REGULAR FREDHOLM PAIRS

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Abstract. In this work it is introduced the notion of regular Fredholm pair, i.e. a Fredholm pair whose operators are regular. The main properties of these objects are studied, and what is more, they are entirely classified. Furthermore, the index of a Fredholm pair turns out to be an extremely useful tool in the description of the aforementioned objects. Finally, regular Fredholm pairs are characterized in terms of regular Fredholm symmetrical pairs, exact chains of multiplication operators, and invertible Banach space operators.

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1. Introduction

There are many ways to extend the notions of Fredholm operator and index to several variable operator theory. For instance, both Fredholm complexes of Banach spaces and the related notion of Fredholm pair have associated an index with good stability properties, see for example [9], [1], [2], [3].

On the other hand, regular maps are a natural generalization of Fredholm operators. However, the boundary maps of a Fredholm complex of Banach spaces are generally not regular. In fact, in order for such a complex to have this property, it must be a split Fredholm complex, see for example [4]. As regard Fredholm pairs, since the operators of such a pair are generally not regular, a similar situation is encountered. The main objective of this work consists in the study of regular Fredholm pairs, i.e. Fredholm pairs whose operators are regular.

In the next section the notion of regular Fredholm pair is introduced. Moreover, some definitions and facts needed for the present work are reviewed, and some preliminary and general results regarding regularity are also proved. In section three the objects under consideration are entirely classified. In section four the index turns out to be an extremely useful tool to describe regular Fredholm pairs. Weyl pairs are also introduced and considered. Finally, in section five three characterizations of regular Fredholm pairs are proved. In fact, these objects are characterized in terms of regular symmetrical Fredholm pairs, exacts chains of multiplication operators, and invertible Banach space operators.

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2. Notations, Definitions and Preliminary Results

From now on X and Y denote two Banach spaces, $L(X,Y)$ the algebra of all linear and continuous operators defined on X with values in Y, and $K(X,Y)$ the closed ideal of all compact operatots of $L(X,Y)$. As usual, when $X=Y$, $L(X,X)$ and $K(X,X)$ are denoted by $L(X)$ and $K(X)$ respectively. For every $S\in L(X,Y)$ the null space of S is denoted by $N(S)=\{x\in X\colon\ S(x)=0\}$, and the range of S by $R(S)=\{y\in Y:\ \exists\, x\in X$ such that $y=S(x)\}$. Next follows the definition of Fredholm pair, see for instance [2].

Definition 2.1. *Let X and Y be two Banach spaces and let $S\in L(X,Y)$, $T\in L(Y,X)$ be such that the following dimensions are finite:*

$$a\colon=\dim N(S)/(N(S)\cap R(T)),\qquad b\colon=\dim R(T)/(N(S)\cap R(T)),$$
$$c\colon=\dim N(T)/(N(T)\cap R(S)),\qquad d\colon=\dim R(S)/(N(T)\cap R(S)).$$

A pair (S,T) with the above properties is called a Fredholm pair.

Let $P(X,Y)$ denote the set of all Fredholm pairs. If $(S,T)\in P(X,Y)$, then the index of (S,T) is defined by the equality

$$\text{ind }(S,T)\colon=a-b-c+d.$$

In particular, if $(S,T)\in P(X,Y)$ is such that $ST=0$ and $TS=0$, that is if $b=d=0$, then (S,T) is said a Fredholm symmetrical pair, see [10]. *Note that in this case (S,T) and (T,S) are Fredholm chains, see* [5, 10.6] *and* [6].

Before going on, several properties of Fredholm pairs are recalled, see [2].

Remark 2.2. First of all, observe that if $S\in L(X,Y)$ is a Fredholm operator, then $(S,0)$ is a Fredholm pair. Furthermore, ind $S=$ ind $(S,0)$. Consequently, the definition of Fredholm pair extends the notion of Fredholm operator to several variable operator theory.

In second place, note that if $(S,T)\in P(X,Y)$, then $(T,S)\in P(Y,X)$ and

$$\text{ind }(T,S)=-\text{ind }(S,T).$$

In third place, if $(S,T)\in P(X,Y)$, then $R(S)$ and $N(T)+R(S)$ are closed subspaces in Y. Similarly, $R(T)$ and $N(S)+R(T)$ are closed subspaces in X. What is more, there are finite dimensional subspaces in X, X_1 and X_2, and in Y, Y_1 and Y_2, such that:

i) $\dim X_1=a$, $\dim X_2=b$, $\dim Y_1=c$, $\dim Y_2=d$,

ii) $N(S)=(N(S)\cap R(T))\oplus X_1$, $R(T)=(N(S)\cap R(T))\oplus X_2$,

iii) $N(T)=(N(T)\cap R(S))\oplus Y_1$, $R(S)=(N(T)\cap R(S))\oplus Y_2$.

In particular,

$$N(S)+R(T)=(N(S)\cap R(T))\oplus(X_1\oplus X_2),$$

and

$$N(T)+R(S)=(N(T)\cap R(S))\oplus(Y_1\oplus Y_2).$$

Moreover, S induces an isomorphism

$$X_2=R(T)/(N(S)\cap R(T))\xrightarrow{\cong}R(ST).$$

Similarly, T induces and isomorphism

$$Y_2=R(S)/(N(T)\cap R(S))\xrightarrow{\cong}R(TS).$$

In particular, $\dim R(ST)=b$ and $\dim R(TS)=d$.

On the other hand, it is easy to prove that if $\tilde{Y}_2$ is another subspace such that $R(S) = (N(T) \cap R(S)) \oplus \tilde{Y}_2$, then $\dim \tilde{Y}_2 = d$ and T induces an isomorphism

$$\tilde{Y}_2 = R(S)/(N(T) \cap R(S)) \xrightarrow{\cong} R(TS).$$

Finally, interchanging S and T, similar properties for the operator S can be proved.

Next follows the definition of regular operator, see for example [5].

Definition 2.3. *Let X and Y be two Banach spaces and let $T \in L(X, Y)$. The operator T is called regular or relatively Fredholm, if there is $T' \in L(Y, X)$ for which*

$$T = TT'T.$$

If T is a regular bounded and linear map, the operator T' in Definition 2.3 is called a *generalized inverse*, or *pseudo inverse*, for T. If, in addition, T is a generalized inverse for T', that is if

$$T' = T'TT',$$

then T' is called a *normalized* generalized inverse, see [5, 3.8] and [6]. It is well known that if T' is a generalized inverse for T, then

$$T'' = T'TT'$$

is a normalized generalized inverse for T, see [5, 3.8] and [6]

On the other hand, when the range of T is closed, the condition of being a regular operator is equivalent to the fact that $N(T)$ and $R(T)$ are complemented subspaces in X and Y respectively, see [5, 3.8.2].

In the following proposition Fredholm pairs whose operators have complemented ranges and null spaces are studied.

Proposition 2.4. *Let X and Y be two Banach spaces and $(S, T) \in P(X, Y)$. The following assertions are equivalent:*

i) $R(T)$ is a complemented subspace in X;
ii) $N(S)$ is a complemented subspace in X;
iii) $N(S) + R(T)$ is a complemented subspace in X;
iv) $N(S) \cap R(T)$ is a complemented subspace in X.

Similarly, the following assertions are equivalent:

i) $R(S)$ is a complemented subspace in Y;
ii) $N(T)$ is a complemented subspace in Y;
iii) $N(T) + R(S)$ is a complemented subspace in Y;
iv) $N(T) \cap R(S)$ is a complemented subspace in Y.

In particular, if S (resp. T) is a regular operator, then T (resp. S) also is a regular operator.

Proof. According to Remark 2.2, all the subspaces involved in the first part of the proposition are closed. Moreover, there are finite dimensional subspaces X_1 and X_2 such that

$$N(S) = (N(S) \cap R(T)) \oplus X_1, \qquad R(T) = (N(S) \cap R(T)) \oplus X_2,$$

and

$$N(S) + R(T) = (N(S) \cap R(T)) \oplus (X_1 \oplus X_2).$$

Now well, it is clear that *i*), *ii*), and *iii*) imply *iv*). On the other hand, according to [5, 6.3.5], *iv*) implies *i*), *ii*) and *iii*).

A similar argument proves the second part of the proposition.

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Next follows the definition of regular Fredholm pair.

Definition 2.5. *Let X and Y be two Banach spaces and let $S \in L(X,Y)$, $T \in L(Y,X)$ be such that $(S,T) \in P(X,Y)$. If the operators S and T are regular, that is if S and T have the equivalent properties of Proposition 2.4, then (S,T) is called a regular Fredholm pair.*

The set of all regular Fredholm pairs is denoted by $RP(X,Y)$.

In particular, if $(S,T) \in RP(X,Y)$ is a Fredhlom symmetrical pair, then (S,T) is said a regular Fredholm symmetrical pair. Note that in this case (S,T) and (T,S) are regular Fredholm chains, see [5, 10.6] *and* [6].

Remark 2.6. Note that, according to Proposition 2.4, if $(S,T) \in P(X,Y)$, then the property of being a regular Fredholm pair is equivalent to the fact that either the operator S or the operator T is regular. Furthermore, if $(S,T) \in P(X,Y)$, then there are closed subspaces $\tilde{X}$ and $\tilde{Y}$, in X and Y respectively, such that

$$\begin{aligned} X &= (R(T) \oplus X_1) \oplus \tilde{X} = (N(S) \oplus X_2) \oplus \tilde{X} = (N(S) + R(T)) \oplus \tilde{X} \\ &= ((N(S) \cap R(T)) \oplus (X_1 \oplus X_2)) \oplus \tilde{X}, \end{aligned}$$

and

$$\begin{aligned} Y &= (R(S) \oplus Y_1) \oplus \tilde{Y} = (N(T) \oplus Y_2) \oplus \tilde{Y} = (N(T) + R(S)) \oplus \tilde{Y} \\ &= ((N(T) \cap R(S)) \oplus (Y_1 \oplus Y_2)) \oplus \tilde{Y}, \end{aligned}$$

where X_j and Y_j, $j = 1, 2$, are the finite dimensional subspaces considered in Remark 2.2.

In addition, if S_2 and $\tilde{S}$ are the restrictions of S to X_2 and to $\tilde{X}$ respectively, and if $\mathcal{S} = S_2 \oplus \tilde{S}$, then according to Remark 2.2,

$$\mathcal{S} \colon X_2 \oplus \tilde{X} \xrightarrow{\cong} R(S), \; R(S) = S(X_2) \oplus S(\tilde{X}) = R(ST) \oplus R(\tilde{S}).$$

Similarly, if T_2 and $\tilde{T}$ are the restrictions of T to Y_2 and to $\tilde{Y}$ respectively, and if $\mathcal{T} = T_2 \oplus \tilde{T}$, then according to Remark 2.2,

$$\mathcal{T} \colon Y_2 \oplus \tilde{Y} \xrightarrow{\cong} R(T), \; R(T) = T(Y_2) \oplus T(\tilde{Y}) = R(TS) \oplus R(\tilde{T}).$$

Remark 2.7. Four examples of regular Fredholm pairs will be considered. In first place, let X and Y be two Banach spaces and (S,T) belong to $P(X,Y)$. According to [2], or Remark 2.2, and to [5, 3.8.2], it is clear that if X and Y are Hilbert spaces, then $P(X,Y) = RP(X,Y)$. On the other hand, according to [5, 6.3.4], if $R(S)$ and $R(T)$ are finite dimensional subspaces of the Banach spaces Y and X respectively, then (S,T) belongs to $RP(X,Y)$.

