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Dynamical symmetries of Markov processes with multiplicative white noise

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Abstract. We analyse various properties of stochastic Markov processes with multiplicative white noise. We take a single-variable problem as a simple example, and we later extend the analysis to the Landau–Lifshitz–Gilbert equation for the stochastic dynamics of a magnetic moment. In particular, we focus on the non-equilibrium transfer of angular momentum to the magnetization from a spin-polarised current of electrons, a technique which is widely used in the context of spintronics to manipulate magnetic moments. We unveil two hidden dynamical symmetries of the generating functionals of these Markovian multiplicative white-noise processes. One symmetry only holds in equilibrium and we use it to prove generic relations such as the fluctuation-dissipation theorems. Out of equilibrium, we take profit of the symmetry-breaking terms to prove fluctuation theorems. The other symmetry yields strong dynamical relations between correlation and response functions which can notably

simplify the numerical analysis of these problems. Our construction allows us to clarify some misconceptions on multiplicative white-noise stochastic processes that can be found in the literature. In particular, we show that a first-order differential equation with multiplicative white noise can be transformed into an additive-noise equation, but that the latter keeps a non-trivial memory of the discretisation prescription used to define the former.

Contents

1. Introduction

Stochastic Markov processes in which the noise acts multiplicatively on a function of the variable of interest are manifold. In physics one finds the diffusion of a colloidal particle close to a wall, in chemistry one counts autocatalytic chemical reactions in which the production of a molecule is enhanced by the presence of the same molecules already produced, in economy the Black and Scholes model provides a theory of option pricing. A detailed discussion of the dynamics of single variable and extended systems with multiplicative noise can be found in [1].

In order to make sense, any Markovian stochastic equation with multiplicative noise, e.g. an overdamped Langevin equation with state-dependent diffusion coefficient, must be given a discretization prescription. For these, the associated Fokker–Planck equation governing the time evolution of the probability distribution function of the stochastic variable(s) depends, in general, on the discretisation prescription parameter, say α, and on the function that multiplies the noise, say g. Noteworthy, its stationary solution also depends on α and g [2–5].

In the presence of a multiplicative noise, Stratonovich noticed that the qualitative behaviour of the stationary probability distribution can change as a function of the noise strength [2], and thus deviate from the usual Gibbs–Boltzmann distribution which would only involve the potential responsible for the deterministic forces acting, say, on the particle. The state-dependence of the function g can have far reaching consequences. For instance, the stationary probability distribution function may develop new extrema that are not the ones set by the deterministic forces. Similar effect on many-body systems can alter the number or the nature of the extrema of the free energy governing the dynamics of an order parameter, and therefore induce so-called noise-induced phase transitions [1]. For instance, models without symmetry breaking potentials can thus exhibit coarsening phenomena.

However, one can adopt a different point of view from the one above. The stochastic differential equation can be modified so that the approach to the usual Gibbs–Boltzmann distribution is ensured for any discretisation prescription parameter α and multiplicative function g. This is achieved by adding a drift term to the stochastic equation (given in equation (2.8)) and necessary even when the common Stratonovich

mid-point prescription is used). With this addition, the α dependence disappears from the Fokker–Planck equation (and the physics in general). Although the dynamics still depend on g, one can show that they converge to a g-independent stationary solution which is now the desired Gibbs–Boltzmann measure.

We present a concise summary of these known, though perhaps not sufficiently assimilated, issues in the two first sections of section 2, that are supplemented by material in appendices A–C, in the framework of a stochastic differential equation on a single variable.

It is sometimes found in the literature that one-dimensional Markov processes with multiplicative noise can be mapped to overdamped Langevin equations with an additive noise, and that once in this new framework all discretisation subtleties can be ignored. This statement is, however, wrong as the new additive-noise equation depends explicitly on the α-prescription used to define the original multiplicative-noise equation. The reason is that the chain rule for the time derivative of a function of the stochastic variable has to be used in the transformation between multiplicative and additive-noise equations, and this chain rule involves α and g. Accordingly, the Fokker–Planck equation associated to the resulting additive-noise Langevin equation and its asymptotic solution depend on α and g. This can be cured by adding a drift term to the additive-noise Langevin equation that is completely equivalent to the one to be used in the multiplicative-noise formalism. We discuss these facts in appendices B and C.

In section 2.3 we recall the path-integral generating-functional formalism for stochastic Langevin processes with multiplicative white noise [6–11]. We explain the apparent differences with the path integrals used in [12], and why we do not agree with the claims in [13, 14].

The functional formulation of stochastic processes is very well-suited to prove model-independent properties of generic physical observables. Although the derivation of fluctuation theorems for white-noise Markov processes has been addressed on general grounds via equation-of-motion formalisms [12, 15, 16], we are still lacking a generic path-integral formulation addressing the case of multiplicative noise. In this paper, we will use a model-independent field transformation in the path-integral formulation to show that the equilibrium fluctuation-dissipation theorems and the out-of-equilibrium fluctuation relations [17–26] hold for multiplicative white-noise Markov processes. Contrary to previous works for which the steady states were governed by non-equilibrium potentials, see e.g. [12–14] and [27, 28], here the approach to a Gibbs–Boltzmann equilibrium is ensured by the presence of a drift term, see also [29].

We will also present another transformation which leaves the action (and the functional measure) invariant and which can be used to derive the Schwinger–Dyson equations governing the coupled dynamics of correlations and linear responses.

For simplicity, we will present detailed derivations in the framework of a single-variable stochastic equation. The generalisation to higher dimensional problems, and field theories, should then be clear. At the end of the paper, we will apply our results to the dynamics of a magnetic moment governed by the stochastic Landau–Lifshitz–Gilbert equation [30, 31] taking advantage of the path-integral formalism developed in [32].

Recapping, the paper is organised as follows. In section 2, we recall the main features of the Langevin, Fokker–Planck and path-integral formulations of stochastic Markov processes with multiplicative white noise in the framework of single-variable

problems. We re-derive Crooks relation [33] between path probabilities for forward and backward stochastic processes, now extended to take into account non-trivial issues due to the discretisation of Markov stochastic processes with multiplicative white noise. We prove equilibrium and out-of-equilibrium fluctuation theorems. Complements to these sections are given in the appendices. Section 3 is devoted to the application of these ideas to the Landau–Lifshitz–Gilbert stochastic equation. Finally, in section 4 we open some ways for future research.

2. Single-variable stochastic Markov processes

In this section, we define the single-variable problem that we use as a framework to recall a number of important features that, sometimes, appear in confusing terms in the literature. We also discuss a time-reversal symmetry of the equilibrium generating functional and we use it to derive equilibrium relations such as the fluctuation-dissipation theorem. Out of equilibrium, this symmetry is broken and we use the resulting symmetry-breaking terms to derive fluctuation relations. We then discuss another symmetry of the generating functional, valid in and out of equilibrium, that is useful to derive the Schwinger–Dyson equations for correlations and linear responses.

2.1. The Langevin equation

Let us consider a real variable x, the dynamics of which is governed by the following Langevin equation

$$\mathrm{d}_t x(t) = f(x) + g(x)\xi(t) \tag{2.1}$$

where $\xi(t)$ is a Gaussian white noise with zero mean $\langle \xi(t) \rangle = 0$ and variance

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t') \text{ with } D = k_B T. \tag{2.2}$$

The noise is said to be *multiplicative* because it acts multiplicatively on $g(x)$, a function of the stochastic variable.

This stochastic differential equation makes sense only when complemented with a discretisation prescription to define at which point $g(x)$ should be evaluated. This is relevant since each pulse $\xi(t)$ yields a discontinuity in x and therefore the value of x at which $g(x)$ is evaluated (and hence the size of the discontinuity) is *a priori* not well defined. Without restricting the generality of the foregoing, we work with the generic α-prescription [3, 34] which corresponds, in discrete time, to

$$x_{n+1} - x_n = f(\bar{x}_n)\mathrm{d}t + g(\bar{x}_n)\mathrm{d}W_n \tag{2.3}$$

with $\mathrm{d}W_n \equiv \xi_n \mathrm{d}t$, $\langle \mathrm{d}W_n \rangle = 0$ and $\langle \mathrm{d}W_n \mathrm{d}W_m \rangle = 2D\delta_{nm}\mathrm{d}t$ for the statistics of the noise, and

$$\bar{x}_n = \alpha x_{n+1} + (1 - \alpha)x_n, \tag{2.4}$$

with α a real parameter $0 \leqslant \alpha \leqslant 1$. Note that the discretisation used in the argument of f is irrelevant in the continuous-time limit. Equation (2.1) can be taken into a form in

which the noise appears additively but in which the dependence on the discretisation parameter α is still present in the new equation, see appendices B and C.

The chain rule for the time derivative of a function F of the variable x depends on the stochastic equation governing the time evolution of x, see [3–5] for Itô ($\alpha = 0$) and Stratonovich ($\alpha = 1/2$) prescriptions and appendix A for a generic α-prescription. In this case, it reads

$$\mathrm{d}_t F(x) = \mathrm{d}_t x\ \partial_x F(x) + (1 - 2\alpha) D g^2(x) \partial_x^2 F(x) \tag{2.5}$$

where $\mathrm{d}_t x \equiv \mathrm{d}x/\mathrm{d}t$. The usual chain rule of conventional calculus is recovered only in the case of the Stratonovich mid-point prescription, $\alpha = 1/2$. Note that the chain rule is independent of the 'force' $f(x)$. In particular, it does not depend on the addition (or not) of a drift term to the Langevin equation such as discussed around equation (2.11) below.

2.2. The Fokker–Planck equation

The Fokker–Planck equation corresponding to the Langevin equation (2.1) in a generic α-prescription reads [3, 4, 7, 28]

$$\partial_t P(x, t) = -\partial_x[(f(x) + 2D\alpha g(x)g'(x))P(x, t)] + D\,\partial_x^2[g^2(x)P(x, t)]. \tag{2.6}$$

(Note the difference between this equation and the one used in [12].) Once supplemented by an initial condition $P_\mathrm{i}(x) = P(x, 0)$, this equation describes the deterministic evolution of the probability density $P(x, t)$ of finding x at time t. It can be written in the form of a continuity equation $\partial_t P + \partial_x J = 0$. Its stationary solution with vanishing current, $J = 0$, is

$$P_\mathrm{st}(x) = Z^{-1}\,[g(x)]^{2(\alpha-1)}\,\mathrm{e}^{\frac{1}{D}\int^x \frac{f(x')}{g^2(x')}} \tag{2.7}$$

where $\int^x$ represents the indefinite integral over x' and Z is a normalisation constant [3, 4, 28]. The approach to this asymptotic form can be proven with the construction of an $\mathcal{H}$-function as in [35], or with the mapping of the Fokker–Planck operator into a Schrödinger operator and the analysis of its eigenvalue problem [36]. Clearly, the fact that P_st depends on α and g shows that these can have highly non-trivial consequences on the transient dynamics as well as the asymptotic stationary properties of the system [1, 2, 37].

However, if we allow ourselves to consider the special 'drift force' [38]

$$f(x) = -g^2(x)V'(x) + 2D(1 - \alpha)g(x)g'(x). \tag{2.8}$$

with the short-hand notation $V' \equiv \partial_x V$ and $g' \equiv \partial_x g$, the Fokker–Planck equation loses any dependence on α,

$$\partial_t P(x, t) = \partial_x\{g^2(x)[V'(x)P(x, t) + D\,\partial_x P(x, t)]\}. \tag{2.9}$$

Importantly, since physical observables are computed using $P(x, t)$, this implies that the physics of equation (2.1) with the 'drift force' in equation (2.8) does not depend on the prescription parameter α. This can also be proven using the BRST symmetry of the

generating functional [27] or with a perturbative analysis [39]. Moreover, the asymptotic solution of the Fokker–Planck equation in equation (2.9) simply reads

$$P_{\mathrm{st}}(x) = Z^{-1}\,\mathrm{e}^{-\frac{1}{D}\,V(x)} = P_{\mathrm{GB}}(x) \tag{2.10}$$

independently of α and g. P_{GB} stands for the Gibbs–Boltzmann equilibrium distribution function in the canonical ensemble of statistical mechanics.

Therefore, in order to describe the Markovian stochastic dynamics of a physical quantity subject to multiplicative white noise and that reaches a usual Gibbs–Boltzmann equilibrium measure, one needs to work with the drifted Langevin equation

$$\mathrm{d}_t x(t) = -g^2(x)\,V'(x) + 2D(1-\alpha)g(x)g'(x) + g(x)\xi(t) \tag{2.11}$$

in a generic α-prescription. Several features of this equation should be remarked. First, there is a non-trivial additional 'drift force' even in the Stratonovich mid-point ($\alpha = 1/2$) prescription. Second, the additional term *is not* equal to the one in the chain rule (2.5). Moreover, this equation is equivalent to the original 'undrifted' equation (2.1) with a post-point prescription $\alpha = 1$ [40–43].

