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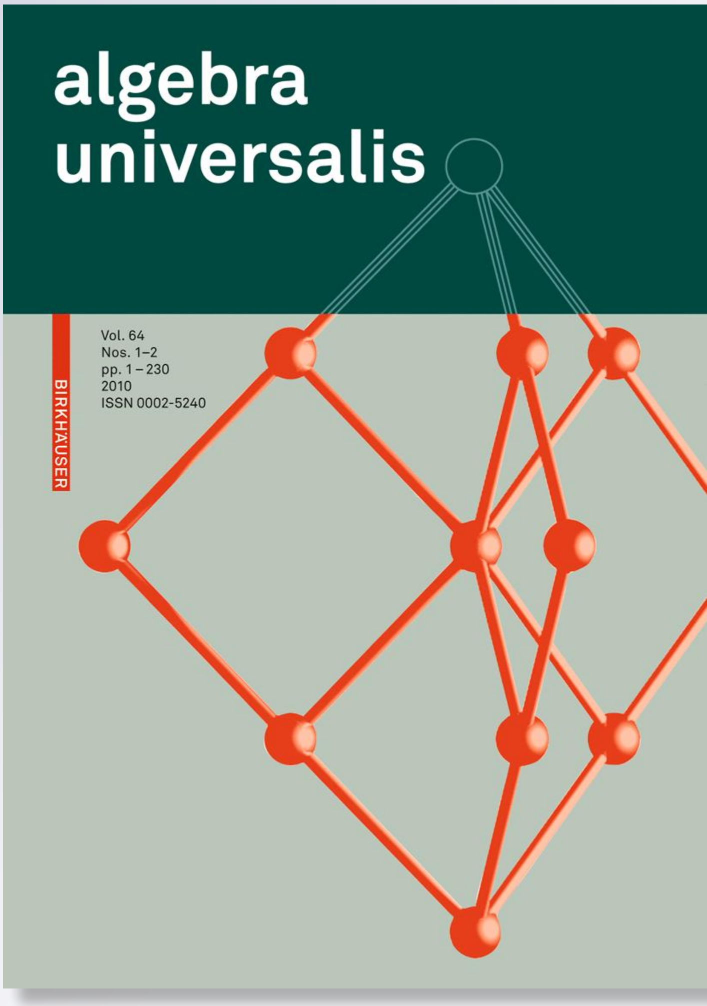


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# MV-closures of Wajsberg hoops and applications

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Abstract. In this paper we construct, given a Wajsberg hoop $\mathbf{A}$, an MV-algebra $\mathbf{MV}(\mathbf{A})$ such that the underlying set $A$ of $\mathbf{A}$ is a maximal filter of $\mathbf{MV}(\mathbf{A})$ and the quotient $\mathbf{MV}(\mathbf{A})/A$ is the two element chain. As an application we provide a topological duality for locally finite Wajsberg hoops based on a previously known duality for locally finite MV-algebras. We also give another duality for $k$-valued Wajsberg hoops based on a different representation of $k$-valued MV-algebras and show the relation to the first duality. We also apply this construction to give a topological representation for free $k$-valued Wajsberg hoops.

## 1. Introduction

Hoops are a particular class $\mathbb{H}$ of algebraic structures which were introduced in an unpublished manuscript by Büchi and Owens in the mid-seventies and later investigated in [2, 9]. They are partially ordered commutative residuated integral monoids satisfying a further divisibility condition. Bosbach showed that $\mathbb{H}$ is a variety.

The variety of hoops includes two classes of algebras that are closely related to familiar algebras of logic: the variety of Brouwerian semilattices, defined relative to $\mathbb{H}$ by the identity $x \odot x \approx x$, and the variety of Wajsberg hoops, defined relative to $\mathbb{H}$ by the axiom $(x \to y) \to y \approx (y \to x) \to x$. It is known that Brouwerian semilattices are the $\{\wedge, \to, 1\}$-subreducts of Heyting algebras, which are the algebraic models of intuitionistic propositional logic, and Wajsberg hoops are the $\{\odot, \to, 1\}$-subreducts (hoop-subreducts) of Wajsberg algebras, which are term equivalent to MV-algebras and are the algebraic models of Łukasiewicz's many-valued logic.

Wajsberg hoops play a fundamental role in the study of the variety of hoops. This is clear since, among others, we have the following result: every subdirectly irreducible hoop $\mathbf{A}$ is an ordinal sum $\mathbf{A} \cong \mathbf{B} \oplus \mathbf{C}$, where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\mathbf{C}$ is a subdirectly irreducible Wajsberg hoop, and consequently, totally ordered. This theorem is a generalization of the well-known characterization of subdirectly irreducible Brouwerian semilattices as the algebras that can be written as $\mathbf{B} \oplus \mathbf{C}_1$, where $\mathbf{B}$ is a Brouwerian semilattice

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and $\mathbf{C}_1$ is the 2-element hoop. In addition, in a sense that it is made precise in [2], the variety of all hoops is the join of the powers of the variety of Wajsberg hoops.

The main aim of this paper is to construct, for a Wajsberg hoop $\mathbf{A}$, an MV-algebra $\mathbf{MV}(\mathbf{A})$ such that $A$ is a maximal filter of $\mathbf{MV}(\mathbf{A})$ and the quotient $\mathbf{MV}(\mathbf{A})/A$ is the two element chain, and to give some applications of this construction to obtain a topological duality for locally finite Wajsberg hoops.

## 2. Preliminaries

In this section, we recall some definitions and collect the properties of hoops and Wajsberg hoops needed in the rest of the paper.

A *hoop* is an algebra $\mathbf{A} = \langle A; \odot, \to, 1\rangle$ such that $\langle A; \odot, 1\rangle$ is a commutative monoid and the following equations are satisfied:

$$x \to x \approx 1,$$
$$x \to (y \to z) \approx (x \odot y) \to z,$$
$$x \odot (x \to y) \approx y \odot (y \to x).$$

As shown here and throughout this article, we will make the distinction between the algebra $\mathbf{A}$ (written in boldface) and its underlying set $A$ (written in italics).

The first systematic study of the structural properties of hoops appeared in Ferreirim's thesis [9]. We refer to [2] for a study of hoops and for further references.

If $\mathbf{A} = \langle A; \odot, \to, 1\rangle$ is a hoop, $\langle A; \odot, 1\rangle$ is a naturally ordered residuated commutative monoid, where the order is defined by $a \leq b$ iff $a \to b = 1$, and residuation means that $a \odot b \leq c$ if and only if $a \leq b \to c$. Any hoop is a meet semilattice and the meet is term-definable as $a \wedge b = a \odot (a \to b)$.

In the following lemma we collect some properties of hoops.

**Lemma 2.1.** *Let $\mathbf{A} = \langle A; \odot, \to, 1\rangle$ be a hoop. For every $a, b, c \in A$, the following holds:*

$$1 \to a = a,$$
$$a \to 1 = 1,$$
$$a \to b \leq (c \to a) \to (c \to b),$$
$$a \leq b \to a,$$
$$a \leq (a \to b) \to b,$$
$$a \to (b \to c) = b \to (a \to c),$$
$$a \to b \leq (b \to c) \to (a \to c),$$
$$a \leq b \text{ implies } b \to c \leq a \to c \text{ and } c \to a \leq c \to b.$$

The variety $\mathbb{WH}$ of *Wajsberg hoops* is defined relative to the variety of hoops by the axiom

$$(T) \quad (x \to y) \to y \approx (y \to x) \to x.$$

The underlying ordering of a Wajsberg hoop is a distributive lattice ordering [9], and the join is term-definable by $a \vee b = (a \to b) \to b$. The lattice operations satisfy the following properties, for every $a, b, c \in A$:

$$(a \vee b) \to c = (a \to c) \wedge (b \to c),$$
$$c \to (a \vee b) = (c \to a) \vee (c \to b),$$
$$(a \wedge b) \to c = (a \to c) \vee (b \to c),$$
$$c \to (a \wedge b) = (c \to a) \wedge (c \to b).$$

A hoop $\mathbf{A} = \langle A; \odot, \to, 1 \rangle$ is *cancellative* if $\langle A; \odot, 1 \rangle$ is cancellative as a monoid. Cancellative hoops form a variety, axiomatized relative to hoops by the equation $x \approx y \to (x \odot y)$ [2]. Any cancellative hoop is a Wajsberg hoop, and cancellative hoops are axiomatized relative to Wajsberg hoops by the equation $x \approx x \to (x \odot x)$.

An *MV-algebra* is a structure $\langle A; \oplus, \neg, 0 \rangle$ such that $\langle A; \oplus, 0 \rangle$ is a commutative monoid and the identities $\neg\neg x \approx x$, $x \oplus \neg 0 \approx \neg 0$, $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$ are satisfied [6]. We will denote the variety of MV-algebras by $\mathbb{MV}$.

A hoop is *bounded* if it has a bottom element with respect to the order. A *Wajsberg algebra* is a bounded Wajsberg hoop in the enriched language $(\odot, \to, 1, 0)$, where 0 is the constant for the bottom element. Wajsberg algebras and MV-algebras are term-wise equivalent [6, 10]. Indeed, if $\langle A; \odot, \to, 1, 0 \rangle$ is a Wajsberg algebra and we define $\neg a = a \to 0$ and $a \oplus b = \neg a \to b$, then $\langle A; \oplus, \neg, 0 \rangle$ is an MV-algebra. Conversely, if $\langle A; \oplus, \neg, 0 \rangle$ is an MV-algebra and we set $1 = \neg 0$, $a \to b = \neg a \oplus b$ and $a \odot b = \neg(\neg a \oplus \neg b)$, then $\langle A; \odot, \to, 1, 0 \rangle$ is a Wajsberg algebra.