Next consider $(\mathcal{X}, d)$ a *complex of Banach spaces* of finite length, that is a sequence

$$0 \to X_n \xrightarrow{d_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{d_1} X_0 \to 0,$$

where X_p are Banach spaces, $p = 0, \ldots, n$, and the bounded operators $d_p \in L(X_p, X_{p-1})$ are such that $d_{p-1}d_p = 0$, $p = 1, \ldots, n$. Define the homology groups of $(\mathcal{X}, d)$ as $H_p(\mathcal{X}, d) = N(d_p)/R(d_{p+1})$, $p = 0, \ldots, n$. A complex $(\mathcal{X}, d)$ is said

Fredholm if all its homology groups are finite dimensional. If this is the case, then it is possible to associate to $(\mathcal{X}, d)$ the integer

$$\text{ind } (\mathcal{X}, d) = \sum_{p=0}^{n} (-1)^p \dim H_p(\mathcal{X}, d),$$

which is called the *index* or the *Euler characteristic* of $(\mathcal{X}, d)$.

In [3] a Fredholm symmetrical pair was associated to each Fredholm complex. In fact, as above consider a complex of Banach spaces $(\mathcal{X}, d)$ and set

$$X = \bigoplus_{p=2k} X_p, \qquad Y = \bigoplus_{p=2k+1} X_p,$$

and define the maps $S \in L(X, Y)$ and $T \in L(Y, X)$ as follows:

$$S = \bigoplus_{p=2k} d_p, \qquad T = \bigoplus_{p=2k+1} d_p,$$

where $k \geq 0$, and $X_p = 0$ when $p \geq n + 1$.

Since $(\mathcal{X}, d)$ is a complex, $TS = 0$ and $ST = 0$. Furthermore, $(\mathcal{X}, d)$ is a Fredholm complex if and only if dim $N(S)/R(T)$ and dim $N(T)/R(S)$ are finite dimensional, which is equivalent to the fact that (S, T) is a Fredholm symmetrical pair. In addition,

$$\text{ind } (\mathcal{X}, d) = \text{ind } (S, T).$$

Now well, a complex of Banach spaces $(\mathcal{X}, d)$ is called *Fredholm split* if there are continuous linear operators

$$X_{p-1} \xrightarrow{h_{p-1}} X_p \xrightarrow{h_p} X_{p+1},$$

such that

$$d_{p+1} h_p + h_{p-1} d_p = I_p - k_p,$$

where $k_p \in K(X_p)$, $p = 0, \ldots, n$. When $k_p = 0$ for $p = 0, \ldots, n$, $(\mathcal{X}, d)$ is said a split complex. According to [4, 2.7], it is easy to prove that a complex $(\mathcal{X}, d)$ is Fredholm split if and only if the above associated pair (S, T) is a regular Fredholm symmetrical pair.

Finally, consider $(\mathcal{K}, \delta)$ a *Fredholm chain of Banach spaces*, that is a sequence

$$0 \to \mathcal{K}_n \xrightarrow{\delta_n} \mathcal{K}_{n-1} \to \ldots \to \mathcal{K}_1 \xrightarrow{\delta_1} \mathcal{K}_0 \to 0,$$

where $\mathcal{K}_p$ are Banach spaces, and the bounded operators $\delta_p \in L(\mathcal{K}_p, \mathcal{K}_{p-1})$ are such that

$$N(\delta_p)/(N(\delta_p) \cap R(\delta_{p+1})) \text{ and } R(\delta_{p+1})/(N(\delta_p) \cap R(\delta_{p+1}))$$

are finite dimensional subspaces of $\mathcal{K}_p$, $p = 0, \ldots, n$.

Recall that in [7] it was introduced the more general concept of *semi-Fredholm chains*. However, since the main concern of this article consists in Fredholm objects, only Fredholm chains will be considered. Furthermore, observe that since $\dim R(\delta_{p-1}\delta_p)$ is finite dimensional, $p = 1, \ldots, n$, a Fredholm chain $(\mathcal{K}, \delta)$ is a particular case of what in [8] was called an *essential complex of Banach spaces*.

As in the case of Fredholm complexes of Banach spaces, it is possible to associate an index to any Fredholm chain. In fact, if $(\mathcal{K}, \delta)$ is such an object, then define its index as

$$\text{ind } (\mathcal{K},\delta) = \sum_{p=0}^{n}(-1)^p(\dim\ N(\delta_p)/(N(\delta_p)\cap R(\delta_{p+1})) - \dim R(\delta_{p+1})/(N(\delta_p)\cap R(\delta_{p+1}))),$$

see [7].

Now well, given a Fredholm chain $(\mathcal{K},\delta)$, define X, Y, $S\in L(X,Y)$ and $T\in L(Y,X)$ as it has been done above for a Fredholm complex of Banach spaces. Then, as in [7], $(\mathcal{K},\delta)$ is a Fredholm chain if and only if the associated pair (S,T) is a Fredholm pair. In addition,

$$\text{ind } (\mathcal{K},\delta) = \text{ind } (S,T).$$

In this work, in order to keep the analogy with complexes of Banach spaces, it will be said that a Fredholm chain $(\mathcal{K},\delta)$ is *split* if $N(\delta_p)$ is a complemented subspaces of $\mathcal{K}_p$, $p=0,\ldots,n$. Note that, as in Proposition 2.4, this condition is equivalent to the fact that $R(\delta_{p+1})$, or $N(\delta_p)+R(\delta_{p+1})$, or $N(\delta_p)\cap R(\delta_{p+1})$ is a complemented subspace of $\mathcal{K}_p$, $p=0,\ldots,n$. Moreover, thanks to [8, 2.3], a split Fredholm chain is a *Fredholm essential complex* in the sense of [8, 2.2].

Now well, it is not difficult to prove that $(\mathcal{K},\delta)$ is a split Fredholm chain if and only if the above associated pair (S,T) is a regular Fredholm pair.

Remark 2.8. Given Banach spaces X and Y, and $T\in L(X,Y)$ a Fredholm operator, then *any* pseudo inverse T' for T is a Fredholm operator and ind $(T') = -$ ind (T), see [5, 6.4.4] and [5, 6.5.5]. Nevertheless, as the following example shows, these results do not hold any more for regular Fredholm pairs. In fact, there is a regular Fredholm pair (S,T), with $S'\in L(Y,X)$ and $T'\in L(X,Y)$ pseudo inverses for S and T respectively, such that (S',T') does not belong to $P(Y,X)$.

Consider finite dimensional Banach spaces X_j and Y_j, $j=1,\,2$, such that $\dim X_2=\dim Y_2$, and Banach spaces $\tilde{X}$, $\tilde{Y}$ and N_j, $j=1,\,2$, such that there are isomorphic operators

$$\tilde{S}\colon \tilde{X}\xrightarrow{\cong} N_2,\qquad \tilde{T}\colon \tilde{Y}\xrightarrow{\cong} N_1.$$

For example, take $\tilde{X}=N_2$ and $\tilde{Y}=N_1$, and S and T the identity map of $\tilde{X}$ and $\tilde{Y}$ respectively.

Define the Banach spaces

$$X = X_1\oplus N_1\oplus X_2\oplus\tilde{X},\qquad Y = Y_1\oplus N_2\oplus Y_2\oplus\tilde{Y},$$

and the linear continuous maps $S\in L(X,Y)$ and $T\in L(Y,X)$ as follows:

$$S\mid_{X_1\oplus N_1}\equiv 0,\qquad S\mid_{X_2}= S_2\colon X_2\to Y_2,\qquad S\mid_{\tilde{X}}=\tilde{S}\colon \tilde{X}\to N_2,$$
$$T\mid_{Y_1\oplus N_2}\equiv 0,\qquad T\mid_{Y_2}= T_2\colon Y_2\to X_2,\qquad T\mid_{\tilde{Y}}=\tilde{T}\colon \tilde{Y}\to N_1,$$

where S_2 and T_2 are any isomorphic maps.

It is easy to prove that $(S,T)\in RP(X,Y)$ and that $\text{ind}\,(S,T)=\dim X_1-\dim Y_1$.

On the other hand, consider the following operators $T'\in L(X,Y)$ and $S'\in L(Y,X)$:

$$T'\mid_{X_1\oplus\tilde{X}}\equiv 0,\qquad T'\mid_{N_1\oplus X_2}= (T_2\oplus\tilde{T})^{-1}\colon N_1\oplus X_2\to Y_2\oplus\tilde{Y},$$
$$S'\mid_{Y_1}\equiv 0,\qquad S'\mid_{N_2\oplus Y_2}= (S_2\oplus\tilde{S})^{-1}\colon N_2\oplus Y_2\to X_2\oplus\tilde{X},$$

and $S' \mid_{\tilde{Y}} : \tilde{Y} \to N_1$ any isomorphism.

An easy calculation proves that T' is a normalized generalized inverse for T and S' is a pseudo inverse for S. However, since

$$R(S') = N_1 \oplus X_2 \oplus \tilde{X},\ N(T') = X_1 \oplus \tilde{X},\ N(T') \cap R(S') = \tilde{X},$$

it is clear that

$$R(S')/(N(T') \cap R(S')) = X_2 \oplus N_1.$$

Therefore, if N_1, which is isomorphic to $\tilde{Y}$, is an infinite dimensional Banach space, then (S', T') does not belong to $P(Y, X)$.

Nevertheless, in the following proposition it is proved that if $(S, T) \in RP(X, Y)$, then there always exist normalized generalized inverses for S and T, S' and T' respectively, such that $(S', T') \in RP(Y, X)$.

Proposition 2.9. *Let X and Y be two Banach spaces and let $(S, T) \in RP(X, Y)$. Then, there is $(S', T') \in RP(Y, X)$ such that:*
i) ind $(S', T') = -$ ind (S, T)*;*
ii) S' and T' are normalized generalized inverses for S and T respectively.

Proof. Consider X, Y, S and T presented as in Remark 2.6, and define the operators S' and T' as follows:

$$S' \mid_{Y_1 \oplus \tilde{Y}} \equiv 0, \qquad S' \mid_{R(S)} = \iota_1 \circ \mathcal{S}^{-1} \colon R(S) \to X,$$
$$T' \mid_{X_1 \oplus \tilde{X}} \equiv 0, \qquad T' \mid_{R(T)} = \iota_2 \circ \mathcal{T}^{-1} \colon R(T) \to Y,$$

where $\iota_1 \colon X_2 \oplus \tilde{X} \to X$ and $\iota_2 \colon Y_2 \oplus \tilde{Y} \to Y$ are the natural inclusion maps.

A straightforward calculation proves that S' and T' are normalized generalized inverses for S and T respectively. In particular, S' and T' are regular operators. Furthermore, since

$$N(S') \cap R(T') = \tilde{Y}, \qquad N(T') \cap R(S') = \tilde{X},$$

it is clear that

$$N(S')/(N(S') \cap R(T')) = Y_1, \qquad R(T')/(N(S') \cap R(T')) = Y_2,$$
$$N(T')/(N(T') \cap R(S')) = X_1, \qquad R(S')/(N(T') \cap R(S')) = X_2.$$

Therefore, (S', T') is a regular Fredholm pair and

$$\text{ind } (S', T') = -\text{ind } (S, T).$$

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Before ending this section, the perturbation properties of regular Fredholm pairs are considered. It is clear that the main results of [2], Theorems 3.1 and 3.2, are still true for regular Fredholm pairs. On the other hand, thanks to [5, 6.3.4], also [2, 2.3] remains true for regular Fredholm pairs.

