## Free-decomposability in <br> D. Castaño <br> J. P. Díaz Varela <br> A. Torrens <br> Varieties of Pseudocomplemented Residuated Lattices


#### Abstract

In this paper we prove that the free pseudocomplemented residuated lattices are decomposable if and only if they are Stone, i.e., if and only if they satisfy the identity $\neg x \vee \neg \neg x=1$. Some applications are given.


Keywords: Pseudocomplemented residuated lattices, free algebras, decomposability, Stone algebras, Boolean elements.

## Introduction

The variety of bounded, integral, residuated, lattice-ordered, commutative monoids (bounded residuated lattices for short) is the algebraic counterpart of Full Lambek logic without contraction and the subvarieties of the former correspond to the axiomatic extensions of the latter. In fact, free residuated lattices are the Lindenbaum algebras of the corresponding extensions. Free algebras are also important in their own right for their algebraic properties, e.g. they contain all the equational information of the variety they generate.

A natural approach to understanding the structure of an algebra consists of decomposing it, whenever possible, into a (direct) Boolean product of indecomposable algebras, thus reducing the problem to the study of those indecomposable stalks. In particular, the decomposability of a bounded residuated lattice depends on the existence of Boolean elements in its lattice reduct (see for example [7], [9], [5] and [4]). In [5] (Theorem 3.1) a representation for every bounded residuated lattice as a weak Boolean product of directly indecomposable algebras is given. In particular free algebras, in any variety of bounded residuated lattices, are weak Boolean products of indecomposable algebras. Hence the main objective of this article is the study of the Boolean skeleton of free algebras in subvarieties of pseudocomplemented residuated lattices.

[^0]The main result of the paper is (as we announced in the abstract): a free algebra in a variety of pseudocomplemented residuated lattices is decomposable if and only if it is a Stone residuated lattice, i.e., if and only if such variety satisfies the Stone identity $\neg x \vee \neg \neg x=1$. In that case the Boolean skeleton turns out to be a retract of the free algebra and, in addition, isomorphic to the free Boolean algebra. This is done in section 2.

In section 1, we collect previous results of bounded residuated lattices and we prove that pseudocomplemented residuated lattices are exactly those bounded residuated lattices whose regular elements form a Boolean algebra. In the last section, we study some examples, such as the subvariety of Heyting algebras.

We assume that the reader has some familiarity with residuated lattices and universal algebra. For residuated lattices we recommend [12], [13] and [10] and the references given there, and for universal algebra we follow the nomenclature given in [3].

## 1. Pseudocomplemented Residuated Lattices

A bounded, integral, residuated, lattice-ordered, commutative monoid, or bounded residuated lattice for short, is an algebra $\boldsymbol{A}=\langle A, *, \rightarrow, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ of type $\langle 2,2,2,2,0,0\rangle$ such that:

- $\langle A, *, \mathbf{1}\rangle$ is a commutative monoid,
- $\boldsymbol{L}(\boldsymbol{A})=\langle A, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ is a bounded lattice with greatest element $\mathbf{1}$ and least element $\mathbf{0}$,
- for any $a, b, c \in A, a * b \leq c$, iff $a \leq b \rightarrow c$, where $\leq$ is the order given by the lattice structure, which is called the natural order of $\boldsymbol{A}$.

It is well known that bounded residuated lattices form a variety, which we shall denote $\mathbb{R L}$. Bounded residuated lattices satisfying the equation $x * x=x$, or equivalently the equation $x * y=x \wedge y$, are called Heyting algebras (see, for instance [14]). We shall represent by $\mathbb{H}$ the variety of all Heyting algebras.

In the next lemma we list, for further reference, some well known properties which we will use throughout this paper.

Lemma 1.1. The following properties hold true in any bounded residuated lattice $\boldsymbol{A}$, where $a, b, c$ denote arbitrary elements of $A$ :
(i) $a \leq b$ if and only if $a \rightarrow b=\mathbf{1}$,
(ii) $a \leq b \rightarrow a$,
(iii) $\mathbf{1} \rightarrow a=a$,
(iv) $(a * b) \rightarrow c=a \rightarrow(b \rightarrow c)$,
$(v) a *(b \vee c)=(a * b) \vee(a * c)$.
On a bounded residuated lattice $\boldsymbol{A}$ we consider the unary operation:

$$
\begin{equation*}
\neg x:=x \rightarrow \mathbf{0}, \quad \text { for all } \quad x \in A \tag{1.1}
\end{equation*}
$$