Although it is conventional to work with Langevin equations of the form of equation (2.1) or equation (2.11) in which the left-hand side (lhs) is solely given by the time derivative, $\mathrm{d}_t x$, it can be illuminating to re-write the latter as

$$k^2(x)\mathrm{d}_t x(t) = -V'(x) - 2D(1-\alpha)\partial_x \ln|k(x)| + k(x)\xi(t), \tag{2.12}$$

where we re-parametrised $g(x) \equiv 1/k(x)$. Equation (2.12) has the form claimed in [44] for the Markovian overdamped dynamics of particles subject to forces deriving from a potential V and interacting with a bath of oscillators via a non-linear coupling $K(x)$ with $K'(x) \equiv k(x)$. The exact integration over the degrees of freedom of the bath gives rise to a viscous friction force, here in the left-hand side (lhs), as well as the multiplicative noise in the right-hand side (rhs). Note that the drift force was not discussed in [44] since the focus in this paper was on non-Markovian dynamics (either because of the presence of a coloured noise or inertia) for which case no drift force is needed to ensure the convergence to the usual Gibbs–Boltzmann distribution. We can therefore re-interpret the time derivative, $\mathrm{d}_t x$, in the lhs of equation (2.1) or equation (2.11) as originating from the dissipative interaction with the same bath that is responsible for the random noise $\xi(t)$.

2.3. The path-integral formulation

The stochastic dynamics of Markov processes governed by Langevin equations can be formulated in terms of path integrals. This approach has been first developed for cases with an additive noise [45–50]. It was later generalised to cases with a multiplicative noise [6] and extended to various discretization schemes and to higher dimensions [7, 8]. Below, we recall the construction of such a path-integral representation on the case of the multiplicative-noise equation (2.1) by following a procedure *à la* Martin–Siggia–Rose–Janssen–deDominicis (MSRJD).

The probability distribution for a given trajectory of x, with initial condition $x(-\mathcal{T})$ at time $-\mathcal{T}$ distributed according to $P_{\mathrm{i}}(x(-\mathcal{T}))$, and governed by equation (2.1) is

$$P[x;\alpha] \propto P_{\mathrm{i}}(x_{-\mathcal{T}}) \left\langle |\mathcal{J}| \int \mathcal{D}[\hat{x}] \, \mathrm{e}^{\int \mathrm{i}\hat{x}_t \, \mathrm{Eq}_t[x,\xi;\alpha]} \right\rangle \tag{2.13}$$

where we used the short-hand notation $\int$ for the time integral in the exponential that runs over the (symmetrised for convenience) time interval $[-\mathcal{T},\mathcal{T}]$, and $x_t = x(t)$ for the time-dependent functions. The brackets denote the statistical average over all possible histories of the noise ξ. We introduced the auxiliary field $\mathrm{i}\hat{x}_t$ to exponentiate the δ-function that imposes that x_t be the solution to the Langevin equation at all times (for a given history of ξ). We wrote the latter constraint in the compact form $\mathrm{Eq}_t[x,\xi;\alpha] = 0$. The Jacobian $\mathcal{J}$ is given by

$$\mathcal{J} \equiv \det_{tt'} \left[\frac{\delta \mathrm{Eq}_t[x,\xi;\alpha]}{\delta x_{t'}} \right]. \tag{2.14}$$

The calculations detailed in appendix D are similar to the ones in [7, 8, 51, 52], and yield for the case of the 'undrifted' Langevin equation (2.1)

$$P[x;\alpha] \propto \int \mathcal{D}[\hat{x}] \, P[x,\mathrm{i}\hat{x};\alpha] \qquad \text{with} \qquad P[x,\mathrm{i}\hat{x};\alpha] = \mathrm{e}^{S[x,\mathrm{i}\hat{x};\alpha]} \tag{2.15}$$

and the Martin–Siggia–Rose–Janssen–deDominicis (MSRJD) action [9, 11]

$$S[x,\mathrm{i}\hat{x};\alpha] \equiv \int [-\mathrm{i}\hat{x}_t(\dot{x}_t - f_t + 2D\alpha g_t' g_t) + D(\mathrm{i}\hat{x}_t)^2 g_t^2 - \alpha f_t'] + \ln P_{\mathrm{i}}(x_{-\mathcal{T}}). \tag{2.16}$$

This action coincides with the form given in [6, 11]. It differs from the one in [12] since the authors used a post-point prescription in the stochastic equation (i.e. $\alpha = 1$) while using a mid-point Stratonovitch prescription in the construction of the path integral formalism. We disagree with the statements made in [13, 14] concerning the invalidity of action functional in [11]. Note that the use of a non-linear change of variable within the path integral is known to be problematic, even when starting from a mid-point discretisation, unless the underlying discretisation is treated with great care. The non-trivial effects of non-linear transformations were already observed in a quantum field theory context [53–56] and they appear within stochastic field theory as well [7, 9]. In other words, covariance of the action functional under general coordinate transformations comes with highly non-trivial treatment of the underlying discretization prescription.

With the addition of the drift force that ensures the approach to the usual Gibbs–Boltzmann equilibrium, see equation (2.8), the MSRJD action reads

$$S[x,\mathrm{i}\hat{x};\alpha] = \int \{-\mathrm{i}\hat{x}_t[\dot{x}_t + g_t^2 V_t' - 2D(1-2\alpha)g_t' g_t] + D(\mathrm{i}\hat{x}_t)^2 g_t^2$$
$$-\alpha \partial_x[-g_t^2 V_t' + 2D(1-\alpha)g_t g_t']\} + \ln P_{\mathrm{i}}(x_{-\mathcal{T}}). \tag{2.17}$$

2.4. Fluctuations

The stochastic nature of the dynamics is responsible for fluctuations of the field (here x_t) and more generally of all the possible physical observables that depend on this field (i.e. any $A(x_t)$). Amongst the few universal results that apply to these dynamics, there

is a class of exact relations between the path probabilities that are very precious since they lead to strong relations between observables. In a functional formalism, these relations can be proven by (1) making use of physical symmetries or broken symmetries of the system or its dynamics, (2) exploiting the invariance of the generating functional under a dummy linear change of integration variables. In this section, we shall discuss these relations in the context of stochastic Markov processes with multiplicative white noise such as the ones defined by equation (2.11).

2.4.1. Relation between path probabilities Let us consider the cases in which the force f depends on a set of externally controlled, possibly time-dependent, parameters λ_t. The stochastic process is characterised by the path integral (2.15) that expresses the joint probability distribution, $P[x, i\hat{x}; \alpha, \lambda]$, of the time series $\{x_t, i\hat{x}_t\}$ of the physical and the auxiliary fields, in the α-prescription, and under the set of parameters λ_t. Following Crooks [33], we ask how does $P[x, i\hat{x}; \alpha, \lambda]$ compare to the probability distribution of the transformed time-dependent variables $\{\mathcal{T}x_t, \mathcal{T}i\hat{x}_t\}$ in another discretisation prescription, $\overline{\alpha}$, and, possibly, under a transformed set of parameters, $\overline{\lambda}_t$. By choosing adequately the transformation rules $\mathcal{T}x$, $\mathcal{T}i\hat{x}$, $\overline{\alpha}$ and $\overline{\lambda}$ we will obtain relations of the type

$$\frac{P[\mathcal{T}x, \mathcal{T}i\hat{x}; \overline{\alpha}, \overline{\lambda}]}{P[x, i\hat{x}; \alpha, \lambda]} = e^{\Delta S[x, i\hat{x}; \alpha, \lambda]}. \tag{2.18}$$

We have distinguished the notation for the transformation of the dynamical fields, $\mathcal{T}x, \mathcal{T}i\hat{x}$, from the changes in the discretisation parameter, $\overline{\alpha}$, and the external time-dependent parameter, $\overline{\lambda}_t$.

The relation (2.18) implies, for the average of a generic function A of the physical and auxiliary fields (but, for simplicity, not of their time derivatives)

$$\int \mathcal{D}[x, \hat{x}] \, A[x, i\hat{x}] \, P[x, i\hat{x}; \overline{\alpha}, \overline{\lambda}]$$
$$= \int \mathcal{D}[\mathcal{T}x, \mathcal{T}\hat{x}] \, A[\mathcal{T}x, \mathcal{T}i\hat{x}] \, P[\mathcal{T}x, \mathcal{T}i\hat{x}; \overline{\alpha}, \overline{\lambda}]$$
$$= \int \mathcal{D}[\mathcal{T}x, \mathcal{T}\hat{x}] \, A[\mathcal{T}x, \mathcal{T}i\hat{x}] \, P[x, i\hat{x}; \alpha, \lambda] \, e^{\Delta S[x, i\hat{x}; \alpha, \lambda]}. \tag{2.19}$$

Moreover, if the measure over the transformed fields can be related to the one over the original ones with a unit Jacobian, and if the domain of integration at each time slice, here the real axis, is unchanged or can be taken back to the real axis, then the relation above becomes

$$\int \mathcal{D}[x, \hat{x}] \, A[x, i\hat{x}] \, P[x, i\hat{x}; \overline{\alpha}, \overline{\lambda}]$$
$$= \int \mathcal{D}[x, \hat{x}] \, A[\mathcal{T}x, \mathcal{T}i\hat{x}] \, e^{\Delta S[x, i\hat{x}; \alpha, \lambda]} \, P[x, i\hat{x}, \alpha, \lambda]. \tag{2.20}$$

Writing $A[\mathcal{T}x, \mathcal{T}i\hat{x}]$ as a new function of the original fields x and $i\hat{x}$, say $B[x, i\hat{x}] \equiv A[\mathcal{T}x, \mathcal{T}i\hat{x}]$, one has a generic relation between averages of different functions:

$$\int \mathcal{D}[x, \hat{x}] \, A[x, i\hat{x}] \, P[x, i\hat{x}; \overline{\alpha}, \overline{\lambda}]$$
$$= \int \mathcal{D}[x, \hat{x}] \, B[x, i\hat{x}] \, e^{\Delta S[x, i\hat{x}; \alpha, \lambda]} \, P[x, i\hat{x}; \alpha, \lambda]. \tag{2.21}$$

With different choices of the function A, and their associated B, one can derive various relation. In particular, choosing $A = 1$,

$$1 = \langle e^{\Delta S[x, i\hat{x}; \alpha, \lambda]} \rangle. \tag{2.22}$$

In section 2.4.2, we will identify the transformation $\mathcal{T}_{\mathrm{eq}}$, associated with the time-reversal invariance of equilibrium dynamics, that leaves the probability density invariant ($\Delta S = 0$) whenever the system is subject to equilibrium conditions, meaning initial conditions drawn from the Gibbs–Boltzmann distribution $P_{\mathrm{GB}} \propto e^{-V/D}$ and dynamics given by the Langevin equation equation (2.11), with a drift force deriving from the same confining potential V, and in contact with a thermal bath at the same temperature such that $k_{\mathrm{B}}T = D$. We later use this invariance to derive generic properties of equilibrium dynamics, such as the fluctuation-dissipation theorem (section 2.4.3). Out of equilibrium, $\Delta S \neq 0$, and we will derive in section 2.4.4 various fluctuation relations that have been extensively studied in recent years [17–26].

2.4.2. The time-reversal transformation We look for the invariance of the generating functional that corresponds to the time-reversal invariance of the equilibrium dynamics. For a clear discussion of time-reversal in the context of Markovian equations of motion, see [57, 58]. Although the action functional in equation (2.17) is relatively cumbersome, the identification of the correct field transformation that leaves it invariant can be simplified by the fact that one expects the time-reversal invariance to hold for the system and its environment separately. In other words, we expect the terms in the action that have their origin in the coupling to the bath to transform independently from the rest of the action. We identify them, see the discussion around equation (2.12), and collect them in

$$S_{\mathrm{diss}}[x, i\hat{x}] \equiv \int i\hat{x}_t [D i\hat{x}_t g_t^2 - \mathrm{d}_t^{(\alpha)} x_t] \tag{2.23}$$

where, to simplify notations, we defined

$$\mathrm{d}_t^{(\alpha)} x_t \equiv \mathrm{d}_t x_t - 2D(1 - 2\alpha) g_t g_t' \tag{2.24}$$

and, we recall, $D = k_{\mathrm{B}}T = \beta^{-1}$. For a field x_t corresponding to a physical quantity x that is even under time-reversal transformation (such as the particle's position), the transformation of the physical field must naturally be $x_t \mapsto x_{-t}$. The expression of S_{diss} in equation (2.23) suggests that we look for a transformation such that $\mathrm{d}_t^{(\alpha)} x_t$ behaves as a usual time derivative under time reversal, i.e.

$$\mathrm{d}_t^{(\alpha)} x_t \mapsto -\mathrm{d}_{-t}^{(\alpha)} x_{-t}. \tag{2.25}$$

This is only true if we simultaneously transform the discretisation parameter $\alpha \mapsto 1 - \alpha$. Altogether, we are led to propose the following transformation of the dynamical field x_t and its associated auxiliary field $i\hat{x}_t$

$$\mathcal{T}_{\mathrm{eq}} = \begin{cases} x_t & \mapsto x_{-t}, \\ i\hat{x}_t & \mapsto i\hat{x}_{-t} - D^{-1} g_{-t}^{-2} \mathrm{d}_{-t}^{(\alpha)} x_{-t}, \end{cases} \tag{2.26}$$

complemented with the transformation of the discretisation parameter

$$\alpha \mapsto \overline{\alpha} \equiv 1 - \alpha. \tag{2.27}$$

It is easy to check that S_{diss} is indeed invariant under this transformation:

$$
\begin{aligned}
S_{\mathrm{diss}}[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; \overline{\alpha}] &= \int \{[\mathrm{i}\hat{x}_{-t} - D^{-1}g_{-t}^{-2}\mathrm{d}_{-t}^{(\alpha)}x_{-t}] \\
&\qquad \times [\mathrm{d}_{-t}^{(\alpha)}x_{-t} + D\, g_{-t}^{2}\mathrm{i}\hat{x}_{-t} - Dg_{-t}^{2}\beta g_{-t}^{-2}\mathrm{d}_{-t}^{(\alpha)}x_{-t}]\} \\
&= \int \{[\mathrm{i}\hat{x}_{-t} - D^{-1}g_{-t}^{-2}\mathrm{d}_{-t}^{(\alpha)}x_{-t}]\, D\, g_{-t}^{2}\mathrm{i}\hat{x}_{-t}\} \\
&= \int \{[D\, g_{t}^{2}\mathrm{i}\hat{x}_{t} - \mathrm{d}_{t}^{(\alpha)}x_{t}]\, \mathrm{i}\hat{x}_{t}\} = S_{\mathrm{diss}}[x, \mathrm{i}\hat{x}; \alpha].
\end{aligned}
\tag{2.28}
$$