In the next section regular Fredholm pairs will be entirely classified.

3. The classification of regular Fredholm pairs

In order to classify regular Fredholm pairs, two sequences of subspaces are introduced.

Definition 3.1. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. The sequences $(R_{S,n})_{n\in\mathbb{N}_0}$ and $(R_{T,n})_{n\in\mathbb{N}_0}$ are defined in the following way:*
If $n=0$, then

$$R_{S,0}=Y, \qquad R_{T,0}=X,$$

and if $R_{S,n}$ and $R_{T,n}$ are defined, then

$$R_{S,n+1}=S(R_{T,n}), \qquad R_{T,n+1}=T(R_{S,n}).$$

Remark 3.2. $(R_{S,n})_{n\in\mathbb{N}_0}$ and $(R_{T,n})_{n\in\mathbb{N}_0}$ are decreasing sequences of Y and X respectively. In fact,

$$R_{S,1}=R(S)\subseteq Y=R_{S,0}, \qquad R_{T,1}=R(T)\subseteq X=R_{T,0}.$$

On the other hand, if $R_{S,n}\subseteq R_{S,n-1}$ and $R_{T,n}\subseteq R_{T,n-1}$, for a fixed $n\geq 1$, then

$$R_{S,n+1}=S(R_{T,n})\subseteq S(R_{T,n-1})=R_{S,n},$$
$$R_{T,n+1}=T(R_{S,n})\subseteq T(R_{S,n-1})=R_{T,n}.$$

Furthermore, since $R_{S,2}=R(ST)$ and $R_{T,2}=R(TS)$, $R_{S,n}$ and $R_{T,n}$ are finite dimensional subspaces of Y and X respectively, $n\geq 2$.

Next follows a description of the above sequences of subspaces.

Proposition 3.3. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Then given $n\in\mathbb{N}$ there are subspaces of X, N^n and X_2^n, and of Y, M^n and Y_2^n, such that:*
i) $R_{S,n}=M^n\oplus Y_2^n$, $R_{T,n}=N^n\oplus X_2^n$;
ii) $M^n=R_{S,n}\cap N(T)$, $N^n=R_{T,n}\cap N(S)$;
iii) $Y_2^n=R_{S,n}\cap Y_2^k$, $X_2^n=R_{T,n}\cap X_2^k$, $k=1,\dots n-1$;
iv) $(M^n)_{n\in\mathbb{N}}$ and $(N^n)_{n\in\mathbb{N}}$ are decreasing sequences of subspaces contained in $N(T)$ and $N(S)$ respectively, moreover, M^n and N^n are finite dimensional subspaces for $n\geq 2$;
v) $(Y_2^n)_{n\in\mathbb{N}}$ and $(X_2^n)_{n\in\mathbb{N}}$ are decreasing sequences of finite dimensional subspaces contained in Y_2 and X_2 respectively;
vi) S (resp. T) induces an isomorphism

$$X_2^n\xrightarrow{\cong}R_{S,n+1} \qquad (resp.\ Y_2^n\xrightarrow{\cong}R_{T,n+1}).$$

Proof. When $n=1$ define

$$M^1=N(T)\cap R(S),\ X_2^1=X_2,\ ,N^1=N(S)\cap R(T),\ Y_2^1=Y_2,$$

where X_2 and Y_2 are the subspaces considered in Remark 2.2 and Remark 2.6.

It is clear that these subspaces verify the above assertions. Next suppose that the propositon is true for $n\geq 1$. Since T (resp. S) induces an isomorphism

$$R_{S,n+1}/(N(T)\cap R_{S,n+1})\xrightarrow{\cong}R_{T,n+2},$$
$$(resp.\ R_{T,n+1}/(N(S)\cap R_{T,n+1})\xrightarrow{\cong}R_{S,n+2},$$

there are finite dimensional subspaces V and W of Y and X respectively, such that

$$R_{S,n+1}=(N(T)\cap R_{S,n+1})\oplus V, \qquad R_{T,n+1}=(N(S)\cap R_{T,n+1})\oplus W,$$

and T (resp. S) induces an isomorphism

$$V\xrightarrow{\cong}R_{T,n+2} \qquad (\text{resp. } W\xrightarrow{\cong}R_{S,n+2}).$$

Observe that, according to an argument similar to one used in Remark 2.2, it is possible to choose $V \subseteq Y_2^n$ and $W \subseteq X_2^n$. Then, define

$$M^{n+1} = N(T) \cap R_{S,n+1},\ N^{n+1} = N(S) \cap R_{T,n+1},\ Y_2^{n+1} = V,\ X_2^{n+1} = W.$$

Clearly $Y_2^{n+1} \subseteq R_{S,n+1} \cap Y_2^n \subseteq R_{S,n+1} \cap Y_2^k \subseteq R_{S,n+1} \cap Y_2$, for $k = 1, \ldots, n$. On the other hand, if $a \in R_{S,n+1} \cap Y_2$, since $R_{S,n+1} = (N(T) \cap R_{S,n+1}) \oplus Y_2^{n+1}$, then there are $m \in N(T) \cap R_{S,n+1}$ and $y \in Y_2^{n+1}$ such that $a = m + y$. However, since $m \in N(T)$ and $a - y \in Y_2$, for $Y_2^{n+1} \subseteq Y_2^n \subseteq Y_2$, then $m = 0$ and $a = y \in Y_2^{n+1}$. Therefore, $Y_2^{n+1} = R_{S,n+1} \cap Y_2^k = R_{S,n+1} \cap Y_2$, for $k = 1, \ldots n$.

Similarly, $X_2^{n+1} = R_{T,n+1} \cap X_2^k = R_{T,n+1} \cap X_2$, for $k = 1, \ldots, n$.

The other points of the proposition are clear.

∎

Our next step consists in the description of the relationship between $R_{S,n}$ and $R_{S,n+1}$, and between $R_{T,n}$ and $R_{T,n+1}$. However, to this end, it is necessary to introduce two new sequences of subspaces.

Definition 3.4. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. The sequences of subspaces of Y and X, $(R_{\tilde{S},n})_{n\in\mathbb{N}}$ and $(R_{\tilde{T},n})_{n\in\mathbb{N}}$ respectively, are defined in the following way.*

If $n = 1$, then

$$R_{\tilde{S},1} = R(\tilde{S}) = S(\tilde{X}), \qquad R_{\tilde{T},1} = R(\tilde{T}) = T(\tilde{Y}),$$

where $\tilde{X}$, $\tilde{Y}$, $\tilde{S}$ and $\tilde{T}$ are the spaces and operators introduced in Remark 2.6, and if $n \geq 2$,

$$R_{\tilde{S},n+1} = S(R_{\tilde{T},n}), \qquad R_{\tilde{T},n+1} = T(R_{\tilde{S},n}).$$

Observe that $R_{\tilde{S},n} \subseteq R_{S,n}$ and $R_{\tilde{T},n} \subseteq R_{T,n}$, for $n \in \mathbb{N}$.

In the next proposition the sequences introduced in Definition 3.4 are characterized.

Proposition 3.5. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Then there are four sequences of subspaces, two of X, $(\mathbb{N}^n)_{n\in\mathbb{N}}$ and $(\mathbb{X}_2^n)_{n\in\mathbb{N}}$, and two of Y, $(\mathbb{M}^n)_{n\in\mathbb{N}}$ and $(\mathbb{Y}_2^n)_{n\in\mathbb{N}}$, such that for $n \in \mathbb{N}$:*

i) $R_{\tilde{S},n} = \mathbb{M}^n \oplus \mathbb{Y}_2^n$, $R_{\tilde{T},n} = \mathbb{N}^n \oplus \mathbb{X}_2^n$;

ii) $\mathbb{M}^n = R_{\tilde{S},n} \cap N(T) \subseteq M^n$, $\mathbb{N}^n = R_{\tilde{T},n} \cap N(S) \subseteq N^n$;

iii) $\mathbb{Y}_2^n = R_{\tilde{S},n} \cap Y_2^n \subseteq Y_2^n$, $\mathbb{X}_2^n = R_{\tilde{T},n} \cap X_2^n \subseteq X_2^n$;

iv) T (resp. S) induces an isomorphism

$$\mathbb{Y}_2^n \xrightarrow{\cong} R_{\tilde{T},n+1}, \qquad (\text{resp. } \mathbb{X}_2^n \xrightarrow{\cong} R_{\tilde{S},n+1});$$

v) $R_{S,n} = R_{S,n+1} \oplus R_{\tilde{S},n}$, $R_{T,n} = R_{T,n+1} \oplus R_{\tilde{T},n}$;

vi) $M^n = M^{n+1} \oplus \mathbb{M}^n$, $N^n = N^{n+1} \oplus \mathbb{N}^n$;

vii) $Y_2^n = Y_2^{n+1} \oplus \mathbb{Y}_2^n$, $X_2^n = X_2^{n+1} \oplus \mathbb{X}_2^n$;

viii) when $n = 1$ there are subspaces of $\tilde{X}$, $\tilde{X}_N$ and $\tilde{X}_2$, and of $\tilde{Y}$, $\tilde{Y}_N$ and $\tilde{Y}_2$, such that $\tilde{X}_2$ and $\tilde{Y}_2$ are finite dimensional,

$$\tilde{X} = \tilde{X}_N \oplus \tilde{X}_2, \qquad \tilde{Y} = \tilde{Y}_N \oplus \tilde{Y}_2,$$

and the following operators are isomorphic maps:

$$\tilde{S}\colon \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1, \qquad \tilde{S}\colon \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1,$$

$$\tilde{T}\colon \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1, \qquad \tilde{T}\colon \tilde{Y}_2 \xrightarrow{\cong} \mathbb{X}_2^1.$$

Proof. Given $n \in \mathbb{N}$, consider the isomorphism induced by T

$$R_{\tilde{S},n}/(N(T) \cap R_{\tilde{S},n}) \xrightarrow{\cong} R_{\tilde{T},n+1}.$$

Since $R_{\tilde{T},n+1} \subseteq R_{T,n+1} \subseteq R_{T,2} = R(TS)$, there is L_n, a finite dimensional subspace of $R_{\tilde{S},n}$, such that $R_{\tilde{S},n} = (N(T) \cap R_{\tilde{S},n}) \oplus L_n$. In addition, T induces an isomorphism

$$L_n \xrightarrow{\cong} R_{\tilde{T},n+1}.$$

Furthermore, since $R_{\tilde{S},n} \subseteq R_{S,n} = M^n \oplus Y_2^n = (R_{S,n} \cap N(T)) \oplus Y_2^n$, according to an argument similar to the one used in Remark 2.2, it is possible to choose $L_n \subseteq Y_2^n$.

Now well, it is clear that $L_n \subseteq R_{\tilde{S},n} \cap Y_2^n$. On the other hand, if $a \in R_{\tilde{S},n} \cap Y_2^n$, then there are $m \in N(T) \cap R_{\tilde{S},n}$ and $l \in L_n$ such that $a = m + l$. However, since $a - l \in Y_2^n \cap (N(T) \cap R_{\tilde{S},n}) \subseteq Y_2^n \cap (N(T) \cap R_{S,n}) = Y_2^n \cap M^n$, $m = 0$ and $a = l \in L_n$. Therefore, $L_n = R_{\tilde{S},n} \cap Y_2^n$.