In any Wajsberg hoop $\mathbf{A} = \langle A; \odot, \to, 1 \rangle$, we can define

$$a \oplus b = (a \to (a \odot b)) \to b.$$

If $\mathbf{A}$ has a least element, then it is easily seen that $\oplus$ is the usual Łukasiewicz sum, while if $\mathbf{A}$ is a cancellative hoop, $a \oplus b = 1$ for every $a, b \in A$ (see [1]).

If $\mathbb{V}$ is a variety of MV-algebras, then the class of *hoop-subreducts* of algebras in $\mathbb{V}$ is a variety of Wajsberg hoops. In particular, $\mathbb{WH}$ is the class of hoop-subreducts of MV-algebras. Furthermore, if $\mathbf{A}$ is an MV-algebra, the variety of hoop-subreducts of $V(\mathbf{A})$ (the variety generated by $\mathbf{A}$) is $V(\mathbf{A}^{\odot,\to})$, where $\mathbf{A}^{\odot,\to}$ denotes the hoop-subreduct of $\mathbf{A}$ (see [1]).

A subset $F$ of a hoop $\mathbf{A}$ is called a *filter* if $1 \in F$ and also $b \in F$ whenever $a \in F$ and $a \to b \in F$. One can easily show that in a hoop $\mathbf{A}$, $F \subseteq A$ is a filter if and only if $F$ is an increasing non-empty subset of $A$ closed under multiplication.

Hoops are congruence 1-regular. For each congruence relation $\theta$ on a hoop $\mathbf{A}$, $1/\theta$ is a filter. Conversely, for any filter $F$ of $\mathbf{A}$ the relation

$$\theta_F = \{(a, b) \in A^2 : a \to b, b \to a \in F\}$$

is a congruence on $\mathbf{A}$ such that $F = 1/\theta_F$. In fact, the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the family of all congruences on $\mathbf{A}$ onto the family of all filters of $\mathbf{A}$, ordered by inclusion. Since any filter $F$ contains 1 and is closed under $\to, \odot$, then it is the universe of a subalgebra $\mathbf{F}$ of $\mathbf{A}$. The bounded distributive lattice of the congruences on $\mathbf{A}$ is algebraic, and hence it is pseudocomplemented. Note that for a bounded Wajsberg hoop $\mathbf{A}$, its congruences do not depend on the fact that we look at $\mathbf{A}$ as an MV-algebra or a Wajsberg hoop.

An important Wajsberg hoop is the hoop-reduct $\mathbf{Ł}_{n,\omega}^{\odot,\to}$ of Chang's algebra $\mathbf{Ł}_{n,\omega}$ (see [5, p. 474]), with universe

$$Ł_{n,\omega} = \{(x,y) : x \in \{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}, y \in \mathbb{Z}\} \\ \cup \{(0,y) : y \in \mathbb{N}\} \cup \{(1,-y) : y \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of non-negative integers, and

$$0_{\mathbf{Ł}_{n,\omega}} = (0,0), \qquad \neg(x,y) = (1-x,-y),$$

$$(x,y) \oplus (z,u) = \begin{cases} (1,0) & \text{if } x+z>1, \\ (1, \min(0,y+u)) & \text{if } x+z=1, \\ (x+z, y+u) & \text{if } x+z<1. \end{cases}$$

Therefore, the hoop operations become

$$1_{\mathbf{Ł}_{n,\omega}^{\odot,\to}} = (1,0),$$

$$(x,y) \odot (z,u) = \neg(\neg(x,y) \oplus \neg(z,u)) = \begin{cases} (0,0) & \text{if } x+z<1, \\ (0, \max(0,y+u)) & \text{if } x+z=1, \\ (x+z-1, y+u) & \text{if } x+z>1, \end{cases}$$

$$(x,y) \to (z,u) = \neg(x,y) \oplus (z,u) = \begin{cases} (1,0) & \text{if } x<z, \\ (1, \min(0,u-y)) & \text{if } x=z, \\ (1-x+z, u-y) & \text{if } x>z. \end{cases}$$

We denote by $\mathbf{Ł}_{\omega}^{\odot,\to}$ the subhoop of $\mathbf{Ł}_{n,\omega}^{\odot,\to}$ whose universe is

$$Ł_{\omega}^{\odot,\to} = \{(1,-y) : y \in \mathbb{N}\}.$$

$\mathbf{Ł}_{\omega}^{\odot,\to}$ generates the variety of cancellative hoops (see [2]).

The finite chains $\mathbf{Ł}_{n}^{\odot,\to}$ are the hoop-reducts of the MV-chains $\mathbf{Ł}_n$. The algebra $\mathbf{Ł}_n$ has universe $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ and

$$0_{\mathbf{Ł}_n} = 0, \qquad \neg\frac{i}{n} = 1 - \frac{i}{n} = \frac{n-i}{n}, \qquad \frac{i}{n} \oplus \frac{j}{n} = \min(1, \frac{i}{n} + \frac{j}{n}).$$

The hoop operations are then given by

$$1_{\mathbf{Ł}_n} = 1, \qquad \frac{i}{n} \odot \frac{j}{n} = \neg(\neg\frac{i}{n} \oplus \neg\frac{j}{n}) = \max(0, \frac{i}{n} + \frac{j}{n} - 1),$$

$$\frac{i}{n} \to \frac{j}{n} = \neg\frac{i}{n} \oplus \frac{j}{n} = \min(1, 1 - \frac{i}{n} + \frac{j}{n}).$$

Aglianò and Panti proved in [1] that the non-trivial subvarieties of $\mathbb{WH}$ are finite joins of $V(\mathbf{Ł}_n^{\odot,\to})$'s, $V(\mathbf{Ł}_{n,\omega}^{\odot,\to})$'s, and $V(\mathbf{Ł}_\omega^{\odot,\to})$'s. These varieties are join irreducible and the order inclusion between them is given by

$V(\mathbf{Ł}_n^{\odot,\to}) \subseteq V(\mathbf{Ł}_m^{\odot,\to})$ iff $n$ is a divisor of $m$,
$V(\mathbf{Ł}_n^{\odot,\to}) \subseteq V(\mathbf{Ł}_{m,\omega}^{\odot,\to})$ iff $n$ is a divisor of $m$,
$V(\mathbf{Ł}_{n,\omega}^{\odot,\to}) \subseteq V(\mathbf{Ł}_{m,\omega}^{\odot,\to})$ iff $n$ is a divisor of $m$,
$V(\mathbf{Ł}_\omega^{\odot,\to}) \subset V(\mathbf{Ł}_{m,\omega}^{\odot,\to})$ for all $m$,
$V(\mathbf{Ł}_\omega^{\odot,\to})$ is incomparable with $V(\mathbf{Ł}_m^{\odot,\to})$ for all $m$.

We set $\mathbb{MV}_k = V(\mathbf{Ł}_k)$ and $\mathbb{WH}_k = V(\mathbf{Ł}_k^{\odot,\to})$, that is, the hoop-subreduct of $\mathbb{MV}_k$. Following [6], we call the algebras in $\mathbb{MV}_k$ *$k$-valued MV-algebras* (or $\mathrm{MV}_k$-algebras for short). Likewise, we call the algebras in $\mathbb{WH}_k$ *$k$-valued Wajsberg hoops*. An equational basis for $\mathbb{MV}_k$ is given in [6, Corollary 8.2.4, Theorem 8.5.1]. The same equational basis characterizes $\mathbb{WH}_k$. In particular, $k$-valued Wajsberg hoops are $k$-potent, i.e., they satisfy the equation $x^k \approx x^{k+1}$.

## 3. MV-closures of Wajsberg hoops

In this section, we construct the announced algebra $\mathbf{MV}(\mathbf{A})$ for a given Wajsberg hoop $\mathbf{A}$ and prove the main results of the paper.