Taking into account that the $\{\rightarrow, \mathbf{0}, \mathbf{1}\}$-reduct of a bounded residuated lattice is a bounded BCK-algebra we have (see, for instance, [8] and [11]):

Lemma 1.2. The following identities and quasi-identities hold true in any residuated lattice:
(a) $x \leq y \Rightarrow \neg y \leq \neg x$,
(b) $\neg x=\neg \neg \neg x$,
(c) $x \leq \neg \neg x$,
(d) $x \rightarrow \neg y=y \rightarrow \neg x$,
(e) $x \rightarrow \neg y=\neg \neg x \rightarrow \neg y$,
(f) $\neg \neg(x \rightarrow \neg y)=x \rightarrow \neg y$,
(g) $x * \neg x=\mathbf{0}$,
$(h) \neg(x \vee y)=\neg x \wedge \neg y$.
An involutive residuated lattice (or integral, commutative Girard monoid [12]) is a bounded residuated lattice satisfying the double negation equation:

$$
\begin{equation*}
\neg \neg x=x \tag{1.2}
\end{equation*}
$$

It follows from (iv) of Lemma 1.1 that in involutive residuated lattices the operations $*$ and $\rightarrow$ are related as follows:

$$
x * y=\neg(x \rightarrow \neg y), \quad x \rightarrow y=\neg(x * \neg y)
$$

The class of involutive residuated lattices is represented by $\mathbb{I R} \mathbb{L}$. Notice that involutive Heyting algebras coincide with Boolean algebras.

We recall that a pseudocomplemented residuated lattice is a bounded residuated lattice that satisfies the equation

$$
\begin{equation*}
x \wedge \neg x=\mathbf{0} \tag{1.3}
\end{equation*}
$$

We shall denote by $\mathbb{P R} \mathbb{L}$ the variety of all pseudocomplemented residuated lattices.

Let $\boldsymbol{A}$ be a bounded residuated lattice. We define the set of regular elements of $\boldsymbol{A}$ as follows:

$$
\operatorname{Reg}(\boldsymbol{A})=\{a \in A: \neg \neg a=a\}=\{\neg a: a \in A\}
$$

It is easy to see that if we consider the binary terms:

$$
x *_{r} y:=\neg \neg(x * y), \quad x \vee_{r} y:=\neg \neg(x \vee y),
$$

then $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})=\left\langle\operatorname{Reg}(\boldsymbol{A}), \wedge, \vee_{r}, *_{r}, \rightarrow, \mathbf{0}, \mathbf{1}\right\rangle$ is an involutive residuated lattice (see [8]). Moreover:

Theorem 1.3. $\boldsymbol{A}$ is pseudocomplemented if and only if $\boldsymbol{\operatorname { R e g }} \boldsymbol{g}(\boldsymbol{A})$ is a Boolean algebra.
Proof. Assume that $\boldsymbol{A}$ is in $\mathbb{P R L}$. Then for any $a \in A, a \wedge \neg a=\mathbf{0}$; by item ( $h$ ) of Lemma 1.2 and (1.3), we obtain

$$
a \vee_{r} \neg a=\neg \neg(a \vee \neg a)=\neg(\neg a \wedge \neg \neg a)=\neg \mathbf{0}=\mathbf{1} ;
$$

and so, by item $(g)$ of Lemma 1.2 and item $(v)$ of Lemma 1.1, we have:

$$
a *_{r} a=\left(a *_{r} a\right) \vee_{r}\left(a *_{r} \neg a\right)=a *_{r}\left(a \vee_{r} \neg a\right)=a *_{r} 1=a .
$$

Then $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is an involutive Heyting algebra, and so it is a Boolean algebra.
Now, assume $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is a Boolean algebra, then for any $a \in A$, since $\neg a, \neg \neg a \in \operatorname{Reg}(\boldsymbol{A})$, we have

$$
a \wedge \neg a \leq \neg \neg a \wedge \neg a=\mathbf{0},
$$

and so $\boldsymbol{A}$ is pseudocomplemented.
In general, in a pseudocomplemented residuated lattice, the Boolean algebra of its regular elements is not a subalgebra. The following improves the results given in [5].

Theorem 1.4. For each $\boldsymbol{A} \in \mathbb{P} \mathbb{R L}$, the following conditions are equivalent,
(i) $\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{A}$,
(ii) $\boldsymbol{A}$ is a Stone residuated lattice, i.e., it satisfies the identity:

$$
\begin{equation*}
\neg x \vee \neg \neg x=1 \tag{1.4}
\end{equation*}
$$

Proof. Assume $(i)$. Since $\operatorname{Reg}(\boldsymbol{A})$ is closed under $\vee$, we have that for any $a \in A, \neg a \vee \neg \neg a \in \operatorname{Reg}(\boldsymbol{A})$. Hence, by Theorem 1.3, $\mathbf{1}=\neg a \vee_{r}$ $\neg \neg a=\neg \neg(\neg a \vee \neg \neg a)=\neg a \vee \neg \neg a$. Therefore (1.4) holds in $\boldsymbol{A}$. The converse implication is proved in [5].

