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ORIGINAL PAPER

Indecomposability of free algebras in some subvarieties of residuated lattices and their bounded subreducts

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Abstract In this paper, we show that free algebras in the variety of residuated lattices and some of its subvarieties are directly indecomposable and show, as a consequence, the direct indecomposability of free algebras for some classes of their bounded implicative subreducts.

Keywords Residuated lattices · Free algebras · Subvarieties · Bounded BCK-algebras

1 Introduction

The variety of residuated lattices is the algebraic counterpart of Full-Lambek logic without contraction ($\mathbf{FL_{ew}}$ for short) and the subvarieties of the former correspond to the axiomatic extensions of the latter. In fact, free residuated lattices are the Lindenbaum algebras of the corresponding extensions. Free algebras are also important in their own right for their algebraic properties, e.g., they contain all the equational information of its generated variety. A natural approach to understanding the structure of an algebra consists of decomposing it, whenever possible, into a (direct) Boolean product of indecomposable algebras, thus reducing the problem to the study of those indecomposable stalks. In particular, the decomposability of a residuated lattice depends on the existence of Boolean elements in its lattice reduct. Therefore, the main objective of this article is the study of Boolean elements in free residuated lattices.

In the first section we collect some basic results about residuated lattices and their implicative filters. In Sect. 2, we review some well-known facts about complemented elements in residuated lattices and their relation to decomposability. We also introduce the notion of *almost complemented* element, and we see some cases in which almost complemented elements coincide with complemented elements. Sections 3 and 4 are the core of the article. We derive the indecomposability of the free residuated lattice using the disjunction property and we extend this result to several subvarieties of residuated lattices. In the last section, we concentrate on extending these indecomposability results to the corresponding $\langle\to,0,1\rangle$-subreducts.

We assume that the reader has some familiarity with residuated lattices and universal algebra. For residuated lattices we recommend Höhle (1995), Kowalski and Ono (2001) and Galatos et al. (2007) and the references given there, and for universal algebra we follow the nomenclature given in Burris and Sankappanavar (1981).

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2 Preliminaries

We start by reviewing some facts about residuated lattices and, in doing so, we set the notation used throughout the article.

An algebra $\boldsymbol{A} = \langle A,\wedge,\vee,\cdot,\to,0,1\rangle$ of type $(2,2,2,2,0,0)$ is a *commutative bounded integral residuated lattice* (briefly *residuated lattice*) provided that:

1. $\boldsymbol{L}(\boldsymbol{A}) = \langle A,\wedge,\vee,0,1\rangle$ is a *bounded lattice* with least element 0 and greatest element 1,

2. $\langle A, \cdot, 1\rangle$ is a *commutative monoid*,
3. for every $a, b, c \in A$, $a \cdot b \leq c$ if and only if $a \leq b \to c$ (residuation condition).

It is easy to see that for any $a, b \in A$, $a \to b$ is uniquely determined by condition (3), in fact,

$$a \to b = \max\{c \in A : c \cdot a \leq b\}.$$

It is well known that residuated lattices form a variety, which we shall denote by $\mathbb{RL}$. Indeed, the residuation condition can be replaced by the following equations:

$$x \approx x \wedge (y \to ((x \cdot y) \vee z)), \tag{1}$$

$$z \approx (y \cdot (x \wedge (y \to z))) \vee z. \tag{2}$$

Following (Monteiro 1980), residuated lattices satisfying the equation $x \cdot y \approx x \wedge y$ will be called *Heyting algebras*. We write $\mathbb{H}$ to denote the variety of all Heyting algebras. Another important subvariety of $\mathbb{RL}$ is the class $\mathbb{MTL}$ of all *MTL-algebras*, i.e., residuated latices satisfying the identity:

$$(x \to y) \vee (y \to x) \approx 1. \tag{3}$$

It is well known that $\mathbb{MTL}$ is the subvariety of $\mathbb{RL}$ generated by totally ordered residuated lattices. More precisely, a residuated lattice satisfies equation (3) if and only if it is a subdirect product of totally ordered residuated lattices (see Höhle 1995 and Esteva and Godo 2001 for details).

In the next lemma we list some general properties of residuated lattices which will be useful later.

Lemma 1.1 *The following properties hold true in any residuated lattice* $\mathbf{A}$, *where* a, b, c *denote arbitrary elements of* A:

1. $a \leq b$ *if and only if* $a \to b = 1$,
2. $1 \to a = a$,
3. $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,
4. $(a \cdot b) \to c = a \to (b \to c)$,
5. $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,
6. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$,
7. $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$. □

On a residuated lattice $\mathbf{A}$ we consider the unary operation:

$$\neg x := x \to 0, \quad \text{for all } x \in A. \tag{4}$$

Since 0 is the least element in $\mathbf{A}$, then the properties listed in the next lemma are easily provable.

