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A CHARACTERIZATION OF ABSOLUTELY MONOTONIC (Δ) FUNCTIONS OF A FIXED ORDER

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Abstract. Absolutely monotonic (Δ) function of order n are characterized in terms of n-dimensional totally increasing functions. Applications to n-copulas are presented.

1. Introduction

Let n be a non-negative integer and let c, d be two real numbers such that $c < d$. Let ϕ be a real function defined in $[c, d]$. Let $t \in [c, d]$ and $h > 0$ be such that $t + nh \in [c, d]$. The *difference of order n and step h of ϕ in t* is defined by

$$\Delta_h^0 \phi(t) = \phi(t), \quad \Delta_h^n \phi(t) = \Delta_h^{n-1} \phi(t+h) - \Delta_h^{n-1} \phi(t).$$

Equivalently,

$$\Delta_h^n \phi(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \phi(t + kh).$$

If $\Delta_h^k \phi(t) \geqslant 0$ for all $t \in [c, d]$, $k = 0, \dots, n$ and $h > 0$ such that $t + kh \leqslant d$, then ϕ is said to be *absolutely monotonic* (Δ) *of order* n (abbreviated: ϕ is $AM(\Delta, n)$). The function ϕ is said to be *absolutely monotonic* (abbreviated: ϕ is AM) if it is $AM(\Delta, n)$ for all n.

AM functions were first introduced by Bernstein [**1**]. He proved that they are necessarily analytic. A function is absolutely monotonic on the negative real axis, if and only if it can be represented there by a Laplace–Stieltjes integral with non-decreasing determining function (see e.g., [**2**] and [**15**]). AM functions appeared first in the construction of n-dimensional distribution functions in connection with Archimedean n-copulas (see [**7**]). They were used then in more general multivariate models: mixtures of powers (see [**8**] and [**4**]) and mixtures of max-infinitely divisible

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distributions (see [**5**]). Absolutely monotonic (D) functions of order n (abbreviated: $AM(D,n)$ functions), i.e., non-negative functions with non-negative derivatives up to and including the n-th, appeared recently with certain transformation of n-copulas (see [**9**]). If a function is $AM(D,n)$ then it is $AM(\Delta,n)$, but the converse is not true. For example, a non differentiable convex increasing and non-negative function is $AM(\Delta,2)$ but not $AM(D,2)$.

In this paper, a characterization of $AM(\Delta,n)$ functions is obtained, and it is shown that for constructing Archimedean n-copulas and the transformed n-copulas considered in [**9**] we can use $AM(\Delta,n)$ functions instead of $AM(D,n)$ functions. In this manner, the multivariate models that can be obtained with these methods is enlarged.

The rest of the paper is divided into sections as follows. In section 2, totally increasing functions are introduced. Then a characterization of $AM(\Delta,n)$ functions in terms of n-dimensional totally increasing functions is obtained. In section 3, we consider $AM(\Delta,n)$ functions in connection with Archimedean n-copulas and the transformed n-copulas considered in [**9**]. Finally, section 4 contains the proofs.

2. A characterization of $AM(\Delta',n)$ functions

A complete discussion of AM functions can be found in Chapter IV of [**16**]. For our purpose we only sate the following result.

Theorem 2.1. [**16**, Chapter IV, Theorem 6] *Let $\phi:[c,d]\to R$ be $AM(\Delta,n)$ with $n\geqslant 1$ and let $s,t\in[c,d]$ be such that $s\leqslant t$. Then $\Delta_h^k\phi(s)\leqslant\Delta_h^k\phi(t)$ for all $k=0,\dots,n-1$ and all $h>0$ such that $y+kh\leqslant d$.*

In the sequel we suppose $n\geqslant 1$. In this section we do not use the non-negative condition for $AM(\Delta,n)$ functions. To simplify exposition, we say that if $\Delta_h^k\phi(t)\geqslant 0$ for all $t\in[c,d]$, $h>0$ such that $t+nh\leqslant d$ and $k=1,\dots,n$, then ϕ is *absolutely monotonic (Δ') of order n* (abbreviated: ϕ is $AM(\Delta',n)$).

We have that a function ϕ is $AM(\Delta',2)$ if and only if ϕ satisfies

$$\phi(t+h)-\phi(t)\geqslant 0 \tag{2.1}$$

for all $t\in[c,d]$ and $h>0$ such that $t+h\in[c,d]$, and

$$\phi(t+2h)-2\phi(t+h)+\phi(t)\geqslant 0 \tag{2.2}$$

for all $t\in[c,d]$ and $h>0$ such that $t+2h\in[c,d]$. Indeed $AM(\Delta',2)$ functions can be characterized in the following alternative manner,

Theorem 2.2. *The function $\phi:[c,d]\to R$ is $AM(\Delta',2)$ if and only if ϕ satisfies*

$$\phi(t_4)-\phi(t_3)-\phi(t_2)+\phi(t_1)\geqslant 0 \tag{2.3}$$

for all $t_1,t_2,t_3,t_4\in[c,d]$ such that $t_1\leqslant t_2\leqslant t_4$, $t_1\leqslant t_3\leqslant t_4$ and $t_4-t_3-t_2+t_1\geqslant 0$.