From now on, $\mathbb{S R L}$ will denoted the variety of all Stone residuated lattices. Notice that Heyting algebras belonging to $\mathbb{S} \mathbb{R} \mathbb{L}$ are known as Stonean Heyting algebras (see, for instance [5]).

Examples of Stone residuated lattices are the pseudocomplemented residuated lattices representable by means of a subdirect product of totally ordered residuated lattices (pseudocomplemented MTL-algebras). Examples of these are Gödel algebras and product algebras.

For a residuated lattice $\boldsymbol{A}, B(\boldsymbol{A})$ stands for the set of all Boolean elements of $\boldsymbol{A}$, i.e., the complemented elements of $\boldsymbol{L}(\boldsymbol{A})$. It is easy to see that if $a \in B(\boldsymbol{A})$, then $\neg a$ is the only complement of $a$; in addition, $\neg \neg a=a$, $a \vee b=\neg a \rightarrow b$ and $a \wedge b=a * b$, for any $b \in A$. Moreover, $a \in B(\boldsymbol{A})$ if and only if $a \vee \neg a=\mathbf{1}$. Then $B(\boldsymbol{A})$ is the universe of a subalgebra of $\boldsymbol{A}$, denoted by $\boldsymbol{B}(\boldsymbol{A})$, which is the greatest Boolean algebra contained in $\boldsymbol{A}$.
$\boldsymbol{B}(\boldsymbol{A})$ is also a subalgebra of $\boldsymbol{\operatorname { R e }} \boldsymbol{g}(\boldsymbol{A})$, but in general they are not equal. From Theorem 1.4 we deduce:

Lemma 1.5. If $\boldsymbol{A} \in \mathbb{P R} \mathbb{L}$, then $\boldsymbol{B}(\boldsymbol{A})=\boldsymbol{\operatorname { R e g }}(\boldsymbol{A})$ if and only if $\boldsymbol{A} \in \mathbb{S} \mathbb{R} \mathbb{L}$.
We recall that an algebra $\boldsymbol{A}$ is called directly indecomposable provided that it has more than one element and whenever $\boldsymbol{A}$ is isomorphic to a direct product of two algebras $\boldsymbol{A}_{1}$ and $\boldsymbol{A}_{2}$, then either $\boldsymbol{A}_{1}$ or $\boldsymbol{A}_{2}$ is the trivial algebra with just one element. It is well known and easy to see that a residuated lattice $\boldsymbol{A}$ is directly decomposable iff $B(\boldsymbol{A}) \neq\{\mathbf{0}, \mathbf{1}\}$ (see for example [13]).

## 2. Free pseudocomplemented residuated lattices

In this section we will analyze the direct indecomposability of free algebras in subvarieties of $\mathbb{P R} \mathbb{L}$.

Lemma 2.1. Let $\boldsymbol{A} \in \mathbb{P} \mathbb{R} \mathbb{L}$, then the following properties hold true:
(a) $\boldsymbol{A}$ satisfies the Glivenko equation:

$$
\begin{equation*}
\neg \neg(\neg \neg x \rightarrow x)=\mathbf{1} \tag{2.5}
\end{equation*}
$$

(b) the map $\neg \neg: x \mapsto \neg \neg x$ is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{\operatorname { R e g }} \boldsymbol{\operatorname { l }}(\boldsymbol{A})$,
(c) if $\boldsymbol{A} \in \mathbb{S} \mathbb{R} \mathbb{L}$, then $\neg \neg$ is a retraction.

Proof. (a): Let $a \in A$, then, by item (ii) of Lemma 1.1, we have $a \leq$ $\neg \neg a \rightarrow a$, and by item $(a)$ of Lemma 1.2 , we deduce $\neg \neg a \leq \neg \neg(\neg \neg a \rightarrow a)$. Moreover, using again item (ii) of Lemma 1.1 and item (a) of Lemma 1.2, we have $\neg a \leq \neg \neg \neg a=\neg \neg a \rightarrow 0 \leq \neg \neg a \rightarrow a \leq \neg \neg(\neg \neg a \rightarrow a)$. Consequently $\neg a \vee \neg \neg a \leq \neg \neg(\neg \neg a \rightarrow a)$, therefore $\mathbf{1}=\neg a \vee_{r} \neg \neg a=\neg \neg(\neg a \vee \neg \neg a) \leq$ $\neg \neg(\neg \neg(\neg \neg a \rightarrow a))=\neg \neg(\neg \neg a \rightarrow a)$.
(b) follows from (a), (see [8]).
$(c)$ : It follows from the fact that the restriction of $\neg \neg$ to $\operatorname{Reg}(\boldsymbol{A})=B(\boldsymbol{A})$ is the identity (cf. [5]).