Next define

$$\mathbb{M}^n = R_{\tilde{S},n} \cap N(T), \qquad \mathbb{Y}_2^n = L_n.$$

It is clear that assertions i)-iv) have been proved for S. A similar argument proves the same points for the operator T.

In order to prove v), an inductive argument wil be used.

According to Remark 2.6,

$$R_{S,1} = R_{S,2} \oplus R_{\tilde{S},1}, \qquad R_{T,1} = R_{T,2} \oplus R_{\tilde{T},1}.$$

Next suppose that the point v) is true for the operators S and T and for a fixed $n \geq 1$. Then, according to Proposition 3.3 i), ii) and vi), and to Propositon 3.5 i), ii) and iv), which have just been proved,

$$\begin{aligned} R_{S,n+1} =& S(R_{T,n}) = S(R_{T,n+1} \oplus R_{\tilde{T},n}) = S((N^{n+1} \oplus X_2^{n+1}) \oplus (\mathbb{N}^n \oplus \mathbb{X}_2^n)) \\ =& S(X_2^{n+1} \oplus \mathbb{X}_2^n) = S(X_2^{n+1}) \oplus S(\mathbb{X}_2^n) = R_{S,n+2} \oplus R_{\tilde{S},n+1}. \end{aligned}$$

Similarly, $R_{T,n+1} = R_{T,n+2} \oplus R_{\tilde{T},n+1}$.

Next, according to Proposition 3.3 i) and to Proposition 3.5 i) and v), which have just been proved, it is easy to conclude that

$$M^n \oplus Y_2^n = (M^{n+1} \oplus \mathbb{M}^n) \oplus (Y_2^{n+1} \oplus \mathbb{Y}_2^n), \; N^n \oplus X_2^n = (N^{n+1} \oplus \mathbb{N}^n) \oplus (X_2^{n+1} \oplus \mathbb{X}_2^n).$$

In particular,

$$\begin{aligned} M^n &= M^{n+1} \oplus \mathbb{M}^n, & N^n &= N^{n+1} \oplus \mathbb{N}^n. \\ Y_2^n &= Y_2^{n+1} \oplus \mathbb{Y}_2^n, & X_2^n &= X_2^{n+1} \oplus \mathbb{X}_2^n. \end{aligned}$$

Finally, consider $n = 1$. According to Remark 2.6,

$$\tilde{S}\colon \tilde{X} \xrightarrow{\cong} R_{\tilde{S},1} = \mathbb{M}^1 \oplus \mathbb{Y}_2^1.$$

Therefore, if

$$\tilde{X}_N = \tilde{S}^{-1}(\mathbb{M}^1), \qquad \tilde{X}_2 = \tilde{S}^{-1}(\mathbb{Y}_2^1),$$

then $\tilde{X} = \tilde{X}_N \oplus \tilde{X}_2$,

$$\tilde{S}\colon \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1, \qquad \tilde{S}\colon \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1,$$

and since $\mathbb{Y}_2^1 \subseteq Y_2$, $\tilde{X}_2$ is finite dimensional subspace of X.

A similar argument proves the case $n = 1$ for the operator T.

∎

As a result of Propositions 3.3 and 3.5, descriptions of X, Y, S, and T are obtained.

Remark 3.6. Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. If $n \in \mathbb{N}$, then according to Remark 2.6 and Propositions 3.3 and 3.5, X and Y may be presented as

$$X = [X_1 \oplus (N^n \oplus \oplus_{i=1}^{n-1} \mathbb{N}^i)] \oplus [X_2^n \oplus \oplus_{i=1}^{n-1} \mathbb{X}_2^i] \oplus [\tilde{X}_N \oplus \tilde{X}_2],$$
$$Y = [Y_1 \oplus (M^n \oplus \oplus_{i=1}^{n-1} \mathbb{M}^i)] \oplus [Y_2^n \oplus \oplus_{i=1}^{n-1} \mathbb{Y}_2^i] \oplus [\tilde{Y}_N \oplus \tilde{Y}_2],$$

and the maps S and T as

$$S \mid_{X_1 \oplus (N^n \oplus \oplus_{i=1}^{n-1} \mathbb{N}^i)} \equiv 0, \qquad S \mid_{X_2^n} : X_2^n \xrightarrow{\cong} R_{S,n+1} = M^{n+1} \oplus Y_2^{n+1},$$
$$S \mid_{\mathbb{X}_2^i} : \mathbb{X}_2^i \xrightarrow{\cong} R_{\tilde{S},i+1} = \mathbb{M}^{i+1} \oplus \mathbb{Y}_2^{i+1}, \qquad S \mid_{\tilde{X}_N} : \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1,\ S \mid_{\tilde{X}_2} : \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1,$$
$$T \mid_{Y_1 \oplus (M^n \oplus \oplus_{i=1}^{n-1} \mathbb{M}^i)} \equiv 0, \qquad T \mid_{Y_2^n} : Y_2^n \xrightarrow{\cong} R_{T,n+1} = N^{n+1} \oplus X_2^{n+1},$$
$$T \mid_{\mathbb{Y}_2^i} : \mathbb{Y}_2^i \xrightarrow{\cong} R_{\tilde{T},i+1} = \mathbb{N}^{i+1} \oplus \mathbb{X}_2^{i+1}, \qquad T \mid_{\tilde{Y}_N} : \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1,\ T \mid_{\tilde{Y}_2} : \tilde{Y}_2 \xrightarrow{\cong} \mathbb{X}_2^1,$$

where $i = 1, \ldots, n-1$.

Remark 3.7. Let X and Y be two Banach spaces and $(S,T) \in RP(X,Y)$. Consider the sequences of subspaces of X and Y $(R_{S,n})_{n\in\mathbb{N}}$ and $(R_{T,n})_{n\in\mathbb{N}}$ respectively. Since $R_{S,n} \subseteq R_{S,2} = R(ST)$ for $n \geq 2$, there is $n_0 \geq 2$ such that $R_{S,n_0} = R_{S,n_0+1}$. Furthermore, according to this observation, it is easy to prove that there is $l \in \mathbb{N}$ such that $R_{S,l} = R_{S,l+k}$ for $k \geq 0$.

Similarly, there is $m \in \mathbb{N}$ such that $R_{T,m} = R_{T,m+k}$ for $k \geq 0$.

Now well, if $R_{S,l} = R_{S,l+k}$ for $k \geq 0$, then $R_{T,l+1} = R_{T,l+1+k}$ for $k \geq 0$. Similarly, if $R_{T,m} = R_{T,m+k}$ for $k \geq 0$, then $R_{S,m+1} = R_{S,m+1+k}$ for $k \geq 0$. Therefore, if p and q denote the first natural numbers such that $R_{S,p} = R_{S,p+k}$ and $R_{T,q} = R_{T,q+k}$ for $k \geq 0$, then p, $q \in \mathbb{N}$, and there are the following possibilities:

i) $p = q$,

ii) if $p < q$, then $q = p + 1$,

iii) if $q < p$, then $p = q + 1$.

The previous remark leads to a definitions which is central in the classification of regular Fredholm pairs.

Definition 3.8. *Let X and Y be two Banach spaces, $(S,T) \in RP(X,Y)$ and p and q as in Remark 3.7. It will be said that the number of the pair (S,T) is n, if $n = min\{p,q\}$, and it will be said that the case of the pair (S,T) is $I - n$ if $p = q$, $II - n$ if $p < q$, and $III - n$ if $q < p$.*

Observe that the above construction is symmetric in X and Y and in S and T. Consequently, in order to study regular Fredholm pairs, interchanging X with Y and S with T if necessary, it is enough to consider only cases $I - n$ and $II - n$.

In the following theorems the classification of regular Fredholm pairs is achieved. Note that the notations of Remark 3.7 will be used. First of all, regular Fredholm pairs whose numbers are equal to 1 are considered.

Theorem 3.9. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Suppose that the number of (S,T) is 1.*

If the case of (S,T) is $I-1$, then the spaces X and Y can be presented as

$$X = X_1 \oplus X_2^2, \qquad Y = Y_1 \oplus Y_2^2,$$

and the operators S and T as

$$S \mid_{X_1} \equiv 0, \qquad S \mid_{X_2^2} : X_2^2 \xrightarrow{\cong} Y_2^2,$$
$$T \mid_{Y_1} \equiv 0, \qquad T \mid_{Y_2^2} : Y_2^2 \xrightarrow{\cong} X_2^2.$$

If the case of (S,T) is $II-1$, then the spaces X and Y can be presented as

$$X = [X_1 \oplus \mathbb{N}^1] \oplus X_2^2, \qquad Y = Y_1 \oplus Y_2^2 \oplus \tilde{Y}_N,$$

and the operators S and T as

$$S \mid_{X_1 \oplus \mathbb{N}^1} \equiv 0, \qquad S \mid_{X_2^2} : X_2^2 \xrightarrow{\cong} Y_2^2,$$
$$T \mid_{Y_1} \equiv 0, \qquad T \mid_{Y_2^2} : Y_2^2 \xrightarrow{\cong} X_2^2, \qquad T \mid_{\tilde{Y}_N} : \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1.$$

If the case of (S,T) is $III-1$, then the spaces X and Y can be presented as

$$X = X_1 \oplus X_2^2 \oplus \tilde{X}_N, \qquad Y = [Y_1 \oplus \mathbb{M}^1] \oplus Y_2^2,$$

and the operators S and T as

$$S \mid_{X_1} \equiv 0, \qquad S \mid_{X_2^2} : X_2^2 \xrightarrow{\cong} Y_2^2, \qquad S \mid_{\tilde{X}_N} : \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1,$$
$$T \mid_{Y_1 \oplus \mathbb{M}^1} \equiv 0, \qquad T \mid_{Y_2^2} : Y_2^2 \xrightarrow{\cong} X_2^2.$$

The spaces involved in the above decompositions are the ones of Remark 3.6.

Proof. Suppose that the case of (S,T) is $I-1$. Since $R(S) = R_{S,1+k}$, $k \geq 0$, according to Proposition 3.5 v), $R_{\tilde{S},k} = 0$ for $k \geq 1$. In particular, $R_{\tilde{S},1} = S(\tilde{X}) = 0$, and since $\tilde{S} \colon \tilde{X} \to R(\tilde{S})$ is an isomorphic map, according to Proposition 3.5 viii), $\tilde{X} = 0$, $\mathbb{M}^1 = 0$ and $\mathbb{Y}_2^1 = 0$.

Similarly, since $1 = p = q$, $R_{\tilde{T},k} = 0$ for $k \geq 1$, $\tilde{Y} = 0$, $\mathbb{N}^1 = 0$ and $\mathbb{X}_2^1 = 0$. Therefore, according to Remark 3.6, X and Y can be presented as

$$X = [X_1 \oplus N^2] \oplus X_2^2, \qquad Y = [Y_1 \oplus M^2] \oplus Y_2^2,$$

and S and T as

$$S \mid_{X_1 \oplus N^2} \equiv 0, \qquad S \mid_{X_2^2} : X_2^2 \xrightarrow{\cong} M^2 \oplus Y_2^2,$$
$$T \mid_{Y_1 \oplus M^2} \equiv 0, \qquad T \mid_{Y_2^2} : Y_2^2 \xrightarrow{\cong} N^2 \oplus X_2^2.$$

Now well, according to the above presentation, $\dim X_2^2 = \dim Y_2^2$, $M^2 = 0$ and $N^2 = 0$.