Proof. Suppose first that ϕ is $AM(\Delta',2)$ and set $t_1,t_2,t_3,t_4\in[c,d]$, such that $t_1\leqslant t_2\leqslant t_4$, $t_1\leqslant t_3\leqslant t_4$ and $t_4-t_3-t_2+t_1\geqslant 0$. Since $t_1\leqslant t_3$, then $t_1+t_4-t_3\leqslant t_4$. Since ϕ is $AM(\Delta',2)$, from Theorem 2.1 we obtain

$$\phi(t_1+t_4-t_3)-\phi(t_1)\leqslant\phi(t_4)-\phi(t_3).$$

We have $t_4 - t_3 - t_2 + t_1 \geqslant 0$ and ϕ is increasing, so

$$\phi(t_2) \leqslant \phi(t_1 + t_4 - t_3).$$

From these two inequalities we obtain (2.3).

Suppose now that ϕ satisfies (2.3) for all $t_1, t_2, t_3, t_4 \in [c, d]$ such that $t_1 \leqslant t_2 \leqslant t_4$, $t_1 \leqslant t_3 \leqslant t_4$ and $t_4 - t_3 - t_2 + t_1 \geqslant 0$. Let first $t \in [c, d]$ and $h > 0$ be such that $t + h \in [c, d]$. Setting $t_1 = t_2 = t$ and $t_3 = t_4 = t + h$ in (2.3), we obtain (2.1). Let now $t \in [c, d]$ and $h > 0$ be such that $t + 2h \in [c, d]$. Setting $t_1 = t$, $t_2 = t_3 = t + h$ and $t_4 = t + 2h$ in (2.3), we obtain (2.2). Thus ϕ is $AM(\Delta', 2)$. □

The above theorem shows that for $AM(\Delta', 2)$ functions we can replace inequalities (2.1) and (2.2) (that involve two and three points in $[c, d]$, respectively, related in a simple manner), by the only inequality (2.3) (that involves four points in $[c, d]$ related in a more complicated manner). The aim of this section is to extend Theorem 2.2 for $AM(\Delta', n)$ functions, with $n \geqslant 2$. To obtain this extension, we introduce a suitable expression for the needed points in $[c, d]$. This is achieved introducing totally increasing functions.

Let $S_1, \ldots, S_n$ be partially ordered sets and, for simplicity, we denote with $\leqslant$ the distinct orders. We consider in $S_1 \times \cdots \times S_n$ the partial order of the product, i. e., if $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in S_1 \times \cdots \times S_n$ then $(a_1, \ldots, a_n) \leqslant (b_1, \ldots, b_n)$ if $a_i \leqslant b_i$ for all $i \in \{1, \ldots, n\}$; and we write $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ when $a_i < b_i$ for all $i \in \{1, \ldots, n\}$. If A and B are sets, then $|A|$ denotes the cardinality of A and $A \setminus B = \{x \in A : x \notin B\}$.

Let $H : S_1 \times \cdots \times S_n \to R$ and let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in S_1 \times \cdots \times S_n$ be such that $(a_1, \ldots, a_n) \leqslant (b_1, \ldots, b_n)$. If $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$, then the *$H$-volume* of $\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ is given by

$$V_H(\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}) = \sum \operatorname{sign}(c_1, \ldots, c_n) H(c_1, \ldots, c_n) \tag{2.4}$$

where the sum is taken over all $(c_1, \ldots, c_n) \in \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ and

$$\operatorname{sign}(c_1, \ldots, c_n) = (-1)^k \text{ with } k = |\{i \in \{1, \ldots, n\} : c_i = a_i\}|.$$

If there exists $i \in \{1, \ldots, n\}$ such that $a_i = b_i$, then $V_H(\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}) = 0$.

Equivalently, the H-volume of $\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ is the nth order difference of H on $\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$, i.e.,

$$\begin{aligned} V_H(\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}) &= \Delta_{(a_1, \ldots, a_n), (b_1, \ldots, b_n)} H(x_1, \ldots, x_n) \\ &= \Delta_{a_1, b_1} \ldots \Delta_{a_n, b_n} H(x_1, \ldots, x_n), \end{aligned} \tag{2.5}$$

where we define the n first order differences of H as

$$\begin{aligned} \Delta_{a_k, b_k} H(x_1, \ldots, x_n) = H(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n) \\ - H(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_n). \end{aligned}$$

Also, by convention, if $k = n$ we set

$$\Delta_{(a_1, \ldots, a_{n-k}), (b_1, \ldots, b_{n-k})} H(x_1, \ldots, x_n) = H(x_1, \ldots, x_n).$$

Definition 2.1. A function $H : S_1 \times \cdots \times S_n \to R$ is said to be n-increasing if $V_H(\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}) \geqslant 0$ for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in S_1 \times \cdots \times S_n$ such that $(a_1, \ldots, a_n) \leqslant (b_1, \ldots, b_n)$.

Definition 2.2. Let $m \in \{1, \ldots, n\}$ and let $F : S_1 \times \cdots \times S_n \to R$. F is said to be m-increasing, if the functions obtained fixing any $n-m$ variables are m-increasing, i.e., if for all m integers $k_1, \ldots, k_m$ such that $1 \leqslant k_1 < \cdots < k_m \leqslant n$, and all $y_i \in S_i$ with $i \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_m\}$, the function $G : S_{k_1} \times \cdots \times S_{k_m} \to R$ given by

$$G(x_{k_1}, \ldots, x_{k_m}) = F(y_1, \ldots, y_{k_1 - 1}, x_{k_1}, y_{k_1+1}, \ldots, y_{k_m - 1}, x_{k_m}, y_{k_m+1}, \ldots, y_n)$$

is m-increasing.

Now we introduce n-dimensional totally increasing functions.