Lemma 1.2 *Let* $\mathbf{A}$ *be a residuated lattice, then for any* $a, b \in A$ *we have*:

1. *if* $a \leq b$, *then* $\neg b \leq \neg a$,
2. $a \leq \neg\neg a$,
3. $\neg a = \neg\neg\neg a$,
4. $\neg(a \cdot b) = \neg(\neg\neg a \cdot \neg\neg b) = a \to \neg b$,
5. $\neg a \cdot a = 0$. □

An *involutive residuated lattice*, also called *integral commutative Girard monoid* (see Höhle 1995), is a residuated lattice satisfying the identity

$$\neg\neg x \approx x. \tag{5}$$

It follows from (4) in Lemma 1.2 that in an involutive residuated lattice the operations $\cdot$ and $\to$ are related as follows:

$$x \cdot y = \neg(x \to \neg y), \tag{6}$$

$$x \to y = \neg(x \cdot \neg y) = \neg y \to \neg x. \tag{7}$$

The class of all involutive residuated lattices is denoted by $\mathbb{IRL}$.

To simplify notation we shall use the following terms, defined recursively:

- $x^0 = 1$, $0x = 0$,
- For any $n \geq 0$, $x^{n+1} = x \cdot x^n$, and $(n+1)x = \neg(\neg x \cdot \neg nx)$.

It is easy to check the properties listed in the next lemma.

Lemma 1.3 *If* $\mathbf{A}$ *is a residuated lattice, then for any* $a, b \in A$, *we have*:

1. *for any* $n \geq 1$, $a^n \to b = \underbrace{a \to (a \to \cdots \to (a}_{n \text{ times}} \to b))$,
2. $a^1 = a$, $1a = \neg\neg a$,
3. *for any* $n \geq 1$, $\neg\neg na = na = n(\neg\neg a) = \neg(\neg a)^n$,
4. *if* $0 \leq k \leq r$, *then* $a^r \leq a^k$ *and* $ka \leq ra$,
5. *if* $a \leq b$, *then for any* $n \geq 0$, $a^n \leq b^n$ *and* $na \leq nb$,
6. *for any* $n \geq 1$, $(a \vee b)^n = \bigvee_{r+k=n} a^k \cdot b^r$. □

By an *implicative filter* or *i-filter* of a residuated lattice $\mathbf{A}$, we mean a subset $F \subseteq A$ satisfying the following conditions:

(F1) $1 \in F$,
(F2) for all $a, b \in A$, if $a \in F$ and $a \leq b$, then $b \in F$,
(F3) if $a, b \in F$, then $a \cdot b \in F$.

Alternatively, *i*-filters may be defined as subsets F of A satisfying **(F1)** and

(F4) if $a, a \to b$ are in F, then $b \in F$.

The family of all *i*-filters of $\mathbf{A}$, which we denote by $\mathcal{F}_i(\mathbf{A})$, is closed under arbitrary intersections and under union of upward directed subfamilies. It is easy to prove that for each $X \subseteq A$, the subset

$$\langle X\rangle = \{a \in A : x_1^{n_1} \cdot \ldots \cdot x_k^{n_k} \leq a,\ k, n_1, \ldots, n_k \geq 0, x_1, \ldots, x_k \in X\} \tag{8}$$

is the smallest *i*-filter containing X, i.e., the intersection of all *i*-filters containing X. For each $x \in A$, we shall write $\langle x\rangle$ instead of $\langle\{x\}\rangle$.

Given an i-filter F of a residuated lattice $\mathbf{A}$, the binary relation

$$\theta(F) := \{(x,y) \in A \times A : x \to y \in F \text{ and } y \to x \in F\}$$

is a congruence on $\mathbf{A}$ such that $F = 1/\theta(F)$, the equivalence class of 1. As a matter of fact, the correspondence $F \mapsto \theta(F)$ is an order isomorphism from $\mathcal{F}_i(\mathbf{A})$ onto $\mathrm{Con}(\mathbf{A})$, the set of all congruence relations of $\mathbf{A}$, both ordered by inclusion. Its inverse is given by the correspondence $\theta \mapsto 1/\theta$. We will write $\mathbf{A}/F$ instead of $\mathbf{A}/\theta(F)$, and a/F instead of $a/\theta(F)$, the equivalence class determined by $a \in A$.

An i-filter F of $\mathbf{A}$ is *proper* provided $F \neq A$. A *maximal i-filter* is a proper i-filter F of $\mathbf{A}$ such that for each i-filter G of $\mathbf{A}$, $F \subsetneq G$ implies $G = A$. We recall the following simple characterization of maximal i-filters.

Lemma 1.4 *A proper i-filter F of $\mathbf{A} \in \mathbb{RL}$ is maximal if and only if for any $a \in A$,*

- $a \notin F$ if and only if there is $n > 0$ such that $\neg a^n \in F$. □

Recall that an algebra is called *simple* provided that it has exactly two congruences (the trivial ones). Hence, a nontrivial residuated lattice $\mathbf{A}$ is simple if and only if $\{1\}$ is the only proper i-filter. By Lemma 1.4, we have the following characterization of simple residuated lattices.