We now have to check that the remaining terms in the action functional are also invariant under the proposed transformation. In the potential case with no time-dependent parameter ($\partial_t \lambda_t = 0$) and a drift force ensuring the convergence to the usual Gibbs–Boltzmann equilibrium measure, the remaining terms are gathered into

$$
S_{\mathrm{det+jac}}[x, \mathrm{i}\hat{x}; \alpha] = \ln P_{\mathrm{i}}(x_{-\mathcal{T}}) - \int \mathrm{i}\hat{x}g_{t}^{2}V_{t}' - \alpha \int \partial_x[-g_{t}^{2}V_{t}' + 2D(1-\alpha)g_t g_t']
$$

and they transform as

$$
\begin{aligned}
S_{\mathrm{det+jac}}&[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; \overline{\alpha}] \\
&= \ln P_{\mathrm{i}}(x_{\mathcal{T}}) + \int [(-\mathrm{i}\hat{x}_{-t} + D^{-1}g_{-t}^{-2}\mathrm{d}_{-t}^{(\alpha)}x_{-t})g_{-t}^{2}V_{-t}' + (1-\alpha)\partial_{x_{-t}}(g_{-t}^{2}V_{-t}')] \\
&\qquad\qquad - 2D\alpha(1-\alpha)\int \partial_x(g_{-t}g_{-t}') \\
&= \ln P_{\mathrm{i}}(x_{\mathcal{T}}) + \int [-\mathrm{i}\hat{x}_t g_t^2 V_t' + D^{-1}\mathrm{d}_t^{(\alpha)}x_t V_t' + (1-\alpha)\partial_{x_t}(g_t^2 V_t')] \\
&\qquad\qquad - 2D\alpha(1-\alpha)\int \partial_x(g_t g_t').
\end{aligned}
\tag{2.29}
$$

We recognise that the first boundary term, $\ln P_{\mathrm{i}}(x_{\mathcal{T}})$, needs to be taken back to the initial time, $-\mathcal{T}$, if one wants to recover the original $S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha]$; the second and last terms are already part of $S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha]$; rewriting $1 - \alpha = \alpha + (1 - 2\alpha)$, the fourth term produces the last piece needed to fully reconstruct the original $S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha]$. All in all, we have

$$
\begin{aligned}
S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; \overline{\alpha}] &= S_{\mathrm{det+jac}}[x, \mathrm{i}\hat{x}, \alpha] + D^{-1}\int \mathrm{d}_t^{(\alpha)}x_t\, V_t' \\
&\quad + (1-2\alpha)\int \partial_{x_t}(g_t^2 V_t') + \ln P_{\mathrm{i}}(x_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}}).
\end{aligned}
\tag{2.30}
$$

Using the explicit form of $\mathrm{d}_t^{(\alpha)}x_t$ in equation (2.24), we can simplify this expression as follows

$$
\begin{aligned}
S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; \overline{\alpha}] &= S_{\mathrm{det+jac}}[x, \mathrm{i}\hat{x}; \alpha] + D^{-1}\int \mathrm{d}_t x_t\, V_t' \\
&\quad + (1-2\alpha)\int g_t^2\, V_t'' + \ln P_{\mathrm{i}}(x_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}}).
\end{aligned}
\tag{2.31}
$$

Replacing the term in $\mathrm{d}_t x_t \, V_t'$ above by using the chain rule of stochastic calculus recalled in equation (2.5) [28],

$$\mathrm{d}_t V_t = \mathrm{d}_t x_t \, V_t' + (1 - 2\alpha) D g_t^2 \, V_t'', \tag{2.32}$$

we obtain

$$\begin{aligned} S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}; \overline{\alpha}] = {}& S_{\mathrm{det+jac}}[x, \mathrm{i}\hat{x}; \alpha] \\ & + D^{-1} \int \mathrm{d}_t V_t + \ln P_{\mathrm{i}}(x_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}}). \end{aligned} \tag{2.33}$$

Finally, with initial conditions drawn from the Gibbs–Boltzmann distribution

$$P_{\mathrm{i}}(x_{-\mathcal{T}}) = Z^{-1} \, \mathrm{e}^{-D^{-1} V(x_{-\mathcal{T}})}, \tag{2.34}$$

we end the proof of the full invariance of the equilibrium action functional in equation (2.17) under the transformation $\mathcal{T}_{\mathrm{eq}}$ given in equation (2.26):

$$S[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}; \overline{\alpha}] = S[x, \mathrm{i}\hat{x}; \alpha] \tag{2.35}$$

and $\Delta S = 0$. Note that to achieve this invariance, there was a subtle interplay between the contributions coming from the deterministic part of the action and the ones coming from the α-dependent Jacobian.

This invariance of the action functional yields the following relation between path probabilities

$$P[x, \mathrm{i}\hat{x}; \alpha] \, \mathcal{D}[x, \hat{x}] = P[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}; \overline{\alpha}] \, \mathcal{D}[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}]. \tag{2.36}$$

After the transformation $\mathcal{T}_{\mathrm{eq}}$, the domain of integration of $\hat{x}_t$ at each time slice of the generating functional is shifted from the real axis to the complex line with a constant imaginary part $\mathrm{i}D^{-1}\mathrm{d}_t x_t$. Using the analyticity of $\exp S[x; \mathrm{i}\hat{x}; \alpha]$, one can return to an integration over the real axis by closing the contour at both infinities and by dropping the contributions of the vertical ends that vanish owing to the term $D(\mathrm{i}\hat{x}_t)^2$. Note also that the Jacobian associated to the change of variables $\{x, \mathrm{i}\hat{x}\} \mapsto \{\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}\}$ is unity. Finally, $\mathcal{D}[x, \hat{x}] = \mathcal{D}[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}]$ and we obtain the following relation between the forward and backward path probabilities

$$\frac{P_{\mathrm{B}}[x, \mathrm{i}\hat{x}]}{P_{\mathrm{F}}[x, \mathrm{i}\hat{x}]} = 1, \tag{2.37}$$

where we defined

$$P_{\mathrm{F}}[x, \mathrm{i}\hat{x}] \equiv P[x, \mathrm{i}\hat{x}, \alpha], \qquad P_{\mathrm{B}}[x, \mathrm{i}\hat{x}] \equiv P[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}, \bar{\alpha}]. \tag{2.38}$$

The relation (2.37) is valid whenever the system is in thermal equilibrium.

2.4.3. The fluctuation-dissipation theorem The fluctuation-dissipation theorem [59–64] is a model-independent relation between the linear response and the correlation of spontaneous equilibrium fluctuations of a given observable. The linear response of x with respect to a previous perturbation is defined as

$$R_\alpha(t, t') = \left.\frac{\delta\langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0}, \tag{2.39}$$

where the infinitesimal perturbation h couples linearly to the field x in such a way that the potential $V \to V_h = V - hx$ and, therefore, $V' \to V'_h = V' - h$. In the path-integral formulation, the linear response is given by

$$R_\alpha(t, t') = \int \mathcal{D}[x, \mathrm{i}\hat{x}] \, x_t \left.\frac{\delta S_h[x, \mathrm{i}\hat{x}; \alpha]}{\delta h_{t'}}\right|_{h=0} \mathrm{e}^{S[x, \mathrm{i}\hat{x}; \alpha]} \tag{2.40}$$

where the action has been modified as

$$S_h[x, \mathrm{i}\hat{x}; \alpha] = S[x, \mathrm{i}\hat{x}; \alpha] + \int h_t \left[\mathrm{i}\hat{x}_t g_t^2 - 2\alpha g_t g'_t\right]. \tag{2.41}$$

Therefore, the linear response is expressed as a correlation function reading

$$R_\alpha(t, t') = \langle x_t[\mathrm{i}\hat{x}_{t'} g_{t'}^2 - 2\alpha g_{t'} g'_{t'}]\rangle_{S[x, \mathrm{i}\hat{x}; \alpha]} \tag{2.42}$$

where the average has to be taken with the measure given by the unperturbed action $S[x, \mathrm{i}\hat{x}; \alpha]$. The subindex α expresses the fact that the stochastic process is defined with a discretisation parameter α. Exchanging momentarily α by $1 - \alpha$ one has

$$R_{1-\alpha}(t, t') = \langle x_t[\mathrm{i}\hat{x}_{t'} g_{t'}^2 - 2(1 - \alpha) g_{t'} g'_{t'}]\rangle_{S[x, \mathrm{i}\hat{x}; 1-\alpha]}. \tag{2.43}$$

Take now the expression in equation (2.43) and perform the change variables in the path integral from $\{x_t, \mathrm{i}\hat{x}_t\}$ to $\{\mathcal{T}_{\mathrm{eq}} x_t, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}_t\}$:

$$R_{1-\alpha}(t, t') = \langle \mathcal{T}_c x_t[\mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}_{t'}\mathcal{T}_{\mathrm{eq}}g_{t'}^2 - 2(1-\alpha)\mathcal{T}_{\mathrm{eq}}g_{t'}\mathcal{T}_{\mathrm{eq}}g'_{t'}]\rangle_{S[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; 1-\alpha]} \tag{2.44}$$

where $\mathcal{T}_{\mathrm{eq}}g = g(\mathcal{T}_{\mathrm{eq}}x_t)$ and similarly for $\mathcal{T}_{\mathrm{eq}}g'$. Using that $S[\mathcal{T}_{\mathrm{eq}}x, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}; 1 - \alpha] = S[x, \mathrm{i}\hat{x}; \alpha]$, and applying the transformation $\mathcal{T}_{\mathrm{eq}}$ defined in equation (2.26) to the function of x_t and $\mathrm{i}\hat{x}_{t'}$ to be averaged, one has

$$
\begin{aligned}
R_{1-\alpha}(t, t') &= \langle x_{-t}\{[\mathrm{i}\hat{x}_{-t'} - D^{-1}g_{-t'}^{-2}\mathrm{d}_{-t'}^{(\alpha)}x_{-t'}g_{-t'}^2 - 2(1-\alpha)g_{-t'}g'_{-t'}]\}\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]} \\
&= \langle x_{-t}[\mathrm{i}\hat{x}_{-t'}g_{-t'}^2 - 2(1-\alpha)g_{-t'}g'_{-t'}]\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]} - D^{-1}\langle x_{-t}\mathrm{d}_{-t'}^{(\alpha)}x_{-t'}]\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]} \\
&= \langle x_{-t}[\mathrm{i}\hat{x}_{-t'}g_{-t'}^2 - 2\alpha g_{-t'}g'_{-t'}]\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]} \\
&\quad - \langle x_{-t}2(1-2\alpha)g_{-t'}g'_{-t'}\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]} - D^{-1}\langle x_{-t}\mathrm{d}_{-t'}^{(\alpha)}x_{-t'}]\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]}
\end{aligned}
$$

Identifying $R_\alpha(-t, -t')$ in the first term in the rhs and using now $\mathrm{d}_{-t}^{(\alpha)}x_{-t} = \mathrm{d}_{-t}x_{-t} - 2D(1 - 2\alpha)g_{-t}g'_{-t}$,

$$R_{1-\alpha}(t, t') = R_\alpha(-t, -t') + D^{-1}\langle x_{-t}\mathrm{d}_{t'}x_{-t'}\rangle_{S[\mathrm{i}\hat{x}, x; \alpha]}. \tag{2.45}$$

Using the fact that the physics cannot depend on the discretisation parameter (see the discussion below the drifted Fokker–Planck equation (2.9)), we can drop the irrelevant index α (or $1 - \alpha$) in the linear response and the correlation function and

$$R(t, t') = R(-t, -t') + D^{-1}\partial_{t'}C(-t, -t'). \tag{2.46}$$

We apply the transformation $\mathcal{T}_{\mathrm{eq}}$ once again on the correlation function in the rhs to show $C(-t, -t') = C(t, t')$. Owing to the time-translational invariance of equilibrium dynamics, $C(t, t') = C(\tau)$ and $R(t, t') = R(\tau)$ where $\tau \equiv t - t'$, and to the causality of the response $R(\tau) = 0$ for $\tau < 0$, we obtain the celebrated fluctuation-dissipation theorem (FDT)

$$R(\tau) = -D^{-1}\Theta(\tau)\mathrm{d}_\tau C(\tau) = -\beta\Theta(\tau)\mathrm{d}_\tau C(\tau). \tag{2.47}$$

Here $\Theta(\tau)$ is the Heaviside step function.