Definition 2.3. If $F : S_1 \times \cdots \times S_n \to R$ is m-increasing for all $m \in \{1, \ldots, n\}$, then F is said to be totally increasing (abbreviated: F is TI).

As an immediate consequence of the previous definitions we have,

Lemma 2.1. *Let $F : S_1 \times \cdots \times S_n \to R$ be n-increasing and let $m \in \{1, \ldots, n-1\}$. Then the m-th differences of F are $(n-m)$-increasing functions.*

Lemma 2.2. *Let $F : S_1 \times \cdots \times S_n \to R$ be n-increasing. Suppose that S_i contains a least element a_i, $i \in \{1, \ldots, n\}$. If $F(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in S_1 \times \cdots \times S_n$ such that $x_i = a_i$ for at least some $i \in \{1, \ldots, n\}$, then F is TI.*

Using Lemma 2.2, we see that a simple and well known example of TI functions, from statistics and probability theory, are n-dimensional distribution functions. It is important to note that Rüschendorf in [**11**] considered TI increasing functions with $S_1 = \cdots = S_n = [0, 1]$ and he called them Δ-monotone functions.

$AM(\Delta', n)$ functions with $n \geqslant 1$, can be characterized in terms of totally increasing n-dimensional functions. As a first result in this direction we have,

Lemma 2.3. *Let $n \geqslant 2$ and let $\phi : [c, d] \to R$. Let $t_0 \in [c, d]$ and let $h > 0$ be such that $t_0 + nh \leqslant d$. Let $S_i = \{a_i, b_i\}$ be ordered sets with $a_i < b_i$, $i \in \{1, \ldots, n\}$. Let $F : S_1 \times \cdots \times S_n \to [c, d]$ be given by*

$$F(x_1, \ldots, x_n) = t_0 + |\{i \in \{1, \ldots, n\} : x_i = b_i\}| h.$$

Then, for all $m \in \{1, \ldots, n\}$ and all m integers $k_1, \ldots, k_m$ such that $1 \leqslant k_1 < \cdots < k_m \leqslant n$ it holds

$$\Delta_{(a_{k_1}, \ldots, a_{k_m}), (b_{k_1}, \ldots, b_{k_m})} \phi \circ F(x_1, \ldots, x_n) = \Delta_h^m \phi(t_0 + |\{i \in \{1, \ldots, n\} \setminus \{k_1, \ldots, k_m\} : x_i = b_i\}| h) \tag{2.6}$$

In particular, ϕ is $AM(\Delta', n)$ if and only if $\phi \circ F$ is TI for all such functions F.

Next appears the principal result of this article.

Theorem 2.3. *Let $\phi : [c, d] \to R$ be $AM(\Delta', n)$. If $F : S_1 \times \cdots \times S_n \to [c, d]$ is TI, then $\phi \circ F$ is TI.*

The proof of Theorem 2.3 (see Section 4) is of elemental algebraic nature but it is very involved.

From Theorem 2.3 and Lemma 2.3 it results the following characterization for $AM(\Delta', n)$ functions in terms of TI functions.

THEOREM 2.4. *A function* $\phi : [c, d] \to R$ *is* $AM(\Delta', n)$ *if and only if for all* TI *functions* $F : S_1 \times \cdots \times S_n \to [c, d]$*, the function* $\phi \circ F$ *is* TI*.*

Theorem 2.4 extends Theorem 2.2. Indeed, for $n = 2$, Theorem 2.4 is a reformulation of Theorem 2.2 in terms of TI functions.

3. Applications to copulas

An n-copula is the restriction to the unit n-cube $[0, 1]^n$ of a multivariate distribution function whose marginals are uniform on $[0, 1]$, more precisely,

DEFINITION 3.1. An n-copula is a function $C : [0, 1]^n \to 0, 1]$ with the following properties:

1. $C(x_1, \ldots, x_n) = 0$ if $x_i = 0$ for any $i = 1, \ldots, n$.
2. $C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$ for each $i = 1, \ldots, n$ and all $x_i \in [0, 1]$.
3. C is n-increasing.

Excellent references about copulas are [**6**], [**10**], [**12**] and [**13**]. In the sequel, we suppose $n \geqslant 2$ when we talk about n-copulas. The notion of copula has been introduced by A. Sklar in response to a question posed by M. Fréchet (see [**14**]). Sklar proved that if H is the joint distribution function of n random variables, $X_1, \ldots, X_n$, and $F_1, \ldots, F_n$ are the distribution functions of $X_1, \ldots, X_n$, respectively, then there exists an n-copula C such that

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{3.1}$$

The n-copula C is uniquely determined on $\text{range}(F_1) \times \cdots \times \text{range}(F_n)$, so that C can be thought of as a description of the way in which a joint distribution function is related to its 1-dimensional marginals.

Because of (3.1),if we have a collection of n-copulas then we automatically have a collection of n-dimensional distributions with whatever 1-dimensional marginal distributions we desire. This is useful in modelling and simulation. Moreover, n-copulas are invariant under strictly increasing transformations of the random variables (see e.g., Theorem 6.5.6 in [**13**]), so they can be used in nonparametric statistic. Most of the dependence structure properties of an n-dimensional distribution function H are in a associated copula C, which does not depend on the marginals, and is often easier to handle than the original H. For these facts, it is very important in statistics to have a great variety of n-copulas. In general, we are interest in models that can lead to parametric families of multivariate distribution functions or copulas with close-form cumulative distribution functions, flexible dependence structure and partial closure under the taking the margins (see [**6**] for more details).