Corollary 1.5 *$\mathbf{A} \in \mathbb{RL}$ is simple if and only if for every $a \in A \backslash \{1\}$, there is $n > 0$ such that $\neg a^n = 1$.* □

It is well known that for each $\mathbf{A} \in \mathbb{RL}$

$$\mathrm{Rad}(\mathbf{A}) = \{a \in A : \forall n > 0, \exists k_n \text{ such that } k_n(a^n) = 1\} \tag{9}$$

is the intersection of all maximal i-filters of $\mathbf{A}$ (see for instance Höhle 1995 and Kowalski and Ono 2001).

The next result, which we will need later, is easy to prove.

Lemma 1.6

(a) *If $h : \mathbf{A}_1 \to \mathbf{A}_2$ is a homomorphism of residuated lattices, then $h[\mathrm{Rad}(\mathbf{A}_1)] \subseteq \mathrm{Rad}(\mathbf{A}_2)$.*
(b) *For any $F \in \mathcal{F}_i(\mathbf{A}), \mathrm{Rad}(\mathbf{A})/F \subseteq \mathrm{Rad}(\mathbf{A}/F)$.* □

An algebra is *semisimple* provided that it is nontrivial and isomorphic to a subdirect product of simple algebras. Thus we have:

Corollary 1.7 *A residuated lattice $\mathbf{A}$ is semisimple if and only if $\mathrm{Rad}(\mathbf{A}) = \{1\}$.* □

3 Complemented elements and semisimplicity

In this section we review the relation between direct decomposability of a residuated lattice and the existence of complemented elements. We also introduce the weaker notion of *almost complemented* element which will be useful in the study of decomposability for subreducts of residuated lattices.

Given a residuated lattice $\mathbf{A}$, $B(\mathbf{A})$ will denote the set of complemented elements of the bounded lattice $\mathbf{L}(\mathbf{A})$. That is, $x \in B(\mathbf{A})$ if and only if there is $y \in A$ such that $x \vee y = 1$ and $x \wedge y = 0$. It is shown in Kowalski and Ono (2001) that $B(\mathbf{A})$ is the universe of a subalgebra of $\mathbf{A}$, denoted $\mathbf{B}(\mathbf{A})$, which is the greatest Boolean algebra contained in $\mathbf{A}$. It is easy to see that if $a \in B(\mathbf{A})$, then $\neg a$ is the only complement of a, and for any $b \in A$ we have that $a \cdot b = a \wedge b$ and for any $c \in B(\mathbf{A})$ $a \to c = \neg a \vee c$. We may also prove that $a \in B(\mathbf{A})$ if and only if $a \vee \neg a = 1$.

In fact, in residuated lattices, complemented elements correspond to factor congruences. A congruence θ of an algebra $\mathbf{A}$ is called *factor*, provided that there is a congruence relation θ' of $\mathbf{A}$, such that $\theta \cap \theta' = \Delta$, the identity, and $\theta \circ \theta' = A^2$. Then $\{\theta, \theta'\}$ is called a factor pair and $\mathbf{A} \cong \mathbf{A}/\theta \times \mathbf{A}/\theta'$. In general, an algebra $\mathbf{A}$ is called *directly indecomposable* if it has more than one element and whenever it is isomorphic to a direct product of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, then either $\mathbf{A}_1$ or $\mathbf{A}_2$ is the trivial algebra with just one element. As a matter of fact, the only factor congruences are the trivial ones. Now, θ is a factor congruence of a residuated lattice $\mathbf{A}$ if and only if there is $a \in B(\mathbf{A})$, such that $\theta = \theta(\langle a \rangle)$. Thus, we have the following result (cf. Kowalski and Ono 2001, Proposition 1.5).

Lemma 2.1 *A residuated lattice $\mathbf{A}$ is directly indecomposable if and only if $\mathbf{B}(\mathbf{A})$ is the two-element Boolean algebra.* □

Lemma 2.2 *If $\mathbf{A}$ is a residuated lattice, then:*

(a) $B(\mathbf{A}) \cap \mathrm{Rad}(\mathbf{A}) = \{1\}$,
(b) $B(\mathbf{A})/\mathrm{Rad}(\mathbf{A}) \subseteq B(\mathbf{A}/\mathrm{Rad}(\mathbf{A}))$,
(c) $\eta_A : a \mapsto a/\mathrm{Rad}(\mathbf{A})$ *is an embedding from* $\mathbf{B}(\mathbf{A})$ into $\mathbf{B}(\mathbf{A}/\mathrm{Rad}(\mathbf{A}))$.

Proof

(a) If $b \in B(\mathbf{A})$, then for any $k, n \geq 1$, $k(b^n) = b$. Hence, $b \in \mathrm{Rad}(\mathbf{A})$ implies $b = 1$.
(b) It follows from the fact that any homomorphism preserves Boolean elements.
(c) Consider $a, b \in B(\mathbf{A})$. If $a/\mathrm{Rad}(\mathbf{A}) = b/\mathrm{Rad}(\mathbf{A})$, then $a \to b$ and $b \to a$ both belong to $\mathrm{Rad}(\mathbf{A}) \cap B(\mathbf{A}) = \{1\}$, so $a = b$. □

As an easy consequence of these properties, we have the following result which will be important in determining the indecomposability of some free residuated lattices.