2.4.4. Broken symmetry and fluctuation theorems There are various ways to drive a system out of equilibrium, e.g. by changing an external parameter in the potential in time, λ_t, or by using initial conditions that are not in equilibrium, $P_{\mathrm{i}}(x_{-\mathcal{T}}) \neq P_{\mathrm{GB}}(x_{-\mathcal{T}})$. In these cases, the associated time-reversal symmetry of the dynamics is broken and the action functional is no longer invariant under the field transformation $\mathcal{T}_{\mathrm{eq}}$. In practice, this means that ΔS defined in equation (2.18) does not vanish and the very same transformation technique that we used earlier can now be used to derive exact out-of-equilibrium relations between path probabilities, the so-called fluctuation relations.

The time-reversed dynamics corresponds to evaluating the action functional in the transformed fields $\mathcal{T}_{\mathrm{eq}}x_t$ and $\mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}_t$, the discretisation prescription parameter $\overline{\alpha} = 1 - \alpha$, and the time-reversed protocol

$$\overline{\lambda}_t = \lambda_{-t}. \tag{2.48}$$

In order to evaluate ΔS, one first notices that the dissipative part of the action S_{diss} does not depend upon the applied force nor the initial condition. Therefore, it remains invariant under $\mathcal{T}_{\mathrm{eq}}$. However, as $S_{\mathrm{det}+\mathrm{jac}}$ depends on both the initial distribution and the force, we expect it to yield $\Delta S \neq 0$. More precisely, the terms contributing to ΔS are given in equation (2.31) where V'_t and V''_t are the first and second derivative with respect to the variable x of the potential V, $V'_t = \partial_x V_t$ and $V''_t = \partial_x^2 V$, respectively. If the potential depends on the time-dependent parameter λ_t, its total time derivative expressed in equation (2.32) acquires an extra term,

$$\mathrm{d}_t V_t = \mathrm{d}_t\lambda \, \partial_\lambda V_t + \mathrm{d}_t x \, \partial_x V_t + (1 - 2\alpha)Dg_t^2 \, \partial_x^2 V_t, \tag{2.49}$$

and we use this new relation to replace $\mathrm{d}_t x \, \partial_x V_t$ in the first term in equation (2.31):

$$\begin{aligned}
S_{\mathrm{det}+\mathrm{jac}}[\mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{x}, \mathcal{T}_{\mathrm{eq}}x; \overline{\alpha}, \overline{\lambda}] &= S_{\mathrm{det}+\mathrm{jac}}[\mathrm{i}\hat{x}, x; \alpha, \lambda] \\
&\quad + \beta\int\mathrm{d}_t V_t - \beta\int\mathrm{d}_t\lambda_t \, \partial_\lambda V_t + \ln P_{\mathrm{i}}(x_{\mathcal{T}}, \lambda_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}}) \\
&= S_{\mathrm{det}+\mathrm{jac}}[\mathrm{i}\hat{x}, x; \alpha, \lambda] - D^{-1}\int\mathrm{d}_t\lambda_t \, \partial_\lambda V_t + D^{-1}V(x_{\mathcal{T}}, \lambda_{\mathcal{T}}) \\
&\quad - D^{-1}V(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}}) + \ln P_{\mathrm{i}}(x_{\mathcal{T}}, \lambda_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}}). \tag{2.50}
\end{aligned}$$

Here, we made explicit the dependence on the time-dependent parameter λ_t of the potential and the initial probability distribution function. The second term in the rhs is related to the work done by the time-dependent potential force

$$W = \int \mathrm{d}_t \lambda_t \; \partial_\lambda V_t. \tag{2.51}$$

For generic P_{i} we cannot simplify further the last four terms in the rhs of equation (2.50) and ΔS is the stochastic entropy, defined as the sum of the Shannon entropy $[\ln P_{\mathrm{i}}(x_{\mathcal{T}}, \lambda_{\mathcal{T}}) - \ln P_{\mathrm{i}}(x_{-\mathcal{T}})]$ and the heat transfer ($\beta \mathcal{Q} = \beta \Delta V - \beta W$ and $\beta = D^{-1}$).

If, instead, we assume that the system is initially prepared in the Gibbs–Boltzmann distribution at temperature $k_{\mathrm{B}} T = D = \beta^{-1}$, under a potential $V(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}})$,

$$P_{\mathrm{i}}(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}}) = Z^{-1}(\lambda_{-\tau}) \; \exp[-\beta V(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}})], \tag{2.52}$$

we find

$$
\begin{aligned}
S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}, \mathcal{T}_{\mathrm{eq}} x; \overline{\alpha}, \overline{\lambda}] = \; & S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha, \lambda] - \beta \int \mathrm{d}_t \lambda_t \; \partial_\lambda V_t \\
& - \ln Z(\lambda_{\mathcal{T}}) + \ln Z(\lambda_{-\mathcal{T}}).
\end{aligned}
\tag{2.53}
$$

The last two terms can be regrouped into

$$\beta \Delta F = \beta[F(\lambda_{\mathcal{T}}) - F(\lambda_{-\mathcal{T}})] = -\ln Z(\lambda_{\mathcal{T}}) + \ln Z(\lambda_{-\mathcal{T}}), \tag{2.54}$$

the free-energy difference between the equilibrium state at the final and initial value of the parameter λ_t. Therefore,

$$S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}, \mathcal{T}_{\mathrm{eq}} x; \overline{\alpha}, \overline{\lambda}] = S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha, \lambda] - \beta W + \beta \Delta F. \tag{2.55}$$

and ultimately

$$
\begin{aligned}
\Delta S &= S_{\mathrm{det+jac}}[\mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}, \mathcal{T}_{\mathrm{eq}} x; \overline{\alpha}, \overline{\lambda}] - S_{\mathrm{det+jac}}[\mathrm{i}\hat{x}, x; \alpha, \lambda] \\
&= -\beta W + \beta \Delta F.
\end{aligned}
\tag{2.56}
$$

In conclusion, we obtain the following relation between the forward and backward path probabilities

$$\frac{P_{\mathrm{B}}[x, \mathrm{i}\hat{x}]}{P_{\mathrm{F}}[x, \mathrm{i}\hat{x}]} = \mathrm{e}^{\Delta S}, \tag{2.57}$$

where we defined

$$P_{\mathrm{F}}[x, \mathrm{i}\hat{x}] \equiv P[x, \mathrm{i}\hat{x}; \alpha, \lambda], \qquad P_{\mathrm{B}}[x, \mathrm{i}\hat{x}] \equiv P[\mathcal{T}_{\mathrm{eq}} x, \mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{x}; \bar{\alpha}, \bar{\lambda}]. \tag{2.58}$$

Multiplying both sides of equation (2.57) by A, a generic observable which can depend on x and $\mathrm{i}\hat{x}$, and summing over all paths, one obtains

$$\langle A(x, \mathrm{i}\hat{x}) \rangle_{\mathrm{B}} = \langle A(x, \mathrm{i}\hat{x}) \mathrm{e}^{\Delta S} \rangle_{\mathrm{F}}, \tag{2.59}$$

where the subscripts F and B stand for averaging with the forward and backward path probability distributions defined in equation (2.58).

In particular, setting $A = 1$, one recovers the Jarzynski relation [17, 65]

$$\langle \mathrm{e}^{-\beta W} \rangle = \mathrm{e}^{-\beta \Delta F}. \tag{2.60}$$

Other fluctuation relations can be found by choosing other observables A.

2.5. Schwinger–Dyson equations

We end the analysis of the single-variable problem by presenting an easy derivation of the Schwinger–Dyson equations which govern the coupled dynamics of correlations and linear responses. The proof is based on the use of another set of transformation rules that leave the action and measure invariant and hold in general.

2.5.1. Out-of-equilibrium symmetry Let us consider the most generic out-of-equilibrium situation, i.e. work with the original Langevin equation (2.1) without making any assumption on the force f, that can possibly be time-dependent. We recall that the MSRJD action functional associated to the dynamics reads

$$S[x, \mathrm{i}\hat{x}; \alpha] = \int [-\mathrm{i}\hat{x}_t(\dot{x}_t - f_t + 2D\alpha g_t g_t' - D g_t^2 \mathrm{i}\hat{x}_t) - \alpha f_t'] + \ln P_\mathrm{i}(x_{-\mathcal{T}}).$$

This action is invariant under the transformation [44]

$$\mathcal{T}_\mathrm{eom} \equiv \begin{cases} x_t & \mapsto & x_t, \\ \mathrm{i}\hat{x}_t & \mapsto & -\mathrm{i}\hat{x}_t + D^{-1} g_t^{-2}(\dot{x}_t - f_t + 2\alpha D g_t g_t'), \end{cases} \tag{2.61}$$

with no need to change the parameter α. We do not reproduce here the proof of invariance as it is rather straightforward.

2.5.2. Ward–Takahashi identities Let us use this invariance of the action functional in the expression of the linear response to a perturbation h such that $f \to f + g^2 h$

$$\begin{aligned} R(t, t') &= \langle x_t(\mathrm{i}\hat{x}_{t'} g_{t'}^2 - 2\alpha g_{t'} g_{t'}') \rangle \\ &= -\langle x_t \mathrm{i}\hat{x}_{t'} g_{t'}^2 \rangle + D^{-1}\langle x_t(\dot{x}_{t'} - f_{t'}) \rangle \\ &= -R(t, t') - 2\alpha \langle x_t g_{t'} g_{t'}' \rangle + D^{-1}\langle x_t(\dot{x}_{t'} - f_{t'}) \rangle. \end{aligned} \tag{2.62}$$

We obtain

$$2DR(t, t') = \partial_{t'} C(t, t') - \langle x_t f_{t'} \rangle - 2\alpha D\langle x_t g_{t'} g_{t'}' \rangle. \tag{2.63}$$

If one uses the drift force given in equation (2.8) to ensure convergence to equilibrium with a Gibbs–Boltzmann measure $P_\mathrm{GB} \propto \mathrm{e}^{-V/D}$, this yields

$$2DR(t, t') = \partial_{t'} C(t, t') + \langle x_t g_{t'}^2 V_{t'}' \rangle - 2D\langle x_t g_{t'} g_{t'}' \rangle. \tag{2.64}$$

2.5.3. Composition of $\mathcal{T}_\mathrm{eq}$ and $\mathcal{T}_\mathrm{eom}$ For equilibrium conditions, when working with the drifted Langevin equation (2.11), both transformations $\mathcal{T}_\mathrm{eq}$ and $\mathcal{T}_\mathrm{eom}$ are symmetries of the generating functional and therefore, their composition

$$\mathcal{T}_{\mathrm{eq}} \circ \mathcal{T}_{\mathrm{eom}} \equiv \begin{cases} x_t & \mapsto x_{-t} \\ \mathrm{i}\hat{x}_t & \mapsto -\mathrm{i}\hat{x}_{-t} + D^{-1}V'_{-t} \end{cases} \tag{2.65}$$

complemented with the transformation of the discretisation parameter

$$\alpha \mapsto \overline{\alpha} \equiv 1 - \alpha. \tag{2.66}$$

is also a symmetry of the generating functional. Starting from the expression of the linear response, we derive

$$\begin{aligned} R(t, t') &= \langle x_t \mathrm{i}\hat{x}_{t'} \rangle - 2\alpha \langle x_t g_{t'} g'_{t'} \rangle \\ &= \langle x_{-t}(-\mathrm{i}\hat{x}_{-t'} + D^{-1}V'_{-t'}) \rangle - 2(1-\alpha)\langle x_{-t} g_{-t'} g'_{-t'} \rangle \\ &= -R(-t, -t') + D^{-1}\langle x_{-t} V'_{-t'} \rangle - 2\langle x_{-t} g_{-t'} g'_{-t'} \rangle. \end{aligned} \tag{2.67}$$

Using the causality of the linear response, applying the transformation one more time, using the time-translational invariance of equilibrium dynamics, and $D = k_{\mathrm{B}}T = \beta^{-1}$ we finally obtain the relation

$$R(t - t') = \Theta(t - t') \left[\beta \langle x_t V'_{t'} \rangle - 2\langle x_t g_{t'} g'_{t'} \rangle \right]. \tag{2.68}$$

This is a generalisation of the relations found in [66–68] for additive-noise stochastic processes, see also [16]. The special interest of this kind of relation is that it allows to compute the linear response, notably in a numerical evaluation, without any need to apply a perturbation, by taking advantage of the expression of R as the sum of two correlation terms.

3. Multi-variable stochastic Markov processes

In section 2, we focused on a stochastic differential equation of a single variable x. In a more general situation, the stochastic variable can be a multi-dimensional vector. In this section, we focus on the stochastic Landau–Lifshitz–Gilbert (sLLG) equation [30, 31], a Langevin equation describing the dynamics of a classical magnetic moment, a $3d$ vector $\mathbf{M}$, in contact with an environment. We analyse the time-reversal transformation of the magnetisation and the auxiliary vector $\hat{\mathbf{M}}$ that leaves the action invariant under equilibrium conditions. We derive some of its consequences, such as the fluctuation-dissipation theorem. Furthermore, we also analyse the out-of-equilibrium dynamics driven by a spin-polarised current of electrons and we derive the corresponding fluctuation relations.

