theta_M{}^{\alpha}(t_{\alpha})_N{}^P \quad \text{with} \quad \Theta_M{}^{\alpha} = 7P_{(\mathbf{912})M}{}^{\alpha, N}{}_{\beta} \tilde{\Omega}_N{}^{\beta}, \quad (6.21)$$

and

$$D_{MN}{}^P = -\vartheta_M\delta_N^P + 8P_{(adj)}{}^P{}_{N}{}^O{}_{M}\vartheta_O, \quad \vartheta_M = -\frac{1}{2}(\tilde{\Omega}_{PM}{}^P - 3\partial_M\Delta). \quad (6.22)$$

The fluxes F involve therefore a projection onto the $\mathbf{912}$ given by the gaugings $X_{MN}{}^P$ plus contributions from the gaugings ϑ_M . As in the DFT case, in the language of gauged supergravity, they correspond to the gauge group generators, i.e. they are contractions of the embedding tensor with the generators of the global symmetry group. For this reason, we will sometimes call them ‘gaugings’. The $X_{MN}{}^P$ piece in (6.20) corresponds to the $\mathbf{912}$ component of the fluxes, satisfying the properties

$$X_{M[NP]} = X_{MP}{}^P = X_{(MNP)} = X_{PM}{}^P = 0, \quad (6.23)$$

which are the well-known conditions satisfied by gaugings in maximal supergravity. The $D_{MN}{}^P$ piece (6.22), on the other hand, contains two terms: one belongs to the $\mathbf{56}$ associated with \mathbb{R}^+ and the other one belongs to the $\mathbf{56}$ in $\mathbf{56} \times \mathbf{133}$. Note, however, that both terms contain the same degrees of freedom in terms of ϑ_M and are therefore not independent. With these results, we are able to express the gauge group generators $(F_M)_N{}^P$ as in [94]

$$F_M = \vartheta_M t_0 + (\Theta_M{}^{\alpha} + 8\vartheta_P(t^{\alpha})_M{}^P)t_{\alpha}. \quad (6.24)$$

In terms of $F_{\bar{A}\bar{B}}{}^{\bar{C}}$, the closure conditions (6.9) evaluated on frames read

$$\begin{aligned} \Delta_{\bar{A}\bar{B}\bar{C}}{}^{\bar{D}} &= -\left([F_{\bar{A}}{}^{\bar{C}}, F_{\bar{B}}{}^{\bar{C}}] + F_{\bar{A}\bar{B}}{}^{\bar{E}}F_{\bar{E}}{}^{\bar{C}}\right)_{\bar{C}}{}^{\bar{D}} \\ &\quad - 2\partial_{[\bar{A}}F_{\bar{B}]}{}^{\bar{C}}{}^{\bar{D}} - 12P_{(adj)}{}^{\bar{D}}{}_{\bar{C}}{}^{\bar{E}}{}_{\bar{F}}\partial_{\bar{E}}F_{\bar{A}\bar{B}}{}^{\bar{F}} + \frac{1}{2}\partial_{\bar{E}}F_{\bar{A}\bar{B}}{}^{\bar{E}}\delta_{\bar{C}}{}^{\bar{D}} = 0. \end{aligned} \quad (6.25)$$

When the fluxes are constant, we recover the quadratic constraints of maximal gauged supergravity. Note that, as it happens in DFT, these constraints can be satisfied through configurations that violate the section condition. This implies necessarily going beyond supergravity, and then gives rise to a novel description of non-geometry in maximal supergravity. This might be useful, for instance, to find an extended geometrical uplift of the new $SO(8)$ gaugings [96], which seem to find obstructions when it comes to uplifts to $D = 11$ supergravity [97]. Conditions (6.25), in turn, imply that the dynamical fluxes in flat indices behave as scalars under the following generalized diffeomorphisms with respect to frame vectors:

$$\delta_{\xi}F_{\bar{A}\bar{B}}{}^{\bar{C}} = \xi^{\bar{D}}\partial_{\bar{D}}F_{\bar{A}\bar{B}}{}^{\bar{C}} + \xi^{\bar{D}}\Delta_{\bar{D}\bar{A}\bar{B}}{}^{\bar{C}}. \quad (6.26)$$

We can now proceed as in the DFT case and look for a geometric construction that gives the *action* from traces of some generalized Ricci tensor. Of course, since we only deal with scalars here, the action will be the scalar potential of the maximal theory. Having defined the generalized notion of the Lie derivative in (6.6), it is natural to look for derivatives that behave

covariantly under such transformations. We begin by defining the covariant derivative of a bein E_A^M as

$$\nabla_M E_A^N = \omega_{M\bar{A}}^{\bar{B}} E_{\bar{B}}^N = \partial_M E_A^N + \Gamma_{MP}^N E_A^P, \quad (6.27)$$

in terms of a Christoffel connection Γ or alternatively a spin connection ω . They are related to the Weitzenböck connection defined in (6.15), which takes values in the algebra of G . In addition, one can relate the gaugings to the Weitzenböck connection through projections, as in equation (6.14). These connections must also transform properly so as to compensate the failure of the derivative to transform as a tensor. Given the fact that the covariant derivative is requested to transform covariantly, so must the spin connection.

We can define the generalized torsion through [44]

$$\mathcal{T}_{\bar{A}\bar{B}}^{\bar{C}} \equiv (E^{-1})_M^{\bar{C}} (\mathcal{L}_{E_{\bar{A}}}^{\nabla} - \mathcal{L}_{E_{\bar{A}}}) E_{\bar{B}}^M, \quad (6.28)$$

where \mathcal{L}^{∇} is defined as in (6.6), but with a partial replaced by a covariant derivative. Using (6.27), we arrive at

$$\mathcal{T}_{\bar{A}\bar{B}}^{\bar{C}} = \omega_{\bar{A}\bar{B}}^{\bar{C}} - 12P_{(adj)}^{\bar{C}}{}_{\bar{B}}{}^{\bar{P}} \bar{Q} \omega_{\bar{P}\bar{A}}^{\bar{Q}} + \frac{1}{2} \omega_{D\bar{A}}^{\bar{D}} \delta_{\bar{B}}^{\bar{C}} - F_{\bar{A}\bar{B}}^{\bar{C}}. \quad (6.29)$$

Since $\sqrt{\mathcal{H}}$ does not transform as a density under the generalized diffeomorphisms (6.1), the proper measure is given by $(\sqrt{\mathcal{H}})^{-1/28} = e^{-2\Delta}$, since

$$\delta_{\xi} e^{-2\Delta} = \partial_P (e^{-2\Delta} \xi^P). \quad (6.30)$$

This can be used to impose compatibility with the determinant of the generalized metric, and together with vanishing torsion they determine the spin connection (which lives in $\mathbf{56} \times \mathbf{133}$) up to a **6480** piece. This piece remains undetermined under these conditions, but a part of it (corresponding to the **63** in $\mathbf{133} = \mathbf{63} + \mathbf{70}$) can be fixed through metric compatibility.