Corollary 2.3 *Let* $\boldsymbol{A}$ *be a residuated lattice. If* $\boldsymbol{A}/\mathrm{Rad}(\boldsymbol{A})$ *is directly indecomposable, then* $\boldsymbol{A}$ *is directly indecomposable.*

An element a of a residuated lattice $\boldsymbol{A}$ is called *almost complemented* provided that it satisfies:

$$a \to \neg a = \neg a \quad \text{and} \quad \neg a \to a = a. \tag{10}$$

It is clear that any complemented element is almost complemented. The following facts justify the above definition.

Lemma 2.4 *Let* $\boldsymbol{A}$ *be a residuated lattice, and let* $a \in A$ *be almost complemented. Then*

(a) $\neg\neg a = a$, and for all $n > 0, \neg a^n = \neg a$ and $na = a$,
(b) $a \vee \neg a \in \mathrm{Rad}(\boldsymbol{A})$.

Proof

(a) Since by definition $\neg\neg a = \neg a \to 0$, and since $\to$ is monotone on the right argument, $\neg a \to 0 \leq \neg a \to a = a$. Then $\neg\neg a \leq a$, and hence, from Lemma 1.2 (2), we get $a = \neg\neg a$. We shall now proceed by induction. From the above, $1a = \neg\neg a = a$. Take $n > 0$ such that $na = a$, then

$$\begin{aligned}(n+1)a &= \neg(a \cdot \neg na) = \neg a \to \neg\neg na \\ &= \neg a \to \neg\neg a = \neg a \to a = a.\end{aligned}$$

Now assume that $\neg a^n = \neg a$, then we have

$$\neg a^{n+1} = \neg(a \cdot a^n) = a \to \neg a^n = a \to \neg a = \neg a.$$

(b) Taking into account Lemma 1.3 (6), Lemma 1.2 (5), Lemma 1.1 (5) and the previous item, we deduce that for any $n \geq 1$:

$$\begin{aligned}\neg(a \vee \neg a)^n &= \neg(a^n \vee (\neg a)^n) = \neg a^n \wedge \neg(\neg a)^n \\ &= \neg a \wedge na = \neg a \wedge a.\end{aligned}$$

Thus for all $n \geq 1$,

$$\begin{aligned}2(a \vee \neg a)^n &= \neg(a \vee \neg a)^n \to \neg\neg(a \vee \neg a)^n \\ &= (a \wedge \neg a) \to \neg(a \wedge \neg a) \\ &= (a \wedge \neg a) \to \neg\neg(a \vee \neg a) = 1.\end{aligned}$$

Hence, $a \vee \neg a \in \mathrm{Rad}(\boldsymbol{A})$. □

From Corollary 1.7 and the above lemma we obtain:

Corollary 2.5 *If* $\boldsymbol{A}$ *is a semisimple residuated lattice, then any almost complemented element is complemented.* □

Another class of residuated lattices for which almost complemented elements are complemented are *MTL-algebras*.

Lemma 2.6 *If* $\boldsymbol{A} \in \mathbb{MTL}$, *then any almost complemented element is complemented.*

Proof This follows immediately since $\mathbb{MTL}$ satisfies the identity:

$$((x \to \neg x) \to \neg x) \wedge ((\neg x \to x) \to x) \approx x \vee \neg x.$$

□

There are examples of residuated lattices with almost complemented elements which are not complemented. For instance, consider the five-element Heyting algebra $\boldsymbol{H}_5 = \langle H_5 = \{0, a, b, c, 1\}, \wedge, \vee, \to, 0, 1\rangle$, whose lattice ordering is given by the Hasse diagram shown in Fig. 1 and whose residuum is given by the table also appearing in that figure.

Observe that $\mathrm{Rad}(\boldsymbol{H}_5) = \{c, 1\}$, hence $\boldsymbol{H}_5$ is not semisimple. Moreover, a and b are almost complemented, but they are not complemented. In fact, $\boldsymbol{H}_5$ is directly indecomposable.

Similarly, we could obtain distributive and/or involutive residuated lattices with almost complemented elements which are not complemented.

4 Free residuated lattices

In this section we use properties of the logic $\mathbf{FL_{ew}}$ and some of its axiomatic extensions to show the direct indecomposability of free residuated lattices. We also extend this result to some other subvarieties.

We first observe that $\mathbf{FL_{ew}}$ (the logic defined by the Full-Lambek sequent calculus without contraction) is strongly algebraizable in the sense of Blok and Pigozzi (1989) and its equivalent algebraic semantics is the variety $\mathbb{RL}$. Moreover, the subvarieties of $\mathbb{RL}$ correspond to the axiomatic extensions of $\mathbf{FL_{ew}}$ (see Komori and Ono 1985, Kowalski and Ono 2001 and Galatos et al. 2007 for example).