Given a variety $\mathbb{V}, \boldsymbol{F}_{\mathbb{V}}(X)$, stands for the $|X|$-free algebra of $\mathbb{V}$. The next theorem follows easily from Lemma 2.1 above:
ThEOREM 2.2. For any subvariety $\mathbb{V}$ of $\mathbb{P R L}$, $\boldsymbol{R e g}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)$ is the $|X|$-free Boolean algebra with $\neg \neg X=\{\neg \neg x: x \in X\}$ as set of free generators.
Corollary 2.3. For any subvariety $\mathbb{V}$ of $\mathbb{S} \mathbb{R} \mathbb{L}$, $\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right) \cong \boldsymbol{F}_{\mathbb{B}}(X)$ and $\boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)$ is a retract of $\boldsymbol{F}_{\mathbb{V}}(X)$.

In what follows we assume that $\mathbb{V}$ is a subvariety of $\mathbb{P R} \mathbb{L}$ and $|X|>0$. Since $\boldsymbol{F}_{\mathbb{V}}(X)$ is a quotient of the $|X|$-term algebra, and each term only depends on a finite set of variables, then we can suppose, without loss of generality, that the cardinality of $X$ is finite. Moreover, if $t\left(x_{1}, \ldots, x_{n}\right)$ is an $n$-ary term, $\bar{t}\left(x_{1}, \ldots, x_{n}\right)$ stands for its image in $\boldsymbol{F}_{\mathbb{V}}\left(\left\{x_{1}, \ldots, x_{n}\right\}\right)$. Let $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $I_{n}=\{1, \ldots, n\}$. For any $I \subseteq I_{n}$, consider the term

$$
a_{I}\left(x_{1}, \ldots, x_{n}\right)=\bigwedge_{i \in I} \neg \neg x_{i} \wedge \bigwedge_{i \notin I} \neg x_{i}
$$

The correspondence $I \mapsto \bar{a}_{I}\left(x_{1}, \ldots, x_{n}\right)$ gives a one-to-one map from $\mathcal{P}\left(I_{n}\right)$, the power set of $I_{n}$, onto the set of all atoms of the free Boolean algebra $\boldsymbol{\operatorname { R e g }}\left(\boldsymbol{F}_{\mathbb{V}}\left(X_{n}\right)\right)$. Hence for any $b \in \operatorname{Reg}\left(\boldsymbol{F}_{\mathbb{V}}\left(X_{n}\right)\right)$, there exists $N \subseteq \mathcal{P}\left(I_{n}\right)$ such that

$$
b=\neg \neg\left(\bigvee_{I \in N} \bar{a}_{I}\left(x_{1}, \ldots, x_{n}\right)\right)
$$

where $N=\left\{I \in \mathcal{P}\left(I_{n}\right): \bar{a}_{I} \leq b\right\}$.
LEMMA 2.4. For any $J \subseteq I_{n}$, consider the $n$-tuple $\overrightarrow{\mathbf{x}}_{J}$ whose $i$-th component is $x$ for $i \in J$, and $\mathbf{1}$ for $i \notin J$. For any $I \subseteq I_{n}$, we get

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)= \begin{cases}\mathbf{1} & \text { if } I=I_{n} \text { and } J=\emptyset \\ \neg \neg x & \text { if } I=I_{n} \text { and } J \neq \emptyset \\ \neg x & \text { if } I=I_{n} \backslash J \text { and } J \neq \emptyset \\ \mathbf{0} & \text { otherwise }\end{cases}
$$

Proof. Let us write the term $a_{I}\left(x_{1}, \ldots, x_{n}\right)$ in the following way:

$$
a_{I}\left(x_{1}, \ldots, x_{n}\right)=\bigwedge_{i \in I \cap J} \neg \neg x_{i} \wedge \bigwedge_{i \in I \backslash J} \neg \neg x_{i} \wedge \bigwedge_{i \in J \backslash I} \neg x_{i} \wedge \bigwedge_{i \notin I \cup J} \neg x_{i}
$$