It can then be shown that a torsionless and metric compatible spin connection has in particular the following determined components:

$$\begin{aligned} W_{PM}{}^P &= -2\vartheta_M \\ P_{(\mathbf{912})QR}{}^S, {}^{MN}{}_P W_{MN}{}^P &= \frac{1}{7} X_{QR}{}^S \\ P_{(\mathbf{56})QR}{}^S, {}^{MN}{}_P W_{MN}{}^P &= -\frac{16}{19} P_{(adj)}{}^S{}_R{}^T \vartheta_T, \end{aligned} \quad (6.31)$$

where the projectors are those of (6.19) contracted with the $E_{7(7)}$ generators. This is analogous to (4.37) and (4.40) in DFT, where the projections there simply amounted to tracing and antisymmetrizing the spin connection.

Finally, following the DFT geometrical construction, a generalized Ricci tensor can be constructed [44] (unlike the DFT case, the definition of a Riemann tensor is less clear)

$$R_{MN} = \frac{1}{2} (R_{MN} + R_{NM} + \Gamma_{RM}{}^P Y_P{}^R{}_S \Gamma_{SN}{}^Q - \Omega_{RM}{}^P Y_P{}^R{}_S \Omega_{SN}{}^Q) \quad (6.32)$$

which is covariant for solutions to the closure constraints. When tracing it with the generalized metric, we can then define a generalized Ricci scalar

$$\mathcal{R} = \frac{1}{4} \mathcal{H}^{MN} \mathcal{R}_{MN} \quad (6.33)$$

which, for any torsionless and metric compatible connection, can be cast in the form

$$\mathcal{R} = \frac{1}{672} (X_{MN}{}^P X_{QR}{}^S \mathcal{H}^{MQ} \mathcal{H}^{NR} \mathcal{H}_{PS} + 7 \mathcal{H}^{MN} X_{MP}{}^Q X_{NQ}{}^P) \quad (6.34)$$

provided the gaugings $\vartheta_M = 0$. Remarkably, this is exactly the scalar potential of maximal supergravity with the very exact overall factor. The relative factor 7 is related to that in (6.31) and can be traced back to the fact that the generalized Lie derivatives are consistent with supersymmetry, in that they generate fluxes in accordance with the linear constraints of the maximal theory.

Finally, note that the Ricci scalar (6.34) can be combined with the measure (6.30) to render a gauge invariant action

$$S = \int dX e^{-2\Delta} \mathcal{R}. \quad (6.35)$$

We have left some important points uncovered here. One of them is the coupling between this ‘scalar’ internal sector with the rest of the theory, i.e. the external spacetime, its metric and vectors (plus the p -form hierarchy of maximal supergravities [95]). The other one is the relation between this setup with string or M-theory degrees of freedom. To establish the correspondence, one then has to provide a proper parameterization of the generalized bein of the generalized metric. Work in this direction was addressed in [40, 44, 46].

7. Worldsheet motivations and approaches to DFT

As discussed in the previous sections, DFT was formulated with the purpose of incorporating T-duality, an essentially *stringy* effect, into a *particle* field theory. Clearly, it would be interesting to deduce DFT from a worldsheet action, much in the same way as supergravity is obtained as the low-energy effective field theory from the two-dimensional description of the string dynamics. In this section, we briefly review some of the attempts that have been followed to construct an $O(D, D)$ covariant two-dimensional worldsheet theory, from which DFT might be explicitly derived.

Before proceeding to DFT, we briefly recall the process leading from the worldsheet theory to supergravity. We refer the reader to the string theory books [4] and references therein for a more detailed and complete discussion of this issue. We then discuss how the procedure has been implemented for DFT.

7.1. The string spacetime action

Strings propagating in backgrounds of massless closed string states are described by an interacting two-dimensional field theory, obtained by exponentiating the vertex operators creating those states. The action is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} [(h^{ab} g_{ij}(x) + i\epsilon^{ab} b_{ij}(x)) \partial_a x^i \partial_b x^j + \alpha' \mathbf{R}\phi(x)]. \quad (7.1)$$

This is a nonlinear sigma model where α' is the square of the string length scale, σ^a , $a = 0, 1$ refer to the worldsheet coordinates τ and σ , respectively, g_{ij} is the spacetime metric, b_{ij} is the antisymmetric tensor, the dilaton involves ϕ and the trace of g_{ij} , and \mathbf{R} is the curvature scalar of the worldsheet.

A consistent string theory requires the two-dimensional quantum field theory to have a local Weyl and Lorentz invariance. This implies that the trace and ϵ^{ab} contraction of the energy–momentum tensor, respectively, should vanish on-shell, which imposes rather non-trivial conditions on the admissible background fields. Actually, to regulate divergences in a quantum theory, one has to introduce a UV cut-off, and therefore, physical quantities typically depend on the scale of a given process after renormalization. Conformal invariance is achieved if the coupling constants do not depend on the scale of the theory. In this case, the couplings are g , b and ϕ and the scale dependence is described by the β -functions of the renormalization group.