In what follows, given a quasivariety $\mathbb{Q}$ we will denote by $\boldsymbol{F}_{\mathbb{Q}}(X)$ its $|X|$-free algebra, where X is a set of free generators with cardinality $|X|$. The aim of this section is to prove the direct indecomposability of $\boldsymbol{F}_{\mathbb{RL}}(X)$ and to extend this result to several of its subvarieties.

Formulas in $\mathbf{FL_{ew}}$ are built, recursively, from a (denumerable) set X of propositional variables, the constant 0 and the binary connectives $\wedge, \vee, \cdot, \to$. The negation and truth constant are defined by $\neg\alpha = \alpha \to 0$ and $1 = \neg 0$. In fact, we can identify the algebra of formulas with the

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

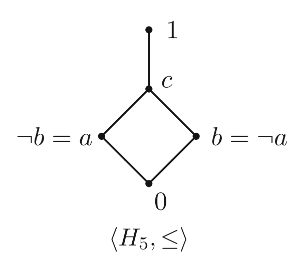

Fig. 1 A Heyting algebra

term-algebra of the language of residuated lattices. It follows from the algebraization of $\mathbf{FL_{ew}}$ that a formula φ is provable in an axiomatic extension $\mathbf{G}$ of $\mathbf{FL_{ew}}$ if and only if the equation $\varphi \approx 1$ holds in its equivalent algebraic semantics $\mathbb{V}_{\mathbf{G}}$, the subvariety of $\mathbb{RL}$ given by the equations $\{\psi \approx 1 : \psi$ is an axiom of $\mathbf{G}\}$. Moreover, the $|X|$-free algebra $\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)$ is the Lindenbaum–Tarski algebra of $\mathbf{G}$.

An axiomatic extension $\mathbf{G}$ of $\mathbf{FL_{ew}}$ has the *disjunction property*, if for any formula α and $\beta, \alpha \vee \beta$ is provable in $\mathbf{G}$ if and only if either α or β is provable in $\mathbf{G}$. This property has an algebraic version: the greatest element 1 is join-irreducible in $\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)$, that is, if $1 = x \vee y$, then $x = 1$ or $y = 1$.

Theorem 3.1 *Let* $\mathbf{G}$ *be an axiomatic extension of* $\mathbf{FL_{ew}}$ *having the disjunction property. Then for any set of variables* $X, \boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)$ *is directly indecomposable.*

Proof Let a be a complemented element in $\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)$. Hence, $a \vee \neg a = 1$. Without loss of generality we can assume that $a \in \boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(Y)$, with $Y \subseteq X$ and Y denumerable. Since 1 is join irreducible in $\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(Y)$, then $a = 1$ or $\neg a = 1$ and so $a \in \{1, 0\}$. This shows that $B(\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)) = \{0, 1\}$ and, by Lemma 2.1, $\boldsymbol{F}_{\mathbb{V}_{\mathrm{G}}}(X)$ is directly indecomposable. □

It is known that $\mathbf{FL_{ew}}$ has the disjunction property, as well as $\mathbf{InFL_{ew}}$, involutive $\mathbf{FL_{ew}}$ denoted $\mathbf{GL}^0$ in Grišin (1981), and the Intuitionistic Propositional Logic **IPC** (see Galatos et al. 2007, Sect. 5.1.1 and Suoma 2007 for example). Since the equivalent algebraic semantics of $\mathbf{InFL_{ew}}$ is $\mathbb{IRL}$, the variety of involutive residuated lattices, and the equivalent algebraic semantics of **IPC** is $\mathbb{H}$, the variety of Heyting algebras, we have:

Theorem 3.2 *The varieties* $\mathbb{RL}, \mathbb{IRL}$ *and* $\mathbb{H}$ *have all their free algebras directly indecomposable.* □

We recall that the varieties $\mathbb{RL}$ and $\mathbb{IRL}$ have all their free algebras semisimple (see Kowalski and Ono 2000, Galatos et al. 2007 and Grišin 1981). However, $\mathbb{H}$ has no semisimple free algebras, because semisimple Heyting algebras are just Boolean algebras.

We know some other varieties of residuated lattices whose free algebras are directly indecomposable. An example is the variety $\mathbb{MV}$ of all *MV*-algebras, which can be defined as involutive residuated lattices satisfying the following identity (see Cignoli and Torrens 2003):

$$x \cdot (x \to y) \approx y \cdot (y \to x), \tag{11}$$

or equivalently, residuated lattices satisfying

$$(x \to y) \to y \approx (y \to x) \to x. \tag{12}$$

Then, it is well known that every free *MV*-algebra is semisimple and directly indecomposable (see for instance Cignoli and Torrens 2003).

Observe that free algebras in the varieties $\mathbb{RL}, \mathbb{IRL}$ and $\mathbb{MV}$ are semisimple, but the varieties themselves are not semisimple, because they contain algebras which are not semisimple.

5 Fully invariant congruences and decomposability

Given an algebra $\boldsymbol{A}$, we say that a congruence relation θ on $\boldsymbol{A}$ is *fully invariant* if, for any endomorphism $h : \boldsymbol{A} \to \boldsymbol{A}$, it satisfies:

$(a, b) \in \theta$ implies $(h(a), h(b)) \in \theta$.