Then

$$
\begin{aligned}
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right) & =\bigwedge_{i \in I \cap J} \neg \neg x \wedge \bigwedge_{i \in I \backslash J} \neg \neg \mathbf{1} \wedge \bigwedge_{i \in J \backslash I} \neg x \wedge \bigwedge_{i \notin I \cup J} \neg \mathbf{1} \\
& =\bigwedge_{i \in I \cap J} \neg \neg x \wedge \bigwedge_{i \in J \backslash I} \neg x \wedge \bigwedge_{i \notin I \cup J} \mathbf{0} .
\end{aligned}
$$

Clearly, if $I \cup J \neq I_{n}, \bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\mathbf{0}$. Thus we may assume that $I \cup J=I_{n}$ and that

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\bigwedge_{i \in I \cap J} \neg \neg x \wedge \bigwedge_{i \in J \backslash I} \neg x
$$

If $I \cap J \neq \emptyset$ and $J \backslash I \neq \emptyset$, then $\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\neg \neg x \wedge \neg x=\mathbf{0}$.
If $I \cap J=\emptyset$, then $I=I_{n} \backslash J$ and

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\bigwedge_{i \in J} \neg x
$$

It follows that $\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\mathbf{1}$ if $J=\emptyset\left(\right.$ and $\left.I=I_{n}\right)$ and $\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\neg x$ if $J \neq \emptyset$. Finally, if $J \backslash I=\emptyset$, then $J \subseteq I$ and

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\bigwedge_{i \in J} \neg \neg x
$$

It follows that $\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\mathbf{1}$ if $J=\emptyset$ and $\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\neg \neg x$ if $J \neq \emptyset$.
Now we can state the main result of the paper.
ThEOREM 2.5. Let $\mathbb{V}$ be a non-trivial subvariety of $\mathbb{P R L}$. Then $\boldsymbol{F}_{\mathbb{V}}(X)$ is directly decomposable if and only if $\mathbb{V} \subseteq \mathbb{S} \mathbb{R} \mathbb{L}$.

Proof. Assume $\boldsymbol{F}_{\mathbb{V}}(X)$ is directly decomposable. Then there exists $\bar{\alpha} \in$ $F_{\mathbb{V}}(X)$ such that $\bar{\alpha} \vee \neg \bar{\alpha}=\mathbf{1}$ and $\bar{\alpha} \neq \mathbf{0}, \mathbf{1}$. We can assume that $\bar{\alpha}=$ $\bar{\alpha}\left(x_{1}, \ldots, x_{n}\right) \in F_{\mathbb{V}}\left(X_{n}\right)$, as above. Since $\bar{\alpha} \in \operatorname{Reg}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)$, there exists $N \subseteq \mathcal{P}\left(I_{n}\right), N \neq \emptyset, \mathcal{P}\left(I_{n}\right)$, such that

$$
\bar{\alpha}\left(x_{1}, \ldots, x_{n}\right)=\neg \neg\left(\bigvee_{I \in N} \bar{a}_{I}\right)
$$

We need to prove that $\neg x \vee \neg \neg x=\mathbf{1}$.
Case 1: Suppose $I_{n} \notin N$. Fix $K \in N$ and let $J=I_{n} \backslash K$. Since $K \neq I_{n}$, $J \neq \emptyset$ and the previous lemma implies that

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)= \begin{cases}\neg x, & \text { if } I=K, \\ \mathbf{0}, & \text { if } I \in N, I \neq K .\end{cases}
$$

It follows that $\bar{\alpha}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\neg \neg(\neg x)=\neg x$. Therefore, as $\bar{\alpha} \vee \neg \bar{\alpha}=\mathbf{1}$, we get $\neg x \vee \neg \neg x=\mathbf{1}$, as desired.
Case 2: Now assume $I_{n} \in N$. Choose $J \subseteq I_{n}$ such that $J \neq I_{n} \backslash I$ for every $I \in N$. Observe that this is possible because $N \neq \mathcal{P}\left(I_{n}\right)$. By the previous lemma we get

$$
\bar{a}_{I}\left(\overrightarrow{\mathbf{x}}_{J}\right)= \begin{cases}\neg \neg x, & \text { if } I=I_{n} \\ \mathbf{0}, & \text { if } I \in N, I \neq I_{n}\end{cases}
$$

Therefore, $\bar{\alpha}\left(\overrightarrow{\mathbf{x}}_{J}\right)=\neg \neg x$ and the equation $\bar{\alpha} \vee \neg \bar{\alpha}=\mathbf{1}$ turns into Stone's equation $\neg \neg x \vee \neg x=\mathbf{1}$.