Let $\mathbf{A} = \langle A; \odot_A, \to_A, 1_A\rangle$ be a Wajsberg hoop. Let

$$\mathbf{MV}(\mathbf{A}) = \langle A \times \{0,1\}; \oplus_{mv}, \neg_{mv}, 0_{mv}\rangle,$$

where

$$0_{mv} := (1_A, 0), \qquad \neg_{mv}(a,i) := (a, 1-i),$$

$$(a,i) \oplus_{mv} (b,j) := \begin{cases} (a \oplus_A b, 1) & \text{if } i = j = 1,\\ (b \to_A a, 1) & \text{if } i = 1 \text{ and } j = 0,\\ (a \to_A b, 1) & \text{if } i = 0 \text{ and } j = 1,\\ (a \odot_A b, 0) & \text{if } i = j = 0.\end{cases}$$

We define $1_{mv} := \neg_{mv} 0_{mv}$, $(a,i) \odot_{mv} (b,j) := \neg_{mv}(\neg_{mv}(a,i) \oplus_{mv} \neg_{mv}(b,j))$, and $(a,i) \to_{mv} (b,j) := \neg_{mv}(a,i) \oplus_{mv} (b,j)$, as usual. It is easily checked that

$$1_{mv} = (1_A, 1),$$

$$(a,i) \odot_{mv} (b,j) = \begin{cases} (a \odot_A b, 1) & \text{if } i = j = 1,\\ (a \to_A b, 0) & \text{if } i = 1 \text{ and } j = 0,\\ (b \to_A a, 0) & \text{if } i = 0 \text{ and } j = 1,\\ (a \oplus_A b, 0) & \text{if } i = j = 0,\end{cases}$$

$$(a,i) \to_{mv} (b,j) = \begin{cases} (a \to_A b, 1) & \text{if } i = j = 1,\\ (a \odot_A b, 0) & \text{if } i = 1 \text{ and } j = 0,\\ (a \oplus_A b, 1) & \text{if } i = 0 \text{ and } j = 1,\\ (b \to_A a, 1) & \text{if } i = j = 0.\end{cases}$$

From this, it is clear that the map $a \mapsto (a, 1)$ is a hoop-embedding from $\mathbf{A}$ into $\mathbf{MV}(\mathbf{A})$. In the sequel, we will identify $a$ and $(a, 1)$ for every $a \in A$, and consider $\mathbf{A}$ a subalgebra of the hoop-reduct of $\mathbf{MV}(\mathbf{A})$. Note also that if $b = (a, 0)$ for some $a \in A$, then $b = \neg_{mv}(a, 1)$. Thus we write $b = \neg_{mv} a$. In this way, we consider $MV(\mathbf{A})$ as the disjoint union of $A$ and $\neg_{mv} A = \{\neg_{mv} a : a \in A\}$.

The order relation on $\mathbf{MV}(\mathbf{A})$ is given by: $(a, i) \leq (b, j)$ if and only if one of the following conditions holds:

$a \leq_A b$ for $i = j = 1$,
$a \oplus_A b = 1_A$ for $i = 0$ and $j = 1$,
$b \leq_A a$ for $i = j = 0$.

Observe that if $\mathbf{A}$ is a cancellative hoop, then $a \oplus_A b = 1_A$ for every $a, b \in A$, so in this case, the elements in $A$ are all above the ones in $\neg_{mv} A$.

Our aim now is to show that $\mathbf{MV}(\mathbf{A})$ is an MV-algebra. Instead of checking that $\mathbf{MV}(\mathbf{A})$ satisfies the MV-axioms, we give a proof that shows some results of independent interest.

Let $\mathbf{Free}_{\mathbb{MV}}(X)$ be the free MV-algebra over the set $X$, and let $\mathbf{Free}_{\mathbb{WH}}(X)$ be the free Wajsberg hoop over $X$. Recall that Wajsberg hoops are the hoop-subreducts of MV-algebras and note that any term in the language of Wajsberg hoops is a term in the language of MV-algebras. These two facts imply that two hoop-terms are equivalent in $\mathbb{WH}$ if and only if they are equivalent in $\mathbb{MV}$. Thus we have that $Free_{\mathbb{WH}}(X) \subseteq Free_{\mathbb{MV}}(X)$.

Let $F(X)$ be the filter generated by $X$ in $\mathbf{Free}_{\mathbb{MV}}(X)$. It may be easily shown that $F(X) \cup \neg F(X)$ is a subuniverse of $\mathbf{Free}_{\mathbb{MV}}(X)$, so we must have $F(X) \cup \neg F(X) = Free_{\mathbb{MV}}(X)$. In addition, we can prove that $F(X) \cap \neg F(X) = \emptyset$. Therefore, $F(X)$ is a maximal filter which is in fact a subuniverse of the hoop-reduct of $\mathbf{Free}_{\mathbb{MV}}(X)$. We denote the corresponding hoop by $\mathbf{F}(X)$. It is clear that $Free_{\mathbb{WH}}(X) \subseteq F(X)$. In fact, we have the following result.

**Theorem 3.1.** $\mathbf{Free}_{\mathbb{WH}}(X) = \mathbf{F}(X)$.

*Proof.* Consider the set $B = Free_{\mathbb{WH}}(X) \cup \neg Free_{\mathbb{WH}}(X) \subseteq Free_{\mathbb{MV}}(X)$. Clearly $1 \in B$ and $B$ is closed under $\neg$. Let us prove that $B$ is closed under $\oplus$. Observe that for any $x, y \in Free_{\mathbb{WH}}(X)$:

$$x \oplus y = (x \to (x \odot y)) \to y \in Free_{\mathbb{WH}}(X) \subseteq B,$$
$$x \oplus \neg y = y \to x \in Free_{\mathbb{WH}}(X) \subseteq B,$$
$$\neg x \oplus \neg y = \neg(x \odot y) \in \neg Free_{\mathbb{WH}}(X) \subseteq B.$$

Therefore $B$ is an MV-subuniverse of $\mathbf{Free}_{\mathbb{MV}}(X)$ that contains $X$. Thus $B = Free_{\mathbb{MV}}(X)$. However, we know that $F(X) \cup \neg F(X) = Free_{\mathbb{MV}}(X)$, $F(X) \cap \neg F(X) = \emptyset$, $Free_{\mathbb{WH}}(X) \subseteq F(X)$, and $\neg Free_{\mathbb{WH}}(X) \subseteq \neg F(X)$. This implies immediately that $\mathbf{Free}_{\mathbb{WH}}(X) = \mathbf{F}(X)$. $\square$

As a consequence we have:

**Corollary 3.2.** $\mathbf{MV}(\mathbf{Free}_{\mathbb{WH}}(X)) \cong \mathbf{Free}_{\mathbb{MV}}(X)$. *In particular, we get* $\mathbf{Free}_{\mathbb{MV}}(X)/Free_{\mathbb{WH}}(X) \cong \mathbf{Ł}_1$, *and* $Free_{\mathbb{WH}}(X) \cap \neg Free_{\mathbb{WH}}(X) = \emptyset$.

**Remark 3.3.** It is well known that $\mathbf{Free}_{\mathbb{MV}}(X)$ is the MV-algebra of McNaughton functions over the $|X|$-cube. If we let $e$ be the vertex of the $|X|$-cube all of whose coordinates are 1, it is trivially verified that $F(X)$ consists of all McNaughton functions $f \in Free_{\mathbb{MV}}(X)$ such that $f(e) = 1$. Thus, as a consequence of the above theorem, we get the representation of the free Wajsberg hoop within $\mathbf{Free}_{\mathbb{MV}}(X)$ given in [1, Theorem 3.1].

Let $\mathbf{A}$ be a Wajsberg hoop and let $f\colon \mathbf{Free}_{\mathbb{WH}}(X) \to \mathbf{A}$ be an onto homomorphism. Define $\widehat{f}\colon \mathrm{Free}_{\mathbb{MV}}(X) \to \mathrm{MV}(\mathbf{A})$ by

$$\widehat{f}(x) = \begin{cases} (f(x), 1) & \text{if } x \in Free_{\mathbb{WH}}(X), \\ (f(\neg x), 0) & \text{if } x \in \neg Free_{\mathbb{WH}}(X). \end{cases}$$

Clearly $\widehat{f}$ is a well-defined onto function.

**Lemma 3.4.** *The function $\widehat{f}$ is in fact an MV-homomorphism.*

*Proof.* (a): $\widehat{f}(0) = (f(1), 0) = (1, 0) = 0_{mv}$.
(b): $\widehat{f}(\neg x) = \neg_{mv}(\widehat{f}(x))$.
Indeed, if $x \in Free_{\mathbb{WH}}(X)$ then

$$\widehat{f}(\neg x) = (f(\neg\neg x), 0) = (f(x), 0) = \neg_{mv}(f(x), 1) = \neg_{mv}\widehat{f}(x).$$

Now let $x \in \neg Free_{\mathbb{WH}}(X)$, with $x = \neg y$, $y \in Free_{\mathbb{WH}}(X)$. Then $\widehat{f}(\neg x) = \widehat{f}(y) = (f(y), 1)$, and $\neg_{mv}\widehat{f}(x) = \neg_{mv}(f(\neg x), 0) = \neg_{mv}(f(y), 0) = (f(y), 1)$.
(c): $\widehat{f}(x \oplus y) = \widehat{f}(x) \oplus_{mv} \widehat{f}(y)$.
We have the following four cases:
(c1): Let $x, y \in Free_{\mathbb{WH}}(X)$. Then $\widehat{f}(x \oplus y) =$

$$(f(x \oplus y), 1) = (f(x) \oplus f(y), 1) = (f(x), 1) \oplus_{mv} (f(y), 1) = \widehat{f}(x) \oplus_{mv} \widehat{f}(y).$$

(c2): Let $x \in Free_{\mathbb{WH}}(X)$ and $y \in \neg Free_{\mathbb{WH}}(X)$, with $y = \neg z$ and with $z \in Free_{\mathbb{WH}}(X)$. Then

$$\widehat{f}(x \oplus y) = \widehat{f}(x \oplus \neg z) = \widehat{f}(z \to x) = (f(z \to x), 1) = (f(z) \to f(x), 1),$$

while $\widehat{f}(x) \oplus_{mv} \widehat{f}(y) = (f(x), 1) \oplus_{mv} (f(z), 0) = (f(z) \to f(x), 1)$.
(c3): Let $x \in \neg Free_{\mathbb{WH}}(X)$ and $y \in Free_{\mathbb{WH}}(X)$. This is analogous to $(c2)$.
(c4): Let $x, y \in \neg Free_{\mathbb{WH}}(X)$, i.e., $x = \neg x_1, y = \neg y_1$, $x_1, y_1 \in Free_{\mathbb{WH}}(X)$. Then

$$\begin{aligned} \widehat{f}(x \oplus y) &= \widehat{f}(\neg x_1 \oplus \neg y_1) = \widehat{f}(\neg(x_1 \odot y_1)) = (f(x_1 \odot y_1), 0) \\ &= (f(x_1) \odot f(y_1), 0) = (f(x_1), 0) \oplus_{mv} (f(y_1), 0) = \widehat{f}(x) \oplus_{mv} \widehat{f}(y). \end{aligned}$$

From $(a)$, $(b)$ and $(c)$, $\widehat{f}$ is an onto homomorphism. □

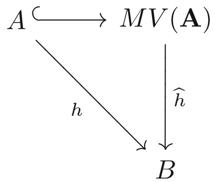
Thus, the following theorem is immediate.