Next suppose that $p = 1$ and $q = p + 1 = 2$. According to Remark 3.7, this is equivalent to the fact that $R(S) = R_{S,1+k}$ and $R_{T,2} = R_{T,2+k}$, where k is a positive integer. Therefore, according to Proposition 3.5 v), $R_{\tilde{S},1+k} = 0$ for $k \geq 0$. In particular, and according to Proposition 3.5 viii), $R_{\tilde{S},1} = 0$, $\tilde{X} = 0$, $\mathbb{M}^1 = 0$ and $\mathbb{Y}_2^1 = 0$. Similarly, and according to Proposition 3.5 v), $R_{\tilde{T},2+k} = 0$ for $k \geq 0$. In particular, $R_{\tilde{T},2} = 0$, and according to Proposition 3.5 vi) and vii), $\mathbb{N}^2 = 0$, $\mathbb{X}_2^2 = 0$,

$N^2 = N^3$ and $X_2^2 = X_2^3$. As a consequence, according to Remark 3.6, the spaces X and Y can be presented as

$$X = [X_1 \oplus (N^2 \oplus \mathbb{N}^1)] \oplus [X_2^2 \oplus \mathbb{X}_2^1], \qquad Y = [Y_1 \oplus M^2] \oplus Y_2^2 \oplus \tilde{Y},$$

and S and T as

$$\begin{array}{lll} S\mid_{X_1\oplus(N^2\oplus\mathbb{N}^1)} \equiv 0, & S\mid_{X_2}: X_2 \xrightarrow{\cong} M^2 \oplus Y_2^2, & \\ T\mid_{Y_1\oplus M^2} \equiv 0, & T\mid_{Y_2^2}: Y_2^2 \xrightarrow{\cong} N^2 \oplus X_2^2, & T\mid_{\tilde{Y}}: \tilde{Y} \xrightarrow{\cong} \mathbb{N}^1 \oplus \mathbb{X}_2^1. \end{array}$$

Now well, since $R_{\tilde{S},2} = 0$, according to Proposition 3.5 iv), $\mathbb{X}_2^1 = 0$. Therefore, $X_2 = X_2^2$, dim X_2^2= dim Y_2^2, $N^2 = 0$, $M^2 = 0$, and according to Proposition 3.5 viii), $\tilde{Y}_2 = 0$.

Interchanging X with Y and S with T, the proof of the case $III - 1$ can be carried out with an argument similar to the one of the case $II - 1$.

∎

Observe that in the case $I - 1$, X and Y are always finite dimensional Banach spaces.

Next, regular Fredholm pairs whose numbers are greater or equal to 2 are classified.

Theorem 3.10. *Let X and Y be two Banach spaces and (S, T) belong to $RP(X, Y)$. Suppose that the case of (S, T) is $I - n$. Then, If $n = p = q \geq 2$ is the number of the pair (S, T), the spaces X and Y can be presented as*

$$\begin{aligned} X &= [X_1 \oplus \oplus_{i=1}^{p-1}\mathbb{N}^i] \oplus [X_2^{p-1} \oplus \oplus_{i=1}^{p-2}\mathbb{X}_2^i] \oplus [\tilde{X}_N \oplus \tilde{X}_2], \\ Y &= [Y_1 \oplus \oplus_{i=1}^{p-1}\mathbb{M}^i] \oplus [Y_2^{p-1} \oplus \oplus_{i=1}^{p-2}\mathbb{Y}_2^i] \oplus [\tilde{Y}_N \oplus \tilde{Y}_2], \end{aligned}$$

and the operators S and T as

$$\begin{array}{lll} S\mid_{X_1\oplus\oplus_{i=1}^{p-1}\mathbb{N}^i} \equiv 0, & S\mid_{X_2^{p-1}}: X_2^{p-1} \xrightarrow{\cong} Y_2^{p-1}, & S\mid_{\mathbb{X}_2^i}: \mathbb{X}_2^i \xrightarrow{\cong} \mathbb{M}^{i+1} \oplus \mathbb{Y}_2^{i+1}, \\ S\mid_{\mathbb{X}_2^{p-2}}: \mathbb{X}_2^{p-2} \xrightarrow{\cong} \mathbb{M}^{p-1}, & S\mid_{\tilde{X}_N}: \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1, & S\mid_{\tilde{X}_2}: \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1, \\ T\mid_{Y_1\oplus\oplus_{i=1}^{p-1}\mathbb{M}^i} \equiv 0, & T\mid_{Y_2^{p-1}}: Y_2^{p-1} \xrightarrow{\cong} X_2^{p-1}, & T\mid_{\mathbb{Y}_2^i}: \mathbb{Y}_2^i \xrightarrow{\cong} \mathbb{N}^{i+1} \oplus \mathbb{X}_2^{i+1}, \\ T\mid_{\mathbb{Y}_2^{p-2}}: \mathbb{Y}_2^{p-2} \xrightarrow{\cong} \mathbb{N}^{p-1}, & T\mid_{\tilde{Y}_N}: \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1, & T\mid_{\tilde{Y}_2}: \tilde{Y}_2 \xrightarrow{\cong} \mathbb{X}_2^1, \end{array}$$

where $i = 1, \ldots, p - 3$, and the spaces involved in the above decomposition are the ones of Remark 3.6.

In addition, if $n = 2$, then $\mathbb{X}_2^1$, $\mathbb{Y}_2^1$, $\mathbb{Y}_2$ and $\mathbb{X}_2$ are null spaces.

Proof. Let $p = q \geq 2$ be the number of the pair (S, T). Since $R_{S,p} = R_{S,p+k}$ and $R_{S,p+k} = R_{S,p+k+1} \oplus R_{\tilde{S},p+k}$ for $k \geq 0$, then $R_{\tilde{S},p+k} = 0$ for $k \geq 0$, that is $\mathbb{M}^{p+k} = 0$ and $\mathbb{Y}_2^{p+k} = 0$ for $k \geq 0$. Furthermore, since according to Proposition 3.5 iv) and vii),

$$S: \mathbb{X}_2^{p-1+k} \xrightarrow{\cong} R_{\tilde{S},p+k}, \qquad X_2^{p-1+k} = X_2^{p+k} \oplus \mathbb{X}_2^{p-1+k},$$

then $\mathbb{X}_2^{p-1+k} = 0$ and $X_2^{p-1} = X_2^{p-1+k}$ for $k \geq 0$. In particular, according to Proposition 3.5 vii),

$$X_2 = X_2^{p-1} \oplus \oplus_{i=1}^{p-2}\mathbb{X}_2^i.$$

On the other hand, since $p=q$, similar properties can be obtained for T and X. Therefore, $\mathbb{N}^{p+k}=0$, $\mathbb{X}_2^{p+k}=0$, $\mathbb{Y}_2^{p-1+k}=0$, $Y_2^{p-1}=Y_2^{p-1+k}$ for $k\geq 0$, and

$$Y_2=Y_2^{p-1}\oplus\oplus_{i=1}^{p-2}\mathbb{Y}_2^i.$$

In addition, since according to Proposition 3.3 i) and vi),

$$S\colon X_2^{p-1}\xrightarrow{\cong}R_{S,p}=M^p\oplus Y_2^p=M^p\oplus Y_2^{p-1},$$
$$T\colon Y_2^{p-1}\xrightarrow{\cong}R_{T,p}=N^p\oplus X_2^p=N^p\oplus X_2^{p-1},$$

it is clear that $M^p=0$, $N^p=0$ and

$$S\colon X_2^{p-1}\xrightarrow{\cong}Y_2^{p-1},\qquad T\colon Y_2^{p-1}\xrightarrow{\cong}X_2^{p-1}.$$

Finally, if $p=q=2$, then $R_{\tilde{S},2}=0$ and $R_{\tilde{T},2}=0$. Consequently, according to Proposition 3.5 i) and viii), $\mathbb{X}_2^1$, $\mathbb{Y}_2^1$, $\tilde{X}_2$ and $\tilde{Y}_2$ are null spaces.

∎

Theorem 3.11. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Suppose that the case of (S,T) is $II-n$. Then, if $n=p=q-1\geq 2$ is the number of the pair (S,T), the spaces X and Y can be presented as*

$$X=[X_1\oplus\oplus_{i=1}^{p}\mathbb{N}^i]\oplus[X_2^{p-1}\oplus\oplus_{i=1}^{p-2}\mathbb{X}_2^i]\oplus[\tilde{X}_N\oplus\tilde{X}_2],$$
$$Y=[Y_1\oplus\oplus_{j=1}^{p-1}\mathbb{M}^j]\oplus[Y_2^{p}\oplus\oplus_{j=1}^{p-1}\mathbb{Y}_2^j]\oplus[\tilde{Y}_N\oplus\tilde{Y}_2],$$

and the operators S and T as

$$S\mid_{X_1\oplus\oplus_{i=1}^{p}\mathbb{N}^i}\equiv 0,\qquad S\mid_{X_2^{p-1}}\colon X_2^{p-1}\xrightarrow{\cong}Y_2^p,\qquad S\mid_{\mathbb{X}_2^i}\colon\mathbb{X}_2^i\xrightarrow{\cong}\mathbb{M}^{i+1}\oplus\mathbb{Y}_2^{i+1},$$
$$S\mid_{\tilde{X}_N}\colon\tilde{X}_N\xrightarrow{\cong}\mathbb{M}^1,\qquad S\mid_{\tilde{X}_2}\colon\tilde{X}_2\xrightarrow{\cong}\mathbb{Y}_2^1,$$
$$T\mid_{Y_1\oplus\oplus_{j=1}^{p-1}\mathbb{M}^j}\equiv 0,\qquad T\mid_{Y_2^p}\colon Y_2^p\xrightarrow{\cong}X_2^{p-1},\qquad T\mid_{\mathbb{Y}_2^j}\colon\mathbb{Y}_2^j\xrightarrow{\cong}\mathbb{N}^{j+1}\oplus\mathbb{X}_2^{j+1},$$
$$T\mid_{\mathbb{Y}_2^k}\colon\mathbb{Y}_2^k\xrightarrow{\cong}\mathbb{N}^{k+1},\qquad T\mid_{\tilde{Y}_N}\colon\tilde{Y}_N\xrightarrow{\cong}\mathbb{N}^1,\qquad T\mid_{\tilde{Y}_2}\colon\tilde{Y}_2\xrightarrow{\cong}\mathbb{X}_2^1,$$

where $i=1,\ldots,p-2$, $j=1,\ldots,p-3$, $k=p-2$, $p-1$, and the spaces involved in the above decomposition are the ones of Remark 3.6.

In addition, if $p=2$, then $\mathbb{X}_2^1$ and $\tilde{Y}_2$ are null spaces.

Proof. Let $p=q-1\geq 2$ be the number of the pair (S,T). Consequently, $q=p+1$, and since $R_{S,p}=R_{S,p+k}$ and $R_{T,p+1}=R_{T,p+1+k}$ for $k\geq 0$, as in Theorem 3.10, $R_{\tilde{S},p+k}=0$ and $R_{\tilde{T},p+1+k}=0$ for $k\geq 0$. In particular, according to Proposition 3.5 i),

$$\mathbb{M}^{p+k}=0,\qquad \mathbb{N}^{p+1+k}=0,$$

for $k\geq 0$.