3.1. The sLLG magnetisation dynamics

In [32], we gave a detailed presentation of the stochastic Landau–Lifshitz–Gilbert (sLLG) equation that describes the dynamics of a magnetic moment $\mathbf{M}$ under the influence of a deterministic local magnetic field $\mathbf{H}_{\mathrm{eff}}$ and a thermal environment responsible for both dissipation and a fluctuating local magnetic field $\mathbf{H}(t)$. Studying this equation in various discretisation prescriptions, we showed that unless the Stratonovich

mid-point is used, a drift term is needed to conserve the magnetisation modulus in the course of time, $|\mathbf{M}| = M_s$, and to ensure the approach to Gibbs–Boltzmann equilibrium in the absence of non-conservative and time-dependent forces. Numerical checks of this fact were discussed in [69]. In [32], we also constructed the path-integral formalism for the generating functional in the Cartesian and spherical coordinate systems. Adapting the results of section 2, i.e. identifying the field transformation that generalises the one in equation (2.26) to the physical problem at hand, we derive the corresponding fluctuation-dissipation theorem as well as the fluctuation relations.

In the so-called Gilbert formulation, the sLLG equation reads [70]

$$\mathrm{D}_t^{(\alpha)}\mathbf{M} = -\gamma_0 \mathbf{M} \wedge \left[\mathbf{H}_{\mathrm{eff}} + \mathbf{H}(t) - \frac{\eta}{M_s}\mathrm{D}_t^{(\alpha)}\mathbf{M} \right]. \tag{3.1}$$

This equation has to be understood in the generic α-prescription: $\overline{\mathbf{M}}_n \equiv \mathbf{M}_n + \alpha(\mathbf{M}_{n+1} - \mathbf{M}_n)$. The α-derivative is defined as

$$\mathrm{D}_t^{(\alpha)} \equiv \mathrm{d}_t + 2D(1-2\alpha)\frac{\gamma_0^2}{1+\eta^2\gamma_0^2}, \tag{3.2}$$

and satisfies $\mathrm{D}_{-t}^{(1-\alpha)} = -\mathrm{D}_t^{(\alpha)}$. The second term in $\mathrm{D}_t^{(\alpha)}$ is necessary to ensure the conservation of the modulus of the magnetic moment, $M_s = |\mathbf{M}|$, and the approach to the Gibbs–Boltzmann equilibrium, see equation (3.7), in the absence of non-conservative forces and time-dependent magnetic fields [32]. We note that $\mathrm{D}_t^{(\alpha)}$ in the last term between the square brackets in equation (3.1) can be simply replaced by d_t as the second term in equation (3.2) does not contribute due to the vector product with $\mathbf{M}$.

The chain rule for the time derivative of a function of the magnetisation vector governed by equation (3.1) reads [32]

$$\mathrm{d}_t U(\mathbf{M}) = \frac{\partial U(\mathbf{M})}{\partial M_i}\mathrm{d}_t M_i + (1-2\alpha)\frac{D\gamma_0^2}{1+\eta^2\gamma_0^2}P_{ij}^{\perp}\frac{\partial^2 U(\mathbf{M})}{\partial M_i \partial M_j}, \tag{3.3}$$

where $P_{ij}^{\perp} \equiv M_s^2\delta_{ij} - M_i M_j$ is the projector on the plane perpendicular to $\mathbf{M}$. A straightforward application to $\mathbf{M}^2$ together with $\mathbf{M} \cdot \mathrm{D}_t^{(\alpha)}\mathbf{M} = 0$ yields $\mathrm{d}_t\mathbf{M}^2 = 0$ and hence the conservation of the modulus.

The sLLG equation depends on several parameters. $\gamma_0 = \gamma\mu_0$ is the product of γ, the gyromagnetic ratio relating the magnetisation to the angular momentum, and μ_0, the vacuum permeability constant. The gyromagnetic factor is given by $\gamma = \mu_{\mathrm{B}}g/\hbar$ (in our convention $\gamma > 0$ for the electronic spin) with μ_{B} Bohr's magneton and g Lande's g-factor.

The Gaussian white noise $\mathbf{H}(t)$ acts multiplicatively on the magnetisation. It has zero average and correlations characterised by the diffusion constant D,

$$\langle H_i(t)\rangle_{\mathbf{H}} = 0, \qquad \langle H_i(t)H_j(t')\rangle_{\mathbf{H}} = 2D\,\delta_{ij}\delta(t-t'). \tag{3.4}$$

We assume that the environment is in equilibrium at the temperature $k_{\mathrm{B}}T \equiv \beta^{-1}$ yielding the Einstein relation

$$D = \frac{\eta k_{\mathrm{B}} T}{M_s V \mu_0} \qquad (3.5)$$

with V the volume of the system and η the friction coefficient that also appears in the last, dissipative, term in the rhs of equation (3.1). Indeed, the term $-\eta \mathrm{D}_t^{(\alpha)} \mathbf{M}$ induces dissipation in the form introduced by Gilbert [71].

The deterministic magnetic field $\mathbf{H}_{\mathrm{eff}}$ collects conservative and non-conservative contributions:

$$\mathbf{H}_{\mathrm{eff}} = \mathbf{H}_{\mathrm{eff}}^{\mathrm{c}} + \mathbf{H}_{\mathrm{eff}}^{\mathrm{nc}}. \qquad (3.6)$$

The former can be derived from a potential energy density U as $\mathbf{H}_{\mathrm{eff}}^{\mathrm{c}} = -\mu_0^{-1} \nabla_{\mathbf{M}} U$ whereas the latter does not admit such a representation. U can possibly have contributions from an externally applied magnetic field $\mathbf{H}_{\mathrm{ext}}$ (that we assume to be constant for simplicity) and from a local magnetic field typically generated by the anisotropy potential of the local crystal structure (the so-called crystal field)

$$U(\mathbf{M}) = -\mu_0 \mathbf{M} \cdot \mathbf{H}_{\mathrm{ext}} + V_{\mathrm{ani}}(\mathbf{M}). \qquad (3.7)$$

We shall only consider the case of time-reversal symmetric potentials, i.e. with the property $U(-\mathbf{M}, -\mathbf{H}_{\mathrm{ext}}) = U(\mathbf{M}, \mathbf{H}_{\mathrm{ext}})$.

A timely example of a non-conservative $\mathbf{H}_{\mathrm{eff}}^{\mathrm{nc}}$ is the so-called spin-torque exchange. In the context of spintronics, the manipulation of the local magnetisation is performed by circulating a spin-polarised current of electrons through the ferromagnetic material. This can exchange angular momentum with the magnetisation via the spin-torque term

$$\mathbf{H}_t^{\mathrm{torq}} = \chi_t \, \mathbf{M}_t \wedge \mathbf{p}_t \qquad (3.8)$$

where χ_t is a time-dependent parameter that is proportional to the externally controlled current J_t, and $\mathbf{p}_t$ is a unit vector indicating the spin polarisation of the incoming electrons that they typically acquire earlier by going through a thick layer of ferromagnetic material with a fixed magnetisation. Dimensional analysis yields $[\chi] = [k_{\mathrm{B}} T/(M_s^2 \mu_0 V)] = [H/M_s] = [(\gamma_0 t M_s)^{-1}]$.

Due to the fact that the magnetic fields appear under a vector product with the magnetisation vector, only their projection on the perpendicular plane to $\mathbf{M}$ have an effect on the magnetic moment dynamics.

For $\mathbf{H}_{\mathrm{eff}}^{\mathrm{nc}} = 0$ and $\mathbf{H}_{\mathrm{eff}}^{\mathrm{c}} = -\mu_0^{-1} \boldsymbol{\nabla}_{\mathbf{M}} U$, the dynamics approach the Gibbs–Boltzmann distribution

$$P_{\mathrm{GB}}(\mathbf{M}) = Z^{-1} \, \mathrm{e}^{-\beta V U(\mathbf{M})}. \qquad (3.9)$$

Note that the partition function $Z = \int \mathrm{d}\mathbf{M} \, \mathrm{e}^{-\beta V U(\mathbf{M})}$ is a function of the inverse temperature, β, the external field, $\mathbf{H}_{\mathrm{ext}}$, and the parameters of the anisotropy potential V_{ani}. One has $Z(\mathbf{H}_{\mathrm{ext}}) = Z(-\mathbf{H}_{\mathrm{ext}})$.

From this stationary distribution, one simply shows the *static fluctuation-dissipation relation* between equilibrium susceptibility and magnetic fluctuation correlations:

$$\frac{\partial \langle M_i \rangle}{\partial H_{\text{ext}\,j}}\bigg|_{\mathbf{H}_{\text{ext}}=0} = \beta \mu_0 V \, \langle (M_i - \langle M_i \rangle)(M_j - \langle M_j \rangle) \rangle. \tag{3.10}$$

We will prove the time-dependent fluctuation-dissipation theorem for this problem below (see also [72]).

For simplicity, we study the dynamics for a system initially prepared in equilibrium with a Gibbs–Boltzmann distribution $P_{\text{i}}(\mathbf{M}_{-\mathcal{T}}) = P_{\text{GB}}(\mathbf{M}_{-\mathcal{T}})$. In particular, we set $\chi_{-\mathcal{T}} = J_{-\mathcal{T}} = 0$. In the absence of a drive, $\mathbf{H}_{\text{eff}}^{\text{nc}} = 0\ \forall t$ and for a time-independent effective field $\mathbf{H}_{\text{eff}}$, the system remains in thermal equilibrium. However, a finite drive $\mathbf{H}_{\text{eff}}^{\text{nc}} \neq 0$ or a time-dependent effective field $\mathbf{H}_{\text{eff}}(t)$ push the magnetic moment out of equilibrium.

3.2. The path-integral formulation

As shown in [32], the generating functional reads

$$\mathcal{Z}[\boldsymbol{\zeta}] = \int \mathcal{D}\,[\mathbf{M}] \int \mathcal{D}[\hat{\mathbf{M}}^\perp] \int \mathcal{D}[\hat{\mathbf{M}}^\parallel] \, \exp\left\{ S[\mathbf{M}, i\hat{\mathbf{M}}] + \int \boldsymbol{\zeta}_t \cdot \mathbf{M}_t \right\}. \tag{3.11}$$

$\int \mathcal{D}\,[\mathbf{M}]$ corresponds to integrating, at all times, over the vector field $\mathbf{M}$ on the 2-sphere of radius M_s, $\int \mathcal{D}\,[\hat{\mathbf{M}}^\perp]$ corresponds to integrating over the auxiliary real vector field $\hat{\mathbf{M}}^\perp$ in the plane perpendicular to $\mathbf{M}$. Correspondingly, $\int \mathcal{D}\,[\hat{\mathbf{M}}^\parallel]$ represents the integration over the auxiliary real vector field in plane parallel to $\mathbf{M}$. As $\mathbf{M} \cdot \mathrm{D}_t^{(\alpha)}\mathbf{M} = 0$, $\hat{\mathbf{M}}^\perp$ and $\mathrm{D}_t^{(\alpha)}\mathbf{M}$ are both perpendicular to $\mathbf{M}$ but not necessarily parallel.

The action functional can be expressed as the sum

$$S = S_{\text{det}} + S_{\text{diss}} + S_{\text{jac}} \tag{3.12}$$

with

$$S_{\text{det}} = -\beta V U(\mathbf{M}_{-\mathcal{T}}, \mathbf{H}_{\text{ext}_{-\mathcal{T}}}) - \ln Z(\mathbf{H}_{\text{ext}_{-\mathcal{T}}}) + \int i\hat{\mathbf{M}}_t^\parallel \cdot \mathrm{D}_t^{(\alpha)}\mathbf{M}_t$$
$$+ \int i\hat{\mathbf{M}}_t^\perp \cdot [M_s^{-2}\mathrm{d}_t\mathbf{M}_t \wedge \mathbf{M}_t + \gamma_0 \mathbf{H}_{\text{eff}\,t}], \tag{3.13}$$

$$S_{\text{diss}} = \gamma_0 \int i\hat{\mathbf{M}}_t^\perp \cdot \left[D\gamma_0\, i\hat{\mathbf{M}}_t^\perp - \frac{\eta}{M_s}\, \mathrm{d}_t\mathbf{M}_t \right]. \tag{3.14}$$

$$S_{\text{jac}} = \frac{\alpha \gamma_0}{1 + \eta^2 \gamma_0^2} \frac{1}{M_s} \int [M_s \epsilon_{ijk} M_{kt} \partial_j H_{\text{eff}_{it}}^{\text{nc},\perp} - \eta \gamma_0 P_{ij\,t}^\perp \partial_j H_{\text{eff}_{it}}^\perp]. \tag{3.15}$$

where $P_{ij}^\perp \equiv M_s^2 \delta_{ij} - M_i M_j$ is the projector on the plane perpendicular to $\mathbf{M}$ and ∂_j is a short-hand notation for ∂_{M_j}. S_{det} encodes the initial conditions and the deterministic forces. S_{diss} encodes the hybridisation with the thermal bath. The term in $i\hat{\mathbf{M}}_t^\parallel$ imposes the conservation of the modulus of the magnetisation vector. S_{jac} stems from

the unicity of the solution to equation (3.1) once the initial conditions and the noise history are specified. For magnetic fields that are independent of the magnetisation vector, $\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{ext}}$, S_{jac} vanishes.