Different methods of constructing n-copulas have been proposed (see e.g. Chapter 3 and Chapter 4 in [**10**]). One of them yields an important class of n-copulas called Archimedean n-copulas. These n-copulas are of the form

$$C(x_1, \dots, x_n) = \lambda^{[-1]}(\lambda(x_1) + \cdots + \lambda(x_n)), \tag{3.2}$$

where $\lambda : [0,1] \to [0,\infty]$ is a continuous strictly decreasing function such that $\lambda(1) = 0$ and $\lambda^{[-1]}$ is the pseudo-inverse of λ given by

$$\lambda^{[-1]}(t) = \begin{cases} \lambda^{-1}(t), & \text{if } 0 \leqslant t \leqslant \lambda(0); \\ 0, & \text{if } \lambda(0) \leqslant t \leqslant \infty. \end{cases} \tag{3.3}$$

The function λ that appears in (3.2) is known as an additive generator of C. Setting $\varphi(t) = \exp(-\lambda(t))$ and $\varphi^{[-1]}(t) = \lambda^{[-1]}(-ln(t))$, (3.2) can be written as

$$C(x_1, \dots, x_n) = \varphi^{[-1]}(\varphi(x_1) \cdots \varphi(x_n)). \tag{3.4}$$

The function φ that appears in (3.4) is known as a multiplicative generator of C. The study of Archimedean n-copulas is fundamentally done using (3.2). It is easy to see that C given by (3.2) satisfies 1 and 2 of Definition 3.1. Condition 3 of Definition 3.1 requires additional properties of λ.

THEOREM 3.1. *C defined by* (3.2) *with $n = 2$ is 2-increasing if and only if λ is convex.*

A proof of the above theorem can be found in Chapter 4 of [**10**]. Kimberling in [**7**] gives necessary and sufficient conditions for a strict generator λ to guarantee that C given by (3.2) to be n-increasing for all $n \geqslant 2$. Indeed the arguments in [**7**] can be used to obtain the following results.

THEOREM 3.2. *If C given by (3.2) is n-increasing, then $\lambda^{[-1]}(-t)$ is $AM(\Delta, n)$.*

THEOREM 3.3. *If $\lambda^{[-1]}(-t)$ is $AM(D, n)$, then C given by (3.2) is n-increasing.*

Note that in Theorem 3.2 and Theorem 3.3 conditions on $\lambda^{[-1]}$ differ. This fact and Theorem 3.1 suggest the possibility of replace in Theorem 3.3, the condition on $\lambda^{[-1]}(-t)$ of being $AM(D, n)$ by the weaker condition of being $AM(\Delta, n)$. In fact, using Theorem 2.3, we can prove,

THEOREM 3.4. *If $\lambda^{[-1]}(-t)$ is $AM(\Delta, n)$, then C given by (3.2) is n-increasing.*

From Theorem 3.2 and Theorem 3.4 we obtain the following characterization of $AM(\Delta, n)$ functions in terms of Archimedean n-copulas.

THEOREM 3.5. *$\lambda^{[-1]}(-t)$ is $AM(\Delta, n)$ if and only if C given by (3.2) is n-increasing.*

Now we present the transformed copulas C_φ. These copulas are given by

$$C_\varphi(x_1, \dots, x_n) = \varphi^{[-1]}(C(\varphi(x_1), \dots, \varphi(x_n)), \tag{3.5}$$

where C is an arbitrary n-copula, $\varphi : [0,1] \to [0,1]$ is a continuous strictly increasing function such that $\varphi(1) = 1$, and $\varphi^{[-1]}$ is the pseudo-inverse of φ defined by

$$\varphi^{[-1]}(t) = \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant \varphi(0); \\ \varphi^{-1}(t), & \text{if } \varphi(0) \leqslant t \leqslant 1. \end{cases} \tag{3.6}$$

If $C(x_1, \ldots, x_n) = x_1 \cdots x_n$, then (3.5) is reduced to (3.4), i.e., in this case (3.5) is the expression for an Archimedean n-copula with multiplicative generator φ.

Copulas C_φ were first considered by Genest and Rivest in [**3**]. In [**9**] conditions that φ must satisfy for C_φ to be an n-copula for every n-copula C are studied and some properties of n-copulas C_φ are explored. It is easy to see that C_φ satisfies conditions 1 and 2 of Definition 3.1. To guarantee that C_φ is n-increasing φ must satisfy additional properties. If $n = 2$ and if φ is concave then C_φ is 2-increasing (see Theorem 2.6 in [**9**]). In general, if $n \geqslant 2$ and if φ is $AM(D, n)$ then C_φ is n-increasing (see Theorem 4.7 in [**9**]).

For copulas C_φ we have a situation similar to that for Archimedean n-copulas. We note that to guarantee that C_φ is n-increasing, the conditions impose to φ for the bivariate case are weaker than for the general case. It is natural to think that we can consider $AM(\Delta, n)$ functions instead of $AM(D, n)$ functions in the general case. Indeed, from Theorem 2.3 we obtain,

THEOREM 3.6. *Let C be an n-copula. Let $\varphi : [0,1] \to [0,1]$ be a continuous strictly increasing function such that $\varphi(1) = 1$. If $\varphi^{[-1]}$ is $AM(\Delta, n)$, then C_φ given by* (3.5) *is n-increasing.*

In the bivariate case, non differentiable generators yield interest copulas. For example, if the additive generator of an Archimedean 2-copula C is piecewise linear, then C is singular. In the general case, in this article it has been shown that $AM(\Delta, n)$ functions can be used instead of $AM(D, n)$ functions to obtain Archimedean n-copulas and n-copulas C_φ. Hereafter, all depend on the number of interest examples of $AM(\Delta, n)$ functions that can be obtained with the aim to have multivariate models with nice properties. This will be the subject of future investigations.