If $\theta(F)$ is fully invariant for an *i*-filter F, we shall also say that F is fully invariant.

Lemma 4.1 *Let* $\boldsymbol{A}$ *be a residuated lattice, then:*

(a) *an i-filter* F *is fully invariant if and only if for any endomorphism* $h : \boldsymbol{A} \to \boldsymbol{A}$, *we have* $h[F] \subseteq F$,
(b) $\mathrm{Rad}(\boldsymbol{A})$ *is fully invariant.*

Proof

(a) Let $h : \boldsymbol{A} \to \boldsymbol{A}$ be an endomorphism. If $\theta(F)$ is fully invariant and $a \in F$, then $(a, 1) \in \theta(F)$ and so $(h(a), 1) \in \theta(F)$; hence $h(a) \in F$. Consequently, $h[F] \subseteq F$. Conversely, if $h[F] \subseteq F$ and $(a, b) \in \theta(F)$, then $a \to b, b \to a \in F$. Hence, $h(a) \to h(b) = h(a \to b), h(b) \to h(a) = h(b \to a) \in F$, and so $(h(a), h(b)) \in \theta(F)$.
(b) It follows from item (a) and Lemma 1.6 (a).

□

We will use the following property of fully invariant congruences. For a more detailed study of the subject we refer the reader to Grätzer (1979), Chap. 4 and Burris and Sankappanavar (1981), Chap. II, Sect. 14. If H, S and P stand, respectively, for closure under homomorphic images, subalgebras and direct products, then

Lemma 4.2 *Let* $\mathbb{V}$ *be a variety and let* θ *be a fully invariant congruence relation on* $\boldsymbol{F}_{\mathbb{V}}(X)$. *Then* $\boldsymbol{F}_{\mathbb{V}}(X)/\theta$ *is a free algebra in* $HSP(\boldsymbol{F}_{\mathbb{V}}(X)/\theta)$, *with* $\{x/\theta : x \in X\}$ *as its set of free generators. Moreover,* $|\{x/\theta : x \in X\}| = |X|$. □

Let $|X| = \aleph_0$, i.e., $X = \{x_n : n \in \omega\}$ is a denumerable set. If, for any variety $\mathbb{V}$, $\boldsymbol{F}_{\mathbb{V}}(\omega)$ stands for $\boldsymbol{F}_{\mathbb{V}}(X)$, then $\mathbb{V} = \mathbb{HSP}(\boldsymbol{F}_{\mathbb{V}}(\omega))$, that is, $\mathbb{V}$ is the variety generated by $\boldsymbol{F}_{\mathbb{V}}(\omega)$. By Lemma 4.1, for any variety $\mathbb{V}$ of residuated lattices, $\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(\omega))$ is fully invariant. Consequently, by Lemma 4.2, if we consider the variety

$$\mathbb{V}^S = HSP(\boldsymbol{F}_{\mathbb{V}}(\omega)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(\omega))),$$

then $\boldsymbol{F}_{\mathbb{V}}(\omega)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(\omega))$ can be taken to be $\boldsymbol{F}_{\mathbb{V}^S}(\omega)$.

If $\mathbb{V}_{sim}$ is the class of all simple algebras in the variety $\mathbb{V}$, we have the following characterization of $\mathbb{V}^S$.

Theorem 4.3 *Let $\mathbb{V}$ be a subvariety of $\mathbb{RL}$. Then:*

(a) *for any set X, $\boldsymbol{F}_{\mathbb{V}}(X)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X))$ is the $|X|$-free algebra of the variety $HSP(\mathbb{V}_{sim})$,*

(b) $\mathbb{V}^S = HSP(\mathbb{V}_{sim})$.

Proof

(a) Let X be a set of free generators. Since $\boldsymbol{F}_{\mathbb{V}}(X)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X))$ is semisimple, it belongs to $HSP(\mathbb{V}_{sim})$. Let $\boldsymbol{S} \in \mathbb{V}_{sim}$. Given a map

$$h : \{x/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X)) : x \in X\} \to S,$$

we define $h_0 : X \to S$ by $h_0(x) := h(x/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X))$. Let H_0 be the homomorphism from $\boldsymbol{F}_{\mathbb{V}}(X)$ to $\boldsymbol{S}$ which extends h_0. Since $\mathrm{Rad}(\boldsymbol{S}) = \{1\}$, by Lemma 1.6 (a), $\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X)) \subseteq 1/(\ker H_0)$. Hence, H_0 induces a homomorphism $H : \boldsymbol{F}_{\mathbb{V}}(X)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X)) \to \boldsymbol{S}$ such that $H(a/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X))) = H_0(a)$ and clearly this homomorphism extends h. Thus, $\boldsymbol{F}_{\mathbb{V}}(X)/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X))$ is free in $HSP(\mathbb{V}_{sim})$ over $\{x/\mathrm{Rad}(\boldsymbol{F}_{\mathbb{V}}(X)) : x \in X\}$.