The β -functions are computed perturbatively. One first expands the fields $x^i(\tau, \sigma)$ around a classical solution $x^i = x_{cl}^i + \pi^i$, where π^i is the quantum fluctuation. The expansion of the Lagrangian then gets quadratic kinetic terms plus interactions of the fluctuations. The theory

has an infinite number of coupling constants: all order derivatives of the background fields evaluated at x_{cl}^i . When all the couplings are small, the theory is then weakly coupled. Assuming the target space has a characteristic length scale R_c , the effective dimensionless couplings are of the order $\alpha'^{1/2}R_c^{-1}$, and then perturbation theory makes sense if R_c is much greater than the string scale. Up to terms involving two spacetime derivatives, the β -functions are given by

$$\begin{aligned}\beta_{ij}^g &= \alpha' R_{ij} + 2\alpha' \nabla_i \nabla_j \phi - \frac{\alpha'}{4} H_{ikl} H_j^{kl} + O(\alpha'), \\ \beta_{ij}^b &= -\frac{\alpha'}{2} \nabla^k H_{kij} + \alpha' H_{kij} \nabla^k \phi + O(\alpha'^2), \\ \beta^\phi &= \frac{D - D_{\text{crit}}}{4} - \frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_k \phi \nabla^k \phi - \frac{\alpha'}{24} H_{ijk} H^{ijk} + O(\alpha'^2),\end{aligned}\quad (7.2)$$

where R_{ij} is the spacetime Ricci tensor, to be distinguished from the worldsheet Ricci tensor \mathbf{R}_{ab} . Terms with more derivatives are of higher order in $\alpha'^{1/2}R_c^{-1}$. Combining (7.2), one then recovers (3.19). The term $D - D_{\text{crit}}$ in β^ϕ is the classical Weyl anomaly, which vanishes in the critical dimension $D_{\text{crit}} = 26$ ($D_{\text{crit}} = 10$) in (super)string theory in flat spacetime, ensuring that the negative norm states decouple.

The vanishing β -function equations, determining the Weyl invariance and UV finiteness of the theory, can be interpreted as the equations of motion derived from the following spacetime action:

$$S = \int d^D x \sqrt{g} e^{-2\phi} \left[-\frac{2(D - D_{\text{crit}})}{3\alpha'} + R - \frac{1}{12} H_{ijk} H^{ijk} + 4\partial_i \phi \partial^i \phi + O(\alpha') \right]. \quad (7.3)$$

We recognize here the action for the bosonic universal gravity sector introduced in (3.17).⁷

This action can alternatively be obtained from the low-energy limit of scattering amplitudes of massless string modes. Low energies here refer to energies much smaller than the string scale, i.e. $E \ll (\alpha')^{-\frac{1}{2}}$, which is equivalent to fixing E and taking the limit $\alpha' \rightarrow 0$. Recalling the mass spectrum of closed strings

$$M^2 = \frac{2}{\alpha'} (N + \tilde{N} - 2), \quad (7.4)$$

where N and \tilde{N} are the number operators for the left and right moving string modes, we see that it is in this regime that massive modes decouple and backgrounds of massive string states can be consistently neglected.

The T-duality symmetry of string scattering amplitudes suggests that a T-duality covariant formulation of the string worldsheet action should exist, from which one could derive a T-duality covariant effective action, following a procedure analogous to the one we have just described for conventional string theory. In the rest of this section, we review various proposals that have been worked out in the literature in order to obtain such formulation and we then discuss their connection with DFT.

7.2. Double string sigma model

Originally, T-duality on the worldsheet was implemented in two-dimensional nonlinear sigma models in backgrounds with n compact dimensions in which the metric and 2-form fields have an isometry along the compact directions [76, 77, 98]. By gauging the isometry through a gauge connection and adding to the action a Lagrange multiplier constraining it to be pure gauge, so that the number of worldsheet degrees of freedom remains the same, one obtains the dual theory.

⁷ The equation of motion (3.21) combined with the trace of (3.19) reproduces β^ϕ here when $D = D_{\text{crit}}$.

Specifically, suppose the nonlinear sigma model (7.1) describing the string dynamics in a metric and a 2-form background

$$S = \frac{1}{2\pi} \int d^2z (g_{ij} + b_{ij}) \partial x^i \bar{\partial} x^j \quad (7.5)$$

is invariant under an isometry acting by the translation of x^κ and the fields g_{ij} and b_{ij} are independent of x^κ (here x^κ refers to one or more spacetime coordinates). Here, we disregard the dilaton and use complex coordinates $z = \sigma + i\tau$, $\bar{z} = \sigma - i\tau$ on a flat worldsheet in units in which $\alpha' = 1/2$. The dual theory can be found from the extended action

$$S' = \frac{1}{2\pi} \int d^2z [(g_{ij} + b_{ij}) D x^i \bar{D} x^j + \tilde{x}_\kappa (\partial \bar{A}^\kappa - \bar{\partial} A^\kappa)], \quad (7.6)$$

where $D x^\kappa = \partial x^\kappa + A^\kappa$, and the Lagrange multiplier \tilde{x}_κ enforces the pure gauge condition $\partial \bar{A}^\kappa - \bar{\partial} A^\kappa = 0$. Gauge fixing $x^\kappa = 0$, one obtains the dual model

$$\tilde{S} = \frac{1}{2\pi} \int d^2z (\tilde{g}_{ij} + \tilde{b}_{ij}) \partial \tilde{x}^i \bar{\partial} \tilde{x}^j \quad (7.7)$$

by integrating out the gauge fields. In this new theory, \tilde{g}_{ij} and \tilde{b}_{ij} are given by Buscher's rules (3.11):

$$\begin{aligned} \tilde{g}_{\kappa\kappa} &= \frac{1}{g_{\kappa\kappa}}, & \tilde{g}_{\kappa i} &= \frac{b_{\kappa i}}{g_{\kappa\kappa}}, & \tilde{g}_{ij} &= g_{ij} - \frac{g_{\kappa i} g_{\kappa j} - b_{\kappa i} b_{\kappa j}}{g_{\kappa\kappa}}, \\ \tilde{b}_{\kappa i} &= \frac{g_{\kappa i}}{g_{\kappa\kappa}}, & \tilde{b}_{ij} &= b_{ij} + \frac{g_{\kappa i} b_{\kappa j} - b_{\kappa i} g_{\kappa j}}{g_{\kappa\kappa}}. \end{aligned} \quad (7.8)$$

Clearly, the background fields are in general completely changed by the duality transformation.