4. Proofs

PROOF OF LEMMA 2.3. The proof proceeds by finite induction. Suppose $m = 1$ and consider $k \in \{1, \ldots, n\}$. By definition of F we have

$$\begin{aligned} \Delta_{a_k, b_k} \phi \circ F(x_1, \ldots, x_n) &= \phi \circ F(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n) \\ &\quad - \phi \circ F(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_n) \\ &= \phi\big(t_0 + (|\{i \in \{1, \ldots, n\} \setminus \{k\} : x_i = b_i\}| + 1)\, h\big) \\ &\quad - \phi\big(t_0 + |\{i \in \{1, \ldots, n\} \setminus \{k\} : x_i = b_i\}| h\big) \\ &= \Delta_h^1 \phi(t_0 + |\{i \in \{1, \ldots, n\} \setminus \{k\} : x_i = b_i\}| h) \end{aligned}$$

Thus (2.6) holds for $m = 1$. Suppose (2.6) is valid for $1 \leqslant m < n$. Let now $m \in \{1, \ldots, n\}$, and consider m integers $k_1, \ldots, k_m$ such that $1 \leqslant k_1 < \cdots < k_m \leqslant n$.

We have

$$\begin{aligned}
&\Delta_{(a_{k_1},\dots,a_{k_m}),(b_{k_1},\dots,b_{k_m})}\phi\circ F(x_1,\dots,x_n)\\
&=\Delta_{(a_{k_1},\dots,a_{k_{m-1}}),(b_{k_1},\dots,b_{k_{m-1}})}\Delta_{a_{k_m},b_{k_m}}\phi\circ F(x_1,\dots,x_n)\\
&=\Delta_{(a_{k_1},\dots,a_{k_{m-1}}),(b_{k_1},\dots,b_{k_{m-1}})}\phi\circ F(x_{k_1},\dots,x_{k_{m-1}},b_{k_m},x_{k_{m+1}},\dots,x_n)\\
&\quad-\Delta_{(a_{k_1},\dots,a_{k_{m-1}}),(b_{k_1},\dots,b_{k_{m-1}})}\phi\circ F(x_{k_1},\dots,x_{k_{m-1}},a_{k_m},x_{k_{m+1}},\dots,x_n)\\
&=\Delta_h^{m-1}\phi\big(t_0+(|\{k\in\{1,\dots,n\}\setminus\{k_1,\dots,k_m\}:x_k=b_k\}|+1)h\big)\\
&\quad-\Delta_h^{m-1}\phi\big(t_0+|\{k\in\{1,\dots,n\}\setminus\{k_1,\dots,k_m\}:x_k=b_k\}|h\big)\\
&=\Delta_h^{m-1}\Delta_h^1\phi\big(t_0+|\{k\in\{1,\dots,n\}\setminus\{k_1,\dots,k_m\}:x_k=b_k\}|h\big)\\
&=\Delta_h^{m}\phi\big(t_0+|\{k\in\{1,\dots,n\}\setminus\{k_1,\dots,k_m\}:x_k=b_k\}|h\big)
\end{aligned}$$

This completes the induction procedure. □

To prove Theorem 2.3 we need the following two auxiliary statements. First of them follows easily from Theorem 2.1.

Lemma 4.1. *Let $\phi:[c,d]\to R$ be $AM(\Delta,n)$ or $AM(\Delta',n)$, and let h be such that $0<h<d-c$. Then $\Delta_h^1\phi:[c,d-h]\to R$ is $AM(\Delta,n-1)$.*

Lemma 4.2. *Let $n\geqslant 2$. Let $S_i=\{a_i,b_i\}$ be ordered sets with $a_i\leqslant b_i$, $i\in\{1,\dots,n\}$. If $F:S_1\times\cdots\times S_n\to R$ is k-increasing for $k=1,2$ and there exists $j\in\{1,\dots,n\}$ such that $F(b_1,\dots,b_n)=F(b_1,\dots,b_{j-1},a_j,b_{j+1},\dots,b_n)$, then*

$$F(x_1,\dots,x_{j-1},b_j,x_{j+1},\dots,x_n)=F(x_1,\dots,x_{j-1},a_j,x_{j+1},\dots,x_n),$$

for all $x_i\in S_i$, $i\in\{1,\dots,n\}\setminus\{j\}$.

Proof. The proof proceeds by induction on n. Let $n=2$. Suppose that $j=2$ (the case $j=1$ is similar). Thus, $F(b_1,b_2)=F(b_1,a_2)$. Since F is k-increasing for $k=1,2$, we have $0\leqslant F(a_1,b_2)-F(a_1,a_2)\leqslant F(b_1,b_2)-F(b_1,a_2)$ Therefore, $F(a_1,b_2)=F(a_1,a_2)$. This verifies the lemma for $n=2$.