On the other hand, according to Proposition 3.5 iv) and vii),

$$S\colon\mathbb{X}_2^{p-1+k}\xrightarrow{\cong}R_{\tilde{S},p+k},\qquad T\colon\mathbb{Y}_2^{p+k}\xrightarrow{\cong}R_{\tilde{T},p+k+1},$$

and

$$X_2^{p-1+k}=X_2^{p+k}\oplus\mathbb{X}_2^{p-1+k},\qquad Y_2^{p+k}=Y_2^{p+k+1}\oplus\mathbb{Y}_2^{p+k}.$$

Consequently,

$$\mathbb{X}_2^{p-1+k}=0,\ \mathbb{Y}_2^{p+k}=0,\ X_2^{p-1}=X_2^{p-1+k},\ Y_2^p=Y_2^{p+k},$$

for $k \geq 0$. Therefore,

$$X_2 = X_2^{p-1} \oplus \oplus_{i=1}^{p-2} \mathbb{X}_2^i, \qquad Y_2 = Y_2^p \oplus \oplus_{j=1}^{p-1} \mathbb{Y}_2^j.$$

Now well, since $X_2^{p-1} = X_2^{p+1}$ and since

$$S\colon X_2^{p-1} \xrightarrow{\cong} R_{S,p} = M^p \oplus Y_2^p, \; T\colon Y_2^p \xrightarrow{\cong} R_{T,p+1} = N^{p+1} \oplus X_2^{p+1},$$

it is clear that $M^p = 0$, $N^{p+1} = 0$, and

$$S\colon X_2^{p-1} \xrightarrow{\cong} Y_2^p, \qquad T\colon Y_2^p \xrightarrow{\cong} X_2^{p-1}.$$

Finally, if $p = 2$, then $R_{\tilde{S},2} = 0$. Consequently, according to Proposition 3.5 i) and viii), $\mathbb{X}_2^1$ and $\tilde{Y}_2$ are null spaces.

∎

Theorem 3.12. *Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Suppose that the case of (S,T) is $III - n$. Then, if $q = p-1 \geq 2$ is the number of the pair (S,T), the spaces X and Y can be presented as*

$$X = [X_1 \oplus \oplus_{i=1}^{q-1} \mathbb{N}^i] \oplus [X_2^q \oplus \oplus_{i=1}^{q-1} \mathbb{X}_2^i] \oplus [\tilde{X}_N \oplus \tilde{X}_2],$$
$$Y = [Y_1 \oplus \oplus_{j=1}^{q} \mathbb{M}^j] \oplus [Y_2^{q-1} \oplus \oplus_{j=1}^{q-2} \mathbb{Y}_2^j] \oplus [\tilde{Y}_N \oplus \tilde{Y}_2],$$

and the operators S and T as

$$S\mid_{X_1 \oplus \oplus_{i=1}^{q-1} \mathbb{N}^i} \equiv 0, \qquad S\mid_{X_2^q}\colon X_2^q \xrightarrow{\cong} Y_2^{q-1}, \qquad S\mid_{\mathbb{X}_2^i}\colon \mathbb{X}_2^i \xrightarrow{\cong} \mathbb{M}^{i+1} \oplus \mathbb{Y}_2^{i+1},$$
$$S\mid_{\mathbb{X}_2^k}\colon \mathbb{X}_2^k \xrightarrow{\cong} \mathbb{M}^{k+1}, \qquad S\mid_{\tilde{X}_N}\colon \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1, \qquad S\mid_{\tilde{X}_2}\colon \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1,$$
$$T\mid_{Y_1 \oplus \oplus_{j=1}^{q} \mathbb{M}^j} \equiv 0, \qquad T\mid_{Y_2^{q-1}}\colon Y_2^{q-1} \xrightarrow{\cong} X_2^q, \qquad T\mid_{\mathbb{Y}_2^j}\colon \mathbb{Y}_2^j \xrightarrow{\cong} \mathbb{N}^{j+1} \oplus \mathbb{X}_2^{j+1},$$
$$T\mid_{\tilde{Y}_N}\colon \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1, \qquad T\mid_{\tilde{Y}_2}\colon \tilde{Y}_2 \xrightarrow{\cong} \mathbb{X}_2^1,$$

where $i = 1, \dots, q-3$, $k = q-2$, $q-1$, $j = 1, \dots, q-2$, and the spaces involved in the above decomposition are the ones of Remark 3.6.

In addition, if $q = 2$, then $\mathbb{Y}_2^1$ and $\tilde{X}_2$ are null spaces.

Proof. Interchanging X with Y and S with T, the proof can be carried out with an argument similar to the one of Theorem 3.11.

∎

Remark 3.13. Let X and Y be Banach spaces and $(S,T) \in RP(X,Y)$. Suppose that the case of (S,T) is $I - 2$. According to Theorem 3.10, X and Y may be described as

$$X = (X_1 \oplus \mathbb{N}^1) \oplus X_2^1 \oplus \tilde{X}_N, \qquad Y = (Y_1 \oplus \mathbb{M}^1) \oplus Y_2^1 \oplus \tilde{Y}_N,$$

and the the operatos S and T can be presented as

$$S\mid_{X_1 \oplus \mathbb{N}^1} \equiv 0, \qquad S\colon X_2^1 \xrightarrow{\cong} Y_2^1, \qquad S\colon \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1,$$
$$T\mid_{Y_1 \oplus \mathbb{M}^1} \equiv 0, \qquad T\colon Y_2^1 \xrightarrow{\cong} X_2^1, \qquad T\colon \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1.$$

Moreover, since $\dim X_2 = \dim X_2^1 = \dim Y_2^1 = \dim Y_2$, ind $(S,T) = \dim X_1 - \dim Y_1$.

Now well, if X' and Y' are Banach spaces and if $(S', T') \in RP(X', Y')$ is a regular Fredholm symmetrical pair, then it is not difficult to prove that

$$X' = X_1' \oplus N \oplus \tilde{X}, \qquad Y' = Y_1' \oplus M \oplus \tilde{Y},$$

where X_1' and Y_1' are finite dimensional subspaces. Furthermore, the operators S' and T' are such that

$$S' \mid_{X_1' \oplus N} \equiv 0,\ S' \colon \tilde{X} \xrightarrow{\cong} M,\ T' \mid_{Y_1' \oplus M} \equiv 0,\ T' \colon \tilde{Y} \xrightarrow{\cong} N,$$

which implies that ind $(S', T') = \dim X_1 - \dim Y_1$. Therefore, a regular Fredholm symmetrical pair is nothing but a very particular type of regular Fredholm pair, that is a pair whose case is $I - 2$ and such that $X_2 = 0$ and $Y_2 = 0$.

Remark 3.14. Observe that if X and Y are Banach spaces and (S, T) belongs to $RP(X, Y)$, then, according to Theorems 3.9 - 3.12, X is a finite dimensional Banach space if and only if Y is.

On the other hand, if X, Y, S and T are constructed as in Theorems 3.9 - 3.12, then $(S, T) \in RP(X, Y)$ and the number and case of (S, T) are the ones considered in the corresponding theorem. Therefore, thanks to Theorems 3.9 - 3.12, regular Fredholm are entirely classified.

4. The index of regular Fredholm pairs and Weyl Pairs

In this section the index of a regular Fredholm pair is studied. It is proved that the index provides a fundamental tool in the description of the spaces and maps of such a pair. Furthermore, Weyl pairs, that is Fredholm pairs whose index is null, are considered.

Theorem 4.1. *Let X and Y be two Banach spaces and (S, T) belong to $RP(X, Y)$. Suppose that the number of the pair (S, T) is greater or equal 2. Consider $n = 2$ and the corresponding decomposition of the spaces X and Y given in Remark 3.6, that is*

$$X = [X_1 \oplus (N^2 \oplus \mathbb{N}^1)] \oplus [X_2^2 \oplus \mathbb{X}_2^1] \oplus [\tilde{X}_N \oplus \tilde{X}_2],$$
$$Y = [Y_1 \oplus (M^2 \oplus \mathbb{M}^1)] \oplus [Y_2^2 \oplus \mathbb{Y}_2^1] \oplus [\tilde{Y}_N \oplus \tilde{Y}_2].$$

Then

$$\begin{aligned} \text{ind } (S, T) &= \dim\ (X_1 \oplus N^2) - \dim\ (Y_1 \oplus \tilde{Y}_2) \\ &= \dim\ (X_1 \oplus \tilde{X}_2) - \dim\ (Y_1 \oplus M^2). \end{aligned}$$

In addition, if the number of the pair (S, T) is 1, then

$$\text{ind } (S, T) = \dim\ X_1 - \dim\ Y_1.$$

Proof. According to Definition 2.1 and Remark 2.2, the index of the pair (S, T) is the number

$$\text{ind } (S, T) = \dim X_1 - \dim X_2 - \dim Y_1 + \dim Y_2.$$

Now well, according to Proposition 3.3 vi), $T \colon Y_2 \xrightarrow{\cong} R_{T,2} = N^2 \oplus X_2^2$. Moreover, according to Proposition 3.5 viii), $T \colon \tilde{Y}_2 \xrightarrow{\cong} \mathbb{X}_2^1$. Consequently

$$\dim Y_2 - \dim X_2 = \dim N^2 - \dim \mathbb{X}_2^1 = \dim N^2 - \dim \tilde{Y}_2,$$

and

$$\text{ind } (S, T) = \dim(X_1 \oplus N^2) - \dim(Y_1 \oplus \tilde{Y}_2).$$

Since ind $(T, S) = -$ ind (S, T), a similar argument proves the second equality.

The last assertion is a consequence of Theorem 3.9.

∎

Remark 4.2. Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Suppose that the number of the pair (S,T) is greater or equal 2. Consider again, as in Theorem 4.1, $n=2$ and the corresponding description of the spaces X and Y given in Remark 3.6.

Now well, according to Proposition 3.5 vii) and viii), $Y_2 = Y_2^2 \oplus \mathbb{Y}_2^1$, and

$$S\colon \tilde{X}_2 \xrightarrow{\cong} \mathbb{Y}_2^1, \qquad S\colon \tilde{X}_N \xrightarrow{\cong} \mathbb{M}^1, \qquad T\colon \tilde{Y}_N \xrightarrow{\cong} \mathbb{N}^1.$$

In addition, according to Proposition 3.3 vi),

$$S\colon X_2 \xrightarrow{\cong} R_{S,2} = M^2 \oplus Y_2^2.$$

Consequently, the subspaces of X and Y that in the above presentation are not related by isomorphic maps are $X_1 \oplus N^2$ and $Y_1 \oplus \tilde{Y}_2$ respectively.

Similarly, interchanging X with Y and S with T, the subspaces of X and Y that in the above presentation are not related by isomorphic maps are $Y_1 \oplus M^2$ and $X_1 \oplus \tilde{X}_2$ respectively.

On the other hand, if the number of the pair is 1, according to Theorem 3.9, the subspaces of X and Y that are not related by isomorphic maps are X_1 and Y_1.

As a result, the index has a fundamental role in the description of regular Fredholm pairs. In fact, the index is a measure of the subspaces of X and Y that in the above decomposition are not related by isomorphisms.