In [99], the (Abelian) T-duality transformations were reformulated in terms of chiral Noether currents associated with the isometries, and it was shown that any dual pair of sigma models can be obtained by gauging different combinations of chiral currents. The equivalence of dual sigma models at the quantum level was analyzed in [100], where it was shown that, while one Lagrangian representation is IR free, the dual one is asymptotically free.

This initial approach to deal with T-duality in the worldsheet theory allows us to map a sigma model action to its T-dual one, but neither of them is manifestly $O(D, D)$ covariant. However, as mentioned above, the T-duality symmetry of string theory suggests that an $O(D, D)$ covariant worldsheet action should exist. A natural guess for such formulation would be a sigma model where the target space coordinates are doubled. Actually, a democratic treatment of momentum and winding modes leads to consider independently the ordinary target space coordinates $x^i = x^i_+(\sigma + \tau) + x^i_-(\sigma - \tau)$ associated with momentum and the dual ones $\tilde{x}^i = x^i_+(\sigma + \tau) - x^i_-(\sigma - \tau)$ associated with winding, or equivalently, the left- and right-moving closed string fields. Moreover, since T-duality mixes the metric and 2-form fields, it is reasonable to expect that these fields combine to form the generalized metric \mathcal{H}_{MN} in (3.22). These heuristic arguments lead to a worldsheet action of the form

$$S = \int dX^M \wedge \star dX^N \mathcal{H}_{MN}, \quad X^M = \begin{pmatrix} \tilde{x}^i \\ x^i \end{pmatrix}, \quad (7.9)$$

where \star is the Hodge dual operation on the worldsheet, which is manifestly duality covariant and two-dimensional Lorentz invariant. However, this action describes twice as many degrees of freedom as the action (7.5), and then, it has to be supplemented with additional constraints in order to eliminate the extra coordinates and be able to reproduce the same physics.

A different approach was followed by Tseytlin, who was able to construct a manifestly $O(D, D)$ covariant worldsheet action for chiral bosons [1]. Evidencing the fact that T-duality

is a canonical transformation of the phase space of string theory [101], $O(D, D)$ covariance is achieved through a first-order action in time derivatives:

$$S = \frac{1}{2} \int d^2\sigma (\mathcal{H}_{MN} \partial_1 X^M \partial_1 X^N - \eta_{MN} \partial_0 X^M \partial_1 X^N), \quad (7.10)$$

which can be naturally interpreted as being expressed in terms of phase-space variables, with the dual fields playing the role of the integrated canonical momenta. This action is invariant under the following sigma-model-type symmetry:

$$X \rightarrow \eta X, \quad \mathcal{H} \rightarrow \eta^T \mathcal{H} \eta, \quad (7.11)$$

transforming both the fields and the couplings.

The price for having duality as a symmetry of the action is the lack of two-dimensional Lorentz invariance. Indeed, introducing the left- and right-moving parts of the string coordinates as independent off-shell fields, one has to face the issue of having to deal with a non-Lorentz invariant action. This is actually the case in any Lagrangian description of chiral scalars, as originally discussed in [102], and in general, the Lorentz invariance is recovered on-shell [1, 13, 103]. Local Lorentz invariance is achieved here if \mathcal{H}_{MN} is either constant or depends only on half of the coordinates (in the language of DFT, the Lorentz invariance requires the strong constraint [37]). The equivalence of the equations of motion following from (7.10) with those of the ordinary sigma model (7.5) was shown in [37].

One way to obtain the action (7.10) starting from (7.1) (setting $\phi = 0$) is to write the Hamiltonian density in a manifestly $O(D, D)$ invariant form, in terms of the canonical momenta p_i conjugate to x^i : $p_i = -g_{ij} \partial_0 x^j + b_{ij} \partial_1 x^j$, namely

$$H = \frac{1}{2} \Psi^M \mathcal{H}_{MN} \Psi^N, \quad \text{with} \quad \Psi^M = \begin{pmatrix} p_i \\ \partial_1 x^i \end{pmatrix}. \quad (7.12)$$

Identifying the momenta p_i with the dual coordinates \tilde{x}_i as $p_i = \partial_1 \tilde{x}_i$, and rewriting the Lagrangian as $L = p_i \partial_0 x^i - H$, the action (7.1) can be recast in terms of the double coordinates X^M as

$$S = \frac{1}{2} \int d^2\sigma (\mathcal{H}_{MN} \partial_1 X^M \partial_1 X^N - \eta_{MN} \partial_0 X^M \partial_1 X^N - \Omega_{MN} \partial_0 X^M \partial_1 X^N), \quad (7.13)$$

where

$$\Omega_{MN} = \begin{pmatrix} 0 & \delta_i^j \\ -\delta^i_j & 0 \end{pmatrix}. \quad (7.14)$$

The Ω -term does not contribute to the field equations and does not affect the classical theory, but it is necessary in the quantum theory [14] and, in particular, to show the equivalence of the doubled to the conventional partition function [35]⁸. The correspondence with the standard formulation of critical string theory only appears after integrating out one of the dual fields. Then, either the original or the dual Lorentz invariant action is recovered.

A similar procedure was followed by Siegel in the so-called two-vielbein formalism [98], where the metric and antisymmetric tensor are combined in two independent vielbeins. Alternatively, as demonstrated in [104], the non-Lorentz invariant doubled action (7.10) can be obtained by fixing the axial gauge in the duality and Lorentz invariant extended action (7.6). This axial gauge fixing was identified in [105] as being responsible for the non-Lorentz invariance of the action, and a non-local gauge fixing condition was proposed in order to get a manifestly Lorentz invariant action.