The two-time correlation between any function of $\mathbf{M}$, $A(\mathbf{M})$, and, say, one component of the magnetisation, M_j, reads

$$C_{AM_j}(t, t') = \langle A(\mathbf{M}(t)) M_j(t') \rangle . \tag{3.16}$$

The linear response of the same observable A measured at time t to a previous perturbation $\widetilde{\mathbf{H}}$ that modifies the potential energy per unit volume U according to $U \mapsto U - \mu_0^{-1} \widetilde{\mathbf{H}} \cdot \mathbf{M}$ (or equivalently $\mathbf{H}_{\text{eff}} \mapsto \mathbf{H}_{\text{eff}} + \widetilde{\mathbf{H}}$) is

$$R_{AM_j}(t, t') = \frac{\delta \langle A(\mathbf{M}(t)) \rangle}{\delta \widetilde{H}_j(t')} \bigg|_{\widetilde{\mathbf{H}}=\mathbf{0}} = \langle A(\mathbf{M}(t)) \gamma_0 \mathrm{i} \hat{M}_j^{\perp}(t') \rangle. \tag{3.17}$$

3.3. Fluctuations

Equilibrium dynamics are ensured as long as the system is initially prepared in equilibrium at a given temperature T and under conservative forces, and that it is subsequently evolved under the same time-independent potential forces and in contact with an environment at the same temperature. In our setup, this corresponds to setting the drive to zero, i.e. $\mathbf{H}_{\text{eff}}^{\text{nc}} = 0$ at all times, and taking $\mathbf{H}_{\text{ext}}$ constant and V_{ani} not explicitly dependent on time.

3.3.1. The time-reversal transformation One can prove that the action in equations (3.12)–(3.15), and more generally the full generating functional, are invariant under the following variable and discretisation parameter transformations

$$\mathcal{T}_{\text{eq}} = \begin{cases} \mathbf{M}_t \mapsto -\mathbf{M}_{-t} , \\ \gamma_0 \, \mathrm{i}\hat{\mathbf{M}}_t^{\perp} \mapsto -\gamma_0 \, \mathrm{i}\hat{\mathbf{M}}_{-t}^{\perp} - \beta V \mu_0 \, \mathrm{d}_t \mathbf{M}_{-t} , \\ \mathrm{i}\hat{\mathbf{M}}_t^{\parallel} \mapsto \mathrm{i}\hat{\mathbf{M}}_{-t}^{\parallel}, \end{cases} \tag{3.18}$$

if one simultaneously changes the discretisation parameter

$$\alpha \mapsto \overline{\alpha} \equiv 1 - \alpha \tag{3.19}$$

and simultaneously reverses all external constant magnetic fields

$$\mathbf{H}_{\text{ext}} \mapsto \overline{\mathbf{H}}_{\text{ext}} \equiv -\mathbf{H}_{\text{ext}}. \tag{3.20}$$

This discrete symmetry of the dynamical action encodes all the features of equilibrium dynamics and it is broken out of equilibrium.

Case without anisotropy potential, $V_{\text{ani}} = 0$ Let us start the proof by treating the simpler case in which $\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{ext}}$. It is easy to see, similarly to the one-dimensional example of section 2, that the dissipative terms in S_{diss} are invariant independently of the other terms of the action functional. We do not reproduce here this calculation as it is quite straightforward. We simply mention that we do not need to use $\alpha \mapsto 1 - \alpha$ since in

this case S_{diss} does not depend on α explicitly. Let us now discuss the invariance of S_{det} and S_{jac}. Since $\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{ext}}$, S_{jac} vanishes and is therefore trivially invariant under $\mathcal{T}_{\text{eq}}$. For S_{det}, we look separately at the terms proportional to $i\hat{\mathbf{M}}_t^{\perp}$ and $i\hat{\mathbf{M}}_t^{\parallel}$. For the former

$$S_{\text{det}}^{\perp}[\mathcal{T}_{\text{eq}}i\hat{\mathbf{M}}, \mathcal{T}_{\text{eq}}\mathbf{M}, -\mathbf{H}_{\text{ext}}] = \beta V \mu_0 \, \mathbf{M}_{\mathcal{T}} \cdot \mathbf{H}_{\text{ext}} - \ln Z(-\mathbf{H}_{\text{ext}})$$
$$+ \int [-i\hat{\mathbf{M}}_{-t}^{\perp} - \beta V \mu_0 \gamma_0^{-1} \, \mathrm{d}_t \mathbf{M}_{-t}] \cdot [M_s^{-2} \mathrm{d}_t \mathbf{M}_{-t} \wedge \mathbf{M}_{-t} - \gamma_0 \mathbf{H}_{\text{ext}}].$$

Given the symmetry property $U(-\mathbf{M}, -\mathbf{H}_{\text{ext}}) = U(\mathbf{M}, \mathbf{H}_{\text{ext}})$, we have $Z(-\mathbf{H}_{\text{ext}}) = Z(\mathbf{H}_{\text{ext}})$. We first change $t \mapsto -t$ as the integration variable in the temporal integrals and we next rearrange terms to write

$$S_{\text{det}}^{\perp}[\mathcal{T}_{\text{eq}}i\hat{\mathbf{M}}, \mathcal{T}_{\text{eq}}\mathbf{M}, -\mathbf{H}_{\text{ext}}] = \beta V \mu_0 \, \mathbf{M}_{\mathcal{T}} \cdot \mathbf{H}_{\text{ext}} - \ln Z(\mathbf{H}_{\text{ext}})$$
$$+ \int [i\hat{\mathbf{M}}_t^{\perp} - \beta V \mu_0 \gamma_0^{-1} \mathrm{d}_t \mathbf{M}_t] \cdot [M_s^{-2} \mathrm{d}_t \mathbf{M}_t \wedge \mathbf{M}_t + \gamma_0 \mathbf{H}_{\text{ext}}]$$
$$= \beta V \mu_0 \, \mathbf{M}_{\mathcal{T}} \cdot \mathbf{H}_{\text{ext}} - \ln Z(\mathbf{H}_{\text{ext}}) + \int i\hat{\mathbf{M}}_t^{\perp} \cdot [M_s^{-2} \mathrm{d}_t \mathbf{M}_t \wedge \mathbf{M}_t + \gamma_0 \mathbf{H}_{\text{ext}}]$$
$$- \beta V \mu_0 \int \mathrm{d}_t \mathbf{M}_t \cdot \mathbf{H}_{\text{ext}}. \tag{3.21}$$

The first integral has the original form. The second integral can be computed directly, as $\mathbf{H}_{\text{ext}}$ is independent of $\mathbf{M}$. One recovers the boundary terms at $\mathcal{T}$ and $-\mathcal{T}$; one cancels the first term in the rhs, the other one builds the initial probability weight.

The term that imposes the spherical constraint is invariant on its own if we use

$$\alpha \mapsto 1 - \alpha \qquad \text{and} \qquad i\hat{\mathbf{M}}_t^{\parallel} \mapsto i\hat{\mathbf{M}}_{-t}^{\parallel} \tag{3.22}$$

Indeed, $\mathrm{D}_t^{(\alpha)} \mapsto \mathrm{D}_t^{(1-\alpha)} = -\mathrm{D}_{-t}^{(\alpha)}$ and

$$\int i\hat{\mathbf{M}}_t^{\parallel} \cdot \mathrm{D}_t^{(\alpha)} \mathbf{M}_t \mapsto \int i\hat{\mathbf{M}}_{-t}^{\parallel} \cdot (-\mathrm{D}_{-t}^{(\alpha)})(-\mathbf{M}_{-t}) = \int i\hat{\mathbf{M}}_t^{\parallel} \cdot \mathrm{D}_t^{(\alpha)} \mathbf{M}_t. \tag{3.23}$$

Note the different transformation rules on $i\hat{\mathbf{M}}^{\perp}$ and $i\mathbf{M}^{\parallel}$. The transformation $\alpha \mapsto 1 - \alpha$ is needed so that the two terms in $\mathrm{D}_t^{(\alpha)}$ be odd under time reversal and $\mathrm{D}_t^{(\alpha)}$ behave as a usual time derivative.

Case with an anisotropy potential, $V_{\text{ani}} \neq 0$ Let us now examine the generic case in which there is an anisotropy potential and S_{jac} no longer vanishes. The analysis of the dissipative part of the action is identical to the one we discussed in the previous section. As for the combined contributions $S_{\text{det}} + S_{\text{jac}}$, we proceed as follows. We start from equation (3.21) conveniently generalised to take into account the fact that $\mathbf{H}_{\text{eff}}^c = -\mu_0^{-1} \nabla_{\mathbf{M}} U$:

$$S_{\text{det}}^{\perp}[\mathcal{T}_{\text{eq}}i\hat{\mathbf{M}}, \mathcal{T}_{\text{eq}}\mathbf{M}, -\mathbf{H}_{\text{ext}}] = -\beta V U(-\mathbf{M}_{\mathcal{T}}, -\mathbf{H}_{\text{ext}}) - \ln Z(\mathbf{H}_{\text{ext}})$$
$$+ \int i\hat{\mathbf{M}}_t^{\perp} \cdot [M_s^{-2} \mathrm{d}_t \mathbf{M}_t \wedge \mathbf{M}_t + \gamma_0 \mathbf{H}_{\text{eff}_t}]$$
$$+ \beta V \int \mathrm{d}_t \mathbf{M}_t \cdot \nabla_{\mathbf{M}_t} U(\mathbf{M}_t, \mathbf{H}_{\text{ext}}). \tag{3.24}$$

We use now equation (3.3)

$$\mathrm{d}_t \mathbf{M} \cdot \nabla_{\mathbf{M}} U = \mathrm{d}_t M_i \, \partial_i U = \mathrm{d}_t U - D(1 - 2\alpha) \frac{\gamma_0^2}{1 + \eta^2 \gamma_0^2} P_{ij}^\perp \partial_i \partial_j U \tag{3.25}$$

in the last term and we obtain $+\beta V U(\mathbf{M}_{\mathcal{T}}, \mathbf{H}_{\mathrm{ext}}) - \beta V U(\mathbf{M}_{-\mathcal{T}}, \mathbf{H}_{\mathrm{ext}})$ after integrating the total time derivative. Using the property $U(-\mathbf{M}_{\mathcal{T}}, -\mathbf{H}_{\mathrm{ext}}) = U(\mathbf{M}_{\mathcal{T}}, \mathbf{H}_{\mathrm{ext}})$, the first term cancels the first term in the rhs in (3.24) and the second term reconstructs the exponential weight in the initial distribution. We therefore have

$$S_{\mathrm{det}}^\perp[\mathcal{T}_{\mathrm{eq}} \mathrm{i}\hat{\mathbf{M}}, \mathcal{T}_{\mathrm{eq}} \mathbf{M}, -\mathbf{H}_{\mathrm{ext}}] = S_{\mathrm{det}}^\perp[\mathrm{i}\hat{\mathbf{M}}, \mathbf{M}, \mathbf{H}_{\mathrm{ext}}]$$
$$-\frac{(1 - 2\alpha)\eta\gamma_0^2}{1 + \eta^2 \gamma_0^2} \frac{1}{M_s \mu_0} \int P_{ij}^\perp \partial_i \partial_j U, \tag{3.26}$$

where we replaced D by its definition in equation (3.5), while

$$S_{\mathrm{jac}}[\mathcal{T}_{\mathrm{eq}} \mathbf{M}, 1 - \alpha] = \frac{[\alpha + (1 - 2\alpha)]\eta\gamma_0^2}{1 + \eta^2 \gamma_0^2} \frac{1}{M_s \mu_0} \int P_{ij}^\perp \partial_i \partial_j U. \tag{3.27}$$

We notice that the first term is what we need to build $S_{\mathrm{jac}}[\mathbf{M}, \alpha]$ and the last term cancels the remaining one in equation (3.26).

The invariance of the term imposing the spherical constraint works in the same way as in the $V_{\mathrm{ani}} = 0$ case.

We have therefore completed the proof of invariance of the action under the transformation in (3.18).

3.3.2. The fluctuation-dissipation theorem Applying this symmetry to the linear response, see equation (3.17) we obtain

$$R_{A\mathbf{M}_j}(t, t') \equiv \langle A(t) \, \gamma_0 \mathrm{i}\hat{M}_j^\perp(t') \rangle$$
$$= -\langle A_{\mathrm{r}}(-t) \, \gamma_0 \mathrm{i}\hat{M}_j^\perp(-t') \rangle - \beta V \mu_0 \, \langle A_{\mathrm{r}}(-t) \mathrm{d}_{t'} M_j(-t') \rangle$$
$$= -R_{A_{\mathrm{r}}\mathbf{M}_j}(-t, -t') - \beta V \mu_0 \, \mathrm{d}_{t'} C_{A_{\mathrm{r}}\mathbf{M}_j}(-t, -t')$$
$$= -R_{A_{\mathrm{r}}\mathbf{M}_j}(-t, -t') + \beta V \mu_0 \, \mathrm{d}_{t'} C_{A\mathbf{M}_j}(t, t'), \tag{3.28}$$

where A_{r} is the time-reversed observable of A. In the last step, we applied the transformation once more to the last term. All averages are taken with the unperturbed action measured in the original variables and with the α parameter, what we would call $S[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}; \alpha]$. Using the causality of the response, $R_{A_{\mathrm{r}}M_j}(-t, -t') = 0$ for $t > t'$ and the time-translational invariance of equilibrium dynamics, one may simplify the expression above to

$$R_{A\mathbf{M}_j}(\tau) = -\beta V \mu_0 \, \Theta(\tau) \, \mathrm{d}_\tau C_{A\mathbf{M}_j}(\tau) \tag{3.29}$$

where we introduced $\tau = t - t'$ and $\Theta(\tau)$ is the Heaviside step function. Note that this relation applies to any observable A. Higher order fluctuation-dissipation relations of this kind can by easily derived.