**Theorem 3.5.** *If* $\mathbf{A}$ *is a Wajsberg hoop, then* $\mathbf{MV}(\mathbf{A})$ *is an MV-algebra. Moreover,* $A$ *is a maximal filter of* $\mathbf{MV}(\mathbf{A})$ *and* $\mathbf{MV}(\mathbf{A})/A \cong \mathbf{Ł}_1$.

We will see that this construction has a universal property. In fact, we will show that $\mathbf{MV}(\mathbf{A})$ is the MV-algebra freely generated over $\mathbf{A}$.

A subset $S$ of an MV-algebra satisfies the *finite product property* (*fpp* for short), provided 0 cannot be obtained with finite products of elements of $S$, that is, the filter generated by $S$ is proper.

**Theorem 3.6.** *Let* $\mathbf{A}$ *be a Wajsberg hoop and* $\mathbf{B}$ *an MV-algebra. If* $h\colon \mathbf{A} \to \mathbf{B}$ *is a hoop-homomorphism, there is a unique MV-homomorphism* $\widehat{h}\colon \mathbf{MV}(\mathbf{A}) \to \mathbf{B}$ *such that* $\widehat{h}\restriction_A = h$.




*In addition, if* $h$ *is injective and* $h[A]$ *has the* fpp *in* $\mathbf{B}$*, then* $\widehat{h}$ *is also injective.*

*Proof.* For each $a \in A$ we define $\widehat{h}(a,1) = h(a)$ and $\widehat{h}(a,0) = \neg h(a)$.

Let us see that $\widehat{h}(\neg_{mv}(a,i)) = \neg\widehat{h}(a,i)$ for $a \in A$ and $i = 0,1$. If $i = 1$, then $\widehat{h}(\neg_{mv}(a,1)) = \widehat{h}(a,0) = \neg h(a) = \neg\widehat{h}(a,1)$. If $i = 0$, then $\widehat{h}(\neg_{mv}(a,0)) = \widehat{h}(a,1) = h(a) = \neg\neg h(a) = \neg\widehat{h}(a,0)$.

Now we prove that $\widehat{h}((a,i) \oplus_{mv} (b,j)) = \widehat{h}(a,i) \oplus \widehat{h}(b,j)$ for $a,b \in A$ and $i,j = 0,1$. If $i = j = 1$, then $\widehat{h}((a,1) \oplus_{mv} (b,1)) =$

$$\widehat{h}(a \oplus b, 1) = h(a \oplus b) = h(a) \oplus h(b) = \widehat{h}(a,1) \oplus \widehat{h}(b,1).$$

If $i = 1$, $j = 0$, we have $\widehat{h}((a,1) \oplus_{mv} (b,0)) =$

$$\widehat{h}(b \to a, 1) = h(b \to a) = h(b) \to h(a) = h(a) \oplus \neg h(b) = \widehat{h}(a,1) \oplus \widehat{h}(b,0).$$

The case $i = 0$, $j = 1$ is analogous to the previous one. Finally, if $i = j = 0$, we get $\widehat{h}((a,0) \oplus_{mv} (b,0)) = \widehat{h}(a \odot b, 0) =$

$$\neg h(a \odot b) = \neg(h(a) \odot h(b)) = \neg h(a) \oplus \neg h(b) = \widehat{h}(a,0) \oplus \widehat{h}(b,0).$$

We also have $\widehat{h}(1,0) = \neg h(1) = \neg 1 = 0$. Thus $\widehat{h}$ is an MV-homomorphism such that $\widehat{h}\restriction_A = h$.

Now assume $h$ is injective and $h[A]$ has the *fpp* in $\mathbf{B}$. Let $(a,i) \in MV(\mathbf{A})$ such that $\widehat{h}(a,i) = 1$. If $i = 1$, then $h(a) = 1$ and, by injectivity of $h$, $a = 1$. If $i = 0$, then $\neg h(a) = 1$, so $h(a) = 0$, which is impossible by the *fpp*. This shows that $\widehat{h}$ is also injective. □

In the last part of the theorem, note that since $h[A]$ is closed under $\odot$, $h[A]$ has the *fpp* if and only if $0 \notin h[A]$.

**Corollary 3.7.** *Let* $h : \mathbf{A}_1 \to \mathbf{A}_2$ *be a homomorphism of Wajsberg hoops; then there is a unique MV-homomorphism* $\widehat{h} : \mathbf{MV}(\mathbf{A}_1) \to \mathbf{MV}(\mathbf{A}_2)$ *such that* $\widehat{h}\restriction_{A_1} = h$*. Moreover, if* $h$ *is injective, surjective, or bijective, so is* $\widehat{h}$*.*

We are now able to define a functor $\mathbb{M}$ from the category $\mathfrak{WH}$ of Wajsberg hoops into the category $\mathfrak{MV}$ of MV-algebras. Given a Wajsberg hoop $\mathbf{A}$, let $\mathbb{M}(\mathbf{A}) = \mathbf{MV}(\mathbf{A})$, and given a hoop-homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, let $\mathbb{M}(h)\colon \mathbf{MV}(\mathbf{A}) \to \mathbf{MV}(\mathbf{B})$ be the corresponding MV-homomorphism $\widehat{h}$ from the last corollary. The functoriality of $\mathbb{M}$ may be readily checked.

We claim that $\mathbb{M}$ is left adjoint to the forgetful functor $\mathbb{U}\colon \mathfrak{MV} \to \mathfrak{WH}$. Indeed, for every Wajsberg hoop $\mathbf{A}$ and MV-algebra $\mathbf{B}$, there is a map

$$\eta_{\mathbf{A},\mathbf{B}}\colon \mathrm{Hom}_{\mathfrak{MV}}(\mathbb{M}(\mathbf{A}), \mathbf{B}) \to \mathrm{Hom}_{\mathfrak{WH}}(\mathbf{A}, \mathbb{U}(\mathbf{B}))$$

given by $\eta_{\mathbf{A},\mathbf{B}}(h) = h\restriction_A$. By Theorem 3.6, it follows easily that $\eta_{\mathbf{A},\mathbf{B}}$ is a bijection. The naturality in $\mathbf{A}$ and $\mathbf{B}$ is a straightforward computation.

This shows that, given a Wajsberg hoop $\mathbf{A}$, the MV-algebra $\mathbf{MV}(\mathbf{A})$ is precisely the MV-algebra freely generated by $\mathbf{A}$ and also guarantees the uniqueness of this construction up to isomorphism. We thus call $\mathbf{MV}(\mathbf{A})$ the *MV-closure* of $\mathbf{A}$.

We now proceed to derive some useful properties of the MV-closure.

For any bounded Wajsberg hoop $\mathbf{A} = \langle A; \odot, \to, 1\rangle$ we let $\mathbf{A}_0 = \langle A; \oplus, \neg, 0\rangle$ be the corresponding MV-algebra with least element 0.

**Lemma 3.8.** *If* $\mathbf{A}$ *is a bounded Wajsberg hoop, then* $\mathbf{MV}(\mathbf{A}) \cong \mathbf{Ł}_1 \times \mathbf{A}_0$*.*

*Proof.* Note that $\{1\}\times A$ is a maximal filter of $\mathbf{Ł}_1\times\mathbf{A}_0$ and $(\mathbf{Ł}_1\times\mathbf{A}_0)/(\{1\} \times A) \cong \mathbf{Ł}_1$. Hence $\mathbf{MV}(\mathbf{1} \times \mathbf{A}) \cong \mathbf{Ł}_1 \times \mathbf{A}_0$, where $\mathbf{1}$ is the trivial Wajsberg hoop. Hence $\mathbf{MV}(\mathbf{A}) \cong \mathbf{Ł}_1 \times \mathbf{A}_0$. □

**Corollary 3.9.** *If* $\mathbf{A}$ *is a finite Wajsberg hoop, then* $\mathbf{MV}(\mathbf{A}) \cong \mathbf{Ł}_1 \times \mathbf{A}_0$.

If $\mathbf{A}$ is a Wajsberg hoop or an MV-algebra, we denote by $Spec(\mathbf{A})$ the set of maximal filters in $\mathbf{A}$.