Tseytlin's formulation can be generalized to allow for background fields with arbitrary dependence on the double coordinates, i.e. other than a generalized metric $\mathcal{H}_{MN}(X)$ generically

⁸ Reference [35] also shows the modular invariance of the one-loop double string theory.

depending on the double coordinates, one can include a symmetric matrix $\mathcal{G}_{MN}(X)$ generalizing η_{MN} and an antisymmetric tensor $\mathcal{C}_{MN}(X)$:

$$S = \frac{1}{2} \int d^2\sigma [-(\mathcal{C}_{MN}(X) + \mathcal{G}_{MN}(X))\partial_0 X^M \partial_1 X^N + \mathcal{H}_{MN}(X)\partial_1 X^M \partial_1 X^N]. \quad (7.15)$$

Demanding the on-shell Lorentz symmetry of this action gives constraint equations for the background fields⁹. Classical solutions of these equations were found in [70]. More general nonlinear sigma models of this form, in which the generalized metric is replaced by a generic symmetric matrix, were analyzed in [106], and it was shown that the solutions to the Lorentz invariance constraints give an action with the form of the Poisson–Lie T-duality action introduced in [107].

For completeness, we list here other approaches that have been followed in the literature to construct double sigma models.

- In backgrounds with a toroidal fiber, the string dynamics can be described by the (partially) doubled formalism introduced by Hull in [14]. This formalism describes a worldsheet embedding into backgrounds that are locally T^n bundles, with coordinates (Y^i, \mathbb{X}^A) , where Y^i are the coordinates of the base and \mathbb{X}^A are the coordinates of the doubled torus fiber. The Lagrangian is a sum of an ordinary sigma model Lagrangian $L(Y)$ like that in (7.1) plus a sigma model for the generalized metric \mathcal{H}_{AB} of the doubled fibers, which crucially only depends on Y^i and there is isometry in all the fiber directions:

$$\mathcal{L} = \frac{1}{4} \mathcal{H}_{AB}(Y) d\mathbb{X}^A \wedge \star d\mathbb{X}^B + \frac{1}{2} \Omega_{AB} d\mathbb{X}^A \wedge d\mathbb{X}^B + L(Y). \quad (7.16)$$

The action must be supplemented with the chirality constraint

$$d\mathbb{X}^A = \eta^{AB} \mathcal{H}_{BC} \star d\mathbb{X}^C, \quad (7.17)$$

ensuring that the fiber directions can be thought of as chiral bosons on the worldsheet, so that the doubling does not increase the number of physical degrees of freedom.

This doubled formalism has been very useful in elucidating the structure of non-geometric backgrounds, such as T-folds.

- In [36], the constraint (7.17) was incorporated into the action, which then reads

$$S = \frac{1}{2} \int d^2\sigma [-\mathcal{G}_{\alpha\beta} \partial_1 X^\alpha \partial_1 X^\alpha + \mathcal{L}_{\alpha\beta} \partial_1 X^\alpha \partial_0 X^\beta + \mathcal{K}_{\alpha\beta} \partial_0 X^\alpha \partial_0 X^\beta], \quad (7.18)$$

where $X^\alpha = (Y^i, \mathbb{X}^A)$ and

$$\mathcal{G}_{\alpha\beta} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \mathcal{H}_{AB} \end{pmatrix}, \quad \mathcal{L}_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \eta_{AB} \end{pmatrix}, \quad \mathcal{K}_{\alpha\beta} = \begin{pmatrix} g_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad (7.19)$$

g_{ij} being the standard sigma model metric for the base.

- For doubled backgrounds which are locally a group manifold, the nonlinear Poisson–Lie sigma model proposed in [107] was rewritten in [108] as

$$S = \frac{1}{2} \oint d^2\sigma (\mathcal{H}_{AB} \partial_1 X^A \partial_1 X^B - \eta_{AB} \partial_1 X^A \partial_0 X^B) + \frac{1}{12} \int_V t_{IJK} dX^I \wedge dX^J \wedge dX^K, \quad (7.20)$$

where V is the volume of the membrane whose boundary is the string worldsheet and t_{IJK} are the structure constants of the gauge algebra.

⁹ The term with coupling $\mathcal{C}_{MN}(X)$ is clearly Lorentz invariant.

7.3. DFT from the double sigma model

We have seen that the vanishing β -function equations can be interpreted as equations of motion derived from the string low-energy effective field theory. It is then through the background field equations of the double sigma model that one expects to make the connection between string theory and DFT. But this raises some conceptual questions. For instance, perturbation theory in the nonlinear sigma model (7.1) is performed around the large volume limit. Can one also define a perturbation theory around the limit of very small substringy sizes of the background? Or even more puzzling, is there a well-defined weak-coupling limit of the T-duality invariant models (7.10) or (7.15)?

These questions have been analyzed by several authors from different viewpoints. From the β -functions' standpoint, the background field method was adapted to the doubled coordinates, expanding them around a classical solution as $X^M = X_{cl}^M + \Pi^M$. Since the fluctuation Π^M does not in general transform as a vector, in order to have a covariant expansion, the expansion parameter is defined as the tangent vector to the geodesic from X_{cl}^M to $X_{cl}^M + \Pi^M$ whose length is equal to that of the geodesic. Since this is a contravariant vector, the expansion is then organized in terms of covariant objects.

The crucial point to elucidate in order to consistently apply this method to the double sigma models is how to define geodesics in double geometries. The simplest options starting from the action (7.15) are to consider geodesics of \mathcal{G}_{MN} or geodesics of \mathcal{H}_{MN} , with the resulting expansions involving covariant derivatives and tensors with respect to the chosen metric. The background field method was first applied to the sigma model of Hull's doubled formalism in [36] using geodesics of \mathcal{H}_{MN} . This was then generalized in [70] for the sigma model (7.15) where the expansion was also performed using geodesics of \mathcal{G}_{MN} and general expressions for the Weyl and Lorentz anomaly terms were found.

UV finiteness and worldsheet Weyl invariance at one loop were shown in [36] to require the vanishing of the generalized Ricci tensor when DFT is restricted to a fibered background of the type that the doubled formalism describes. In [37], the vanishing $\beta^{\mathcal{H}}$ -function equation from the sigma model (7.10), with \mathcal{H}_{MN} arbitrarily depending on any of the doubled coordinates, subject to the strong constraint, was found to match the equation of motion for the generalized metric obtained from the DFT action (3.71). Hence, the conformal invariance of the double chiral sigma model (7.10) under the strong constraint corresponds to the generalized Ricci flatness equation, and this implies that DFT is the spacetime effective field theory of the double worldsheet action. A preliminary similar result was also found in [37] for the generalized dilaton.