This shows that $\mathbb{V}$ satisfies the Stone identity.
Finally we can summarize the above results in the next corollary.
Corollary 2.6. For each non-trivial variety $\mathbb{V}$ of pseudocomplemented residuated lattices, the following properties are equivalent:
(i) $\mathbb{V}$ is a variety of Stone residuated lattices,
(ii) for some non-empty set $X, \boldsymbol{F}_{\mathbb{V}}(X)$ is directly decomposable,
(iii) for some non-empty set $X, \operatorname{Reg}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)=B\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)$,
(iv) for some non-empty set $X, \boldsymbol{B}\left(\boldsymbol{F}_{\mathbb{V}}(X)\right)$ is isomorphic to the $|X|$-free Boolean algebra.

In the last corollary, observe that the decomposability of any free algebra in $\mathbb{V}$ or, equivalently, the existence of a non-trivial Boolean element in any free algebra in $\mathbb{V}$, implies the decomposability of every free algebra in $\mathbb{V}$.

## 3. Some examples

### 3.1. Heyting algebras

In this section we apply the results obtained for free-decomposability to subvarieties of Heyting algebras.

Let $\Lambda(\mathbb{H})$ be the lattice of subvarieties of $\mathbb{H}$. Let $\mathbb{V}_{1}, \mathbb{V}_{2} \in \Lambda(\mathbb{H})$. The pair $\left(\mathbb{V}_{1}, \mathbb{V}_{2}\right)$ splits $\Lambda(\mathbb{H})$ if $\mathbb{V}_{1} \nsubseteq \mathbb{V}_{2}$ and for any $\mathbb{V} \in \Lambda(\mathbb{H})$, either $\mathbb{V}_{1} \subseteq \mathbb{V}$ or $\mathbb{V} \subseteq \mathbb{V}_{2}$. It easy to see that $\mathbb{V}_{1}$ is completely join irreducible and $\mathbb{V}_{2}$ is completely meet irreducible. Hence it is immediate that $\mathbb{V}_{1}=V(\boldsymbol{A})$, i.e., $\mathbb{V}_{1}$ is the variety generated by some subdirectly irreducible algebra $\boldsymbol{A}$. Such an $\boldsymbol{A}$ is called a splitting algebra in $\mathbb{H}$, and $\mathbb{V}_{1}=V(\boldsymbol{A})$ is called a splitting variety. The variety $\mathbb{V}_{2}$ is called the cosplitting variety of $\mathbb{V}_{1}$.

In [2, Corollary 3.2] it is proved that if $\mathbb{V}$ is of finite type and has EDPC (equationally definable principal congruences), then every finite subdirectly irreducible member of $\mathbb{V}$ is splitting, and thus splitting algebras in such varieties are just all finite subdirectly irreducible ones.

Consider the five-element Heyting algebra $\boldsymbol{H}_{5}=\langle\{\mathbf{0}, a, b, c, \mathbf{1}\}, \wedge, \vee$, $\rightarrow, \mathbf{0}, \mathbf{1}\rangle$, whose lattice order is given by the Hasse diagram below and whose residuum is given in Table ( $H 5$ ).

| $\rightarrow$ | $\mathbf{0}$ | $a$ | $b$ | $c$ | $\mathbf{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{0}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $a$ | $b$ | $\mathbf{1}$ | $b$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $b$ | $a$ | $a$ | $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $c$ | $\mathbf{0}$ | $a$ | $b$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $\mathbf{1}$ | $\mathbf{0}$ | $a$ | $b$ | $c$ | $\mathbf{1}$ |

Table (H5)

$\left\langle H_{5}, \leq\right\rangle$

Let $\mathbb{S H}=\mathbb{S} \mathbb{R} \mathbb{L} \cap \mathbb{H}$, the variety of all Stonean Heyting algebras. We can easily see that $\boldsymbol{H}_{5}$ is subdirectly irreducible and $\boldsymbol{H}_{\mathbf{5}} \notin \mathbb{S H}$ (take $a \in H_{5}$ and note that $\neg a \vee \neg \neg a=c \neq 1)$.

Lemma 3.1. Let $\boldsymbol{A} \in \mathbb{H}$. $\boldsymbol{A} \notin \mathbb{S H}$ if and only if $\boldsymbol{H}_{\mathbf{5}}$ is isomorphic to a subalgebra of $\boldsymbol{A}$.