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3.3.3. Broken symmetry and fluctuation theorems In this three-dimensional vector problem, the spin-torque due to a spin-polarised current $\chi_t \neq 0$ for $t > -\mathcal{T}$ is a non-conservative force that drives the magnetic moment out of equilibrium. It gives rise to the following contribution to the deterministic action,

$$S_{\text{det}}^{\text{torq}} = \gamma_0 \int i\hat{\mathbf{M}}_t^{\perp} \cdot \chi_t \, (\mathbf{M}_t \wedge \mathbf{p}_t), \tag{3.30}$$

and to an extra term in the Jacobian,

$$S_{\text{jac}}^{\text{torq}} = \frac{2\alpha\gamma_0}{1 + \eta^2\gamma_0^2} \int \chi_t \, \mathbf{M}_t \cdot \mathbf{p}_t. \tag{3.31}$$

When evaluated with the time-reversed variables

$$\overline{\chi}_t = -\chi_{-t} \qquad\qquad \overline{\mathbf{p}}_t = -\mathbf{p}_{-t}, \tag{3.32}$$

(ensuring notably that $\mathbf{H}^{\text{torq}}$ is odd under time reversal), the deterministic part of the action functional corresponding to the time-reversed dynamics reads

$$\begin{aligned} S_{\text{det}}^{\text{torq}}[\mathcal{T}_{\text{eq}}\mathbf{M}, \mathcal{T}_{\text{eq}}i\hat{\mathbf{M}}; \overline{\chi}, \overline{\mathbf{p}}] &= \int [-\gamma_0 i\hat{\mathbf{M}}_{-t}^{\perp} - \beta V\mu_0 d_t\mathbf{M}_{-t}] \cdot \chi_{-t} \, (-\mathbf{M}_{-t} \wedge \mathbf{p}_{-t}) \\ &= \int [\gamma_0 i\hat{\mathbf{M}}_t^{\perp} \cdot \chi_t \, (\mathbf{M}_t \wedge \mathbf{p}_t) + \beta V\mu_0 d_{-t}\mathbf{M}_t \cdot \chi_t \, (\mathbf{M}_t \wedge \mathbf{p}_t)] \\ &= S_{\text{det}}^{\text{torq}}[\mathbf{M}, i\hat{\mathbf{M}}; \chi, \mathbf{p}] - \beta V\mu_0 \int d_t\mathbf{M}_t \cdot \chi_t \, (\mathbf{M}_t \wedge \mathbf{p}_t). \end{aligned} \tag{3.33}$$

We see that the term that is generated cannot be partially integrated away, as in the potential case, since $\mathbf{H}^{\text{torq}}$ is not the gradient of a potential. The spin-torque contribution to the Jacobian transforms as

$$\begin{aligned} S_{\text{jac}}^{\text{torq}}[\mathcal{T}_{\text{eq}}\mathbf{M}; \overline{\alpha}, \overline{\chi}, \overline{\mathbf{p}}] &= -\frac{2(1-\alpha)\gamma_0}{1 + \eta^2\gamma_0^2} \int \chi_t \, \mathbf{M}_t \cdot \mathbf{p}_t \\ &= S_{\text{jac}}^{\text{torq}}[\mathbf{M}; \alpha, \chi, \mathbf{p}] + \Delta S_{\text{jac}}^{\text{torq}}[\mathbf{M}; \chi, \mathbf{p}] \end{aligned} \tag{3.34}$$

with

$$\Delta S_{\text{jac}}^{\text{torq}}[\mathbf{M}; \chi, \mathbf{p}] = -\frac{2\gamma_0}{1 + \eta^2\gamma_0^2} \int \chi_t \, \mathbf{M}_t \cdot \mathbf{p}_t. \tag{3.35}$$

The rest of the action remains invariant under this transformation as it was in the absence of the spin-torque term and with no time-dependent parameter dependencies in the effective field $\mathbf{H}_{\text{ext}}$. Ultimately, we obtain

$$\Delta S \equiv \Delta S_{\text{jac}}^{\text{torq}}[\mathbf{M}; \chi, \mathbf{p}] - \beta V\mu_0 W[\mathbf{M}; \chi, \mathbf{p}]. \tag{3.36}$$

with

$$W[\mathbf{M}; \chi, \mathbf{p}] \equiv \int \chi_t \, \mathbf{p}_t \cdot (d_t \mathbf{M}_t \wedge \mathbf{M}_t) \tag{3.37}$$

the work performed by the spin-torque term. This result implies the relation

$$\frac{P_{\mathrm{B}}[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}]}{P_{\mathrm{F}}[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}]} = \mathrm{e}^{\Delta S} \tag{3.38}$$

where we defined the forward and backward path probability distributions

$$P_{\mathrm{F}}[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}] \equiv P[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}; \alpha, \mathbf{H}_{\mathrm{ext}}, \chi, \mathbf{p}], \tag{3.39}$$

$$P_{\mathrm{B}}[\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}] \equiv P[\mathcal{T}_{\mathrm{eq}}\mathbf{M}, \mathcal{T}_{\mathrm{eq}}\mathrm{i}\hat{\mathbf{M}}; \overline{\alpha}, \overline{\mathbf{H}}_{\mathrm{ext}}, \overline{\chi}, \overline{\mathbf{p}}]. \tag{3.40}$$

Multiplying the identity (3.38) by a generic observable A which can depends on $\mathbf{M}$, $\mathrm{i}\hat{\mathbf{M}}$ and $\mathbf{H}_{\mathrm{ext}}$, one obtains

$$\langle A(\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}, \mathbf{H}_{\mathrm{ext}})\rangle_{\mathrm{B}} = \langle A(\mathbf{M}, \mathrm{i}\hat{\mathbf{M}}, \mathbf{H}_{\mathrm{ext}}) \, \mathrm{e}^{\Delta S}\rangle_{\mathrm{F}} \tag{3.41}$$

where the subscripts F and B stand for averaging with the forward and backward path probability distributions defined in equations (3.39) and (3.40), respectively. In particular, for $A = 1$, this boils down to the Jarzynski equality [17, 65] that reads in this case

$$\langle \mathrm{e}^{\Delta S_{\mathrm{jac}}^{\mathrm{torq}}[\mathbf{M}; \chi, \mathbf{p}] - \beta V \mu_0 W[\mathbf{M}; \chi, \mathbf{p}]}\rangle_{\mathrm{F}} = 1. \tag{3.42}$$

4. Conclusions and outlook

In this paper, we studied Markov stochastic processes with multiplicative white noise and adequately drifted to ensure their approach to the usual Gibbs–Boltzmann measure under equilibrium conditions. In this respect, our viewpoint is different from the one in [28] where no drift force was added to the stochastic differential equation and the dynamics approached a non-Gibbs–Boltzmann stationary state.

In recent years, thermodynamic relations and concepts in out-of-equilibrium stochastic processes have been searched for. In this paper, we proposed a field-theoretical derivation of fluctuation theorems for Markov stochastic processes with multiplicative white noise. This approach, based on a particular symmetry breaking of the path integral representation of the generating functional, extends our previous work [44] by showing that a single model-independent field transformation, $\mathcal{T}_{\mathrm{eq}}$, can generate all the fluctuation theorems for any stochastic evolution, in the presence of white or colored, additive or multiplicative noise.

One could ask whether the effective temperature idea [73] applies to multiplicative-noise processes that are not able to reach equilibrium with their surroundings and whether once set out of equilibrium they would satisfy fluctuation theorems in the way discussed in [74] for additive-noise processes. Exploring the *stochastic thermodynamics* and energetics [25, 75] proposals for multiplicative-noise Markov processes should also be an interesting research project.

Acknowledgments

Appendix A. Stochastic calculus

A.1. Chain rule

We examine the time derivative of a generic function, $[F(x_{n+1}) - F(x_n)]/\mathrm{d}t$, with the stochastic variable x governed by the Langevin equation with multiplicative white noise,

$$x_{n+1} - x_n = f(\bar{x}_n)\mathrm{d}t + g(\bar{x}_n)\mathrm{d}W_n, \tag{A.1}$$

with $(\mathrm{d}W_n)^2 \simeq 2D\,\mathrm{d}t$.

If we expand x_{n+1} in $F(x_{n+1})$, and x_n in $F(x_n)$, around the generic α point $\bar{x}_n = \alpha x_{n+1} + (1-\alpha)x_n$ we obtain

$$F(x_{n+1}) - F(x_n) = F(\bar{x}_n + (1-\alpha)(x_{n+1} - x_n)) - F(\bar{x}_n - \alpha(x_{n+1} - x_n))$$

$$= (x_{n+1} - x_n)F'(\bar{x}_n) + \frac{1}{2}(1-2\alpha)(x_{n+1} - x_n)^2 F''(\bar{x}_n) + \mathcal{O}(\mathrm{d}x^3)$$

with $\mathrm{d}x = x_{n+1} - x_n$. Using now equation (A.1) to replace $(x_{n+1} - x_n)^2$ by $2Dg(\bar{x}_n)^2\mathrm{d}t + \mathcal{O}(\mathrm{d}t^{3/2})$,

$$F(x_{n+1}) - F(x_n) = (x_{n+1} - x_n)F'(\bar{x}_n) + (1-2\alpha)Dg(\bar{x}_n)^2 F''(\bar{x}_n)\mathrm{d}t + \mathcal{O}(\mathrm{d}x^3)$$

that in the limit $\mathrm{d}t \to 0$ becomes

$$\frac{F(x_{n+1}) - F(x_n)}{\mathrm{d}t} = \frac{x_{n+1} - x_n}{\mathrm{d}t}F'(\bar{x}_n) + (1-2\alpha)Dg(\bar{x}_n)^2 F''(\bar{x}_n). \tag{A.2}$$

This expression is written as the generalised chain rule [3, 4]

$$\mathrm{d}_t F(x) = \mathrm{d}_t x\, F'(x) + (1-2\alpha)Dg^2(x)F''(x). \tag{A.3}$$

A.2. From the α to the Stratonovich prescription

One can transform a stochastic equation in the generic α-prescription into one in the Stratonovich mid-point prescription by simply expanding the arguments

of f and g around the latter points. More precisely, let us start from an equation in the α-prescription

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)\xi \;\Leftrightarrow\; x_{n+1} - x_n = f(\bar{x}_n)\mathrm{d}t + g(\bar{x}_n)\mathrm{d}W_n, \tag{A.4}$$

i.e. $\bar{x}_n = x_n + \alpha(x_{n+1} - x_n)$. The Stratonovich mid-points are $x_n^S = x_n + \frac{1}{2}(x_{n+1} - x_n)$, and α and S points are related by

$$\bar{x}_n = x_n^S - \frac{1}{2}(1 - 2\alpha)(x_{n+1} - x_n). \tag{A.5}$$

Expanding now equation (A.4) around the S mid-points one finds

$$x_{n+1} - x_n = [f(x_n^S) - (1 - 2\alpha)Dg(x_n^S)g'(x_n^S)]\mathrm{d}t + g(x_n^S)\mathrm{d}W_n \tag{A.6}$$

where we dropped contributions of $\mathcal{O}(\mathrm{d}t^{3/2})$ and we used $(\mathrm{d}W_n)^2 = 2D\mathrm{d}t$. The function g that multiplies $\mathrm{d}W_n$ is evaluated now at the S point x_n^S and in this sense this is an equation in the Stratonovich prescription.

This same strategy can be followed to transform an equation from the α to the α'-prescription at the price of modifying the force with an adequate drift term.

Appendix B. From multiplicative to additive noise

It is often found in the literature that a multiplicative-noise process can be mapped to an additive-noise process, and that in the latter formulation all subtleties linked to the discretisation prescription can be simply forgotten. Here we show that, while indeed such a mapping exists [76], the discretisation used to define the original multiplicative-noise process enters the additive-noise process in the form of a non-trivial drift force.

Let us re-parametrise the original equation of motion, equation (2.1) with $g(x) \equiv 1/k(x)$, such that the origin of the multiplicative noise and of the velocity in the lhs can be tracked back to a non-linear coupling to a thermal bath of oscillators, see the discussion below equation (2.12). We write it as the equation (in the α-prescription)

$$k^2(x)\mathrm{d}_t x(t) = f(x) + k(x)\xi(t), \tag{B.1}$$

where we also re-parametrized f by $f(x)k^2(x) \mapsto f(x)$ such that f can now be thought of as a true force, in units of Newtons, possibly deriving from a potential $f(x) = -V'(x)$. If we now divide this equation by $k(x)$, we obtain an equation in which the noise appears additively. However, it has to be treated with great care because the term $k(x)\mathrm{d}_t x$ hides subtleties associated with the discretisation.