Although imposing the strong constraint means the theory is no longer truly doubled, the appearance of the generalized Ricci tensor in this context is non-trivial and seems evidence not only of an effective double geometry, but also of a string theory origin of DFT. Nevertheless, it would be interesting to investigate if these conclusions still hold beyond the strong constraint, in a truly double space. Indeed, as extensively discussed in the preceding sections, DFT with the strong constraint is equivalent to the standard field theory description of the massless modes of the string. Actually, the strong constraint implies one can perform an $O(D, D)$ rotation, so that the fields only depend on x^i and, since all $O(D, D)$ indices in the action are contracted properly, its form is preserved under such a rotation.

Furthermore, we have seen that in standard string theory, perturbation theory makes sense in the background field expansion of the action (7.1) if $\alpha'^{1/2} \ll R_c$ and in this regime one can also neglect the massive string states. In order to analyze the validity of perturbation theory in the double space, since $O(n, n)$ duality is a symmetry of string theory on an n -torus with a constant b_{ij} background, which survives in the effective field theory when

it is dimensionally reduced on T^n , it is convenient to recall the mass spectrum of closed strings in these backgrounds. In terms of the quantized canonical momenta $p_i = \frac{n_i}{R}$ and winding numbers \tilde{p}^i , the mass in $d = D - n$ dimensions is given by

$$\begin{aligned} m^2 &= \frac{1}{2\alpha'^2} g_{ij} (v_L^i v_L^j + v_R^i v_R^j) + \frac{2}{\alpha'} (N + \tilde{N} - 2), \\ v_{L,R}^i &= \alpha' g^{ij} \left(\frac{n_j}{R} + b_{jk} \tilde{p}^k R \right) \pm \tilde{p}^i R, \end{aligned} \quad (7.21)$$

where N and \tilde{N} are the number operators for the left- and right-moving oscillators, respectively, of all the coordinates: compact and non-compact (we are assuming, for simplicity, the same radius R for all compact dimensions). DFT deals with massless states of the D -dimensional theory, i.e. having $N = \tilde{N} = 1$, and then it includes all the momentum and winding modes of the lower dimensional theory, which are massive. Since it truncates the massive levels (of the decompactified theory), one wonders whether this corresponds to a consistent truncation. Then, a better understanding of this issue seems necessary in order to strengthen the link between string theory and DFT. By the same token, given that a T-duality symmetric description treats the compactification scale and its inverse on an equal footing, it seems important to clarify what is the rank of parameters for which the coupling constants are small and the theory is weakly coupled, so that the perturbative expansion can be trusted.

Another way to tackle these issues is through the computation of scattering amplitudes describing the interactions of winding and momentum states in geometric and non-geometric backgrounds. The first step in this direction was taken in [109], where scattering amplitudes of closed string tachyons in an R -flux background were computed and a very interesting non-associative behavior of the spacetime coordinates was found. We shall review this work in the next section, but here we point out that an effective field theory analysis of these kinds of scattering amplitudes, which would give an alternative approach to this question, is not yet available.

In the absence of a better comprehension, it is important to note that the background-field equations in a particular duality frame are the same as for the usual string. *A priori* this does not have to be the case since, as we have seen, the string winding modes could in principle correct the usual β -functions. Moreover, given that T-duality is corrected by worldsheet instantons and the doubled space contains the naive T-dual, corrections to the double geometry could arise. It is then reasonable to expect that once double geometry is understood, one will be able to elucidate these questions. In this sense, a higher loop calculation of the β -functionals would be important since the full generalized Riemann tensor is expected to appear in case the analogy with ordinary string theory goes through. As a matter of fact, as discussed in section 4, although a duality covariant generalized Riemann tensor has been constructed whose contractions give the generalized Ricci tensor and scalar, it cannot be completely determined from the physical fields of DFT as in ordinary Riemannian geometry. Better understanding the link between string theory and DFT might also help to uncover the geometry of the double space.

8. Other developments and applications

DFT has proven to be a powerful tool to explore string theoretical features beyond the supergravity limit and Riemannian geometry. Over the last few years there has been a great deal of progress on these issues, growing largely out of the systematic application of symmetries and dualities. We certainly do not have a complete understanding of DFT, but an increasing number of promising directions have opened following the original works and several encouraging ideas have been put forward. We cannot discuss all of them in detail here but, besides the

topics covered above and by way of conclusion, we would like to comment on some recent developments and open problems.

8.1. Non-commutative/non-associative structures in closed string theory

In the presence of a constant 2-form field, the coordinates of the end points of open strings attached to a D -brane become non-commutative [110]. Moreover, in the background of a non-trivial H -flux, the coordinates are not only non-commutative but also non-associative [111]. This behavior is revealed by scattering amplitudes of open string states and usually interpreted as a consequence of the structure of interactions in open string theory, which involve Riemann surfaces with boundaries. To compute scattering amplitudes, the vertex operators creating open string states must be inserted on the boundaries, and then a background that is sensitive to ordering might distinguish the insertion points.

In contrast, the sum over worldsheets defining interactions of closed strings contains Riemann surfaces with no boundaries, in which the vertex operators are inserted in the bulk. Therefore, one would not expect to have non-commutative coordinates in closed string theory because no unambiguous notion of ordering can be defined in scattering amplitudes. However, it has been argued that in presence of non-geometric fluxes, the coordinates of closed strings can become non-commutative or even non-associative [112, 113, 109].