Suppose $n\geqslant 3$ and the lemma is valid for $n-1$. Suppose that $j=n$ (the other cases are similar). Thus, $F(b_1,\dots,b_n)=F(b_1,\dots,b_{n-1},a_n)$. Clearly, the function $F':S_1\times\cdots\times S_{n-2}\times S_n\to R$ given by

$$F'(x_1,\dots,x_{n-2},x_n)=F(x_1,\dots,x_{n-2},b_{n-1},x_n)$$

satisfies the hypothesis of the lemma (with $j=n$). Then, by our supposition

$$F(x_1,\dots,x_{n-2},b_{n-1},b_n)=F(x_1,\dots,x_{n-2},b_{n-1},a_n)\tag{4.1}$$

for all $x_i\in S_i, i\in\{1,\dots,n-2\}$. Since F is k-increasing for $k=1,2$,

$$\begin{aligned}0&\leqslant F(x_1,\dots,x_{n-2},a_{n-1},b_n)-F(x_1,\dots,x_{n-2},a_{n-1},a_n)\\&\leqslant F(x_1,\dots,x_{n-2},b_{n-1},b_n)-F(x_1,\dots,x_{n-2},b_{n-1},a_n)\end{aligned}\tag{4.2}$$

for all $x_i\in S_i, i\in\{1,\dots,n-2\}$. By (4.1) and (4.2),

$$F(x_1,\dots,x_{n-2},a_{n-1},b_n)=F(x_1,\dots,x_{n-2},a_{n-1},a_n)\tag{4.3}$$

for all $x_i \in S_i, i \in \{1, \ldots, n-2\}$. By (4.1) and (4.3),

$$F(x_1, \ldots, x_{n-1}, b_n) = F(x_1, \ldots, x_{n-1}, a_n)$$

for all $x_i \in S_i, i \in \{1, \ldots, n-1\}$. This conclude the induction procedure. □

Now we are in position to prove Theorem 2.3.

PROOF OF THEOREM 2.3. The proof proceeds by induction on n. For $n = 1$ the theorem is trivially valid. Suppose $n \geqslant 2$ and the theorem valid for $n-1$.

Since ϕ is $AM(\Delta', n)$, then it is $AM(\Delta', n-1)$. Thus we have,

ASSERTION 1. *If $T_1, \ldots, T_{n-1}$ are ordered sets and $G : T_1 \times \cdots \times T_{n-1} \to [c, d]$ is TI, then $\phi \circ G$ is TI.*

By Assertion 1, since F is m-increasing for $m \in \{1, \ldots, n-1\}$, then $\phi \circ F$ is m-increasing for $m \in \{1, \ldots, n-1\}$. So, it only remains to see that $\phi \circ F$ is n-increasing. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in S_1 \times \cdots \times S_n$ be such that $(a_1, \ldots, a_n) \leqslant (b_1, \ldots, b_n)$. Consider the function $G_{a_n}^{b_n}$ defined on $S_1 \times \cdots \times S_{n-1}$ by

$$G_{a_n}^{b_n}(x_1, \ldots, x_{n-1}) = \Delta_{a_n, b_n} F(x_1, \ldots, x_n).$$

Since F is TI, then $G_{a_n}^{b_n}$ is TI. We have,

$$\begin{aligned} (4.4)\quad & \Delta_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} \phi \circ F(x_1, \ldots, x_n) \\ &= \Delta_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \big[\phi \circ F(x_1, \ldots, x_{n-1}, b_n) - \phi \circ F(x_1, \ldots, x_{n-1}, a_n)\big] \\ &= \Delta_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \big[\phi\big(F(x_1, \ldots, x_{n-1}, a_n) + G_{a_n}^{b_n}(x_1, \ldots, x_{n-1})\big) \\ &\qquad - \phi \circ F(x_1, \ldots, x_{n-1}, a_n)\big] \end{aligned}$$

There are three cases:

Case 1. $G_{a_n}^{b_n}(b_1, \ldots, b_{n-1}) = 0$. By Lemma 4.2, $G_{a_n}^{b_n}(x_1, \ldots, x_{n-1}) = 0$ for all $x_i \in S_i$, $i \in \{1, \ldots, n-1\}$. Therefore, by (4.4),

$$\Delta_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} \phi \circ F(x_1, \ldots, x_n) = 0.$$

Case 2. $G_{a_n}^{b_n}(b_1, \ldots, b_{n-1}) = d - c$. Since $F(x_1, \ldots, x_n) \in [c, d]$ and F is 1-increasing, then $F(x_1, \ldots, x_{n-1}, a_n) = c$ for all $x_i \in S_i$, $i \in \{1, \ldots, n-1\}$. Then, by (4.4),

$$\begin{aligned} & \Delta_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} \phi \circ F(x_1, \ldots, x_n) \\ &= \Delta_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \big[\phi\big(F(x_1, \ldots, x_{n-1}, b_n)\big) - \phi(c)\big] \\ &= \Delta_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \phi\big(F(x_1, \ldots, x_{n-1}, b_n)\big) \end{aligned}$$

By Assertion 1 and the previous equality, since $F_{b_n} : S_1 \times \cdots \times S_{n-1} \to [c, d]$ given by $F_{b_n}(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_{n-1}, b_n)$ is TI, then

$$\Delta_{(a_1,\ldots,a_n),(b_1,\ldots,b_n)} \phi \circ F(x_1, \ldots, x_n) \geqslant 0.$$

Case 3. $0 < G_{a_n}^{b_n}(b_1, \ldots, b_{n-1}) < d - c$. Set $h = G_{a_n}^{b_n}(b_1, \ldots, b_{n-1})$. By Lemma 4.1, since $\phi : [c, d] \to R$ is $AM(\Delta', n)$, then $\Delta_h^1 \phi : [c, d-h] \to R$ is