Remark 4.3. Let X and Y be two Banach spaces and (S,T) belong to $RP(X,Y)$. Suppose that the number of the pair (S,T) is greater or equal 2. Consider, as in Theorem 4.1, $n=2$ and the corresponding decompositon of X and Y given in Remark 3.6. Next suppose that $\operatorname{ind}(S,T)=0$. According to Theorem 4.1, this is equivalent to the fact that $X_1 \oplus N^2$ is isomorphic to $Y_1 \oplus \tilde{Y}_2$ and $X_1 \oplus \tilde{X}_2$ to $Y_1 \oplus M^2$. However, since according to Proposition 3.5 viii), $\mathbb{N}^1$ is isomorphic to $\tilde{Y}_N$ and $\mathbb{M}^1$ to $\tilde{X}_N$, then $N(S)$ is isomorphic $Y/R(S)$ and $N(T)$ to $X/R(T)$. Consequently, according to [5, 3.8.6], S and T are *decomposably regular or relatively Weyl operators*, that is S and T are regular maps which have isomorphic pseudoinverses $S' \in L(Y,X)$ and $T' \in L(X,Y)$ respectively, see [5, 3.8.5]. Similarly, if the number of the pair (S,T) is 1, and if $\operatorname{ind}(S,T)=0$, then, according to Theorems 3.9 and 4.1, S and T are decomposably regular operators. As an analogy to Weyl operators, Weyl pairs are introduced, see [5, 6.5].

Definition 4.4. *Let X and Y be two Banach spaces and (S,T) belong to $P(X,Y)$. The pair (S,T) is said a Weyl pair, if* $\operatorname{ind}(S,T)=0$. *The set of all Weyl pairs is denoted by $W(X,Y)$. In addition, if (S,T) belongs to $W(X,Y) \cap RP(X,Y)$, then (S,T) is called a regular Weyl pair. The set of all regular Weyl pairs is denoted by $RW(X,Y)$.*

Remark 4.5. Let X and Y be two Banach spaces and $S \in L(X,Y)$. According to Remark 2.2, it is clear that if S is a Weyl operator, then $(S,0)$ belongs to $RW(X,Y)$.

On the other hand, it is well known that in order for $S \in L(X,Y)$ to be a Weyl operator it is necessary and sufficient that S is Fredholm and decomposably regular, see for example [5, 6.5.2]. However, as the following example shows, there are regular Fredholm pairs whose operators are decomposably regular and whose index is not null.

Let I and J arbitrary infinite disjoint sets and consider the Hilbert spaces $N = l^2(I)$ and $M = l^2(J)$. Let I_1 be a finite non void set such that $I_1 \cap I = \emptyset = I_1 \cap J$, and consider the Hilbert space $X_1 = l^2(I_1)$. Next define the Hilbert spaces

$$X = X_1 \oplus N \oplus M, \qquad Y = M \oplus N,$$

and the operators $S \in L(X,Y)$ and $T \in L(Y,X)$

$$S\mid_{X_1\oplus N}\equiv 0,\ S = I_M\colon M \to M,\ T\mid_M\equiv 0,\ T = I_N\colon N \to N,$$

where I_M and I_N denotes the identity maps of M and N respectively.

It is clear that S and T are regular operators. Moreover, (S,T) belongs to $RP(X,Y)$, actually, (S,T) is a regular Fredholm symmetrical pair, and $\operatorname{ind}(S,T) = \dim X_1$, which is non null, for I_1 is a non void set.

Now well, since I_1 is a finite set and I is an infinite set, $X_1 \oplus N$ is isomorphic to N. Consequently, $N(S)$ is isomorphic to $Y/R(S)$, that is S is a decomposably regular operator, see [5, 3.8.6]. Similarly, $X_1 \oplus M$ is isomorphic to M. Therefore, $N(T)$ is isomorphic to $X/R(T)$, that is T is a decomposably regular operator, see [5, 3.8.6].

5. Characterizations of Regular Fredholm Pairs

In this section three characterizations of regular Fredholm pairs are proved. In the first one such objects are characterized in terms of regular Fredholm symmetrical pairs. This characterization plays a central role in the proof of the second one, where regular Fredholm pairs are characterized in terms of exact chains of multiplication operators. Finally, in the third one, the objects under consideration are characterized in terms of invertible Banach space operators.

In order to prove the first characterization some preparation is needed.

Remark 5.1. Consider X and Y two Banach spaces, and $S \in L(X,Y)$ and $T \in L(Y,X)$ two operators such that $R(ST)$ and $R(TS)$ are finite dimensional subspaces of Y and X respectively. Then, it is possible to define the Banach spaces $\mathcal{X} = X/R(TS)$ and $\mathcal{Y} = Y/R(ST)$, and the linear and continuous maps $\overline{S} \in L(\mathcal{X},\mathcal{Y})$ and $\overline{T} \in L(\mathcal{Y},\mathcal{X})$, the factorizations of S and T through the respective invariant subspaces. It is clear that $\overline{S}\circ\overline{T} = 0$ and $\overline{T}\circ\overline{S} = 0$. Furthermore, according to [2, 2.1],

$$N(\overline{S})/R(\overline{T}) \cong N(S)/(N(S)\cap R(T)),\ N(\overline{T})/R(\overline{S}) \cong N(T)/(N(T)\cap R(S)).$$

Therefore, the pair (S,T) belongs to $P(X,Y)$ if and only if $(\overline{S},\overline{T})$ is a Fredholm symmetrical pair, see [2, 2.1].

Theorem 5.2. *Let X and Y be two Banach spaces and (S,T) belong to $P(X,Y)$. Then, with the notations of Remark 5.1, (S,T) belongs to $RP(X,Y)$ if and only if $(\overline{S},\overline{T})$ is a regular Fredholm symmetrical pair.*

Proof. First of all, note that if $(S,T) \in RP(X,Y)$, then according to Remark 5.1, in order to prove that $(\overline{S},\overline{T})$ is a regular Fredholm symmetrical pair, it is enough to prove that $\overline{S}$ and $\overline{T}$ are regular operators.

Consider $n = 2$ and the corresponding decomposition of X and Y of Remark 3.6, that is

$$X = (X_1 \oplus N^2 \oplus \mathbb{N}^1) \oplus (X_2^2 \oplus \mathbb{X}_2^1) \oplus \tilde{X},\ Y = (Y_1 \oplus M^2 \oplus \mathbb{M}^1) \oplus (Y_2^2 \oplus \mathbb{Y}_2^1) \oplus \tilde{Y},$$

and recall that

$$R(ST) = R_{S,2} = M^2 \oplus Y_2^2, \qquad R(TS) = R_{T,2} = N^2 \oplus X_2^2.$$

Therefore, $X/R(TS)$ and $Y/R(ST)$ can be identified with

$$\mathcal{X} = X_1 \oplus \mathbb{N}^1 \oplus \mathbb{X}_2^1 \oplus \tilde{X}, \qquad \mathcal{Y} = Y_1 \oplus \mathbb{M}^1 \oplus \mathbb{Y}_2^1 \oplus \tilde{Y}.$$

Moreover, since

$$R(\overline{S}) = R(S)/R(ST),\ R(\overline{T}) = R(T)/R(TS),$$
$$N(\overline{S}) = S^{-1}(R(ST))/R(TS) = (N(S) + R(T))/R(TS),$$
$$N(\overline{T}) = T^{-1}(R(TS))/R(ST) = (N(T) + R(S))/R(ST),$$

these spaces can be identified with

$$R(\overline{S}) = \mathbb{M}^1 \oplus \mathbb{Y}_2^1, \qquad N(\overline{S}) = X_1 \oplus \mathbb{N}^1 \oplus \mathbb{X}_2^1$$
$$R(\overline{T}) = \mathbb{N}^1 \oplus \mathbb{X}_2^1, \qquad N(\overline{T}) = Y_1 \oplus \mathbb{M}^1 \oplus \mathbb{Y}_2^1.$$

Therefore, according to [5, 3.8.2], $\overline{S}$ and $\overline{T}$ are regular operators.

On the other hand, suppose that $\overline{S}$ is a regular operator. Then, there is V, a closed linear subspace of $\mathcal{X}$, such that

$$N(\overline{S}) \oplus V = \mathcal{X}.$$

Let $\pi\colon X \to \mathcal{X}$ be the canonical projection and $V_1 = \pi^{-1}(V) \cap R(TS)$. Since V_1 is a finite dimensional subspace of the Banach space $\pi^{-1}(V)$, there is a closed linear subspace $W_1 \subseteq \pi^{-1}(V)$ such that

$$V_1 \oplus W_1 = \pi^{-1}(V).$$

Now well, since π is a surjective map and since $\pi(V_1) = 0$,

$$\pi(W_1) = \pi(\pi^{-1}(V)) = V.$$

Furthermore, according to [2, 2.1],

$$\pi(N(S) + R(T) + W_1) = \pi(N(S) + R(T)) + \pi(W_1) = N(\overline{S}) + V = \mathcal{X}.$$

Consequently, $N(S)+R(T)+W_1+R(TS) = X$. However, since $R(TS) \subseteq R(T)$, $(N(S) + R(T)) + W_1 = X$.

Next consider $L = (N(S)+R(T))\cap W_1$. Since $\pi(L) \subseteq N(\overline{S})\cap V = 0$, $L \subseteq R(TS)$. Consequently, $L \subseteq W_1 \cap R(TS) = 0$. Therefore,

$$(N(S) + R(T)) \oplus W_1 = X.$$

Similarly, if $\overline{T}$ is a regular operator, then there is W_2, a closed subspace of Y, such that

$$(N(T) + R(S)) \oplus W_2 = Y.$$

Finally, since $(S, T) \in P(X, Y)$, according to Proposition 2.4, (S, T) is a regular Fredholm pair. ∎

Next follows the preparation needed for the second characterization.

Remark 5.3. Let X and Y be two Banach spaces and $S \in L(X, Y)$. Then, given another Banach space Z, it is possible to define the left and right multiplication operators induced by S, that is

$$L_S\colon L(Z, X) \to L(Z, Y), \qquad L_S(V) = SV,$$
$$R_S\colon L(Y, Z) \to L(X, Z), \qquad R_S(W) = WS,$$

where $V \in L(Z, X)$ and $W \in L(Y, Z)$.

It is clear that $\| L_S \| \leq \| S \|$. Furthermore, since $L_S(K(Z,X)) \subseteq K(Z,Y)$ and $R_S(K(Y,Z)) \subseteq K(X,Z)$, it is possible to introduce the operators

$$\tilde{L}_S\colon C(Z,X) \to C(Z,Y), \qquad \tilde{R}_S\colon C(Y,Z) \to C(X,Z),$$

where

$$C(Z,X) = L(Z,X)/K(Z,X), \qquad C(Z,Y) = L(Z,Y)/K(Z,Y),$$
$$C(Y,Z) = L(Y,Z)/K(Y,Z), \qquad C(X,Z) = L(X,Z)/K(X,Z),$$

and the maps $\tilde{L}_S$ and $\tilde{R}_S$ are the factorizations of L_S and R_S through the respective closed ideal of compact operators.

Similarly, if $T \in L(Y,X)$, then, given another Banach space Z, it is possible to define L_T and R_T, the left and right multiplication operators induced by T, that is

$$L_T\colon L(Z,Y) \to L(Z,X), \qquad L_T(V) = TV,$$
$$R_T\colon L(X,Z) \to L(Y,Z), \qquad R_T(W) = WT,$$

where $V \in L(Z,Y)$ and $W \in L(X,Z)$. Furthermore, as above, it is also possible to define

$$\tilde{L}_T\colon C(Z,Y) \to C(Z,X), \qquad \tilde{R}_T\colon C(X,Z) \to C(Y,Z),$$

the factorizations of L_T and R_T through the respective closed ideal of compact operators.