Actually, non-geometric fluxes twist the Poisson structure of the phase space of closed strings and the non-vanishing equal time commutator of closed string coordinates in a Q -flux background has been conjectured to be given by

$$\lim_{\sigma \rightarrow \sigma'} [x^i(\tau, \sigma), x^j(\tau, \sigma')] = \oint_{C_k} Q^{ij}_k dx^k, \tag{8.1}$$

where C_k is a cycle around which the closed string wraps, while non-associativity has been argued to arise in an R -flux background in which

$$\lim_{\sigma', \sigma'' \rightarrow \sigma} ([x^i(\tau, \sigma), [x^j(\tau, \sigma'), x^k(\tau, \sigma'')]] + \text{cyclic}) = R^{ijk}. \tag{8.2}$$

In particular, non-commutativity has been studied in the three-dimensional background with Q -flux that is dual to the flat 3-torus with H -flux discussed in sections 5.2 and 5.3. Recall that one can use Buscher's rules (3.11) to map the flat 3-torus with H -flux to a twisted torus with zero H -flux in which the twist is related to a geometric flux ω . A further T-duality then yields the non-geometric Q -flux background in which the metric and 2-form are locally but not globally well defined. In the simple case in which C_k is a circle and the Q -flux is constant: $Q^{ij}_k = Q\epsilon^{ij}_k$, the commutator (8.1) becomes

$$\lim_{\sigma \rightarrow \sigma'} [x^i(\tau, \sigma), x^j(\tau, \sigma')] = 2\pi Q\epsilon^{ij}_k \tilde{p}^k, \tag{8.3}$$

and then we see that non-commutativity is a non-local effect related to winding.

Non-associativity of the string coordinates was first observed in the theory of closed strings moving on the 3-sphere S^3 in the presence of an H -flux background [112]. This theory is described by the exactly solvable $SU(2)_k$ WZW model, and then a conformal field theory computation can be performed. A non-vanishing equal-time, equal-position cyclic double commutator of the spacetime coordinates, independent of the worldsheet coordinates, was found. More recently, a non-trivial cyclic three product was also found in [109] from the scattering amplitudes of closed string tachyon vertex operators in an R -flux background. The three-tachyon correlator gets a non-trivial phase in R -space depending on the operators ordering, before enforcing momentum conservation. The non-vanishing cyclic 3-bracket of the coordinates appears then to be consistent with the structure of two-dimensional conformal field theory.

The non-associative geometry probed by closed strings in flat non-geometric R -flux backgrounds has also been studied in [72] from a different perspective. Starting from a Courant sigma model on an open membrane, regarded as a topological sector of closed string dynamics in R -space, the authors derive a twisted Poisson sigma model on the boundary of the membrane. For the constant R -flux, they obtain closed formulas for the corresponding non-associative star product and its associator.

Recall that starting from a geometrical background and performing three T-dualities in these three-dimensional backgrounds, one runs out of isometric directions. In particular, in the R -flux background, the notion of locality is completely lost in the conventional space. In DFT instead, the resulting background depends on a dual coordinate, and these global and local issues can be avoided. Thus, by naturally incorporating all the T-dual backgrounds in a covariant picture through a double space, DFT provides a convenient framework for analyzing non-commutativity/non-associativity. Actually, as discussed in [113], in the doubled phase-space, T-duality would exchange commutators among the conventional spacetime coordinates with others among the dual ones. If coordinates commute in the first setting while the duals do not, the situation gets exchanged after T-duality.

8.2. Large gauge transformations in DFT

While all the results of this review are based on the infinitesimal generalized diffeomorphisms (3.39), finite gauge transformations were considered by Hohm and Zwiebach in [114] under the imposition of the strong constraint. They are defined through exponentiations of the generalized Lie derivatives, and are interpreted as generalized coordinate transformations in the doubled space. In [114], a formula for large gauge transformations was proposed and tested, which is written in terms of derivatives of the coordinate maps. Successive generalized coordinate transformations give a generalized coordinate transformation that differs from the direct composition of the original two: it is constructed using the C-bracket. Interestingly, although these transformations form a group when acting on fields, they do not associate when acting on coordinates, and then one wonders whether this can be related to the works in [112].

By now, it is not completely known how to construct a non-trivial patching of local regions of the doubled manifold leading to non-geometric configurations. As we reviewed, the notion of a T-fold is based on the idea that field configurations on overlaps can be glued with the use of T-duality transformations. In order to address questions of this type in DFT, we need a clear picture of the finite gauge transformations. This is a very interesting line of research.

8.3. New perspectives on α' corrections

The effective supergravity action is nicely covariantized under the T-duality group and generalized diffeomorphisms. One can then wonder if a similar covariantization occurs for the α' corrections to the action. This question was posed in [29], where a first step in this direction was given. Specifically, within a generalized metric formulation, it was shown that the Riemann-squared scalar $R_{ijkl}R^{ijkl}$, familiar in α' corrections to the low-energy effective action of string theory, is not obtained (after the proper implementation of the strong constraint in the supergravity frame $\tilde{\partial}^i = 0$) from any covariant expression built out of the generalized metric and generalized dilaton (and setting $b_{ij} = \phi = 0$), and quartic in generalized derivatives. For the sake of concreteness, let us be more specific. This obstruction appears due to a problematic contribution in the expansion, taking the form $g^{mp}g^{iq}g^{kl}g^{mt}g^{rs}\partial_k g_{mr}\partial_i g_{ns}\partial_q\partial_l g_{pt}$. It was shown that there is no possible covariant combination giving rise to a term like this, and the origin of this problem can be traced back to the $O(D, D)$ structure of the generalized metric.

To understand the significance of this result, suppose one had succeeded in constructing such a covariant combination related to $R_{ijkl}R^{ijkl}$. Then, one could have written a general four-derivative action from linear combinations of the squares of the generalized curvatures. Being constructed from covariant objects, any of them would be invariant. As argued in [29], this would be unexpected because the field redefinitions $g_{ij} \rightarrow g_{ij} + \alpha'(a_1 R_{ij} + a_2 g_{ij} R)$ that respect diffeomorphism invariance, map α' -corrected actions into each other, and alter the coefficients of Ricci-squared and R-squared terms. After these field redefinitions, the T-duality transformation of g_{ij} would be α' -corrected, in conflict with the original assumption.

Although things are not as easy as one would have liked them to be, a better understanding of α' corrections in DFT would help to understand the contributions of the 2-form and dilaton to the corrections, as commented in [115], and also possibly help to find new patterns, based on duality arguments. Also, a better understanding of this problem might shed light on the mysterious unphysical components of the generalized Riemann tensor and vice versa. In any case, α' -corrections to supergravity in the context of DFT seem to be a very promising line of research, where plenty of things remain to be done and learnt.