(b) It follows from (*a*) and the remarks given above.

□

We say that a variety is *free-indecomposable* if all its free algebras are directly indecomposable. Then from the above theorem and Corollary 2.3, we get the following result concerning indecomposability of free algebras.

Corollary 4.4 *Let $\mathbb{V}$ be a subvariety of $\mathbb{RL}$, then if $\mathbb{V}^S$ is free-indecomposable, then $\mathbb{V}$ is free-indecomposable.* □

For instance, consider $\mathbb{GRL}$ the class of all Glivenko residuated lattices, i.e., residuated lattices satisfying the equation

$$\neg\neg(\neg\neg x \to x) \approx 1. \tag{13}$$

It follows from the results given in (Cignoli and Torrens 2004) that $\mathbb{GRL}^S = \mathbb{IRL}$, and since $\mathbb{IRL}$ is free-indecomposable, then $\mathbb{GRL}$ is also free-indecomposable.

If $\mathbb{BL}$ is the class of all *BL*-algebras, i.e., the class of residuated lattices satisfying Eqs. 11 and 3, then $\mathbb{BL}^S = \mathbb{MV}$, and so $\mathbb{BL}$ is free-indecomposable (cf. Cignoli and Torrens 2003).

6 Free bounded *BCK*-algebras

We now combine the knowledge of some free-indecomposable varieties of residuated lattices with the notion of almost complemented element to derive the free-indecomposability of some quasivarieties of the corresponding subreducts.

Given a class $\mathbb{K}$ of algebras in the language $\mathcal{L}$ and given a sublanguage $\mathcal{L}_0 \subseteq \mathcal{L}$, we denote by $\mathbb{K} \restriction \mathcal{L}_0$ the class of $\mathcal{L}_0$-reducts of algebras in $\mathbb{K}$. Therefore, $S(\mathbb{K} \restriction \mathcal{L}_0)$ is the class of $\mathcal{L}_0$-subreducts of algebras in $\mathbb{K}$. If we consider a quasivariety $\mathbb{Q}$, it is straightforward to verify that $S(\mathbb{Q} \restriction \mathcal{L}_0)$ is also a quasivariety. In addition, it is also easy to see that the free algebra in $S(\mathbb{Q} \restriction \mathcal{L}_0)$ over a set X of free generators is precisely the subalgebra of $\boldsymbol{F}_{\mathbb{Q}}(X) \restriction \mathcal{L}_0$ generated by X.

We will study the indecomposability of free algebras in quasivarieties of $\{\to, 0, 1\}$-subreducts of $\mathbb{RL}$. It is known that $S(\mathbb{RL} \restriction \{\to, 0, 1\})$ is the quasivariety $b\mathbb{BCK}$ of all bounded *BCK*-algebras (see for instance Idziak 1984, Cignoli and Torrens 2004, and the references given there). Bounded *BCK*-algebras can be defined as $\{\to, 0, 1\}$-algebras of type $(2, 0, 0)$ satisfying the following equations and quasiequation:

$$(x \to y) \to ((y \to z) \to (x \to z)) \approx 1 \tag{14}$$

$$1 \to x \approx x \tag{15}$$

$$x \to 1 \approx 1 \tag{16}$$

$$0 \to x \approx 1 \tag{17}$$

$$(x \to y \approx 1) \& (y \to x \approx 1) \Rightarrow x \approx y \tag{18}$$

Moreover, it is shown in (Wroński 1983) that $b\mathbb{BCK}$ is not a variety (see also Wroński 1983).

Given an $\{\to, 0, 1\}$-algebra $\boldsymbol{B}$, $\mathrm{Con}_{b\mathbb{BCK}}(\boldsymbol{B})$ will represent the family of all $b\mathbb{BCK}$-congruences, i.e., congruences θ of $\boldsymbol{B}$ such that $\boldsymbol{B}/\theta \in b\mathbb{BCK}$. Then for any $\boldsymbol{A} \in \mathbb{RL}$, it is easily verified that

$$\begin{aligned}\mathrm{Con}(\boldsymbol{A}) &= \{\theta \in \mathrm{Con}(\langle A, \to, 0, 1\rangle) : \langle A, \to, 0, 1\rangle/\theta \in b\mathbb{BCK}\} \\ &= \mathrm{Con}_{b\mathbb{BCK}}(\langle A, \to, 0, 1\rangle).\end{aligned}$$

Let $\boldsymbol{B}$ be a bounded *BCK*-algebra. An *i*-filter of $\boldsymbol{B}$ is a subset satisfying conditions (**F1**) and (**F4**) described after Lemma 1.3. Furthermore, the correspondence $F \mapsto \theta(F)$ defined after Lemma 1.3 gives an order isomorphism from the set of all *i*-filters of $\boldsymbol{B}$ onto $\mathrm{Con}_{b\mathbb{BCK}}(\boldsymbol{B})$, both ordered by inclusion. An element a in B is called *factor* if $\theta(\langle a\rangle)$ is a factor congruence of $\boldsymbol{B}$, where $\langle a\rangle$ is the *i*-filter generated by $\{a\}$. In fact, all factor congruences are of this form, i.e., any factor congruence is given by a factor element of $\boldsymbol{B}$ (see Gispert and Torrens 2008, Lemma 3.1 for details). Furthermore, factor elements in $\boldsymbol{B}$ form a Boolean algebra $B_F(\boldsymbol{B})$ which is a subalgebra of $\boldsymbol{B}$. In Gispert and Torrens (2008), Lemma 2.2, it is also shown that any factor element a in a bounded *BCK*-algebra satisfies $a \to \neg a = \neg a$ as well as $\neg a \to a = a$. These simple facts together with the following lemma will be useful in the sequel.