Equation (B.1) is defined in the α-prescription, for which unusual rules of calculus apply. Re-writing it momentarily as

$$\mathrm{d}_t x = f(x)/k^2(x) + 1/k(x)\,\xi(t), \tag{B.2}$$

one can show that the associated chain rule reads (see the proof in Gardiner's book [3] for $\alpha = 0$ or $1/2$ and recalled in equation (2.5) for any α)

$$\mathrm{d}_t K(x) = K'(x)\mathrm{d}_t x + D(1 - 2\alpha)\,K''(x)/K'(x)^2. \tag{B.3}$$

where we introduced K such that $K'(x) \equiv k(x)$. Using equation (B.3), equation (B.2) can be re-written as

$$\mathrm{d}_t K - D(1 - 2\alpha)K''(x)/K'(x)^2 = f(x)/K'(x) + \xi(t). \qquad (B.4)$$

Let us now perform the change of variable from x to u according to the non-linear transformation

$$u \equiv K(x) \qquad (B.5)$$

to get the additive-noise process

$$\mathrm{d}_t u = D(1 - 2\alpha)K''(x(u)) / K'(x(u))^2 + \tilde{f}(u) + \xi(t) \qquad (B.6)$$

where we introduced $\tilde{f}(u) \equiv f(x(u))/K'(x(u))$. We finished mapping the original multiplicative-noise process in equation (B.1) to an additive-noise Langevin equation with a simple time-derivative $\mathrm{d}_t u$ in the lhs, but the first term in the rhs is non-trivially inherited from the discretisation of the original multiplicative-noise process and depends explicitly on α and k. It only vanishes for $\alpha = 1/2$ (Stratonovich calculus) or $k' = 0$ (additive noise in the original Langevin equation).

If we re-parametrize back to the original notations of equation (2.1), i.e. $k \mapsto 1/g$ and $f \mapsto f/g^2$, we obtain the mapping of equation (2.1) to

$$\mathrm{d}_t u = -D(1 - 2\alpha)g'(x(u)) + f(x(u))/g(x(u)) + \xi(t) . \qquad (B.7)$$

Appendix C. Stationary distribution and drift term

We revisit here the need for a drift term to ensure the approach to the Gibbs–Boltzmann distribution in the asymptotic long-time limit by working with the Langevin equation approach exclusively. We show that, given the generic multiplicative-noise equation (B.1), the stationary distribution is not of Boltzmann form unless a force is added to the conservative force, consistently with what we found with the Fokker–Planck approach in section 2.2.

C.1. Case of a Stratonovich prescription

Let us first treat the simpler case of $\alpha = 1/2$. We learned above that equation (B.1) can be re-written as the additive-noise equation of motion

$$\mathrm{d}_t u = \tilde{f}(u) + \xi(t) \qquad (C.1)$$

with

$$u \equiv K(x), \quad K'(x) \equiv k(x), \quad \tilde{f}(u) \equiv f(x)/k(x). \qquad (C.2)$$

The stationary probability distribution of the stochastic variable u governed by equation (C.1) is

$$\widetilde{P}_{\mathrm{st}}(u) = N \mathrm{e}^{-\beta \widetilde{V}(u)} \qquad \text{with} \qquad \widetilde{V}(u) \equiv -\int^u \mathrm{d}u' \tilde{f}(u') \qquad (C.3)$$

and N a normalisation constant. Switching back to the stochastic variable x governed by equation (B.2), the corresponding probability distribution can be recovered as

$$P_{\rm st}(x) = \left| \frac{{\rm d}u(x)}{{\rm d}x} \right| \widetilde{P}_{\rm st}(u(x)) = N|k(x)|\,{\rm e}^{\beta \int^{u(x)} {\rm d}u' \tilde{f}(u')}$$

$$= N|k(x)|\,{\rm e}^{\beta \int^x {\rm d}x' f(x')}. \tag{C.4}$$

This corresponds to the usual Gibbs–Boltzmann distribution if we allow ourselves to work with

$$f = -V' - k_{\rm B}T\, k'/k. \tag{C.5}$$

With this choice, the equation of motion becomes

$$k^2(x){\rm d}_t x(t) = -V'(x) - k_{\rm B}Tk'(x)/k(x) + k(x)\xi(t). \tag{C.6}$$

If we re-parametrize back to the original notations of equation (2.1), i.e. $k \mapsto 1/g$ and $f \mapsto f/g^2$, we obtain

$$ {\rm d}_t x(t) = -g^2(x)V'(x) + k_{\rm B}Tg'(x)g(x) + g(x)\xi(t) \tag{C.7}$$

which is equation (2.11) in the case $\alpha = 1/2$.

C.2. Generic α-prescription

The generalisation to the generic α case is straightforward. Equation (B.1) can be re-written as equation (B.6) with a stationary probability distribution $\widetilde{P}_{\rm st}(u) = N{\rm e}^{-\beta \widetilde{V}_k(u)}$ for u and

$$\widetilde{V}(u) \equiv -\int^u {\rm d}u'[\, \tilde{f}(u') + D(1-2\alpha)K''(x(u'))/K'(x(u'))^2]. \tag{C.8}$$

Returning to the the stochastic variable x, via $u = K(x)$, the corresponding stationary probability distribution is

$$P_{\rm st}(x) = N|k(x)|{\rm e}^{(1-2\alpha)\ln k(x) + \beta \int^x {\rm d}x' f(x')}. \tag{C.9}$$

This corresponds to the usual Gibbs–Boltzmann distribution if we work with

$$f(x) = -V'(x) - 2k_{\rm B}T(1-\alpha)k'(x)/k(x). \tag{C.10}$$

The Langevin equation becomes

$$k^2(x){\rm d}_t x(t) = -V'(x) - 2k_{\rm B}T(1-\alpha)\,k'(x)/k(x) + k(x)\xi(t) \tag{C.11}$$

If we re-parametrize back to the original notations of equation (2.1), i.e. $k \mapsto 1/g$ and $f \mapsto f/g^2$, we obtain

$$ {\rm d}_t x(t) = -g^2(x)V'(x) + 2D(1-\alpha)\,g'(x)g(x) + g(x)\xi(t) \tag{C.12}$$

that is the same drifted equation that we had obtained from a Fokker–Planck analysis in section 2.2, see equation (2.11).

Appendix D. The path integral

Following a route similar to the ones in [7, 8, 51, 52], we sketch the construction of the path integral for the Langevin equation of motion with multiplicative white noise:

$$\mathrm{Eq}[x(t), \xi(t)] \equiv \mathrm{d}_t x - f(x) - g(x)\xi(t) = 0. \tag{D.1}$$

In the construction, we use a continuous time notation with the discretisation subtleties being encoded in the choice of the value of the Heaviside theta-function at zero, $\Theta(0) = \alpha$. Later, we specify the definition of the path-integral measure.

D.1. Path integral construction

The explicit calculation of the Jacobian yields

$$\mathcal{J} \equiv \det_{tt'}\left[\frac{\delta \mathrm{Eq}[x(t), \xi(t)]}{\delta x(t')}\right] = \det_{tt'}\left[\mathrm{d}_t \delta(t - t') + A(x, \xi)\delta(t - t')\right] \tag{D.2}$$

with $A(x, \xi) \equiv -f'(x) - g'(x)\xi(t)$, $f'(x) = \mathrm{d}_x f(x)$ and $g'(x) = \mathrm{d}_x g(x)$. Note that if $g(x) \neq \mathrm{ct}$ the noise appears explicitly in the functional under $\det_{tt'}$. After some simple algebra, $\mathcal{J}$ can be factorised as

$$\mathcal{J} \equiv \det_{tt'}\left[\mathrm{d}_t \delta(t - t')\right] \det_{tt'}\left[\delta(t - t') + \Theta(t - t')A(x, \xi)\right], \tag{D.3}$$

and the first factor can be discarded in the normalisation. We can now re-write the second factor with the help of the identity $\det(1 + C_\xi) = \exp\mathrm{Tr}\ln(1 + C_\xi)$ with the causal function $C_\xi(x, t, t') = \Theta(t - t')A(x, \xi)$, where we highlighted the dependence of C_ξ on the noise by adding a subscript ξ to C. The $\ln(1 + C_\xi)$ can now be expanded in Taylor series. Usually, the causal structure of C (that is also usually noise-independent) truncates the series at first order in C. However, in this explicitly noise dependent case one needs to be careful and also keep the quadratic order [10]:

$$\mathcal{J} \propto \exp\mathrm{Tr}_{tt'}\left[\Theta(t - t')A(x, \xi) - \frac{1}{2}C_\xi^2(x, t, t')\right]$$
$$= \exp\int\mathrm{d}t\left[\Theta(0)A(x, \xi) - \frac{1}{2}C_\xi^2(x, t, t)\right] \tag{D.4}$$

where $C_\xi^2(x, t, t') \equiv \int \mathrm{d}t''\Theta(t - t'')A(x(t), \xi(t))\,\Theta(t'' - t')A(x(t''), \xi(t''))$. Using now $\Theta(0) = \alpha$, and simplifying notations such as $\dot{x} = \mathrm{d}_t x$, $g'(x(t)) = g'$, $C_\xi^2(x, t, t') = C_\xi^2$ and $\int \mathrm{d}t = \int$, $P[x]$ reads

$$P[x] \propto \int \mathcal{D}[\xi]\mathcal{D}[\hat{x}]\,\mathrm{e}^{\alpha\int A(x,\xi) - \frac{1}{2}\mathrm{Tr}_{tt'}C_\xi^2 - \int \mathrm{i}\hat{x}[\dot{x} - f - g\xi] - \frac{1}{4D}\int \xi^2}. \tag{D.5}$$

(To alleviate the notation we do not write here the time-dependence of the functions in the action, as we do in the main text.) Before performing the integration over ξ that involves

$$\int \mathcal{D}[\xi]\,\mathrm{e}^{-\frac{1}{4D}\int \xi^2 + \int (\mathrm{i}\hat{x}g - \alpha g')\xi - \frac{1}{2}\mathrm{Tr}_{tt'}C_\xi^2}, \tag{D.6}$$

let us translate the noise by a function of the variables x and $\mathrm{i}\hat{x}$, $\xi \mapsto \xi + 2D(\mathrm{i}\hat{x}g - \alpha g')$, in the functional integral. Notice that $\xi \in \mathbb{R}$ but $\mathrm{i}\hat{x} \in \mathrm{i}\mathbb{R}$. We can restore the original integration domain using the analyticity of the exponential that is zero on the boundary thanks to the term $-(4D)^{-1}\int \xi^2$. The functional integral in (D.6) transforms into a new path integral

$$\mathrm{e}^{-D\int(\mathrm{i}\hat{x}g - \alpha g')^2} \int \mathcal{D}[\xi]\ \mathrm{e}^{-\frac{1}{4D}\int \xi^2 - \frac{1}{2}\mathrm{Tr}_{tt'} C^2_{\xi + 2D(\mathrm{i}\hat{x}g - \alpha g')}}. \tag{D.7}$$

Keeping the terms in $C^2_{\xi + 2D(\mathrm{i}\hat{x}g - \alpha g')}$ that are quadratic in the noise and yield a $\delta(t - t')$ contribution within the $\mathrm{Tr}_{tt'}$ under the noise average, and using the notation $\langle \dots \rangle = \int \mathcal{D}[\xi]\, \mathrm{e}^{-(4D)^{-1}\int \xi^2} \dots$ one has

$$\left\langle \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}_{tt'} C^2_{\xi + 2D(\mathrm{i}\hat{x}g - \alpha g')}} \right\rangle = \mathrm{e}^{-\frac{1}{2}\left\langle \mathrm{Tr}_{tt'} C^2_{\xi + 2D(\mathrm{i}\hat{x}g - \alpha g')} \right\rangle}$$
$$= \mathrm{e}^{-\frac{1}{2}\iint \mathrm{d}t\, \mathrm{d}t' \Theta(t - t')\Theta(t' - t)g'(x(t))g'(x(t'))\langle \xi(t)\xi(t')\rangle} = \mathrm{e}^{-D\alpha^2 \int g'^2}.$$

Altogether we obtain $P[x] \propto \int \mathcal{D}[\hat{x}]\, \mathrm{e}^{S[x, \mathrm{i}\hat{x}]}$ with the action

$$S[x, \mathrm{i}\hat{x}] = -\int [\mathrm{i}\hat{x}(\dot{x} - f + 2D\alpha g'g - D\mathrm{i}\hat{x}g^2) + \alpha f'] \tag{D.8}$$

(to which we should add the contribution from the initial measure). This action is consistent with the results reported in [11] who used a slightly different approach in which the equation of motion was reformulated as

$$\mathrm{Eq}[x(t), \xi(t)] \equiv \frac{\mathrm{d}_t x - f(x)}{g(x)} - \xi(t) = 0. \tag{D.9}$$

This is convenient because the noise does not appear explicitly in the Jacobian, although its effect subtly re-appears along the calculation

D.2. Path integral measure

We work with a symmetric time interval $t \in [-\mathcal{T}, \mathcal{T}]$ which is divided in N discrete time intervals, $t_n \equiv -\mathcal{T} + n\Delta t$ with $n = 0, \dots, N$ and increment $\Delta t \equiv 2\mathcal{T}/N$. The continuous time limit is performed by sending N to infinity while keeping $\mathcal{T}$ finite. We define the path integral over trajectories on the time interval $[-\mathcal{T}, \mathcal{T}]$ as

$$\int \mathcal{D}[x] = \lim_{N \to \infty} \prod_{n=0}^{N} \int \mathrm{d}x_n \tag{D.10}$$

and for the auxiliary field

$$\int \mathcal{D}[\hat{x}] = \lim_{N \to \infty} \prod_{n=1}^{N} \int \frac{\mathrm{d}\hat{x}_n}{2\pi}. \tag{D.11}$$

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