$AM(\Delta, n-1)$. Since $F_{a_n} : S_1 \times \cdots \times S_{n-1} \to [c, d-h]$ given by $F_{a_n}(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_{n-1}, a_n)$ is TI, then $(\Delta_h^1 \phi) \circ F_{a_n}$ is TI. Therefore,

$$\begin{aligned} &\Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},b_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-1})\big) \\ &\qquad - \phi \circ F(x_1,\ldots,x_{n-2},b_{n-1},a_n)\big] \\ &\geqslant \Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},a_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-1})\big) \\ &\qquad - \phi \circ F(x_1,\ldots,x_{n-2},a_{n-1},a_n)\big] \end{aligned} \tag{4.5}$$

Since F and $G_{a_n}^{b_n}$ are TI, then for all k such that $2 \leqslant k \leqslant n$, the function $H : S_1 \times \cdots \times S_{n-1} \to R$ given by

$$\begin{aligned} H(x_1,\ldots,x_{n-1}) = {} & F(x_1,\ldots,x_{n-k},a_{n-k+1},x_{n-k+2},\ldots,x_{n-1},a_n) \\ & + G_{a_n}^{b_n}(b_1,\ldots,b_{n-k},x_{n-k+1},\ldots,x_{n-1}), \end{aligned} \tag{4.6}$$

is TI. Thus, from Assertion 1 and (4.5), we obtain

Assertion 2. *For $2 \leqslant k \leqslant n$ we have*

$$\begin{aligned} &\Delta_{(a_1,\ldots,a_{n-k},a_{n-k+2},\ldots,a_{n-1}),(b_1,\ldots,b_{n-k},b_{n-k+2},\ldots,b_{n-1})} \\ &\quad \big[\phi\big(F(x_1,\ldots,x_{n-k},b_{n-k+1},x_{n-k+2},\ldots,x_{n-1},a_n) \\ &\qquad + G_{a_n}^{b_n}(b_1,\ldots,b_{n-k+1},x_{n-k+2},\ldots,x_{n-1})\big) \\ &\quad - \phi \circ F(x_1,\ldots,x_{n-k},b_{n-k+1},x_{n-k+2},\ldots,x_{n-1},a_n)\big] \\ &\geqslant \Delta_{(a_1,\ldots,a_{n-k},a_{n-k+2},\ldots,a_{n-1}),(b_1,\ldots,b_{n-k},b_{n-k+2},\ldots,b_{n-1})} \\ &\quad \big[\phi\big(F(x_1,\ldots,x_{n-k},a_{n-k+1},x_{n-k+2},\ldots,x_{n-1},a_n) \\ &\qquad + G_{a_n}^{b_n}(b_1,\ldots,b_{n-k},a_{n-k+1},x_{n-k+2},\ldots,x_{n-1})\big) \\ &\quad - \phi \circ F(x_1,\ldots,x_{n-k},a_{n-k+1},x_{n-k+2},\ldots,x_{n-1},a_n)\big] \end{aligned} \tag{4.7}$$

To see Assertion 2 we proceed by finite induction. For $k = 2$, the function $H : S_1 \times \cdots \times S_{n-1} \to [c, d]$ given by (4.6) takes the form

$$H(x_1,\ldots,x_{n-1}) = F(x_1,\ldots,x_{n-2},a_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-2},x_{n-1})$$

By Assertion 1, since H is TI, then $\phi \circ H$ is TI. Thus,

$$\begin{aligned} &\Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},a_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-1})\big) \\ &\qquad -\phi \circ F(x_1,\ldots,x_{n-2},a_{n-1},a_n)\big] \\ &\geqslant \Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},a_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-2},a_{n-1})\big) \\ &\qquad -\phi \circ F(x_1,\ldots,x_{n-2},a_{n-1},a_n)\big] \end{aligned} \tag{4.8}$$

By (4.5) and (4.8),

$$\begin{aligned} &\Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},b_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-1})\big) \\ &\qquad -\phi \circ F(x_1,\ldots,x_{n-2},b_{n-1},a_n)\big] \\ &\geqslant \Delta_{(a_1,\ldots,a_{n-2}),(b_1,\ldots,b_{n-2})}\big[\phi\big(F(x_1,\ldots,x_{n-2},a_{n-1},a_n) + G_{a_n}^{b_n}(b_1,\ldots,b_{n-2},a_{n-1})\big) \\ &\qquad -\phi \circ F(x_1,\ldots,x_{n-2},a_{n-1},a_n)\big] \end{aligned}$$

This verifies (4.7) for $k=2$. Suppose (4.7) is valid for $2 \leqslant k < n$ and rewrite it to obtain

$$
\begin{aligned}
(4.9)\qquad & \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big] \\
& \geqslant \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big]
\end{aligned}
$$

For $k+1$, the function $H : S_1 \times \dots \times S_{n-1} \to [c,d]$ given by (4.6) takes the form

$$
\begin{aligned}
H(x_1,\dots,x_{n-1}) &= F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
&\quad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)},x_{n-(k+1)+1},\dots,x_{n-1})
\end{aligned}
$$

By Assertion 1, since H is TI then $\phi\circ H$ is TI. Thus,

$$
\begin{aligned}
(4.10)\qquad & \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big] \\
& \geqslant \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big]
\end{aligned}
$$

From (4.9) and (4.10) we obtain

$$
\begin{aligned}
& \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},b_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big] \\
& \geqslant \Delta_{(a_1,\dots,a_{n-(k+1)},a_{n-(k+1)+2},\dots,a_{n-1}),(b_1,\dots,b_{n-(k+1)},b_{n-(k+1)+2},\dots,b_{n-1})} \\
& \quad \big[\phi\big(F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n) \\
& \qquad + G_{a_n}^{b_n}(b_1,\dots,b_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1})\big) \\
& \qquad - \phi\circ F(x_1,\dots,x_{n-(k+1)},a_{n-(k+1)+1},x_{n-(k+1)+2},\dots,x_{n-1},a_n)\big]
\end{aligned}
$$