Lemma 5.1 (Gispert and Torrens 2008, Lemma 3.3) *A bounded BCK-algebra* $\boldsymbol{B}$ *is directly indecomposable if and only if the only factor elements are* $\{0,1\}$. □

Theorem 5.2 *Let* $\mathbb{Q}$ *be a subquasivariety of* $\mathbb{RL}$ *such that* $\boldsymbol{F}_{\mathbb{Q}}(X)$ *is directly indecomposable and semisimple. Then* $\boldsymbol{F}_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ *is directly indecomposable.*

Proof Suppose that $\boldsymbol{F}_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ is directly decomposable. Then there exists a factor element $a \in F_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ such that $a \neq 0,1$. This element satisfies $a \to \neg a = \neg a$ and $\neg a \to a = a$. Since $\boldsymbol{F}_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ is a $\{\to,0,1\}$-subalgebra of $\boldsymbol{F}_{\mathbb{Q}}(X)$, a is almost complemented in $\boldsymbol{F}_{\mathbb{Q}}(X)$. By Corollary 2.5, a is complemented which contradicts the indecomposability of $\boldsymbol{F}_{\mathbb{Q}}(X)$. Thus $\boldsymbol{F}_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ is directly indecomposable. □

For instance, since $\boldsymbol{F}_{\mathbb{RL}}(X)$ is semisimple and directly indecomposable, it follows that $\boldsymbol{F}_{b\mathbb{BCK}}(X)$ is directly indecomposable. Analogously, since $S(\mathbb{IRL}\restriction\{\to,0,1\}) = \mathbb{IBCK}$, the class of involutive *BCK*-algebras, we get that $\boldsymbol{F}_{\mathbb{IBCK}}(X)$ is directly indecomposable.

Lemma 2.6 gives us another source of free-indecomposable quasivarieties of $b\mathbb{BCK}$. The proof of the following theorem is analogous to the proof of Theorem 5.2.

Theorem 5.3 *Let* $\mathbb{Q}$ *be a subquasivariety of* $\mathbb{MTL}$ *such that* $\boldsymbol{F}_{\mathbb{Q}}(X)$ *is directly indecomposable. Then* $\boldsymbol{F}_{S(\mathbb{Q}\restriction\{\to,0,1\})}(X)$ *is directly indecomposable.*

Since $\mathbb{BL}$ is free-indecomposable, it follows from the above lemma that $S(\mathbb{BL}\restriction\{\to,0,1\})$ is free indecomposable. From the results of Aglianò et al. (2007) it is easy to prove that $S(\mathbb{BL}\restriction\{\to,0,1\})$ is the variety of all *bounded basic BCK-algebras*, i.e., bounded *BCK*-algebras satisfying the following equations:

$$(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z) \tag{19}$$

$$((x \to y) \to z) \to (((y \to x) \to z) \to z) \approx 1 \tag{20}$$

Indeed, it is immediate that any $\{\to,0,1\}$-subreduct of a *BL*-algebra is a bounded basic *BCK*-algebras. Conversely, given a bounded basic *BCK*-algebra $\boldsymbol{B} = \langle B,\to,0,1\rangle$, the reduct $\langle B,\to,1\rangle$ is a *basic BCK-algebra* (i.e., a *BCK*-algebra satisfying Eqs. 19 and 20). By the results in Aglianò et al. (2007), there exists a *BL*-algebra $\boldsymbol{C} = \langle C,\wedge,\vee,\cdot,\to,0',1\rangle$ such that $\langle B,\to,1\rangle$ is a subalgebra of $\boldsymbol{C}\restriction\{\to,1\}$. The element $0 \in B$ need not coincide with $0'$. However, we may consider the algebra $\boldsymbol{C}_0 = \langle [0),\wedge,\vee,\cdot_0,\to,0,1\rangle$ where $[0) = \{c \in C : 0 \leq c\}$ and

$$x \cdot_0 y = (x \cdot y) \vee 0, \qquad \text{for every } x,y \in [0).$$

It is straightforward to show that $\mathbf{C}_0$ is also a *BL*-algebra and that $\mathbf{B}$ is a subalgebra of $\mathbf{C}_0\restriction\{\to,0,1\}$.

Remark 5.4 Finally, we should remark that the procedure used in this article for bounded *BCK*-algebras is not applicable to *BCK*-algebras. The main reason is that, in general, in *BCK*-algebras the universal congruence is not principal.

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