Proof. Suppose that $\boldsymbol{A} \notin \mathbb{S H}$. Then there is $a \in A$ such that $\neg a \vee \neg \neg a \neq \mathbf{1}$. Let $B=\{\mathbf{0}, \neg a, \neg \neg a, \neg a \vee \neg \neg a, \mathbf{1}\}$. Then $B$ is a subuniverse of $A$. Indeed, clearly $B$ is a sublattice of $A$. Let us see that $B$ is closed under implication,

- Since $\neg a \wedge(\neg a \rightarrow \neg \neg a)=\mathbf{0}$, we have $\neg \neg a \leq \neg a \rightarrow \neg \neg a \leq \neg \neg a$. Thus $\neg a \rightarrow \neg \neg a=\neg \neg a$.
- Since $\neg \neg a \wedge(\neg \neg a \rightarrow \neg a)=\mathbf{0}$, we have $\neg a \leq \neg \neg a \rightarrow \neg a \leq \neg a$. Thus $\neg \neg a \rightarrow \neg a=\neg a$.
- $(\neg \neg a \vee \neg a) \rightarrow \mathbf{0}=\mathbf{0}$.
- $(\neg \neg a \vee \neg a) \rightarrow \neg a=a \rightarrow \neg(\neg \neg a \vee \neg a)=a \rightarrow \mathbf{0}=\neg a$.
- $(\neg \neg a \vee \neg a) \rightarrow \neg \neg a=\neg a \rightarrow \neg(\neg \neg a \vee \neg a)=\neg a \rightarrow \mathbf{0}=\neg \neg a$.

Then $B$ is a subuniverse of $\boldsymbol{A}$ and $\boldsymbol{B}$ is isomorphic to $\boldsymbol{H}_{5}$.
The converse implication is immediate.
Corollary 3.2. The pair $\left(\mathbb{V}\left(\boldsymbol{H}_{\mathbf{5}}\right), \mathbb{S H}\right)$ splits $\Lambda(\mathbb{H})$. Therefore $\mathbb{S H}$ is a cosplitting variety.

Combining Lemma 3.1 and Corollary 2.6 we get the following result.
Corollary 3.3. For any non-trivial variety $\mathbb{V} \in \Lambda(\mathbb{H})$, the following conditions are equivalent:
(a) $\boldsymbol{F}_{\mathbb{V}}(X)$ is directly indecomposable.
(b) $\mathbb{V} \nsubseteq \mathbb{S H}$.
(c) $\boldsymbol{H}_{\mathbf{5}}$ is isomorphic to a subalgebra of $\boldsymbol{F}_{\mathbb{V}}(X)$.

The pseudocomplemented distributive lattices are the $\{\wedge, \vee, \neg, \mathbf{0}, \mathbf{1}\}$ subreducts of Heyting algebras (see for example [1]), and they form a variety that we will denoted by $\mathbb{P L}$. The lattice of subvarieties $\Lambda(\mathbb{P L})$ is an $(\omega+1)$ chain, and for each non-trivial subvariety there is $n \geq 0$, such that it is generated by the algebra $\boldsymbol{B}_{\boldsymbol{n}}=\mathbf{2}^{\boldsymbol{n}} \oplus \mathbf{1}$, the $n$-atom Boolean algebra with an element added on the top. Then $\boldsymbol{B}_{\mathbf{0}}$ is the two-element chain (Boolean algebra) and $\boldsymbol{B}_{\mathbf{1}}$ the three-element chain (the Stone algebra in the terminology of $[1])$, and $\boldsymbol{B}_{\mathbf{2}}$ is the $\{\wedge, \vee, \neg, \mathbf{0}, \mathbf{1}\}$-reduct of $\boldsymbol{H}_{\mathbf{5}}$. The varieties in $\Lambda(\mathbb{P L})$ are

$$
\mathbb{T} \subsetneq V\left(\boldsymbol{B}_{0}\right)=\mathbb{B} \subsetneq V\left(\boldsymbol{B}_{1}\right)=\mathbb{S P L} \subsetneq V\left(\boldsymbol{B}_{2}\right) \subseteq \ldots \subseteq V\left(\boldsymbol{B}_{\omega}\right)=\mathbb{P L},
$$

where $\mathbb{T}$ is the trivial variety, $\mathbb{B}$ is the variety of Boolean algebras, $\mathbb{S P L}$ is the variety of Stone pseudocomplemented distributive lattices (remember that the Stone equation only involves the operations $\neg$ and $\vee$ ) and $V\left(\boldsymbol{B}_{\mathbf{2}}\right)$ is the variety generated by the pseudocomplemented lattice reduct of the Heyting algebra $\boldsymbol{H}_{5}$.