This shows Assertion 2. Now, by Assertion 2 with $k = n$, we have

$$\begin{aligned}\Delta_{(a_2,\dots,a_n),(b_2,\dots b_n)}\big[\phi\big(F(b_1,x_2,\dots,x_{n-1},a_n)+G_{a_n}^{b_n}(b_1,x_2,\dots,x_{n-1})\big)\\ -\phi\circ F(b_1,x_2,\dots,x_{n-1},a_n)\big]\\ \geqslant \Delta_{(a_2,\dots,a_n),(b_2,\dots b_n)}\big[\phi\big(F(a_1,x_2,\dots,x_{n-1},a_n)+G_{a_n}^{b_n}(a_1,x_2,\dots,x_{n-1})\big)\\ -\phi\circ F(a_1,x_2,\dots,x_{n-1},a_n)\big]\end{aligned}$$

or equivalently,

$$\begin{aligned}\Delta_{(a_1,\dots,a_{n-1}),(b_1,\dots,b_{n-1})}\big[\phi\big(F(x_1,\dots,x_{n-1},a_n)+G_{a_n}^{b_n}(x_1,\dots,x_{n-1})\big)\\ -\phi\circ F(x_1,\dots,x_{n-1},a_n)\big]\geqslant 0.\end{aligned}$$

Thus, by (4.4),

$$\Delta_{(a_1,\dots,a_n),(b_1,\dots,b_n)}\phi\circ F(x_1,\dots,x_n)\geqslant 0.$$

This shows that F is n-increasing. □

Proof of Theorem 3.4. Let $(a_1,\dots,a_n),(b_1,\dots,b_n)\in[0,1]^n$ be such that $(a_1,\dots,a_n)<(b_1,\dots,b_n)$. Set $B=\{a_1,b_1\}\times\cdots\times\{a_n,b_n\}$ and consider the following two cases:

Case 1. $\lambda(0)<\infty$; or $\lambda(0)=\infty$ and $a_i\neq 0$ for all $i\in\{1,\dots,n\}$. Let $c=\max\{\lambda(x_1)+\cdots+\lambda(x_n):(x_1,\dots,x_n)\in B\}$. Clearly $c<\infty$ and since λ is decreasing, then the function $F:B\to[-c,0]$ given by

$$F(x_1,\dots,x_n)=-[\lambda(x_1)+\cdots+\lambda(x_n)]$$

is TI. By Theorem 2.3, since $\lambda^{[-1]}(-t)$ is $AM(\Delta,n)$ then $\lambda^{[-1]}(-F)$ is TI. In particular, $\lambda^{[-1]}(-F)$ is n-increasing, and consequently $V_C(B)\geqslant 0$.

Case 2. $\lambda(0)=\infty$ and $a_i=0$ for some $i\in\{1,\dots,n\}$. For simplicity, suppose that $a_1=\cdots=a_m=0$ and $a_{m+1}>0,\dots,a_n>0$ for some $m\in\{1,\dots,n\}$. Consider the set

$$B_\varepsilon=\{\varepsilon,b_1\}\times\cdots\times\{\varepsilon,b_m\}\times\{a_{m+1},b_{m+1}\}\times\cdots\times\{a_n,b_n\}$$

where $0<\varepsilon<\min\{b_1,\dots,b_n\}$. By case 1, $V_C(B_\varepsilon)\geqslant 0$. Since C is continuous, $V_C(B_\varepsilon)$ tends to $V_C(B)$ when ε tends to 0. Thus, $V_C(B)\geqslant 0$.

From the previous analysis it results that C is n-increasing. □

Proof of Theorem 3.6. Consider the function $F:[0,1]^n\to[0,1]$ given by $F(x_1,\dots,x_n)=C(\varphi(x_1),\dots,\varphi(x_n))$. Let $k\in\{1,\dots,n\}$ and let $K\subseteq\{1,\dots,n\}$ with $|K|=k$. For simplicity, suppose $K=\{1,\dots,k\}$. Consider the set

$$B=\{a_1,b_1\}\times\cdots\times\{a_k,b_k\}$$

where $0\leqslant a_i<b_i\leqslant 1$, $i\in\{1,\dots,k\}$. Since φ is strictly increasing, then $(\varphi(a_1),\dots,\varphi(a_k))<(\varphi(b_1),\dots,\varphi(b_k))$. Consider now the set

$$B'=\{\varphi(a_1),\varphi(b_1)\}\times\cdots\times\{\varphi(a_k),\varphi(b_k)\}$$

Clearly,

$$\Delta_{(a_1,\dots,a_k),(b_1,\dots,b_k)}F(x_1,\dots,x_n)=\Delta_{(\varphi(a_1),\dots,\varphi(a_k)),(\varphi(b_1),\dots,\varphi(b_k))}C(x_1,\dots,x_n)$$

By Lemma 2.2, C is TI. Then $\Delta_{(a_1,\dots,a_k),(b_1,\dots,b_k)}F(x_1,\dots,x_n)\geqslant 0$.

This shows that F is TI. Now, by Theorem 2.3, $C_\varphi = \varphi^{[-1]} \circ F$ is TI and, in particular, is n-increasing. □

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