**Lemma 3.10.** *Let* $\mathbf{A}$ *be a bounded Wajsberg hoop. If* $M$ *is a maximal filter of* $\mathbf{A}$*, then there is a maximal filter* $M'$ *of* $\mathbf{MV}(\mathbf{A})$ *such that* $M' \cap A = M$*. Moreover the correspondence* $M \mapsto M\cap A$ *gives a bijection from* $Spec(\mathbf{MV}(\mathbf{A}))\backslash\{A\}$ *onto* $Spec(\mathbf{A})$*.*

*Proof.* It is a consequence of Lemma 3.8, since $\mathbf{MV}(\mathbf{A}) \cong \mathbf{Ł}_1 \times \mathbf{A}_0$ and the maximal filters of $\mathbf{Ł}_1 \times \mathbf{A}_0$ are $\{1\} \times A$ and $Ł_1 \times U$, where $U$ is a maximal filter of $\mathbf{A}$. □

A proper filter $P$ in a Wajsberg hoop or MV-algebra $\mathbf{A}$ is prime if for $a, b \in A$, either $a \to b \in P$ or $b \to a \in P$. Clearly $P$ is prime in $\mathbf{A}$ if and only if $\mathbf{A}/P$ is a chain. Observe that this is equivalent to the condition: if $a \vee b \in P$ then either $a \in P$ or $b \in P$.

**Lemma 3.11.** *Let $P$ be a prime filter in a Wajsberg hoop $\mathbf{A}$. Then $\mathbf{A}/P$ is bounded if and only if $P$ is not prime in $\mathbf{MV}(\mathbf{A})$.*

*Proof.* For every $a \in A$, let $[a]_P$ be the class of $a$ in $\mathbf{A}/P$. Observe that the class of $a$ in $\mathbf{MV}(\mathbf{A})/P$ coincides with $[a]_P$. Indeed, $[a]_P$ does not contain elements of the form $\neg c$, $c \in A$: if $\neg c$ is in the class of $a$ in $\mathbf{MV}(\mathbf{A})/P$, then $a \to \neg c \in P \subseteq A$. But $a \to \neg c = \neg(a \odot c)$, which belongs to $\neg A$. Therefore, $\mathbf{MV}(\mathbf{A})/P \cong \mathbf{MV}(\mathbf{A}/P)$.

If the chain $\mathbf{A}/P$ is not bounded, then $\mathbf{A}/P$ is cancellative (see [1, Proposition 2.1]) and $\mathbf{MV}(\mathbf{A}/P) \cong \mathbf{MV}(\mathbf{A})/P$ is a chain, so $P$ is prime in $\mathbf{MV}(\mathbf{A})$.

Conversely, if $\mathbf{A}/P$ is bounded, $\mathbf{MV}(\mathbf{A})/P \cong \mathbf{MV}(\mathbf{A}/P) \cong \mathbf{Ł}_1 \times \mathbf{A}_0/P$, which is not a chain. So $P$ is not prime in $\mathbf{MV}(\mathbf{A})$. □

Let $Prim(\mathbf{A})$ denote the set of prime filters in a Wajsberg hoop or an MV-algebra $\mathbf{A}$.

**Theorem 3.12.** *Let $\mathbf{A}$ be a Wajsberg hoop and $P$ a prime filter of $\mathbf{A}$. Then there exists a prime filter $P'$ of $\mathbf{MV}(\mathbf{A})$ such that $P' \cap A = P$. Moreover the correspondence $P' \mapsto \varphi(P') = P' \cap A$ gives a bijection from the set $Prim(\mathbf{MV}(\mathbf{A})) \setminus \{A\}$ onto $Prim(\mathbf{A})$.*

*Proof.* Let us see first that $\varphi$ is onto. Let $P$ be a prime filter in $\mathbf{A}$. If $P$ is prime in $\mathbf{MV}(\mathbf{A})$, then it suffices to consider $P' = P$. If $P$ is not prime in $\mathbf{MV}(\mathbf{A})$, then by Lemma 3.11, $\mathbf{A}/P$ is bounded and there is an isomorphism $f\colon \mathbf{MV}(\mathbf{A})/P \to \mathbf{Ł}_1 \times (\mathbf{A}/P)_0$. Let $P' = \pi_P^{-1}(f^{-1}(\{(0,1),(1,1)\}))$, where $\pi_P\colon \mathbf{MV}(\mathbf{A}) \to \mathbf{MV}(\mathbf{A})/P$ is the canonical epimorphism. $P'$ is a prime filter of $\mathbf{MV}(\mathbf{A})$. Moreover, $P' = P \cup \neg C$ where $C$ is the least element of $\mathbf{A}/P$. This shows that $P \subseteq P'$ and $\varphi(P') = P' \cap A = P$, as was to be proved.

Let us prove that $\varphi$ is injective. Let $P$ a prime filter in $\mathbf{A}$ such that $P$ is prime in $\mathbf{MV}(\mathbf{A})$. We have that $P \subseteq A$ and $A$ is a maximal filter in $\mathbf{MV}(\mathbf{A})$. But in an MV-algebra, each prime filter is contained in a unique maximal filter. So if $Q$ is a prime filter such that $P \subseteq Q$, then $Q \subseteq A$. If $\varphi(Q) = P$, then $Q = P$. Suppose now that $P$ in not prime in $\mathbf{MV}(\mathbf{A})$. Let $Q$ be a prime filter in $\mathbf{MV}(\mathbf{A})$ such that $P \subseteq Q$ and $Q \cap A = P$. Let $P' = \pi_P^{-1}(f^{-1}(\{(0,1),(1,1)\}))$. It is easy to see that the only prime filter in $\mathbf{MV}(\mathbf{A})/P \cong \mathbf{Ł}_1 \times \mathbf{A}_0/P$ which does not contain $\{(0,1),(1,1)\}$ is $\{1\} \times A/P$. Since $Q \neq A$, $f[\pi_P[Q]] \neq \{1\} \times A/P$, so $P' \subseteq Q$. If $Q \neq P'$, there is an element $q \in Q \setminus P'$. Since $q \notin P'$, $f([q]_P) \notin \{(0,1),(1,1)\}$. Then $f([q]_P) = (r,[a]_P)$, where $a \in A$, $a \notin P$. Since $(r,[a]_P) \leq (1,[a]_P)$, then $(1,[a]_P) \in f[\pi_P[Q]]$. But then $a \in Q \cap A$, contradicting the fact that $Q \cap A = P$. Therefore, $Q = P'$. □

## 4. Applications

**4.1. Topological duality for locally finite Wajsberg hoops.** In [7] the authors give a dual equivalence between the category of locally finite MV-algebras and a certain category of multisets. We will see that the construction given in the previous section allows us to derive a topological duality for locally finite Wajsberg hoops by making minor changes to the corresponding duality for MV-algebras.

We start by reviewing the duality given in [7].

A *supernatural number* is a function $\nu\colon P \to \{0, 1, 2, \ldots, \infty\}$, where $P$ stands for the set of positive prime numbers. Given two supernatural numbers $\nu, \mu$, we write $\nu \leq \mu$ if $\nu(p) \leq \mu(p)$ for each $p \in P$. It is clear that $\leq$ is a partial order on the set $G$ of supernatural numbers.

We associate with each natural number $n$ a corresponding supernatural number $\nu_n$ such that $\nu_n(p)$ is the exponent of the prime $p$ in the decomposition of $n$ as a product of prime powers. Thus, if we identify $n$ with $\nu_n$, it is clear that $G$ is a generalization of the set of natural numbers and that the partial order defined on $G$ is an extension of the divisibility relation.

We also define a topology on $G$ whose basis consists of the set $G$ itself together with the sets $U_n = \{\nu \in G : \nu > \nu_n\}$ for each natural number $n$.

We are now ready to define the category $\mathfrak{C}$ of multisets. An object in $\mathfrak{C}$ or *multiset* is a pair $\langle X, \sigma\rangle$, where $X$ is a Stone space and $\sigma\colon X \to G$ is a continuous mapping from $X$ into the space of supernatural numbers. A morphism $\varphi\colon \langle X, \sigma\rangle \to \langle Y, \rho\rangle$ is given by a continuous mapping $\varphi\colon X \to Y$ such that $\rho(\varphi(x)) \leq \sigma(x)$ for every $x \in X$.

In [7], the authors find a dual equivalence between the category $\mathfrak{C}$ and the category $\mathfrak{MV}_{lf}$ of locally finite MV-algebras. By virtue of the construction of the MV-closure of a Wajsberg hoop given in the previous section, we may now modify the previous duality to obtain a corresponding duality for the category of locally finite Wajsberg hoops.

We define the category $\mathfrak{C}^*$ whose objects are triples $\langle X, \sigma, x_0\rangle$, where $\langle X, \sigma\rangle$ is a multiset, $x_0 \in X$, and $\sigma(x_0) = \nu_1$. A morphism $\varphi\colon \langle X, \sigma, x_0\rangle \to \langle Y, \rho, y_0\rangle$ is given by a morphism $\varphi\colon \langle X, \sigma\rangle \to \langle Y, \rho\rangle$ in the category $\mathfrak{C}$ such that $\varphi(x_0) = y_0$.

We also define an auxiliary category $\mathfrak{MV}^*_{lf}$ whose objects are pairs $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is a locally finite MV-algebra and $F$ is a filter of $\mathbf{A}$ such that $\mathbf{A}/F \cong \mathbf{Ł}_1$. In this category, a morphism $h\colon \langle \mathbf{A}, F\rangle \to \langle \mathbf{B}, G\rangle$ is an MV-homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ such that $h(F) \subseteq G$.

Following the work in [7], it may be easily shown that the same dual equivalence between the categories $\mathfrak{C}$ and $\mathfrak{MV}_{lf}$ defines a dual equivalence between the categories $\mathfrak{C}^*$ and $\mathfrak{MV}^*_{lf}$. Hence, it only remains to show that $\mathfrak{MV}^*_{lf}$ is equivalent to $\mathfrak{WH}_{lf}$, the category of locally finite Wajsberg hoops.