As in the case of Heyting algebras and pseudocomplemented residuated lattices we have the following theorem.
Theorem 3.4. For a variety $\mathbb{V} \in \Lambda(\mathbb{P L})$, the following conditions are equivalent:
(a) $\boldsymbol{F}_{\mathbb{V}}(X)$ is directly indecomposable,
(b) $\mathbb{V} \nsubseteq \mathbb{S P L}$,
(c) $\boldsymbol{H}_{\mathbf{5}}$ is isomorphic to a subalgebra of $\boldsymbol{F}_{\mathbb{V}}(X)$.

An interesting description for free Stone pseudocomplemented distributive lattices is given in [6].

### 3.2. Distributive non-Stone pseudocomplemented residuated lattices

In this section we exhibit some examples of distributive pseudocomplemented residuated lattices, which are not Stone residuated lattices.

We recall that a residuated lattice is called distributive when its lattice reduct is distributive. Since distributivity can be described by means of an identity, the class $\mathbb{D} \mathbb{R} \mathbb{L}$ of all distributive residuated lattices is a variety. We also recall that for $n>0$, residuated lattices satisfying the equation
$\left(\mathrm{E}_{n}\right) x^{n}=x^{n+1}$,
are called $n$-potent. The variety of all $n$-potent pseudocomplemented residuated lattices will be denoted by $\mathbb{P} \mathbb{R} \mathbb{L}_{n}$. Moreover, $\mathbb{D P R} \mathbb{L}$ and $\mathbb{D} \mathbb{P R} \mathbb{L}_{n}$ respectively stand for $\mathbb{D} \mathbb{R} \mathbb{L} \cap \mathbb{P} \mathbb{R L}$ and $\mathbb{D} \mathbb{R} \mathbb{P} \cap \mathbb{P R L}{ }_{n}$.

Consider the distributive lattice $\boldsymbol{L}\left(\boldsymbol{D}_{n}\right)=\left\langle D_{n}, \wedge, \vee, \mathbf{0}, \mathbf{1}\right\rangle$, where

$$
D_{n}=\{\mathbf{1}\} \cup\{r s: 0 \leq r, s \leq n,\}
$$

and the lattice order is given by the diagram depicted in Figure 1.
We define the operation $*$ on $D_{n}$ as follows: for any $a, b \in D_{n}$
$a * b=b * a=\left\{\begin{array}{cl}a & \text { if } b=\mathbf{1} \\ 0 & \text { if } \begin{cases}a \in\{0 s: 0 \leq s \leq n\} \\ b \in\{r 0: 0 \leq r \leq n\},\end{cases} \\ 0 \min \left\{s+s^{\prime}, n\right\} & \text { if }\left\{\begin{array}{l}a \in\{0 s: 0 \leq r \leq n\} \\ b \in\left\{r^{\prime} s^{\prime}: 0<r^{\prime}, s^{\prime} \leq n\right\},\end{array}\right. \\ \min \left\{r+r^{\prime}, n\right\} \min \left\{s+s^{\prime}, n\right\} & \text { if }\left\{\begin{array}{l}a=r s, \\ b=r^{\prime} s^{\prime},\end{array} \text { and } 0<r, s, r^{\prime}, s^{\prime} \leq n .\right.\end{array}\right.$
For example, in $D_{4}$ we have $31 * 21=\min \{3+2,4\} \min \{1+1,4\}=42$, $02 * 11=03$ and $20 * 31=40$. It is a straightforward to see that for any $a, b, c \in D_{n}, a *(b \vee c)=(a * b) \vee(a * c)$, hence since $\boldsymbol{L}\left(\boldsymbol{D}_{\boldsymbol{n}}\right)$ is a finite lattice, then for any $a, b \in D_{n}, a \rightarrow b=\max \left\{c \in D_{n}: a * c \leq b\right\}$ exists. And so $\boldsymbol{D}_{\boldsymbol{n}}=\left\langle D_{n}, \wedge, \vee, *, \rightarrow, \mathbf{0}, \mathbf{1}\right\rangle$, is a residuated lattice (see, for instance [13] and [10]). Moreover, it is distributive and $n$-potent. The reader can verify with a simple calculation that for any $n>0, V\left(\boldsymbol{D}_{\boldsymbol{n}}\right) \subsetneq V\left(\boldsymbol{D}_{\boldsymbol{n}+\mathbf{1}}\right)$. Observe that for all $n>0, \boldsymbol{R e} \boldsymbol{g}\left(\boldsymbol{D}_{\boldsymbol{n}}\right)$ is the four element Boolean algebra.


Figure 1. The lattice reduct of $\boldsymbol{D}_{n}$

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