8.4. Geometry for non-geometry

As we have seen, T-duality appears to imply that the geometrical structure underlying string theory goes beyond the usual framework of differential geometry and suggests an extension of the standard diffeomorphism group of General Relativity. A new geometrical framework to describe the non-geometric structures was developed in [64, 69]. The idea of these pieces of work is to provide a general formulation to study non-geometric backgrounds in conventional higher dimensional spacetime, in a formalism that facilitates the treatment of global issues that are problematic in standard supergravity.

When the generalized metric has the form (3.22), it is said to be in the *geometric frame*. A general $O(D, D)$ transformation mixes the usual metric and 2-form fields in a complicated way and a (T-duality inspired) field redefinition is convenient to re-parameterize the generalized metric such that the description of non-geometric backgrounds becomes more natural (this can be named the *non-geometric frame*). The field redefinition makes a dual metric and a bivector β^{ij} enter the game, and these now become the fields of the geometric action for non-geometric fluxes. Performing these field redefinitions in the supergravity action (3.17) makes the non-geometric fluxes appear in such a way that the new actions are well defined in terms of the new fields. DFT provides a natural framework to interpolate between these two frames, in which geometric and non-geometric backgrounds are better described.

The new actions can be interpreted as coming from the differential geometry of Lie algebroids. These are generalizations of Lie algebras where the structure constants can be spacetime dependent. Lie algebroids give a natural generalization of the familiar concepts of standard Riemannian geometry, such as covariant derivatives, torsion and curvatures. A detailed account of the relation between these conventional objects and those appearing for Lie algebroids is presented in [69].

These are very nice results that specialize in the geometry and dynamics of non-geometric backgrounds.

8.5. Beyond supergravity: DFT without strong constraint

As we explained, in order for the generalized Lie derivative to generate closed transformations, the fields and gauge parameters of the theory must be constrained, i.e. DFT is a restricted theory (3.44). One possibility of solving the constraints is to impose an even stronger restriction: the

strong constraint (3.30). This possibility is the most explored one and allows for a generic form of fields and gauge parameters, but with a strong restriction in their coordinate dependence: they can only depend on a (un-doubled) slice of the double space. This enables a direct relation to supergravity and puts DFT in a safe and controlled place. There are however other solutions [65, 66, 34, 53–55, 30] in which the *shape* of the fields is restricted, but not the coordinate dependence, which can then be truly double. As we extensively reviewed, in this situation the fields adopt the form of a Scherk–Schwarz reduction ansatz, and this facilitates to make contact with gauged supergravities in lower dimensions. The double coordinate dependence here is encoded in the gaugings, which cover the corners of the configuration space that are not reached from standard supergravity compactifications.

These doubled solutions correspond to the first attempts of consistently going beyond supergravity in DFT. Whether these extensions live within string theory is a question that remains unanswered. This seems most likely to be the case, because these extensions are precisely governed by the symmetries of string theory. In any case, DFT provides a (stringy-based) scenario in which supergravity is only a particular limit, and many explorations beyond this limit still have to be done.

8.6. (Exotic) brane orbits in DFT

In the open string sector, T-duality exchanges Dirichlet and Neumann boundary conditions, and then relates D -branes to different dimensionalities. This situation was nicely depicted in the double torus in [116]. After evaluating the one-loop beta function for the boundary gauge coupling, the effective field theory for the double D -branes was obtained, and is described by a T-duality covariant DBI action of double fields.

In the NS-NS sector, the NS5-brane and KK5-monopole were also considered in the double torus [117]. Both configurations are related by T-duality, and the orbit is known to continue. By applying a further T-duality, one obtains the S^2_2 -brane (see [118, 119] for detailed discussions) which looks like a T-fold, and is a special case of a Q -brane [120]. DFT allows us to T-dualize further in order to obtain an R -brane. The picture is analogous to that of duality orbits on non-geometric fluxes. The *exotic* Q - and R -branes are nicely accommodated in DFT, and the frameworks of [64] and [69] suitably describe their underlying geometry. Being sources of non-geometric fluxes, they exhibit an interesting non-associative/non-commutative behavior [120].

The NS5 and KK5 source BIs on their world volumes [121] and their exotic T-duals are likely to source the corresponding T-dual BIs for non-geometric fluxes [68]. Interestingly, these BIs are naturally identified with the consistency constraints of DFT (3.64).

Brane orbits have been extensively discussed in [122] and [118], and we refer to those papers for a general discussion on the topic. There are still plenty of unanswered questions, for example, regarding the existence (and validity) of bound states of geometric and non-geometric branes described by configurations that violate the strong constraint. This exciting area of research is just beginning, and DFT seems to be a suitable framework for exploration.

8.7. New possibilities for upliftings, moduli fixing and dS vacua

Only a subset of all the possible deformations (gaugings or fluxes) in gauged supergravities in four dimensions can be reached from standard compactifications of $D = 10, 11$ supergravity, as we explained. The rest of them (the non-geometric orbits), on the other hand, do not admit supergravity uplifts, and then one has to appeal to duality arguments in order to make sense of them from a lower dimensional perspective. DFT (and the more general

U-duality covariant frameworks) provides a suitable scenario to uplift non-geometric orbits in an extended geometrical sense [67]. As we explained, non-geometric fluxes seem to be necessary ingredients in purely flux-based moduli stabilization surveys [83]. The same happens in dS vacua explorations. Although there is beautiful recent progress in the quest for classical (meta)stable dS vacua with non-geometric fluxes [83], their uplift to extended geometry (in particular DFT) or the ten-dimensional geometric actions for non-geometric fluxes [64, 69] is still an open question.

Once again, DFT seems to provide a suitable framework to uplift the gauged supergravities with non-geometric fluxes that give rise to desired phenomenological features. Progress in this direction was achieved in some particular gauged supergravities [67] through consistent relaxations of the strong constraint.

Also in this direction, the extended geometry of [55] might shed light on the uplifts of the new $SO(8)$ maximal supergravities [96], which seem to find obstructions in their uplift to $D = 11$ supergravity [97].

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