Let $\mathbb{F}\colon \mathfrak{MV}^*_{lf} \to \mathfrak{WH}_{lf}$ be the functor such that $\mathbb{F}(\langle \mathbf{A}, F\rangle)$ is the hoop $\mathbf{F}$ and given a morphism $h\colon \langle \mathbf{A}, F\rangle \to \langle \mathbf{B}, G\rangle$ the morphism $\mathbb{F}(h)\colon \mathbf{F} \to \mathbf{G}$ is simply the restriction of $h$. .

Conversely, for any locally finite Wajsberg hoop $\mathbf{L}$, let $\mathbb{G}\colon \mathfrak{WH}_{lf} \to \mathfrak{MV}^*_{lf}$ be the functor given by $\mathbb{G}(\mathbf{L}) = \langle \mathbf{MV}(\mathbf{L}), L\rangle$, and such that $\mathbb{G}(f) = \widehat{f}$ for any hoop-homomorphism $f$ (see Corollary 3.7). The local finiteness of $\mathbf{MV}(\mathbf{L})$ is easily derived from the local finiteness of $\mathbf{L}$.

It is now very simple to show that the functors $\mathbb{F}$ and $\mathbb{G}$ define an equivalence between the categories $\mathfrak{WH}_{lf}$ and $\mathfrak{MV}^*_{lf}$. We have thus proved the following result.

**Theorem 4.1.** *The categories $\mathfrak{WH}_{lf}$ and $\mathfrak{C}^*$ are dually equivalent.*

**4.2. Another topological duality for $k$-valued Wajsberg hoops.** It is easy to derive a topological duality for $k$-valued MV-algebras as a restriction of the duality given in the previous section for locally finite MV-algebras. In fact, as stated in [7], it is immediate that the category $\mathfrak{MV}_k$ of MV$_k$-algebras is dually equivalent to the full subcategory of $\mathfrak{C}$ whose objects are multisets $\langle X, \sigma\rangle$ such that $\sigma(x) \leq \nu_k$ for every $x \in X$. We will show that this duality is essentially the same as the one that may be obtained from the representation of MV$_k$-algebras given in [8]. We start by reviewing this representation and extending it to a categorial duality.

Given an integer $k \geq 1$, by $Div(k)$ we will denote the set of positive divisors of $k$, and by $Div^*(k)$ the proper positive divisors of $k$, that is, $Div^*(k) = \{n \in Div(k) : n < k\}$. The set $Div(k)$ has a natural structure of distributive lattice under the divisibility order, and $Div^*(k)$ has a meet semilattice structure under the inherited order.

Given $k \geq 1$, a *$k$-valued Boolean space* is a pair $\langle X, \rho\rangle$, such that $X$ is a Boolean space and $\rho$ is a meet-homomorphism from the lattice of positive divisors of $k$ into the lattice of closed subsets of $X$, such that $\rho(k) = X$.

If the set $Ł_k$ is equipped with the discrete topology, and $\langle X, \rho\rangle$ is a $k$-valued Boolean space, then $\mathbf{C}_k(X, \rho)$ denotes the MV$_k$-algebra formed by the continuous functions $f$ from $X$ into $Ł_k$ such that $f(\rho(d)) \subseteq Ł_d$ for $d \in Div^*(k)$, with the algebraic operations defined point-wise. Clearly, for a given clopen $N$ and characteristic function $\gamma_N$, the correspondence $N \mapsto \gamma_N$ defines an isomorphism from $Clop(X)$ onto $B(\mathbf{C}_k(X, \rho))$, the set of Boolean elements of $\mathbf{C}_k(X, \rho)$.

Given two $k$-valued Boolean spaces $\langle X_1, \rho_1\rangle$ and $\langle X_2, \rho_2\rangle$, we say that a function $h\colon \langle X_1, \rho_1\rangle \to \langle X_2, \rho_2\rangle$ is a *$k$-valued Stone function* if $h$ is continuous and satisfies the condition $h(\rho_1(d)) \subseteq \rho_2(d)$ for $d \in Div(k)$.

Let $\mathfrak{X}_k$ be the category whose objects are $k$-valued Boolean spaces and whose morphisms are $k$-valued Stone functions, and let $\mathfrak{MV}_k$ be the category whose objects are $k$-valued MV-algebras with homomorphisms. We will now extend the topological representation for MV$_k$-algebras given in [8] to a topological duality between the categories $\mathfrak{X}_k$ and $\mathfrak{MV}_k$.

For an MV$_k$-algebra $\mathbf{A}$, let $X(\mathbf{A})$ be the set of homomorphisms $\chi\colon \mathbf{A} \to \mathbf{Ł}_k$. If we consider the set $Ł_k$ endowed with the discrete topology and $Ł_k^A$ with the corresponding product topology, then $X(\mathbf{A}) \subseteq Ł_k^A$ inherits the topology of $Ł_k^A$. The sets $W_{a,i} = \{\chi \in X(\mathbf{A}) : \chi(a) = \frac{i}{k}\}$ for $a \in A$ and $0 \leq i \leq k$, form a subbasis for the topology on $X(\mathbf{A})$. It is known that $X(\mathbf{A})$ is homeomorphic to the Stone space of $B(\mathbf{A})$. If we define $\rho(d) = \{\chi \in X(\mathbf{A}) : \chi(A) \subseteq Ł_d\}$ for $d \in Div(k)$, it is clear that $\mathbb{X}_k(\mathbf{A}) = \langle X(\mathbf{A}), \rho\rangle$ is a $k$-valued Boolean space. In addition, for every homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$, let $\mathbb{X}_k(h)\colon \langle X(\mathbf{A}_2), \rho_{\mathbf{A}_2}\rangle \to \langle X(\mathbf{A}_1), \rho_{\mathbf{A}_1}\rangle$ be given by $\mathbb{X}_k(h)(\chi) = \chi \circ h$ for every $\chi \in X(\mathbf{A}_2)$. It may be proved that $\mathbb{X}_k$ is a contravariant functor from $\mathfrak{MV}_k$ into $\mathfrak{X}_k$.

Conversely, for each $k$-valued Boolean space $\langle X, \rho\rangle$, we define $\mathbb{C}_k(\langle X, \rho\rangle) = \mathbf{C}_k(X, \rho)$ and for each $k$-valued Stone function $h\colon \langle X_1, \rho_1\rangle \to \langle X_2, \rho_2\rangle$, we define $\mathbb{C}_k(h)\colon \mathbf{C}_k(X_2, \rho_2) \to \mathbf{C}_k(X_1, \rho_1)$ by $\mathbb{C}_k(h)(f) = f \circ h$ for every $f \in C_k(X_2, \rho_2)$. Then it is easily shown that $\mathbb{C}_k$ is a contravariant functor from $\mathfrak{X}_k$ into $\mathfrak{MV}_k$.

The following lemma is proved in [8, Theorem 1.5].

**Lemma 4.2.** *Given an* $\mathrm{MV}_k$*-algebra* $\mathbf{A}$*, there exists an isomorphism* $\alpha_A\colon \mathbf{A} \to \mathbb{C}_k(\mathbb{X}_k(\mathbf{A}))$ *given by* $\alpha_A(a)(\chi) = \chi(a)$ *for every* $\chi \in X(\mathbf{A})$ *and* $a \in A$.

We will now derive a similar result for $k$-valued Boolean spaces. Note that we refer to isomorphisms in the category $\mathfrak{X}_k$ as *$k$-valued homeomorphisms*.

Recall that it is possible to define one-variable terms $\sigma_i^k(x)$ for $i = 1, \ldots, k$ in the language of MV-algebras such that

$$\sigma_i^k\left(\frac{j}{k}\right) = \begin{cases} 1 & \text{if } i + j > k, \\ 0 & \text{otherwise,} \end{cases}$$

for every $\frac{j}{k} \in Ł_k$. These terms are called *Moisil terms* [8] and satisfy several properties collected in the following lemma.

**Lemma 4.3.** *For any* $\mathrm{MV}_k$*-algebra* $\mathbf{A}$ *and any* $a, b \in A$*:*

(*a*) $a \in B(\mathbf{A})$ *if and only if* $a = \sigma_i^k(a)$ *for some* $1 \le i \le k$ *if and only if* $a = \sigma_i^k(a)$ *for all* $1 \le i \le k$*,*
(*b*) $\sigma_i^k$ *is a lattice homomorphism from* $\mathbf{A}$ *into* $B(\mathbf{A})$*, for each* $1 \le i \le k$*,*
(*c*) $\sigma_i^k(\sigma_j^k(a)) = \sigma_j^k(a)$ *for all* $1 \le i, j \le k$*,*
(*d*) $\sigma_1^k(a) \le \sigma_2^k(a) \le \cdots \le \sigma_k^k(a)$*,*
(*e*) *if* $\sigma_i^k(a) = \sigma_i^k(b)$ *for all* $1 \le i \le k$*, then* $a = b$*.*

We can now state and prove the result analogous to Lemma 4.2 that we promised above.

**Lemma 4.4.** *Given a $k$-valued Boolean space* $\langle X, \rho\rangle$*, there exists a $k$-valued homeomorphism* $\delta_X\colon \langle X, \rho\rangle \to \mathbb{X}_k(\mathbb{C}_k(X, \rho))$ *given by* $\delta_X(x)(f) = f(x)$ *with every* $f \in C_k(X, \rho)$ *and* $x \in X$.