Next suppose that $R(ST)$ and $R(TS)$ are finite dimensional subspaces of Y and X respectively. Then

$$\tilde{L}_S\tilde{L}_T = \tilde{L}_{ST} = 0, \qquad \tilde{L}_T\tilde{L}_S = \tilde{L}_{TS} = 0,$$

that is the pairs of operators $(\tilde{L}_S, \tilde{L}_T)$ and $(\tilde{L}_T, \tilde{L}_S)$ are chains, see [5, 10.3] or [6].

Finally, consider $U \in L(X_2, X_3)$ and $V \in L(X_1, X_2)$, where X_1, X_2 and X_3 are three Banach spaces, and suppose that (U,V) is a chain, that is $UV = 0$. The chain (U,V) is called *exact*, if $R(V) = N(U)$. In addition, it is said that (U,V) is an *invertible chain*, if there are continuous linear maps $V_1 \in L(X_2, X_1)$ and $U_1 \in L(X_3, X_2)$ such that

$$U_1U + VV_1 = I,$$

where I denotes the identity map of X_2, see [5, 10.3.1] or [6].

Next follows the second characterization of regular Fredholm pairs.

Theorem 5.4. *Let X and Y be two Banach spaces, and consider $S \in L(X,Y)$ and $T \in L(Y,X)$ two operators such that $R(ST)$ and $R(TS)$ are finite dimensional subspaces of Y and X respectively. With the same notations of Remark 5.3, the following assertions are equivalent:*

i) the pair (S,T) belongs to $RP(X,Y)$;

ii) the operators S and T are regular, and $(\tilde{L}_S, \tilde{L}_T)$ (resp. $(\tilde{L}_T, \tilde{L}_S)$) is an invertible chain for any Banach space Z;

iii) the operators S and T are regular, and $(\tilde{L}_S, \tilde{L}_T)$ (resp. $(\tilde{L}_T, \tilde{L}_S)$) is an invertible chain for the Banach space X (resp. Y);

iv) the operators S and T are regular, and $(\tilde{L}_S, \tilde{L}_T)$ (resp. $(\tilde{L}_T, \tilde{L}_S)$) is an exact chain for the Banach space X (resp. Y) .

Similarly, the following assertions are equivalent:

i) the pair (S,T) belongs to $RP(X,Y)$;

ii) the operators S and T are regular, and $(\tilde{R}_S, \tilde{R}_T)$ (resp. $(\tilde{R}_T, \tilde{R}_S)$) is an invertible chain for any Banach space Z;

iii) the operators S and T are regular, and $(\tilde{R}_S, \tilde{R}_T)$ (resp. $(\tilde{R}_T, \tilde{R}_S)$) is an invertible chain for the Banach space Y (resp. X) ;

iv) the operators S and T are regular, and $(\tilde{R}_S, \tilde{R}_T)$ (resp. $(\tilde{R}_T, \tilde{R}_S)$) is an exact chain for the Banach space Y (resp. X) .

Proof. First of all, observe that since $R(ST)$ and $R(TS)$ are finite dimensional Banach space, there exist $\mathcal{X}$ and $\mathcal{Y}$, two closed subspaces of X and Y respectively, such that $X = \mathcal{X} \oplus R(TS)$ and $Y = \mathcal{Y} \oplus R(ST)$. Moreover, if S and T are presented as matrices, that is if

$$S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix},$$

and if $X/R(TS)$ and $Y/R(ST)$ are identified with $\mathcal{X}$ and $\mathcal{Y}$ respectively, then the maps $\overline{S}$ and $\overline{T}$ in Remark 5.1 can be identified with S_{11} and T_{11} respectively.

In addition, if Z is an arbitrary Banach space, since

$$L(Z, X) = L(Z, \mathcal{X}) \oplus L(Z, R(TS)), \qquad L(Z, Y) = L(Z, \mathcal{Y}) \oplus L(Z, R(ST)),$$
$$K(Z, X) = K(Z, \mathcal{X}) \oplus K(Z, R(TS)), \qquad K(Z, Y) = K(Z, \mathcal{Y}) \oplus K(Z, R(ST)),$$

then,

$$C(Z, X) = C(Z, \mathcal{X}), \qquad C(Z, Y) = C(Z, \mathcal{Y}).$$

Furthermore, it is clear that

$$\tilde{L}_S = \tilde{L}_{S_{11}}, \qquad \tilde{L}_T = \tilde{L}_{T_{11}}.$$

Now well, if (S, T) belongs to $RP(X, Y)$, then according to Theorem 5.2 and to the above identifications, (S_{11}, T_{11}) is a regular Fredholm symmetrical pair. Therefore, according to [5, 10.6.2], there are operators S_1 and S_2 in $L(\mathcal{Y}, \mathcal{X})$, T_1 and T_2 in $L(\mathcal{X}, \mathcal{Y})$, and two operators with finite dimensional rank, $K_1 \in L(\mathcal{X})$ and $K_2 \in L(\mathcal{Y})$, such that

$$S_1 S_{11} + T_{11} T_1 = I_1 - K_1, \qquad T_2 T_{11} + S_{11} S_2 = I_2 - K_2,$$

where I_1 and I_2 denote the identity maps of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Consequently,

$$\tilde{L}_{S_1} \tilde{L}_{S_{11}} + \tilde{L}_{T_{11}} \tilde{L}_{T_1} = \mathcal{I}_1, \qquad \tilde{L}_{T_2} \tilde{L}_{T_{11}} + \tilde{L}_{S_{11}} \tilde{L}_{S_2} = \mathcal{I}_2,$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the identity maps of $C(Z, \mathcal{X})$ and $C(Z, \mathcal{Y})$ respectively.

However, since $C(Z, \mathcal{X}) = C(Z, X)$, $C(Z, \mathcal{Y}) = C(Z, Y)$, $\tilde{L}_{S_{11}} = \tilde{L}_S$ and $\tilde{L}_{T_{11}} = \tilde{L}_T$, then $(\tilde{L}_S, \tilde{L}_T)$ and $(\tilde{L}_T, \tilde{L}_S)$ are invertible chains in the sense of [5, 10.3.1] or [6].

It is clear that ii) implies iii) and iii) implies iv).

Next suppose that iv) holds. Since S and T are regular maps, there are operators $S' \in L(Y, X)$ and $T' \in L(X, Y)$ such that $S = SS'S$ and $T = TT'T$.

Now well, if, as above, X and Y are decomposed as direct sums

$$X = \mathcal{X} \oplus R(TS), \qquad Y = \mathcal{Y} \oplus R(ST),$$

and if S and S' are presented as matrices, that is if

$$S = \begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}, \qquad S' = \begin{pmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{pmatrix},$$

then a straightforward calculation proves that

$$S_{11} = S_{11}S'_{11}S_{11} + S_{11}S_1,$$

where $S_1 \in L(\mathcal{X})$ is an operator whose range is finite dimensional.

On the other hand, since $C(\mathcal{X}) = C(\mathcal{X}, \mathcal{X}) = C(X, X) = C(X)$, $C(\mathcal{X}, \mathcal{Y}) = C(X, Y)$, $\tilde{L}_{S_{11}} = \tilde{L}_S$ and $\tilde{L}_{T_{11}} = \tilde{L}_T$, the chain $(\tilde{L}_{S_{11}}, \tilde{L}_{T_{11}})$ is exact. However, if $\mathcal{I}$ denotes the identity of $C(\mathcal{X})$, since

$$\tilde{L}_{S_{11}}(\mathcal{I} - [S'_{11}S_{11}]) = 0,$$

then there is $B \in L(\mathcal{X}, \mathcal{Y})$ such that

$$\tilde{L}_{T_{11}}([B]) = \mathcal{I} - [S'_{11}S_{11}],$$

that is

$$T_{11}B + S'_{11}S_{11} = I - K,$$

where I is the identitiy map of $\mathcal{X}$, and $K \in L(\mathcal{X})$ is a compact operator.

Now well, since (S_{11}, T_{11}) is a chain, that is $R(T_{11}) \subseteq N(S_{11})$, and since

$$T_{11}B(N(S_{11})) = (I - K)(N(S_{11})),$$

then $(I - K)(N(S_{11})) \subseteq N(S_{11})$. However, $N(S_{11})$ is a Banach space and $I - K$ is a Fredholm operator in $N(S_{11})$. Consequently, $\dim N(S_{11})/(I-K)(N(S_{11}))$ is finite dimensional, and since $(I-K)(N(S_{11})) \subseteq R(T_{11}) \subseteq N(S_{11})$, then $\dim N(S_{11})/R(T_{11})$ is finite.

A similar argument proves that $\dim N(T_{11})/R(S_{11})$ is finite. Therefore, (S_{11}, T_{11}) is a Fredholm symmetrical pair. However, according to the above identifications and to Remark 5.1, $(S, T) \in P(X, Y)$, and since S and T are regular operators, (S, T) is a regular Fredholm pair.

Similar arguments prove the second part of the theorem.

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Theorem 5.5. *Let X and Y be two Banach spaces and consider $S \in L(X, Y)$ and $T \in L(Y, X)$, two regular operators such that $R(ST)$ and $R(TS)$ are finite dimensional subspaces of Y and X respectively. Then, with the same notations of Remark 5.3 and Theorem 5.4, if S' and T' are generalized inverses for S and T respectively, necessary and sufficient for (S, T) to belong to $RP(X, Y)$ is that*

$$\tilde{L}_{S'}\tilde{L}_S + \tilde{L}_T\tilde{L}_{T'} \text{ and } \tilde{L}_{T'}\tilde{L}_T + \tilde{L}_S\tilde{L}_{S'}$$

are invertible operators for any Banach space Z.

Similarly, necessary and sufficient for (S, T) to belong to $RP(X, Y)$ is that

$$\tilde{R}_{S'}\tilde{R}_S + \tilde{R}_T\tilde{R}_{T'} \text{ and } \tilde{R}_{T'}\tilde{R}_T + \tilde{R}_S\tilde{R}_{S'}$$

are invertible operators for any Banach space Z.

Proof. Since $R(ST)$ and $R(TS)$ are finite dimensional Banach spaces, $(\tilde{L}_S, \tilde{L}_T)$ and $(\tilde{L}_T, \tilde{L}_S)$ are chains for any Banach space Z. Furthermore, since $S = SS'S$ and $T = TT'T$,

$$\tilde{L}_S = \tilde{L}_S\tilde{L}_{S'}\tilde{L}_S, \qquad \tilde{L}_T = \tilde{L}_T\tilde{L}_{T'}\tilde{L}_T,$$

that is $\tilde{L}_{S'}$ and $\tilde{L}_{T'}$ are generalized inverses for $\tilde{L}_S$ and $\tilde{L}_T$ respectively.

Now well, acording to [6, 1], necessary and sufficient for

$$\tilde{L}_{S'}\tilde{L}_S + \tilde{L}_T\tilde{L}_{T'} \text{ and } \tilde{L}_{T'}\tilde{L}_T + \tilde{L}_S\tilde{L}_{S'}$$

to be invertible is the fact that $(\tilde{L}_S, \tilde{L}_T)$ and $(\tilde{L}_T, \tilde{L}_S)$ are invertible chains. However, according to Theorem 5.4, this last assertion is equivalent to the fact that (S,T) belongs to $RP(X,Y)$.

A similar argument proves the second part of the theorem.

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