*Proof.* It is trivially verified that $\delta_X$ is a well-defined mapping from $X$ into $X(C_k(X, \rho))$.

Let $x_1, x_2 \in X$ for $x_1 \neq x_2$. Since $X$ is a totally disconnected topological space, there exists a clopen $N$ in $X$ such that $x_1 \in N$ but $x_2 \notin N$. Consider the characteristic function $\gamma_N \in C_k(X, \rho)$. Then $\delta_X(x_1)(\gamma_N) = \gamma_N(x_1) = 1$ but $\delta_X(x_2)(\gamma_N) = \gamma_N(x_2) = 0$. This shows that $\delta_X$ is injective.

Consider $h \in X(C_k(X,\rho))$, that is, $h\colon \mathbf{C}_k(X,\rho) \to \mathbf{Ł}_k$ is a homomorphism. We will show that there exists $x \in X$ such that $\delta_X(x) = h$. This is equivalent to showing that there exists $x \in X$ such that $h(f) = f(x)$ for every $f \in C_k(X,\rho)$. First suppose that for every $x \in X$ there is a clopen $N_x$ in $X$ such that $x \in N_x$ and $h(\gamma_{N_x}) = 0$. By compactness, there exist $x_1,\dots,x_n \in X$ such that $X = N_{x_1} \cup \dots \cup N_{x_n}$. Therefore,

$$1 = h(\gamma_X) = h(\gamma_{N_{x_1}} \oplus \dots \oplus \gamma_{N_{x_n}}) = h(\gamma_{N_{x_1}}) \oplus \dots \oplus h(\gamma_{N_{x_n}}) = 0,$$

a contradiction. This shows that there exists $x^* \in X$ such that for every clopen $N$ with $x^* \in N$, $h(\gamma_N) = 1$ (recall that $h(\gamma_N)$ must be a Boolean element in $\mathbf{Ł}_n$ since $\gamma_N$ is a Boolean element in $\mathbf{C}_k(X,\rho)$). Note also that if $N$ is clopen and $x^* \notin N$, then $X \setminus N$ is also clopen and $x^* \in X \setminus N$, so $h(\gamma_{X\setminus N}) = 1$ and $h(\gamma_N) = 0$. This proves that $h(\gamma_N) = \gamma_N(x^*)$ for every clopen $N$ in $X$. Since $B(\mathbf{C}_k(X,\rho)) = \{\gamma_N : N \in Clop(X)\}$, we can rephrase the last statement as: $h(f) = f(x^*)$ for every $f \in B(\mathbf{C}_k(X,\rho))$. Now consider an arbitrary element $f \in C_k(X,\rho)$. For $1 \le i \le k$ we have

$$\sigma_i^k(h(f)) = h(\sigma_i^k(f)) = \sigma_i^k(f)(x^*) = \sigma_i^k(f(x^*)).$$

Hence, $h(f) = f(x^*)$, as was to be proved.

We now prove that $\delta_X$ is continuous. Since a subbasis for the topology on $\mathbb{X}_k(\mathbb{C}_k(X,\rho))$ is given by $W_{f,i} = \{h \in X(C_k(X,\rho)) : h(f) = \frac{i}{k}\}$ for $f \in C_k(X,\rho)$ and $0 \le i \le k$, we only need to show that $\delta_X^{-1}(W_{f,i})$ is open in $X$. But $\delta_X^{-1}(W_{f,i}) = f^{-1}(\frac{i}{k})$ which is indeed open in $X$.

We have thus shown that $\delta_X$ is a continuous bijection. Since the spaces considered are compact and Hausdorff, it follows immediately that $\delta_X$ is a homeomorphism.

Let $\rho'$ be the meet-homomorphism from $Div(k)$ into the lattice of closed sets in $\mathbb{X}_k(\mathbb{C}_k(X,\rho))$; $\rho'$ is given by $\rho'(d) = \{\chi \in X(C_k(X,\rho)) : \chi(C_k(X,\rho)) \subseteq Ł_d\}$ for $d \in Div(k)$.

First we show that $\delta_X(\rho(d)) \subseteq \rho'(d)$ for every $d \in Div^*(k)$. In fact, if $x \in \rho(d)$, then $\delta_X(x)(f) = f(x) \in Ł_d$ for every $f \in C_k(X,\rho)$. So $\delta_X(x)(C_k(X,\rho)) \subseteq Ł_d$ and $\delta_X(x) \in \rho'(d)$.

We also have to prove that $\delta_X^{-1}(\rho'(d)) \subseteq \rho(d)$ for every $d \in Div^*(k)$. To show that, let $x \notin \rho(d)$. Let $m$ be the greatest common divisor of the set $\{n \in Div(k) : x \in \rho(n)\}$. Then $\rho(m) = \bigcap\{\rho(n) : x \in \rho(n), n \in Div(k)\}$, so $x \in \rho(m)$. Note that for every $n \in Div(k)$ we have $x \in \rho(n)$ iff $m$ divides $n$. Let $U = \bigcap\{X \setminus \rho(n) : m \text{ does not divide } n, n \in Div(k)\}$. Then $U$ is open and $x \in U$. Since $X$ has a basis of clopen sets, there is a clopen $N$ such that $x \in N$ and $N \subseteq U$. Consider $f\colon X \to Ł_k$ given by

$$f(z) = \begin{cases} \frac{1}{m} & \text{if } z \in N, \\ 0 & \text{if } z \notin N. \end{cases}$$

Since $N$ is clopen, $f$ is continuous. Let us see that $f(\rho(n)) \subseteq Ł_n$ for every $n \in Div^*(k)$. Indeed, if $m$ divides $n$, then $Ł_m \subseteq Ł_n$, so $f(\rho(n)) \subseteq \{0, \frac{1}{m}\} \subseteq Ł_n$. If $m$ does not divide $n$, then $U \subseteq X \setminus \rho(n)$, so $N \subseteq X \setminus \rho(n)$, and thus

$\rho(n) \subseteq X \backslash N$. In this case, $f(\rho(n)) = \{0\} \subseteq \text{Ł}_n$. This shows that $f \in C_k(X, \rho)$. Now since $x \notin \rho(d)$, $m$ does not divide $d$ and $\delta_X(x)(f) = f(x) = \frac{1}{m} \notin \text{Ł}_d$. Hence $\delta_X(x) \notin \rho'(d)$, so $x \notin \delta_X^{-1}(\rho'(d))$. □

We can now derive the following theorem.

**Theorem 4.5.** *The contravariant functors $\mathbb{C}_k$ and $\mathbb{X}_k$ define a dual equivalence between the categories $\mathfrak{X}_k$ and $\mathfrak{MV}_k$.*

Based on this result and using the MV-closure of a Wajsberg hoop, we can now derive a topological duality for the category $\mathfrak{WH}_k$ of $k$-valued Wajsberg hoops. To this end, we define the category $\mathfrak{X}_k^*$ whose objects are triples $\langle X, \rho, u\rangle$, where $\langle X, \rho\rangle$ is a $k$-valued Boolean space and $u$ is a fixed element in $X$ such that $u \in \rho(1)$. A morphism in $\mathfrak{X}_k^*$ between the objects $\langle X_1, \rho_1, u_1\rangle$ and $\langle X_2, \rho_2, u_2\rangle$ is simply a $k$-valued Stone function $f\colon \langle X_1, \rho_1\rangle \to \langle X_2, \rho_2\rangle$ such that $f(u_1) = u_2$. Extending the duality given in Theorem 4.5, we may derive, in a natural way, a dual equivalence between the category $\mathfrak{X}_k^*$ and the category $\mathfrak{MV}_k^*$, i.e., the category whose objects are pairs $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is a $k$-valued MV-algebra and $F$ is a maximal filter of $\mathbf{A}$ such that $\mathbf{A}/F \cong \mathbf{Ł}_1$, and whose morphisms $h\colon \langle \mathbf{A}_1, F_1\rangle \to \langle \mathbf{A}_2, F_2\rangle$ are homomorphisms $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ such that $h(F_1) \subseteq F_2$. Again, as in the previous section, the category $\mathfrak{MV}_k^*$ is easily seen to be equivalent to the category $\mathfrak{WH}_k$ of $k$-valued Wajsberg hoops and homomorphisms. Composing both equivalences, we get a dual equivalence between the categories $\mathfrak{X}_k^*$ and $\mathfrak{WH}_k$. We state this result as a separate theorem.

**Theorem 4.6.** *The categories $\mathfrak{WH}_k$ and $\mathfrak{X}_k^*$ are dually equivalent.*

Now, let $\mathfrak{C}_k$ denote the full subcategory of $\mathfrak{C}$ whose objects are multisets $\langle X, \sigma\rangle$ such that $\sigma(x) \leq \nu_k$ for every $x \in X$. Since both categories $\mathfrak{C}_k$ and $\mathfrak{X}_k$ are dually equivalent to $\mathfrak{MV}_k$, it follows immediately that $\mathfrak{C}_k$ and $\mathfrak{X}_k$ are equivalent. Analogously, the categories $\mathfrak{C}_k^*$ and $\mathfrak{X}_k^*$ must also be equivalent, being both dually equivalent to $\mathfrak{WH}_k$.

For the sake of completeness, we will now describe explicitly the equivalence between $\mathfrak{C}_k$ and $\mathfrak{X}_k$, leaving the corresponding equivalence between $\mathfrak{C}_k^*$ and $\mathfrak{X}_k^*$ as an immediate corollary.

We will define a pair of functors $\mathbb{A}\colon \mathfrak{C}_k \to \mathfrak{X}_k$ and $\mathbb{B}\colon \mathfrak{X}_k \to \mathfrak{C}_k$ that define the equivalence mentioned. The proof that these functors define indeed an equivalence is straightforward.

For every object $\langle X, \sigma\rangle$ in $\mathfrak{C}_k$, let $\mathbb{A}(X, \sigma) = \langle X, \rho\rangle$, where $\rho$ is the function from $Div(k)$ into the lattice of closed subsets of $X$ given by

$$\rho(n) = \{x \in X : \sigma(x) \leq \nu_n\}.$$

If $f\colon \langle X, \rho\rangle \to \langle X', \rho'\rangle$ is a morphism in $\mathfrak{C}_k$, it is easy to show that $f$ itself is also a morphism in $\mathfrak{X}_k$. Hence we define $\mathbb{A}(f) = f$.

Conversely, given an object $\langle X, \rho\rangle$ in $\mathfrak{X}_k$, we define $\mathbb{B}(X, \rho) = \langle X, \sigma\rangle$, where, for every $x \in X$, $\sigma(x)$ is the supernatural number corresponding to the greatest common divisor of the set $\{n \in Div(k) : x \in \rho(n)\}$. For each morphism $f\colon \langle X, \rho\rangle \to \langle X', \rho'\rangle$ in $\mathfrak{X}_k$, we define $\mathbb{B}(f) = f$, since it may be easily shown that $f$ itself is a morphism in $\mathfrak{C}_k$.

This completes the definition of the equivalence between $\mathfrak{C}_k$ and $\mathfrak{X}_k$.

**4.3. Free $k$-valued Wajsberg hoops.** As another application of the MV-closure for a Wajsberg hoop $\mathbf{A}$, we are going to characterize $\mathbf{Free}_{\mathbb{WH}_k}(X)$, the free algebra in the variety $\mathbb{WH}_k$ over a set $X$. For finite $X$, we get the expression for $\mathbf{Free}_{\mathbb{WH}_k}(X)$ as a product of algebras $\mathbf{Ł}_d$, extending a result of A. Monteiro.

Recall that a Boolean algebra $\mathbf{B}$ is said to be *free over a poset* $Y \subseteq B$ if for each Boolean algebra $\mathbf{C}$ and for each non-decreasing function $f\colon Y \to C$, $f$ can be uniquely extended to a homomorphism from $\mathbf{B}$ into $\mathbf{C}$.

The next theorem is proved in [3].

**Theorem 4.7.** $\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_k}(X))$ *is the free Boolean algebra over the poset* $Y = \{\sigma_i^k(x) : x \in X, i = 1, \dots, k\}$.

The proof of [3, Lemma 3.6] can be easily adapted to give

**Lemma 4.8.** *Consider the poset* $Y = \{\sigma_i^k(x) : x \in X, i = 1, \dots, k\}$. *The correspondence that assigns to each upward closed subset* $S \subseteq Y$ *the Boolean ultrafilter* $U_S$ *generated by the set* $S \cup \{\neg y : y \in Y \setminus S\}$, *defines a bijection from the set of upward closed subsets of* $Y$ *onto the ultrafilters of* $\mathbf{B}(\mathbf{Free}_{MV_k}(X))$.

Given a set $X$ and the poset $Y = \{\sigma_i^k(x) : x \in X, i = 1, \dots, k\}$, we define chains $K_j^k(x) := \{\sigma_i^k(x) : j \le i \le k\}$ for $1 \le j \le k$ and $x \in X$. Clearly, $K_i^k(x) \cap K_j^k(y) = \emptyset$ if $x \ne y$ and $1 \le i, j \le k$. We also have $Y = \bigcup_{x \in X} K_1^k(x)$.

Let $R_k$ be the set of upward closed subsets of $Y$. Let

$$Ch_x = \{K_j^k(x) : j = 1, \dots, k\} \cup \{\emptyset\}.$$

For each $d \in Div^*(k)$ we define the set

$$H_d^x = \{C \in Ch_x : \frac{\#C}{k} \in Ł_d\}$$

where $\#C$ denotes the cardinal of the set $C$.

For each $d \in Div^*(k)$, we define $R_d \subseteq R_k$ such that $S \in R_d$ iff $S = \bigcup_{x \in X} C_x$ for $C_x \in H_d^x$.

We are now able to state the main result of [4].

**Theorem 4.9.** $\mathbf{Free}_{\mathbb{MV}_k}(X)$ *is isomorphic to the algebra of continuous functions* $f$ *from the Stone space of the free Boolean algebra over the poset* $Y = \{\sigma_i^k(x) : x \in X, 1 \le i \le k\}$ *into* $\mathbf{Ł}_k$ *such that for each* $d \in Div^*(k)$ *and each* $S \in R_d$, $f(U_S) \in Ł_d$.

We define, for each $x \in X$ and any upward closed subset $S$ of $Y$,

$$j_{x,S} = \begin{cases} 0 & \text{if } \sigma_1^k(x) \in S, \\ \max\{i \in \{1, \dots, k\} : \sigma_i^k(x) \notin S\} & \text{otherwise.} \end{cases}$$

Note that for each $x \in X$, the function $\widehat{x}\colon Spec(\mathbf{B}(\mathbf{Free}_{\mathbb{MV}_k}(X))) \to \mathbf{Ł}_k$ corresponding to $x$ by Theorem 4.9 is defined as follows:

$$\widehat{x}(U_S) = \frac{k - j_{x,S}}{k}.$$

In Section 2 we showed that $\mathbf{MV}(\mathbf{Free}_{\mathbb{WH}}(X)) = \mathbf{Free}_{\mathbb{MV}}(X)$. In the same way we obtain the following result.

**Theorem 4.10.** *For every $k \in \mathbb{N}$, $\mathbf{MV}(\mathbf{Free}_{\mathbb{WH}_k}(X)) \cong \mathbf{Free}_{\mathbb{MV}_k}(X)$. Moreover, $\mathbf{Free}_{\mathbb{WH}_k}(X) \cong \mathbf{F}(X)$ where $F(X)$ is the filter generated by $X$ within $\mathbf{Free}_{\mathbb{MV}_k}(X)$.*

Combining Theorems 4.9 and 4.10 we get:

**Theorem 4.11.** *$\mathbf{Free}_{\mathbb{WH}_k}(X)$ is isomorphic to the algebra of continuous functions $f$ from the Stone space of the free Boolean algebra over the poset $Y = \{\sigma_i^k(x) : x \in X, 1 \leq i \leq k\}$ into $\mathbf{Ł}_k$ such that $f(U_Y) = 1$ and for each $d \in Div^*(k)$ and each $S \in R_d$, we have $f(U_S) \in Ł_d$.*

*Proof.* Set $\widehat{X} = \{\widehat{x} : x \in X\}$ and let $F(\widehat{X})$ be the filter in the functional representation of $\mathbf{Free}_{\mathbb{MV}_k}(X)$ given by Theorem 4.9 corresponding to $F(X)$. Note that for every $x \in X$, $\widehat{x}(U_Y) = 1$. Hence $f(U_Y) = 1$ for every $f \in F(\widehat{X})$. Conversely, if $f \notin F(\widehat{X})$, then $f \in \neg F(\widehat{X})$, so $\neg f(U_Y) = 1$ and $f(U_Y) = 0$. This shows that $F(\widehat{X})$ is characterized precisely by the fact that $f(U_Y) = 1$. □

For finite $X$ we can obtain an even more concrete characterization for $\mathbf{Free}_{\mathbb{WH}_k}(X)$. First we state the corresponding result for $\mathbf{Free}_{\mathbb{MV}_k}(X)$ given in [4, Theorem 4.1].

**Theorem 4.12.** *If $X$ is finite,*

$$\mathbf{Free}_{\mathbb{MV}_k}(X) \cong \prod_{d \in Div(k)} \mathbf{Ł}_d^{\alpha_d},$$

*where for each $d \in Div(k)$, $\alpha_d = \#\left(R_d \setminus \bigcup_{k \in Div^*(d)} R_k\right)$.*

The characterization of finitely generated free $k$-valued Wajsberg hoops now follows immediately.

**Theorem 4.13.** *If $X$ is finite,*

$$\mathbf{Free}_{\mathbb{WH}_k}(X) \cong \mathbf{Ł}_1^{\alpha_1 - 1} \times \prod_{\substack{d \in Div(k) \\ d \neq 1}} \mathbf{Ł}_d^{\alpha_d}.$$

**Remark 4.14.** If $\#X = r \geq 1$, then $Y$ is a union of $r$ chains of length 2. Then $\#R_2 = 3^r$ and $\#R_1 = 2^r$. It follows that $\mathbf{Free}_{\mathbb{MV}_2}(X) \cong \mathbf{Ł}_1^{2^r} \times \mathbf{Ł}_2^{3^r - 2^r}$ which is the formula given by A. Monteiro in the early sixties. As a corollary we obtain $\mathbf{Free}_{\mathbb{WH}_2}(X) \cong \mathbf{Ł}_1^{2^r - 1} \times \mathbf{Ł}_2^{3^r - 2^r}$.

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